Eigenfunction Local Coordinates and the Local Riemann mapping Theorem

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Many thanks to the IPAM Staff and the Program Organizers!

March 30, 2007

Outline of the Talk

1. Intro to Diffusion Geometry
2. Eigenfunctions and Coordinates
3.The Riemann Mapping Theorem
4 Statement of the Main Result
5. Heat Kernels
6. How to find good eigenfunctions

There is a long history to this type of result and many names have been unjustly deleted from the talk.

Eigenfunction Coordinates (Due to < Euclid)

Consider the circle with eigenfunctions sin(nθ) and cos(nθ). The eigenfunction sin(θ) provides coordinates on [π/2,-π/2]. But those coordinates are not Bi-Lipschitz because the derivative of sin is cos, which vanishes at π/2. However, sin(θ) is a Bi-Lipschitz homeo of

[-3π/8,π3/8] onto [sin(-3π/8), sin(3π/8)].

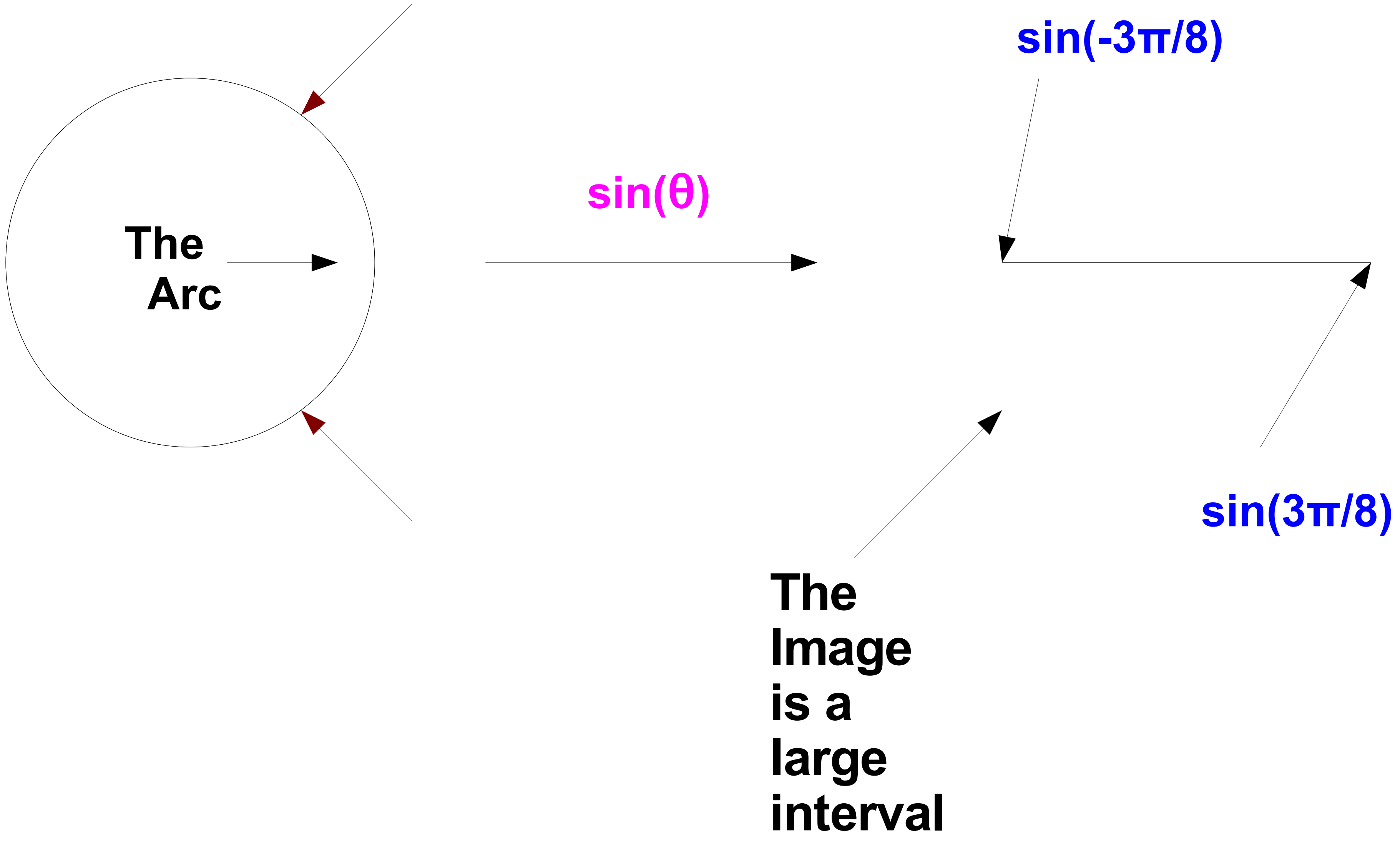
Using this and other coordinates from the cosine we easily cover the circle with 4 good coordinate patches. For simple reasons

No coordinate patch from the sines and cosines can cover more than ½ of the circle.

