

*Laplacian transport towards
irregular interfaces:
the mathematics*

Denis S. GREBENKOV

Laboratoire de Physique de la Matière Condensée
CNRS – Ecole Polytechnique, France

Email: denis.grebenkov@polytechnique.edu

Web: <http://pmc.polytechnique.fr/pagesperso/dg>

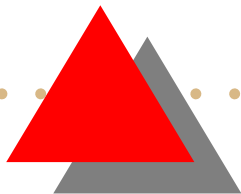




Plan

- Laplacian transport phenomena
- Theoretical approach
- Numerical and experimental study
- Conclusion, questions and perspectives

Oxygen diffusion towards and through alveolar membranes



Oxygen diffusion towards and through alveolar membranes

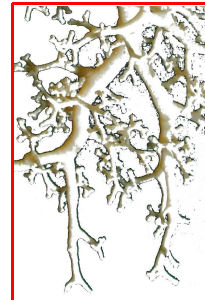
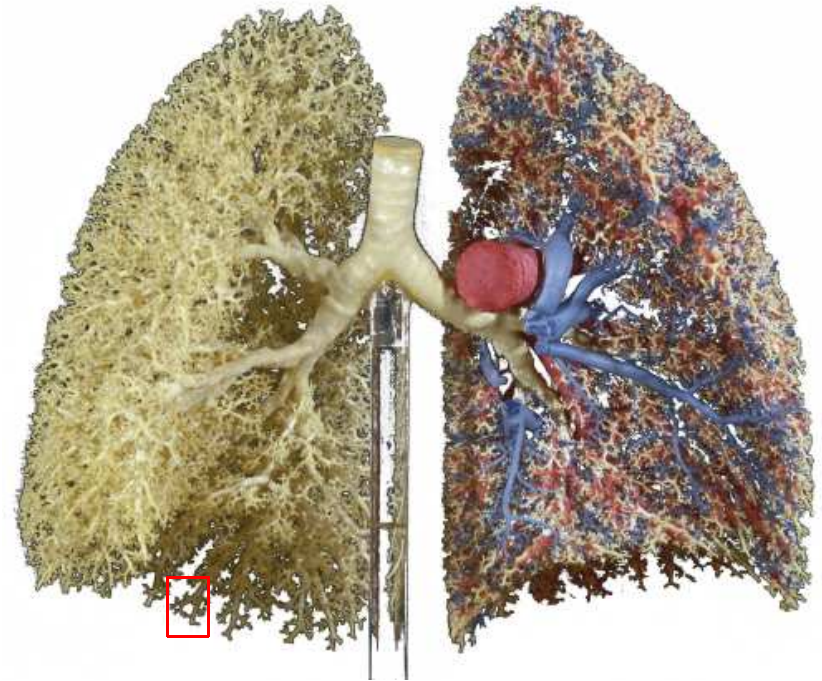
Dichotomic tree

Volume $\sim 4 - 5$ liters

Area $\sim 100 - 150$ m²

23 generations :
convection ~ 15
diffusion ~ 8

Human lung



Oxygen diffusion towards and through alveolar membranes

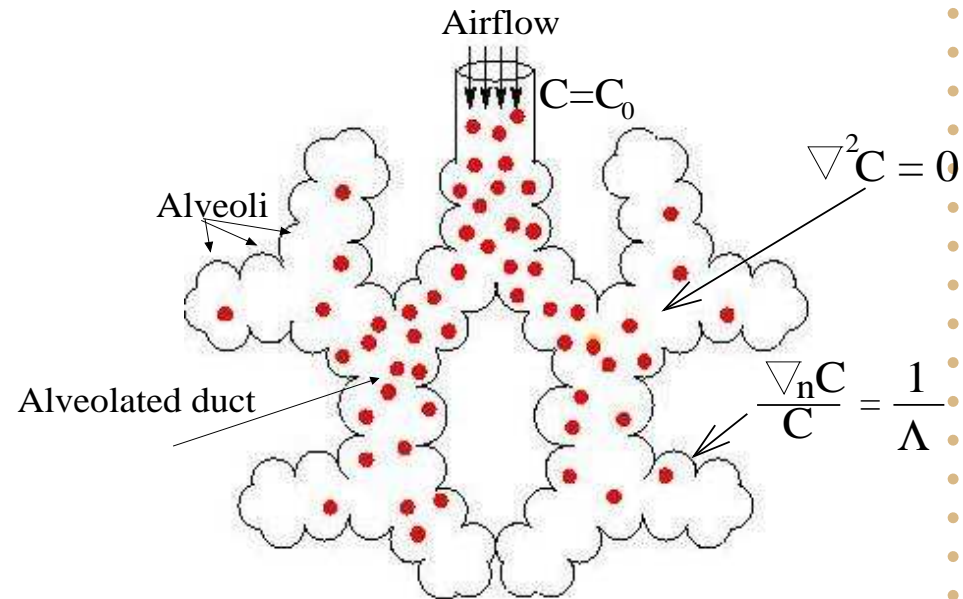
Laplacian transport

$\Delta C = 0$ in the volume

$C = C_0$ at the entrance

$\mathbf{J} = -D\nabla C$ Fick's law

$J_n = WC$ on the membrane



$\Lambda = D/W$ characteristic length of the problem

D diffusion coefficient

W membrane permeability

Oxygen diffusion towards and through alveolar membranes

Laplacian transport

$$\Delta C = 0$$

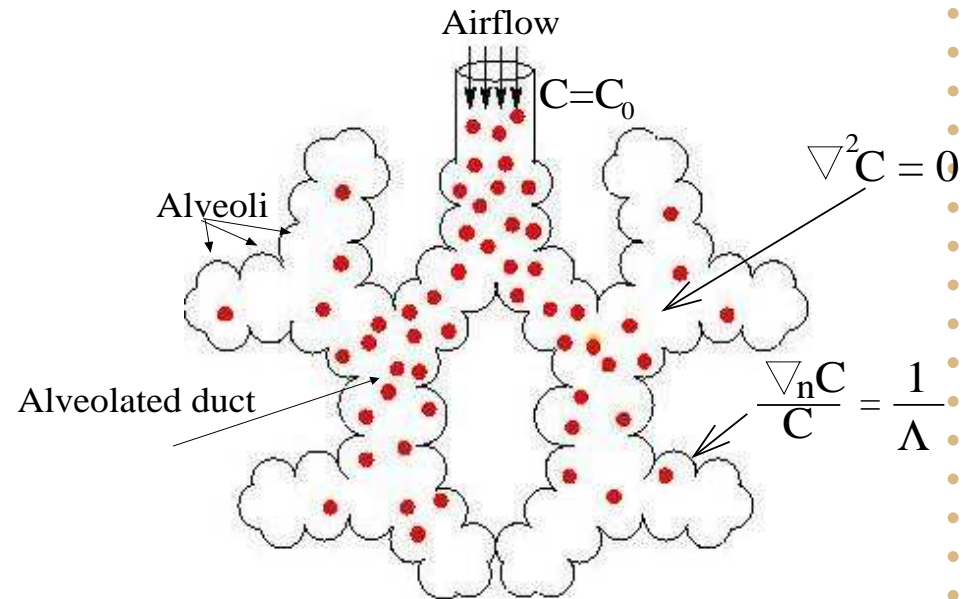
in the volume

$$C = C_0$$

at the entrance

$$\frac{\partial C}{\partial n} = \frac{C}{\Lambda}$$

on the membrane



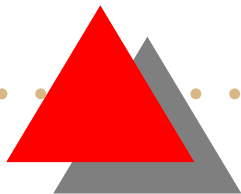
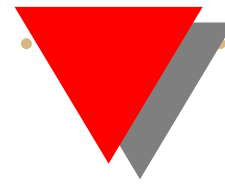
$$\Lambda = D/W$$

characteristic length of the problem

D diffusion coefficient

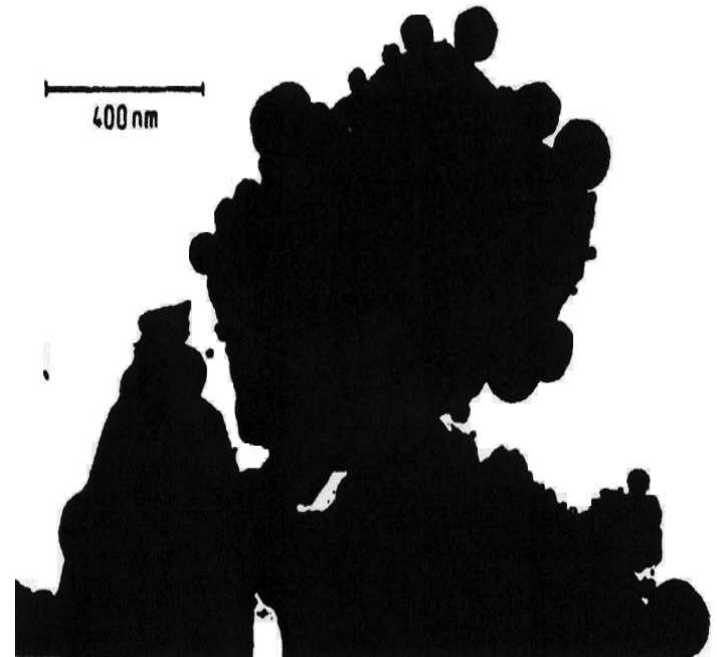
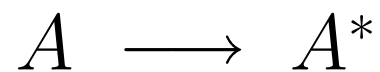
W membrane permeability

Heterogeneous catalysis



Heterogeneous catalysis

Chemical reaction



Heterogeneous catalysis

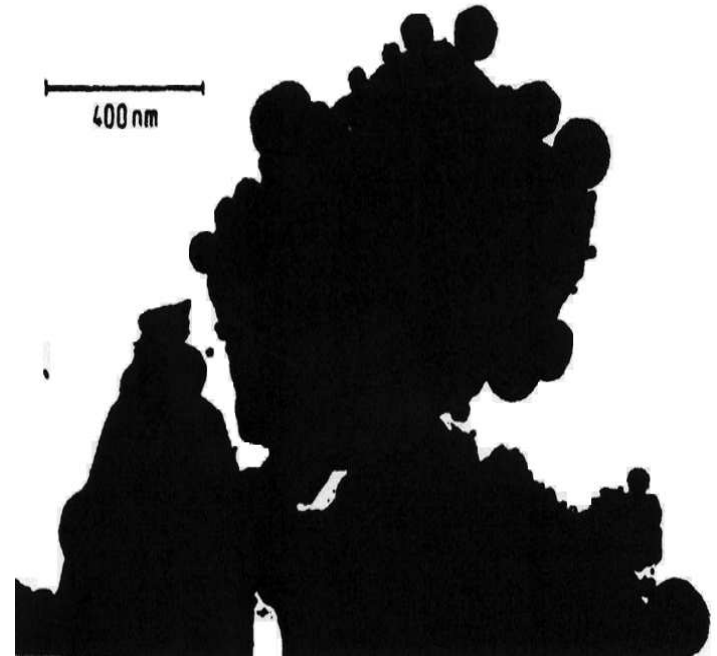
Description

$$\Delta C = 0$$

$$C = C_0$$

$$\frac{\partial C}{\partial n} = \frac{C}{\Lambda}$$

in the volume
on the source
on the surface



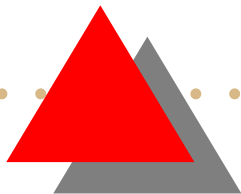
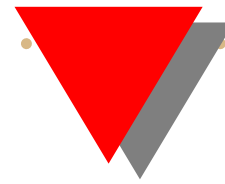
$$\Lambda = D/K$$