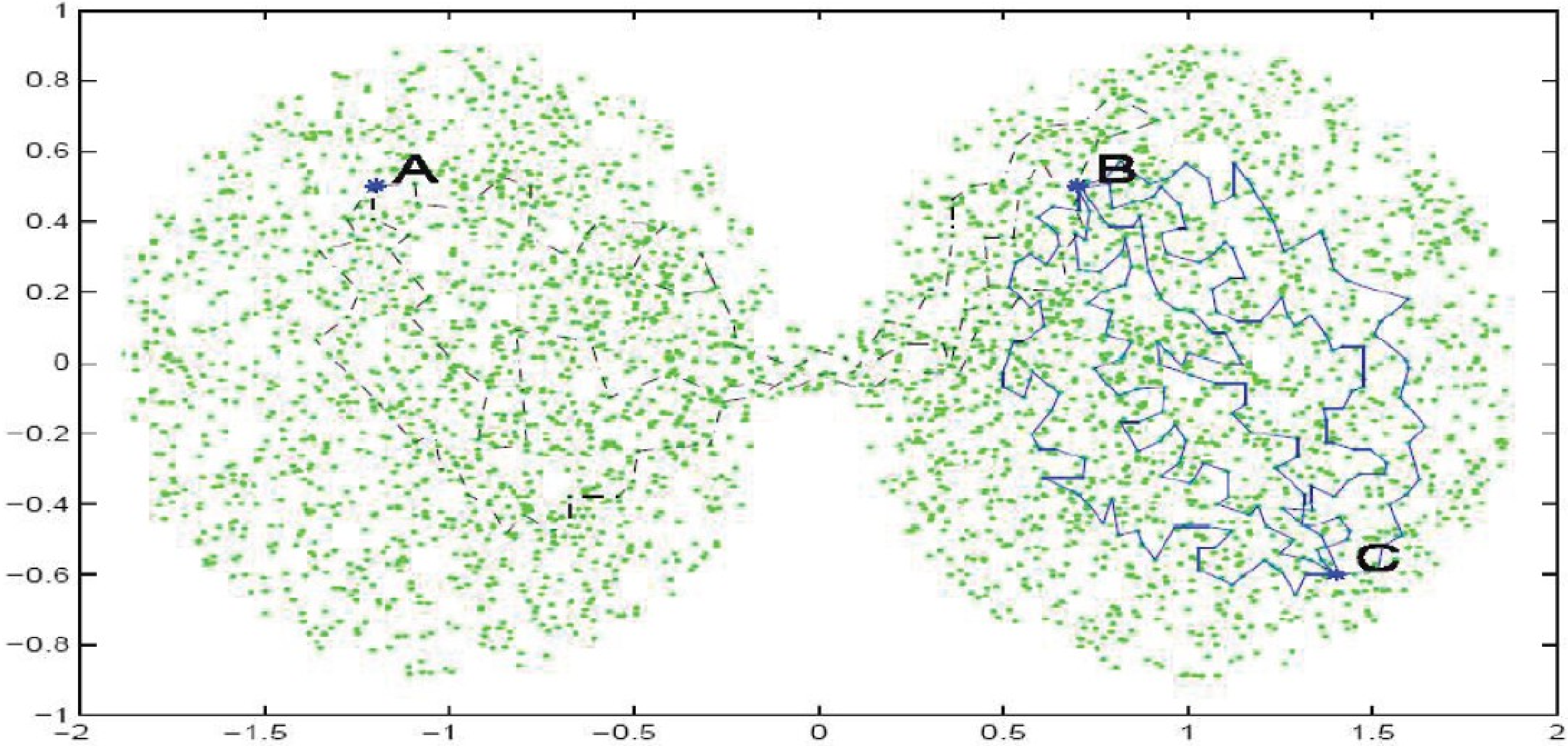
On the torus [0,1]x[0,1] the eigenfunctions of Laplace are (times 2) {sin(mπx)sin(nπy), sin(mπx)cos(nπy), cos(mπx)sin(nπy), cos(mπx)cos(nπy)}. Again we can use a small number of eigenfunction coordinate patches, e.g. (x,y) → (sin(πx),sin(πy)), to cover the torus, and

These are Bi-Lipschitz coordinates onto (essentially) the unit disk.
(We actually get squares.)

Pre-Euclidean Coordinates

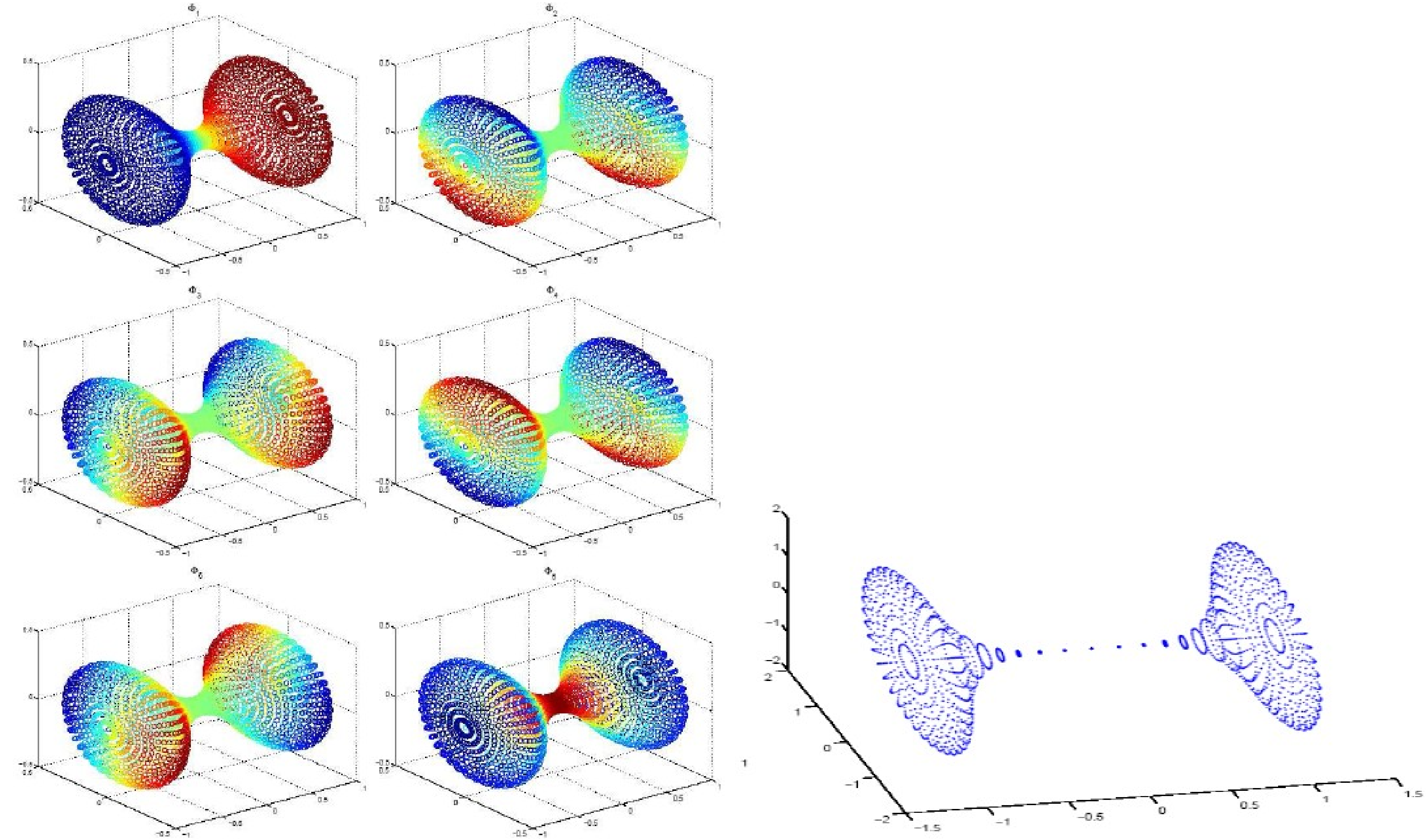


Diffusion vs. Geodesic Distance



$d_{geod.}(A,B) \sim d_{geod.}(C,B)$, however $d^{(t)}(A,B) >> d^{(t)}(C,B)$.

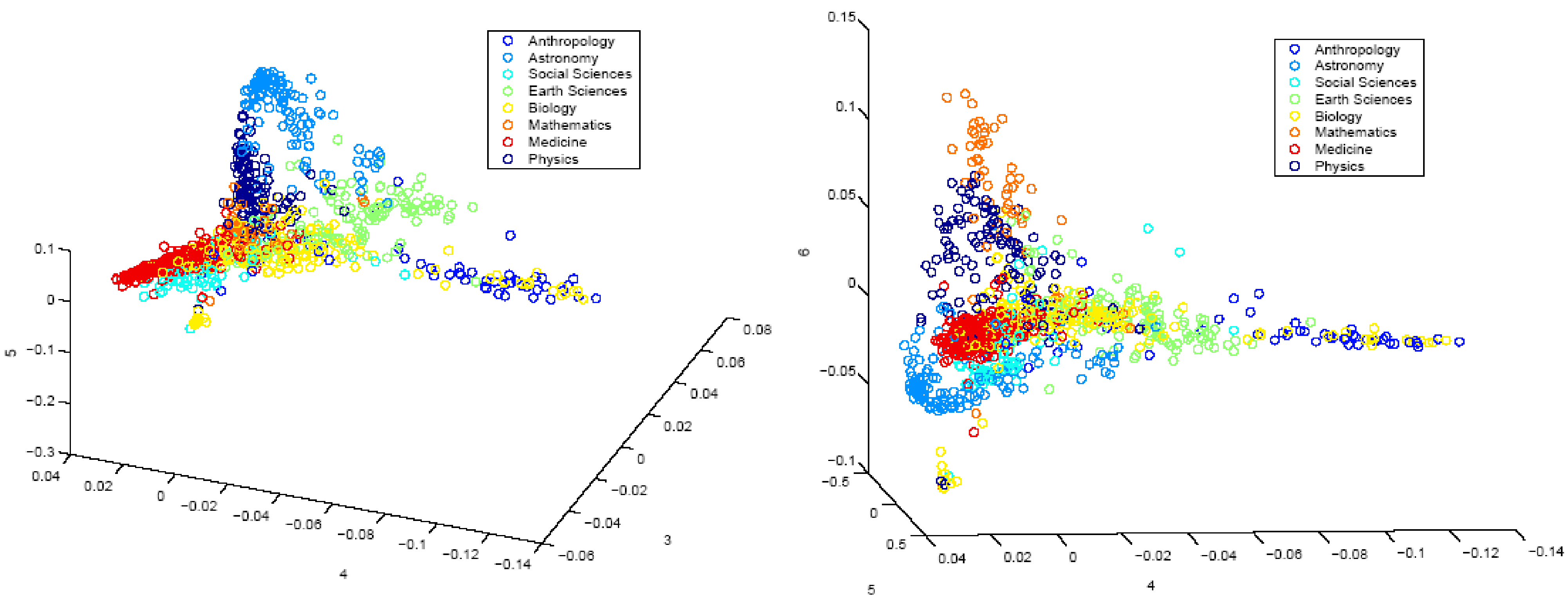
Diffusion maps: Example



Eigenfunctions on a dumbell-shaped manifold, and corresponding diffusion map; pictures courtesy of Stephane Lafon.

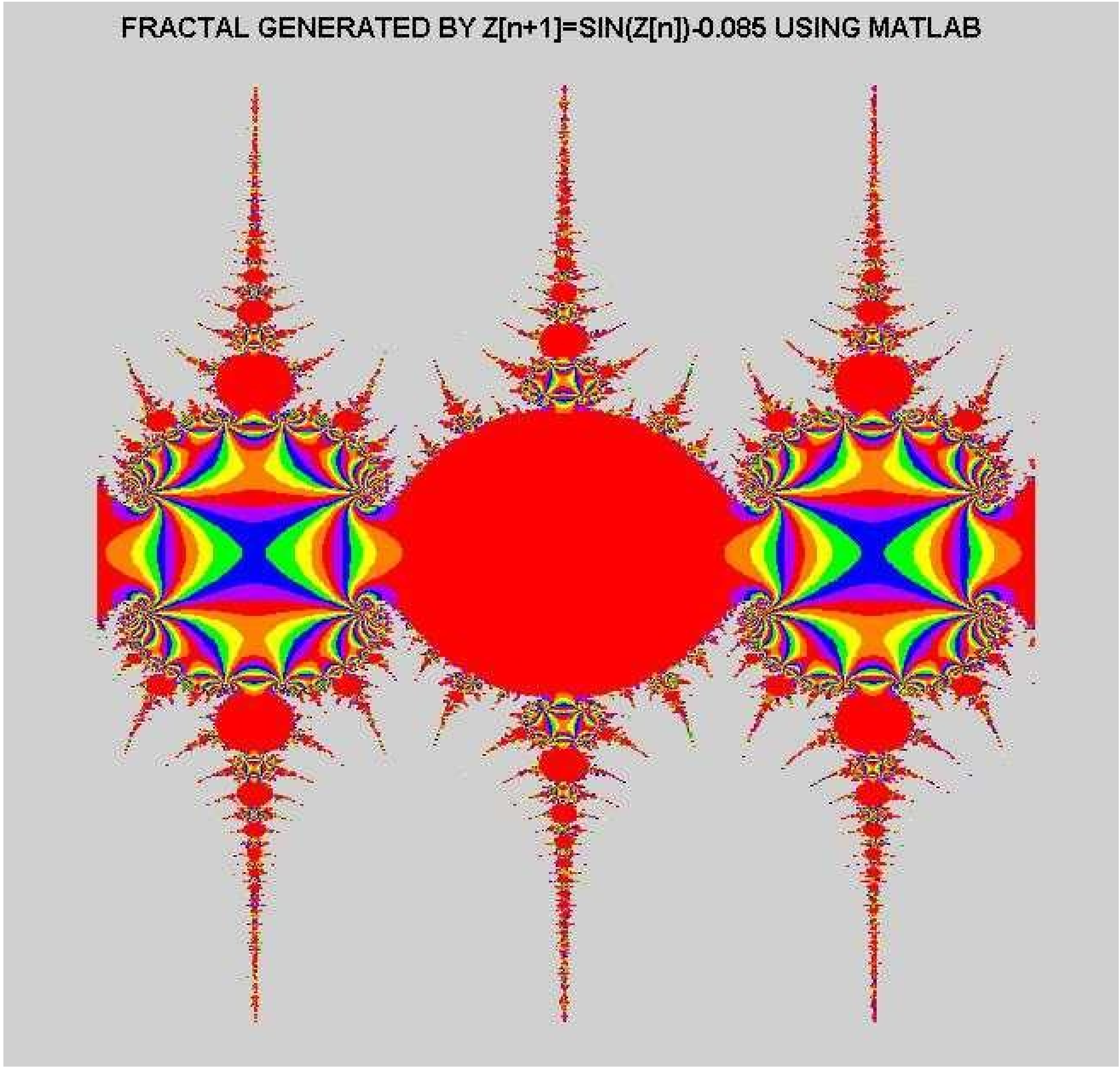
Application to text document classification

Consider about 1000 Science News articles, from 8 different categories. For each we compute about 10000 coordinates, the i-th coordinate of document d representing the frequency in document d of the i-th word in a fixed dictionary. The diffusion map gives the embedding below.