characteristic length of the problem

D diffusion coefficient

K catalyst's reactivity

Electric transport through an irregular electrode

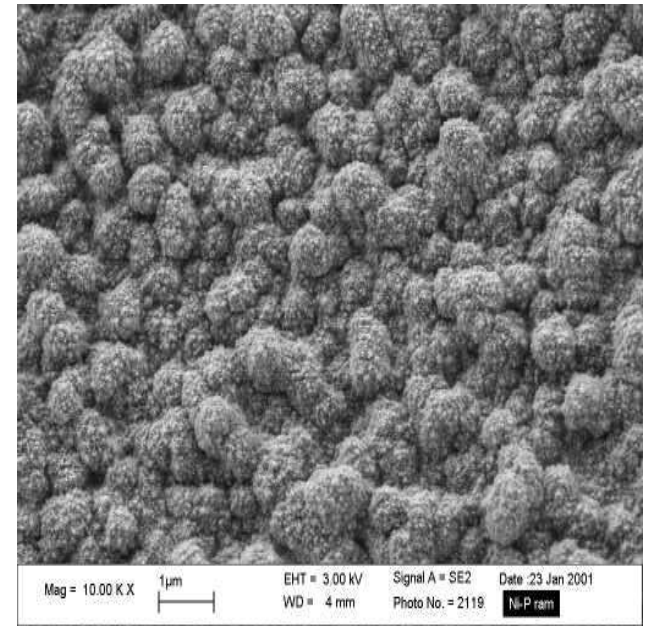


Electric transport through an irregular electrode

Microroughness
(of metallic surface)

Irregular geometry
on different scales

Impedance ?



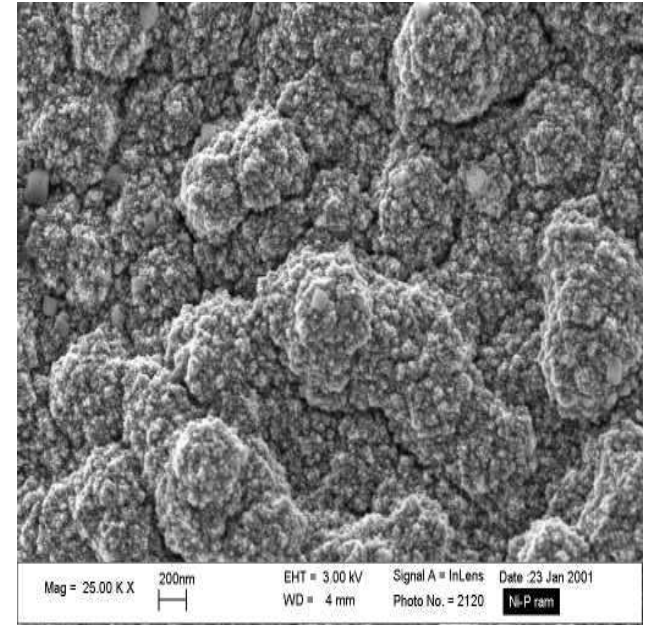
1 μm

Electric transport through an irregular electrode

Microroughness
(of metallic surface)

Irregular geometry
on different scales

Impedance ?



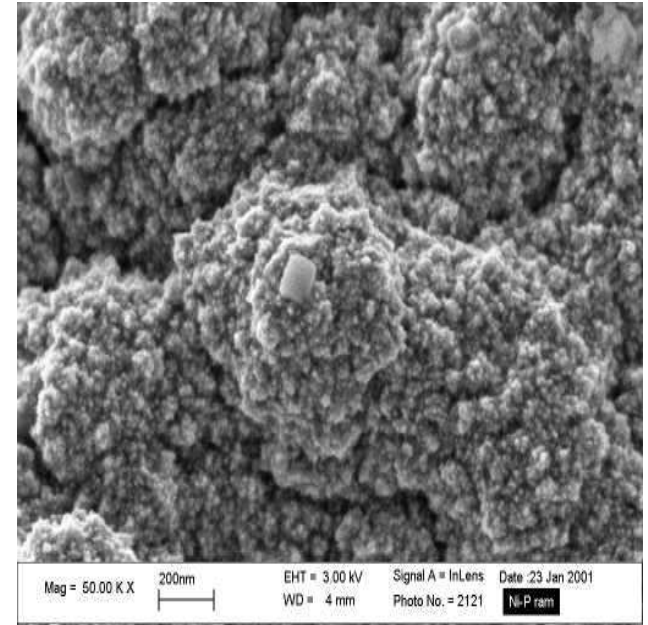
1 μm

Electric transport through an irregular electrode

Microroughness
(of metallic surface)

Irregular geometry
on different scales

Impedance ?

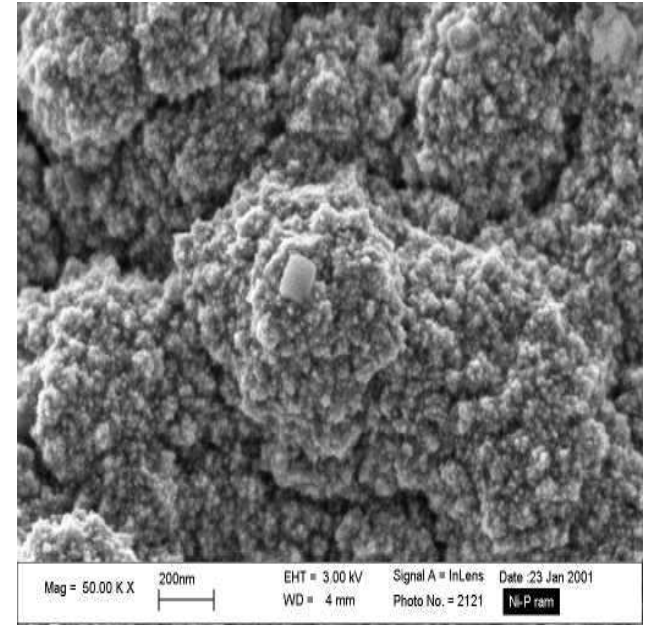


1 μm

Electric transport through an irregular electrode

Electric transport:
(direct constant potential)