Embedding $\Xi_6^{(0)}(x) = (\xi_1(x), \ldots, \xi_6(x))$: on the left coordinates $3, 4, 5$, and on the right coordinates $4, 5, 6$.

Post-Euclidean Eigenfunction



The Riemann Mapping Theorem:

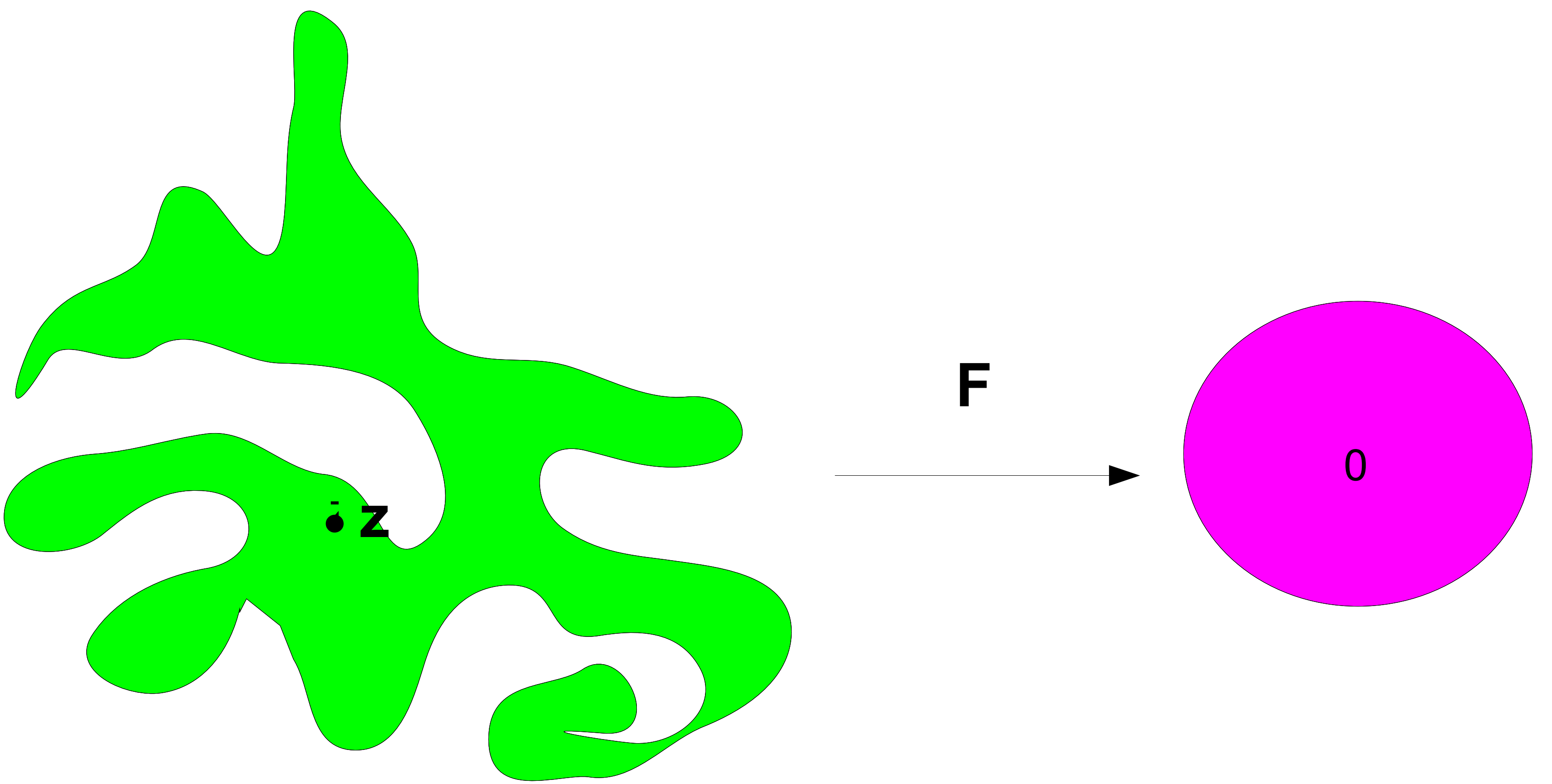
(For the Plane)

Let D be a simply connected domain. Then if z is any chosen point in D, there is a choice F of Riemann Mapping Function, such that

$$F: D \to B(0,1) \qquad (= \text{Unit Disk}),$$

and

$$F(z) = 0.$$



The Theorem (First Version -There is also a version for manifolds)

Let *D* be a domain in $\mathbb{R}^d$ with Volume ≤ 1. Let $B(x,r) \subseteq D$ be a ball of radius *r*. Then there are d choices of Laplace eigenfunctions,

$$\{\varphi_1, \varphi_2, \ldots\ldots\ldots, \varphi_d\} = \Phi, \text{ such that}$$

1. $\Phi : B(x,\epsilon r) \to \mathbb{R}^d$ is a local coordinate system.

2. $B(\Phi(x), \delta) \subseteq \Phi(B(x,\epsilon r))$. (Big image = almost unit Ball)

3. There are good Bi-Lipschitz estimates for Φ.

4. $\lambda_1,\ldots,\lambda_d \sim 1/r^2$

Here all constants are independent of D.

Estimates for the Heat Kernel = *K* on *D*

Suppose *x* is a point in *D* with distance *r* to the boundary. The since Brownian motion moves about distance *r* in time r^2, the heat kernel *K* for *D* satisfies

$$K(x, y, t) \sim k(x, y, t)$$

and

$$\nabla K(x, y, t) \sim \nabla k(x, y, t) \qquad (\nabla = \text{gradient})$$

when $t < c\, r^2$. Here *k* is the heat kernel for Euclidean space.

Proof: (This could be for either the Dirichlet or Neumann heat kernel.) Use the path integral representation of *K*.

"HEAT TRIANGULATION" (Also for manifolds)

Let D be a domain in $\mathbb{R}^d$ with Volume ≤ 1. Let $B(x,r) \subseteq D$ be a ball of radius r. Then there are d choices of points x_j (the d "cardinal points of the compass" at distance ~ r from x) such that for the heat kernels $K_j = K(x_j, y, t)$ with base at these points, and with time $t \sim r^2$,

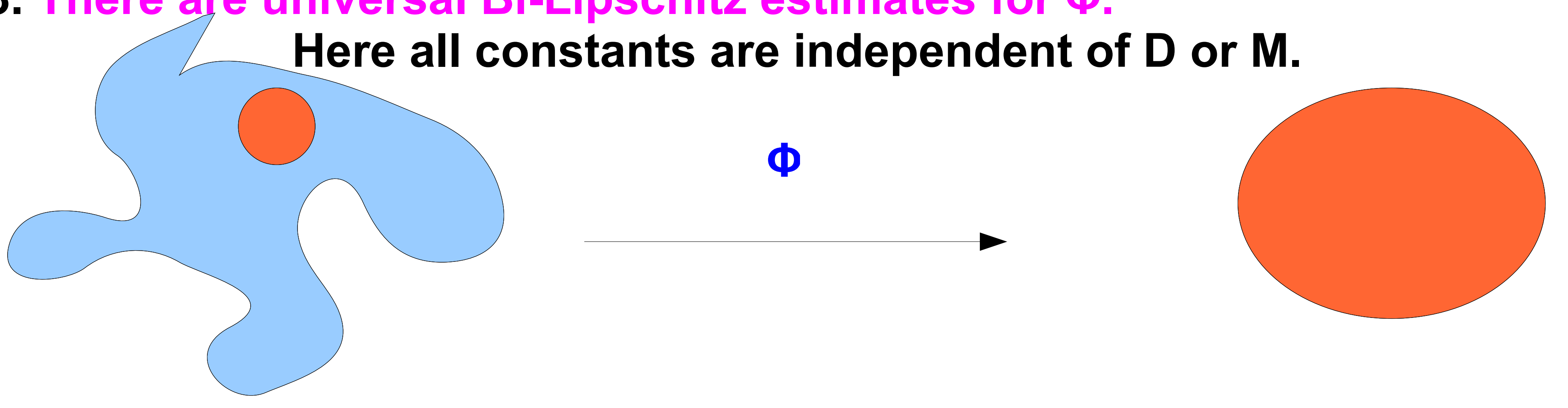
$$\{K_1, K_2, \ldots\ldots\ldots, K_d\} = \Phi$$

1. $\Phi : B(x, \epsilon r) \to \mathbb{R}^d$ is a local coordinate system.

2. The image ~ $B(0, r^{-d})$

3. There are universal Bi-Lipschitz estimates for Φ.

Here all constants are independent of D or M.



These results are also true in the **$C^{1+\alpha}$ category for manifolds** of volume 1. **Lipschitz is DIFFERENT**.

For the EIGENFUNCTION THEOREM, a careful statement of the distortion is that after multiplying DOWN, the map has universal distortion estimates (large dilation followed by universally controlled BiLipschitz constant). There are values

$$0 < \delta_j \leq 1 \text{ such that for}$$

$$\{\delta_1\varphi_1, \delta_2\varphi_2, \ldots\ldots\ldots, \delta_d\varphi_d\} = \Phi$$

$\Phi : B(x,\epsilon r) \rightarrow \mathbb{R}^d$ is a local coordinate system that blows up the ball of radius ~ ϵ to a ball of size ~ 1 WITH UNIVERSALLY BOUNDED DISTORTION ESTIMATES. Without the choices of δ_j it is possible (due to localization of eigenfunctions) that one blows up to an "ellipse" of large eccentricity. (This would give apparently bad distortion.)

CAN WE TAKE ALL THE $\delta_j = 1$?

Which Eigenfunctions are allowed?

Answer: One can use EITHER Dirichlet or Neumann Eigenfunctions. BUT WE ALWAYS USE L^2 NORM = 1.

Dirichlet: The eigenfunctions vanish on the boundary. (Same as studying Random Walk absorbed at the boundary.)

Neumann: (These are the ones used in Diffusion Geometry.) The normal derivative of the eigenfunctions vanish on the boundary. (Same as studying Random Walk reflecting off the boundary.)

Why do we need VOLUME (D) ≤ 1?

Example: The cube in dimension d with side-length $L >> 1$. The eigenfunctions in two dimensions are

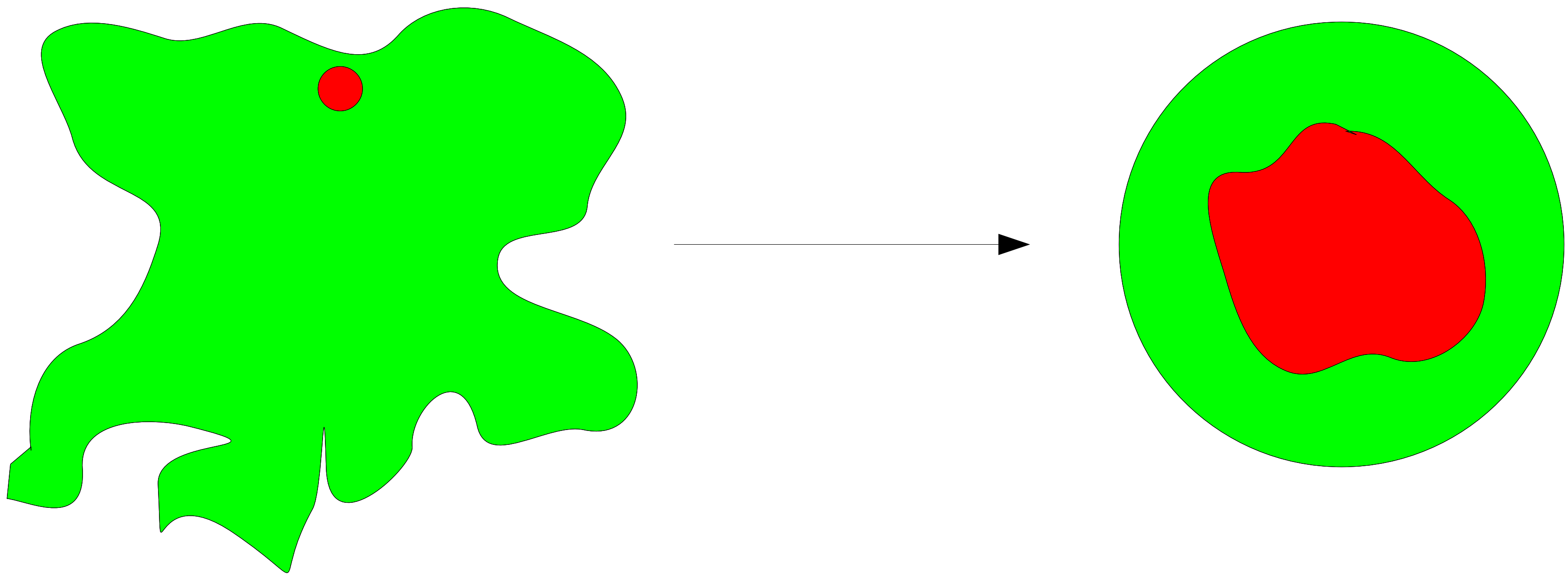
$$\{2L^{-1}\sin(m\pi x)\sin(n\pi y)\}.$$

These eigenfunctions are too small to satisfy the conclusions of the theorem. If $L \leq 1$ we are ok.
(The case of the torus is the same as is the Neumann case.)

The Distortion Theorem(s) (1920's)

If F is a Riemann Mapping from D to the Unit Disk , $F(z_0) = 0$, and r = distance(z_0 , boundary D), then F is a **BEAUTIFUL** Bi-Lipschitz mapping from $B(z_0, r/2)$ to a **BIG** region containing the origin. All constants are **universal**. (One result = Kobe ¼ Theorem)

($|F'| \sim 1/r$ on the small disk.)