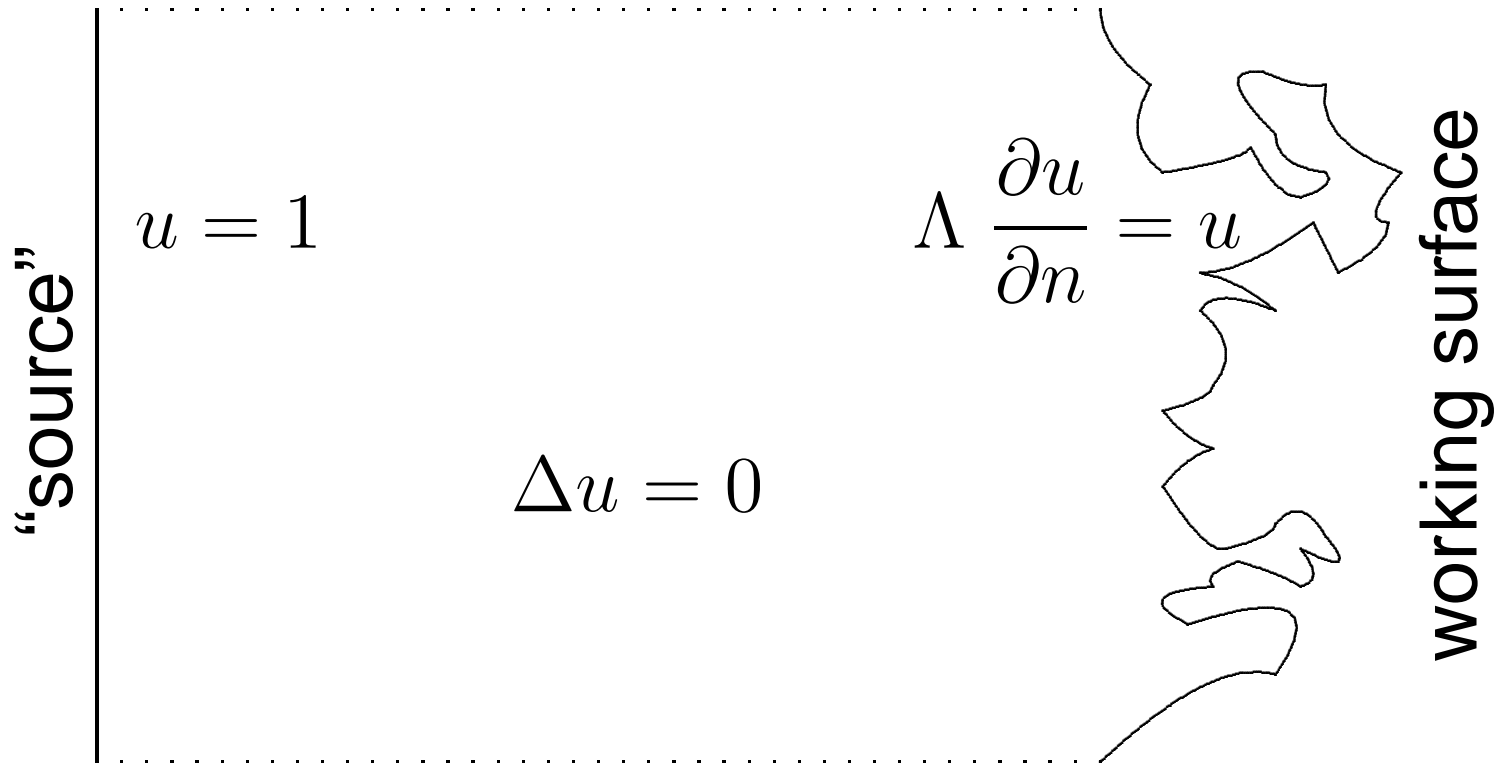
$$\Delta V = 0 \quad \text{in the volume}$$
$$V = V_0 \quad \text{on the counter-electrode}$$
$$\frac{\partial V}{\partial n} = \frac{1}{\Lambda} V \quad \text{on the working electrode}$$