Let's Reexamine Riemann

He certainly (?) knew this!

Log $|F(z)| = G(z, z_0)$ = Green's Function with pole at z_0. (This function ~ $\text{Log}|z - z_0|$ near z_0 and $= 0$ on bdry(D). Now apply the fundamental relation to the heat kernel:

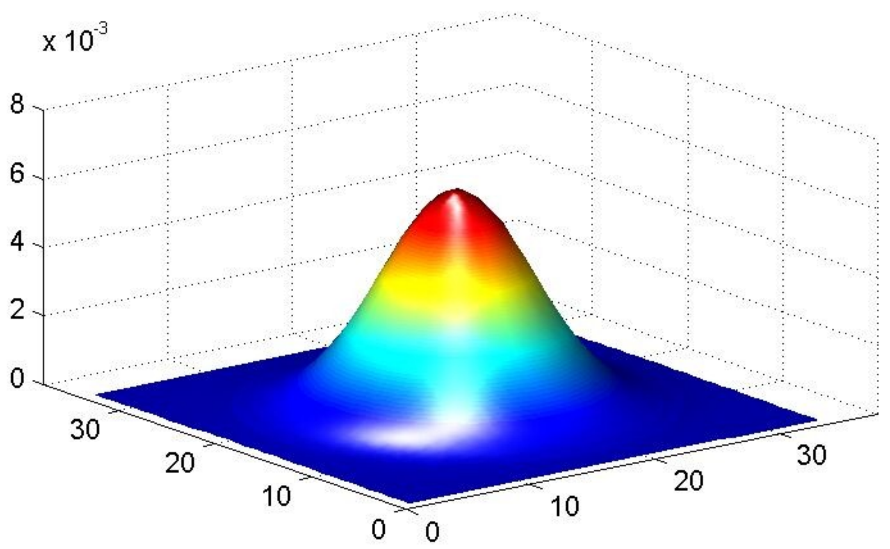
$$G(z, z_0) = \int_0^\infty K(z, z_0, t)\, dt$$

where K is the heat kernel with base at z_0, and at time t. NOW (!)

$$K(x, y, t) = \sum \varphi_j(x)\, \varphi_j(y)\, e^{-\lambda_j t} \quad ,$$

where the sum is over all DIRICHLET eigenfunctions.

The Heat Kernel in the plane (Gaussian)



Ingredients in the proof

- **Heat kernel and its gradient are large around *x*.**
  This follows by comparing with heat kernel in $\mathbb{R}^d$: the heat kernel on *D* not too different at this distance from boundary and this space and time scale.
- **Write heat kernel in terms of eigenfunctions.**
  Can control which eigenfunction really participate in the expansion around *x*, at this space and time scale. Weyl's theorem.
- **Want to use "pigeon-hole" principle to find eigen-function which has large gradient in a given direction around *x* (uniformly and universally).**
- **Recurse in d independent directions to find the full mapping.**

COUNTING EIGENFUNCTIONS WITH WEYL

Weyl's Theorem: Let D be a domain in $\mathbb{R}^d$ with finite volume. (Manifold = same.) Then the number of eigenvalues λ_j (or eigenfunctions) less than T satisfies

$$\#\{ j \mid \lambda_j < T \} \sim \mathrm{VOL}(D)\, T^{d/2}$$

Example: Consider the torus (or square). The eigenfunctions are (times 2) {$\sin(m\pi x)\sin(n\pi y)$, $\sin(m\pi x)\cos(n\pi y)$, $\cos(m\pi x)\sin(n\pi y)$, $\cos(m\pi x)\cos(n\pi y)$}. The eigenvalue for a pair (m,n) is $\pi^2(m^2 + n^2)$. Now just count.

PROOF OF WEYL: Use the Heat Kernel! The same estimates and philosophy will be used by us later. (This gives two-sided estimates for the approximation ~ above.)

Estimates for the Heat Kernel = *K* on *D*

Suppose *x* is a point in *D* with distance *r* to the boundary. The since Brownian motion moves about distance *r* in time r^2, the heat kernel *K* for *D* satisfies

$$K(x, y, t) \sim k(x, y, t)$$

and

$$\nabla K(x, y, t) \sim \nabla k(x, y, t) \qquad (\nabla = \text{gradient})$$

when $t < c\, r^2$. Here *k* is the heat kernel for Euclidean space.

Proof: (This could be for either the Dirichlet or Neumann heat kernel.) Use the path integral representation of *K*.

OUTLINE OF PROOF FOR d = 2 and Volume = 1

Combine the estimates

$$\nabla K(x, y, t) \sim \nabla k(x, y, t) \quad \text{for } t \sim r^2$$

and

$$\#\{ j \mid \lambda_j < T \} \sim \mathrm{VOL}(D)\, T^{d/2} \sim T$$

Apply the fundamental formula for the Heat Kernel:

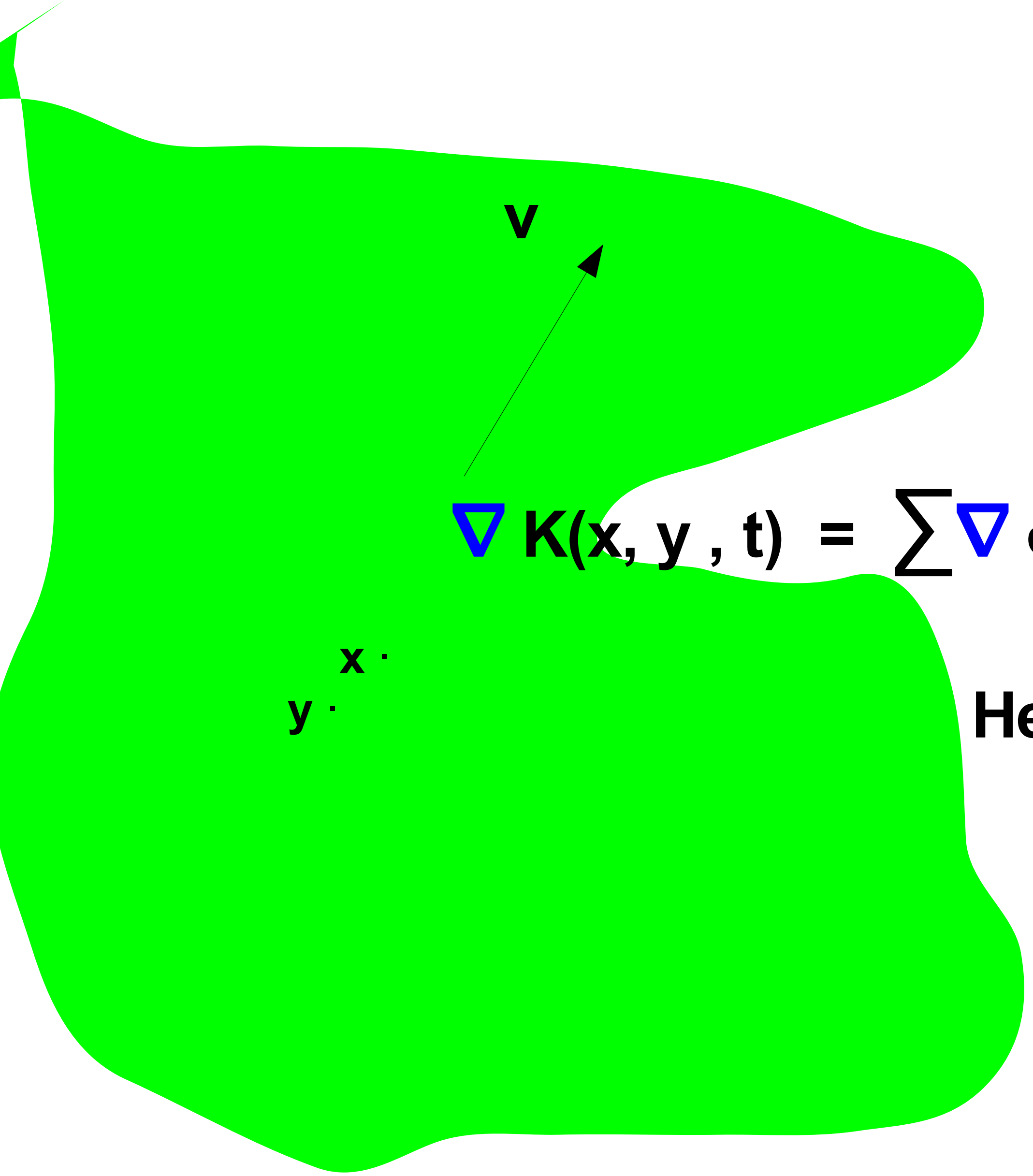
$$K(x, y, t) = \sum \varphi_j(x)\, \varphi_j(y)\, e^{-\lambda_j t},$$

("standard" estimates)

$$\sim \sum_A \text{''}$$

Here $A = \{ \lambda_j \sim r^{-2} \}$ ("Smallish", but not too small eigenvalues) AND THE SAME HOLDS FOR THE GRADIENT.

Pick a Direction v by picking x and y in D (d = 2)



$$\nabla K(x, y, t) = \sum \nabla \varphi_j(x)\, \varphi_j(y)\, e^{-\lambda_j t} \sim r^{-3}$$

Here $|x - y| \sim r$, the sum is over A

Now Count Eigenfunctions: Remember there are r-1 in the collection A

Standard Estimates on the gradient for $\lambda \sim r^{-2}$:

$$|\nabla \varphi(x)|^2 \leq C\, r^{-4} \iint_{B(x,r)} |\varphi(z)|^2 dA(z)$$

By counting arguments with this estimate and the last slide there exists an eigenfunction $\varphi = \varphi_j$ with eigenvalue λ_j such that

$$| \quad |\nabla \varphi(x)| \sim \left(C\, r^{-4} \iint_{B(x,r)} |\varphi(z)|^2 dA(z)\right)^{1/2} \geq c'\, r^{-1}$$

So the directional gradient is LARGE. **NOW........**

We need to check the gradient in the direction v is large on B(x,cr).

But the derivatives of eigenfunctions are eigenfunctions (at least locally, different boundary conditions) and from the standard gradient estimate

$$|\partial_w \nabla_v \varphi(x)|^2 \leq C \lambda r^{-2} \iint_{B(x,r)} |\nabla_v \varphi(z)|^2 dA(z)$$

We get

$$|\partial_w \nabla_v \varphi(\,.\,)| \leq C' r^{-2} \quad \text{on } B(x,cr).$$

Therefore the directional gradient of $\varphi_j(x)$ does not get small on B(x,r).

Linear Algebra

Induction argument at step k: Given a d – k +1 dimensional subspace S_k of $\mathbb{R}^d$, pick an eigenfunction whose gradient in that direction, ∂_k, is large in one direction v_k in that subspace.

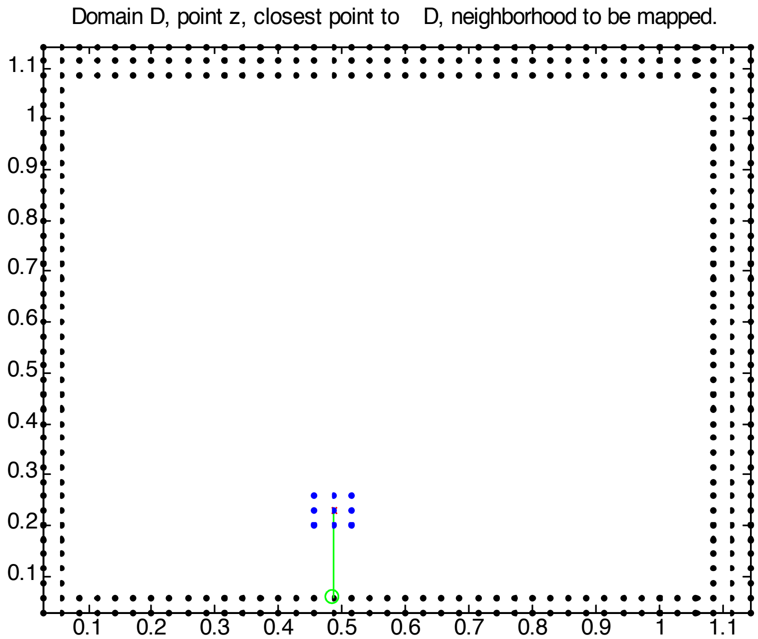
Then define $S_k = \mathbb{R}^d \ominus \{v_1, v_2, \ldots\ldots\ldots, v_k\}$.

Then $\{v_1, v_2, \ldots\ldots\ldots, v_d\}$ will span $\mathbb{R}^d$. (And they will do so with good estimates.) Therefore the system

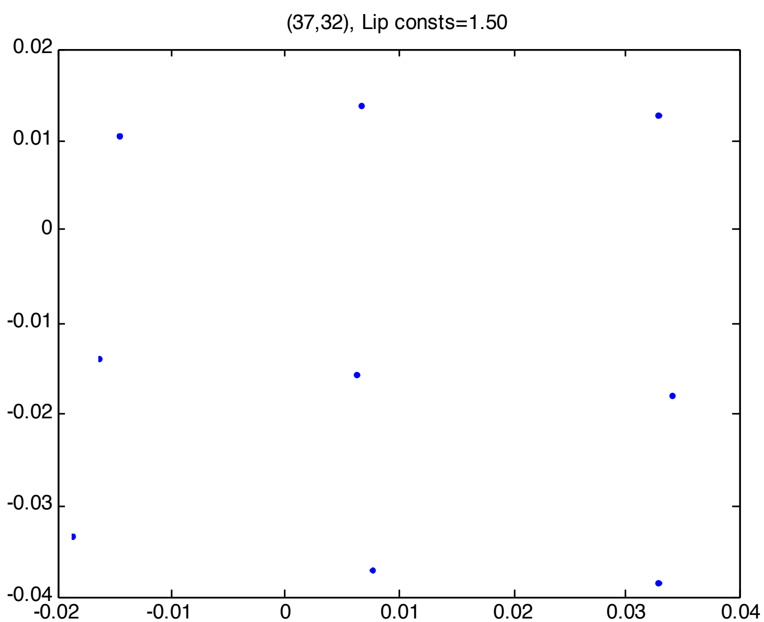
$$\{\varphi_1, \varphi_2, \ldots\ldots\ldots, \varphi_d\} = \Phi$$

is a good coordinate system.

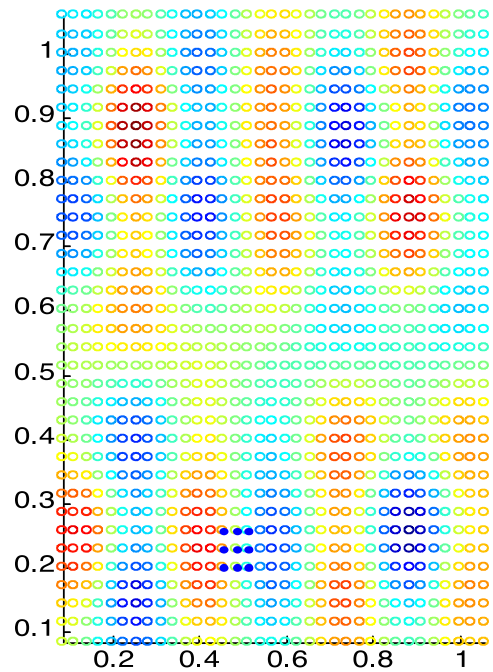
Example 1

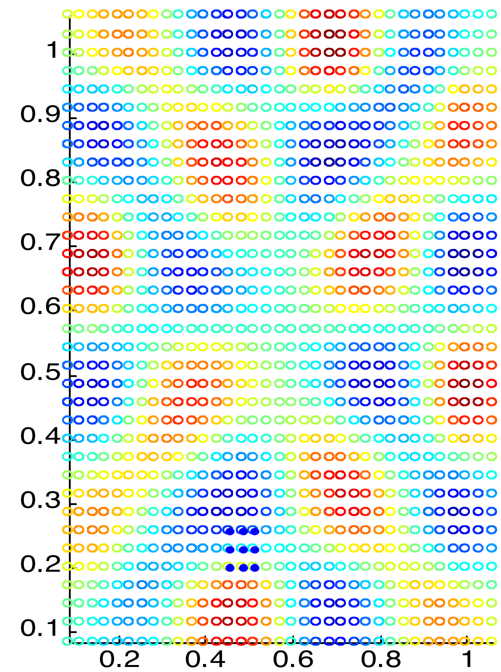


Domain D, point z, closest point to D, neighborhood to be mapped.

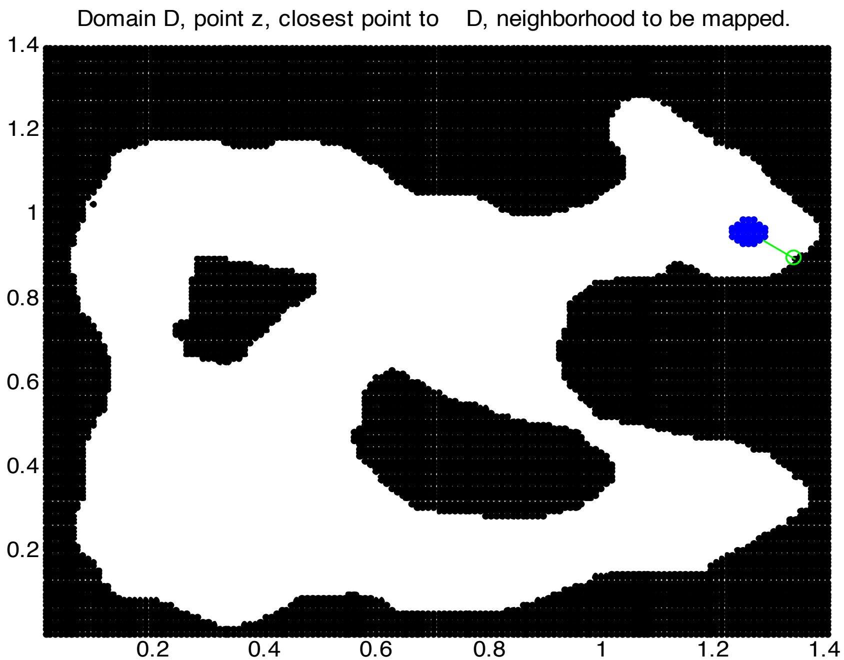


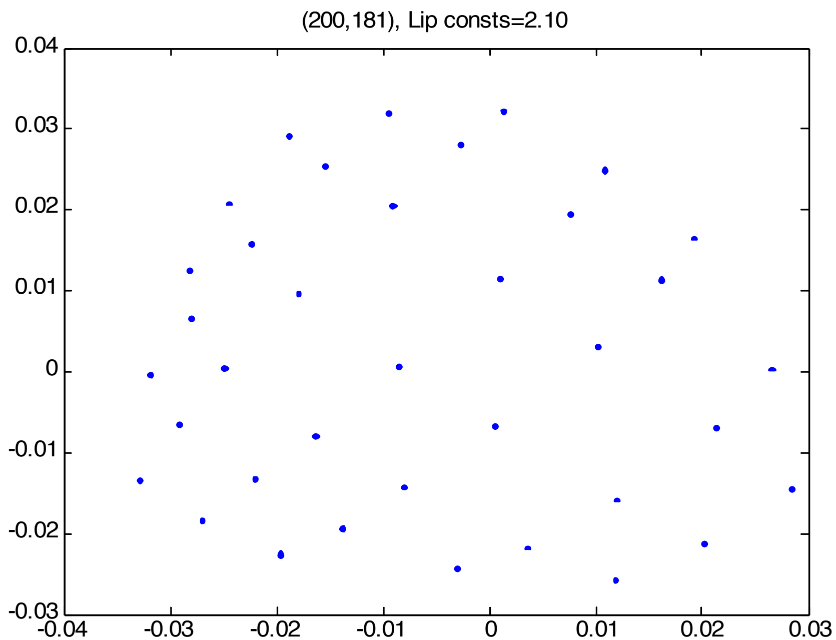
(37,32), Lip consts=1.50





Example 3


Domain D, point z, closest point to D, neighborhood to be mapped.


(200,181), Lip consts=2.10