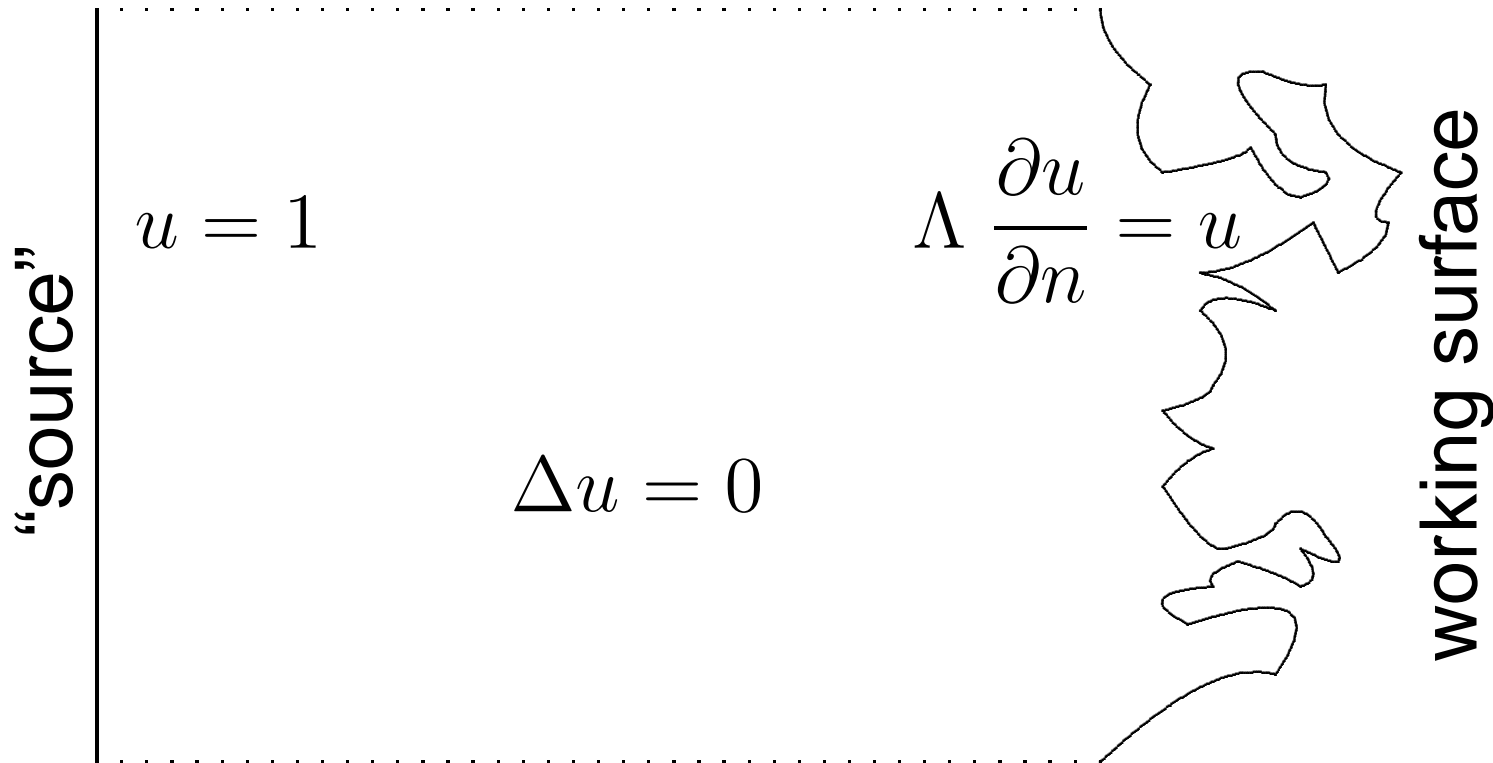
$1 \mu m$

$\Lambda = r / \rho$ characteristic length of the problem
 r resistance of the working electrode
 ρ resistivity of the electrolyte

Laplacian transport



Laplacian transport

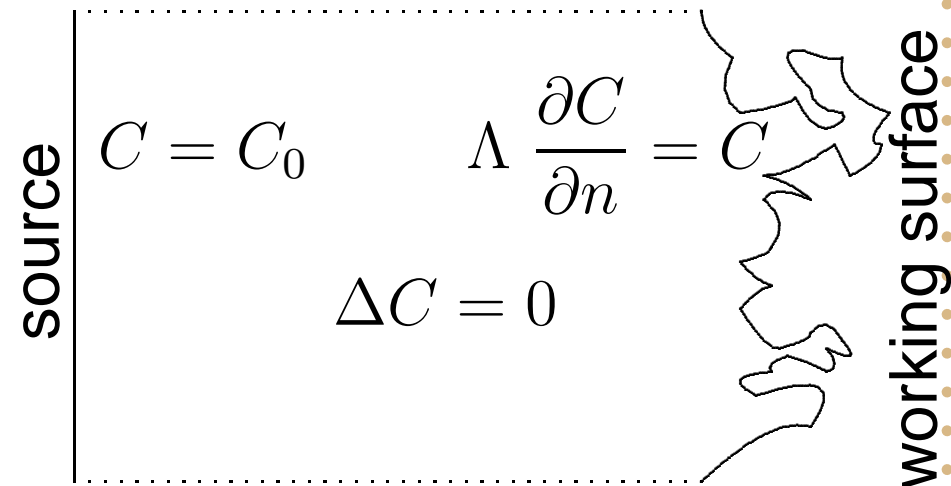


How does a geometrical irregularity influence the transport properties of the system?

Spectroscopic impedance

Mathematical model:
mixed boundary value
problem for Laplace
equation

Diffusive transport

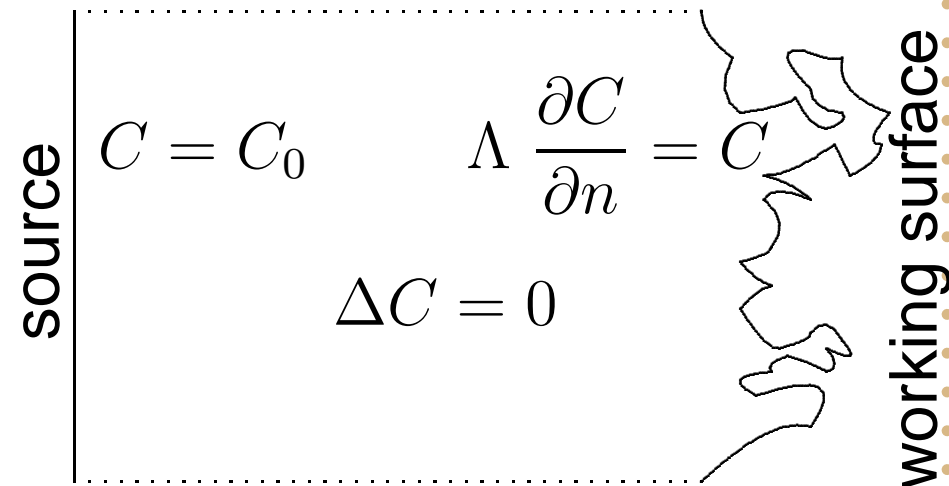


Spectroscopic impedance

Mathematical model:
mixed boundary value
problem for Laplace
equation

We are interested
in the total flux Φ_Λ
through the working
surface:

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

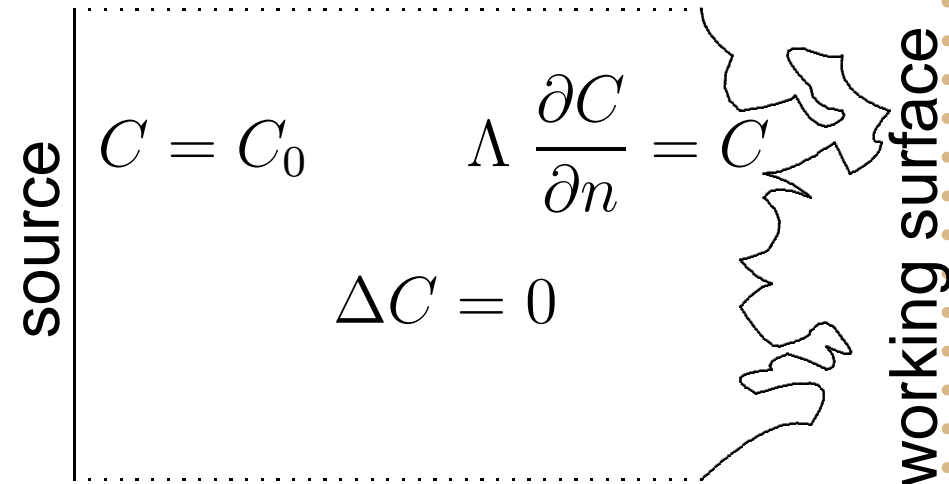
Spectroscopic impedance

Mathematical model:
mixed boundary value
problem for Laplace
equation

We are interested
in the total flux Φ_Λ
through the working
surface:

Cell impedance:

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z_{cell}(\Lambda) = \frac{C_0}{\Phi_\Lambda}$$

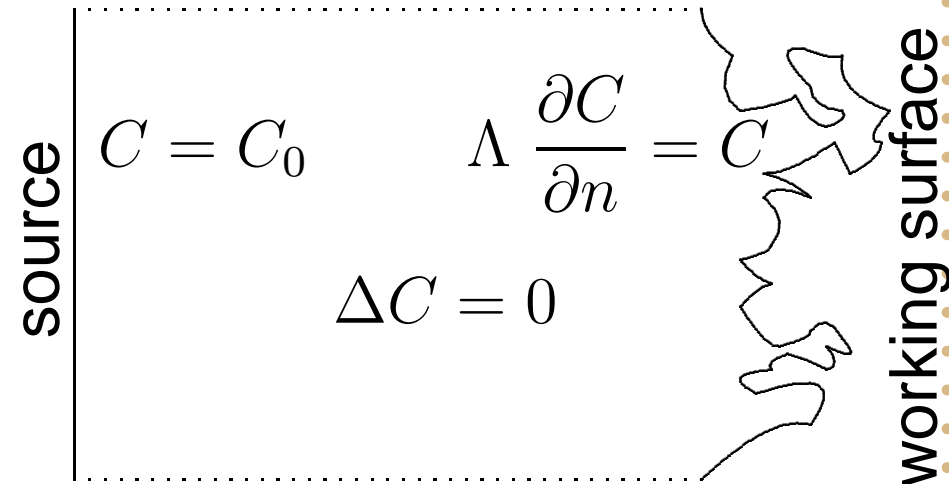
Spectroscopic impedance

Mathematical model:
mixed boundary value
problem for Laplace
equation

We are interested
in the total flux Φ_Λ
through the working
surface:

Spectroscopic
impedance:

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

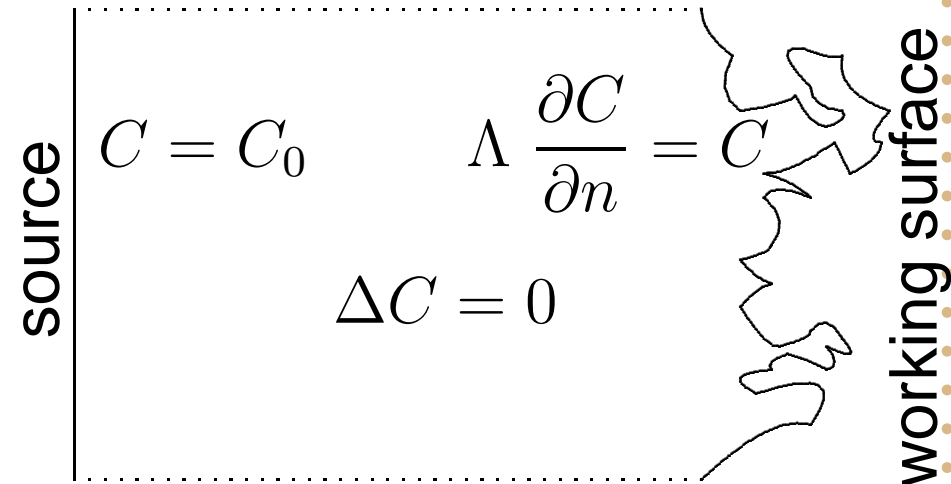
$$Z(\Lambda) = Z_{cell}(\Lambda) - Z_{cell}(0)$$

Spectroscopic impedance

Dirichlet mode:

$$\Lambda \longrightarrow 0$$

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

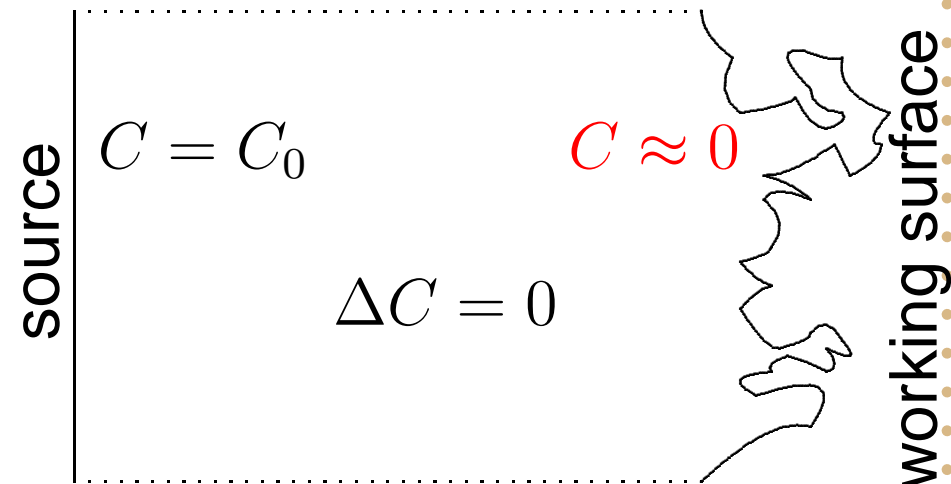
Spectroscopic impedance

Dirichlet mode:

$$\Lambda \longrightarrow 0$$

Absorbing surface

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

Spectroscopic impedance

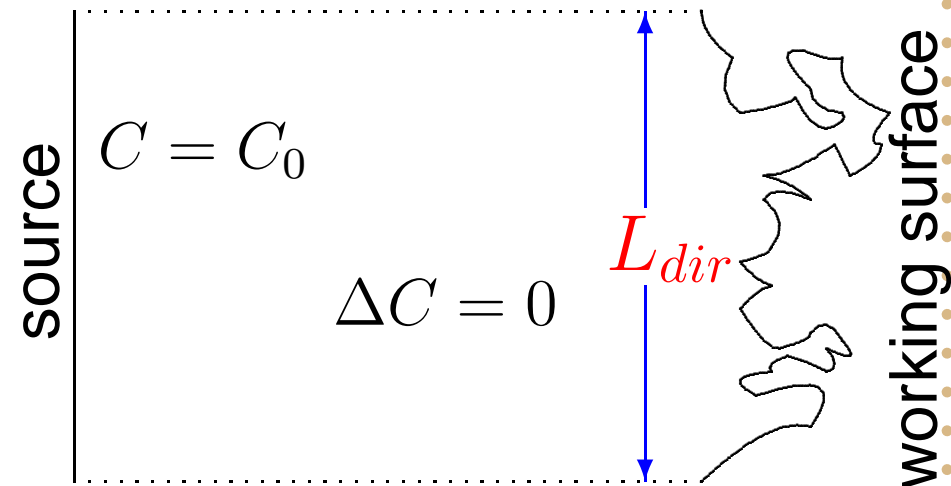
Dirichlet mode:

$$\Lambda \longrightarrow 0$$

Absorbing surface

$$Z(\Lambda) \simeq \frac{\Lambda}{DL_{dir}}$$

Diffusive transport



$$\Phi_{\Lambda} = D \int_{\partial\Omega} ds \frac{\partial C_{\Lambda}}{\partial n} \quad \Lambda = \frac{D}{W}$$

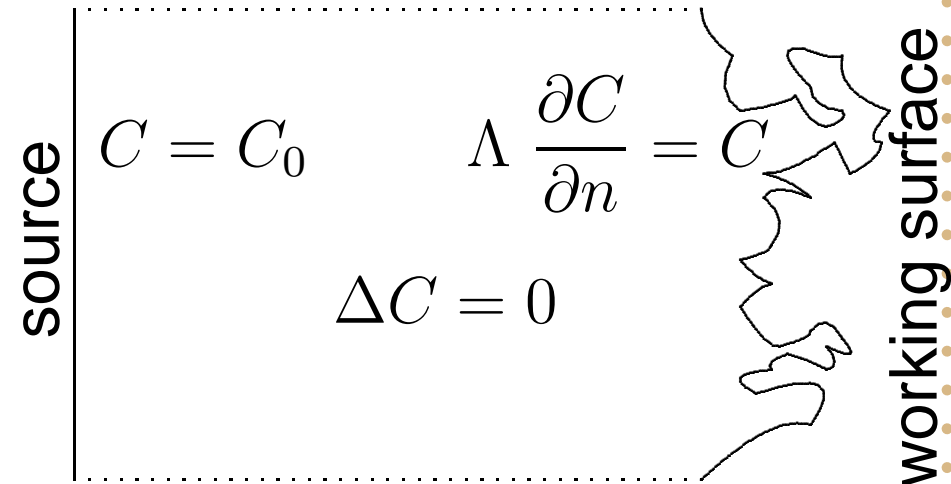
$$Z(\Lambda) = \frac{C_0}{\Phi_{\Lambda}} - \frac{C_0}{\Phi_0}$$

Spectroscopic impedance

Neumann mode:

$$\Lambda \longrightarrow \infty$$

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

Spectroscopic impedance

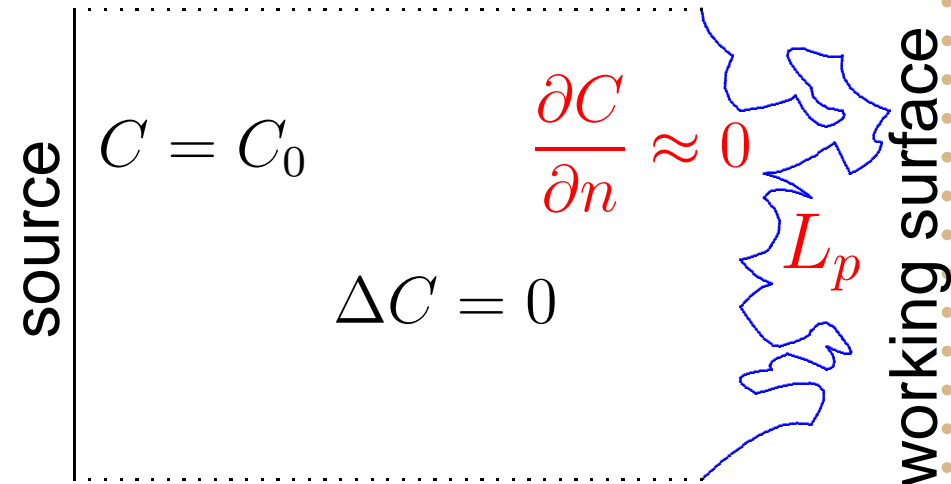
Neumann mode:

$$\Lambda \longrightarrow \infty$$

Reflecting surface

$$Z(\Lambda) \simeq \frac{\Lambda}{DL_p}$$

Diffusive transport



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

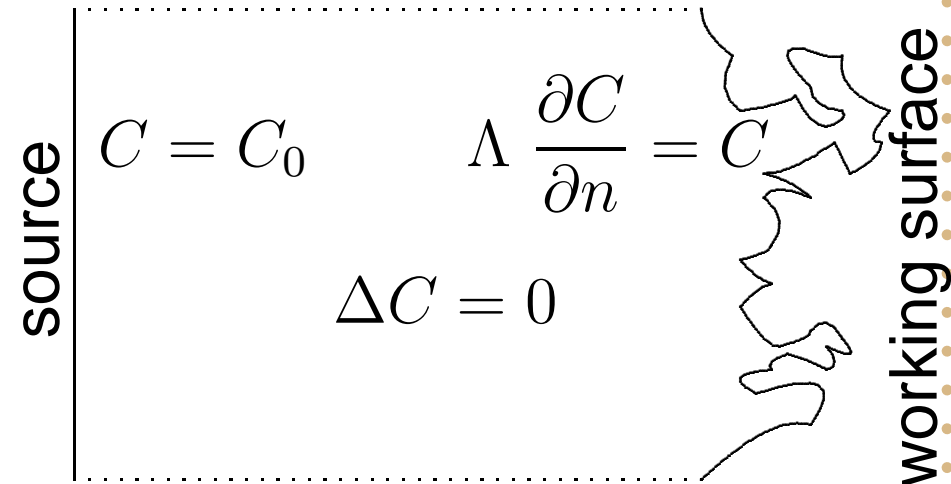
Spectroscopic impedance

Intermediate mode:

Diffusive transport

Constant phase angle behavior

$$Z(\Lambda) \sim \Lambda^\beta$$



$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

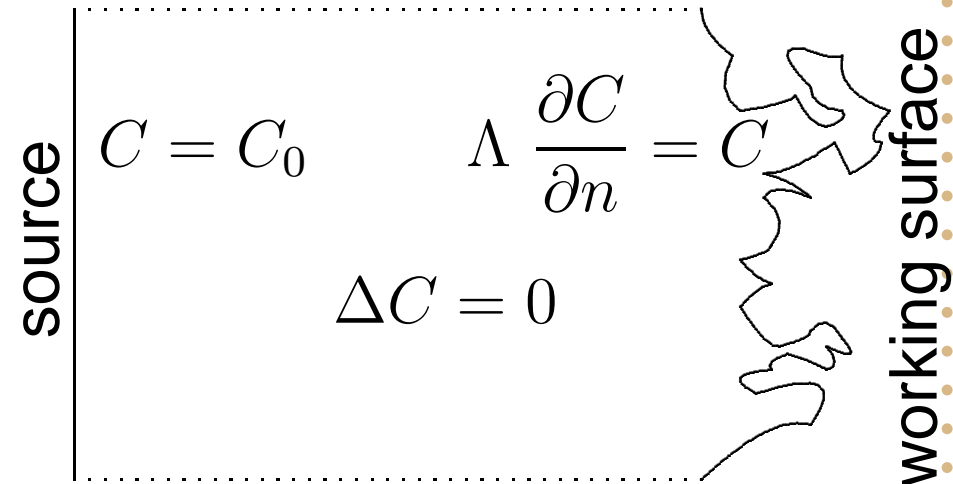
Spectroscopic impedance

Intermediate mode:

Diffusive transport

Constant phase angle behavior

$$Z(\Lambda) \sim \Lambda^\beta$$



Fractal boundary

$$\beta = \frac{1}{D_f} \quad \beta = \frac{\tau(2)}{D_f}$$

$$\Phi_\Lambda = D \int_{\partial\Omega} ds \frac{\partial C_\Lambda}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

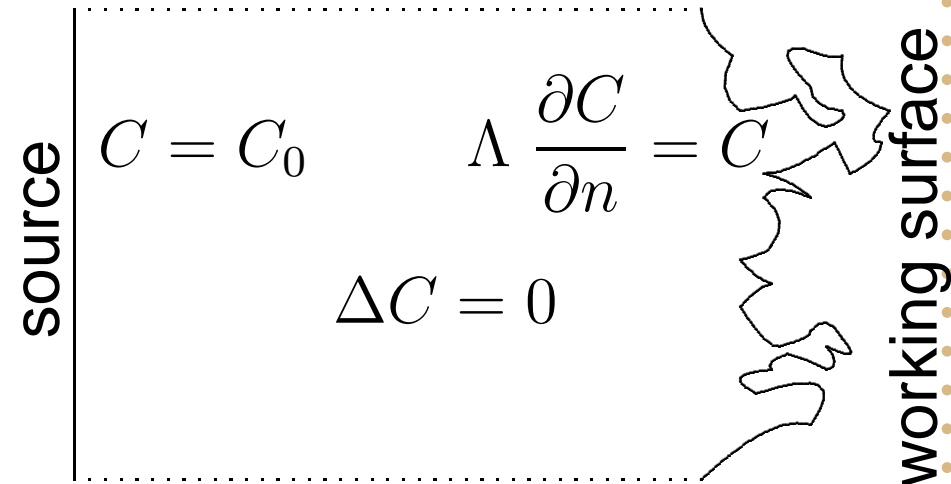
Spectroscopic impedance

Intermediate mode:

Diffusive transport

Arbitrary surface

How does the geometrical irregularity influence transport properties?



$$\Phi_{\Lambda} = D \int_{\partial\Omega} ds \frac{\partial C_{\Lambda}}{\partial n} \quad \Lambda = \frac{D}{W}$$

$$Z(\Lambda) = \frac{C_0}{\Phi_{\Lambda}} - \frac{C_0}{\Phi_0}$$



Plan

- Laplacian transport phenomena
- **Theoretical approach**
- Numerical and experimental study
- Conclusion, questions and perspectives

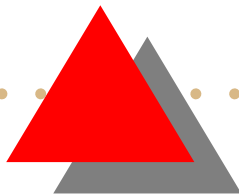
Theoretical description

- Laplace equation + mixed condition:
Description in average



Theoretical description

- Laplace equation + mixed condition:
Description in average
- Partially reflected Brownian motion:
Random trajectory of each particle

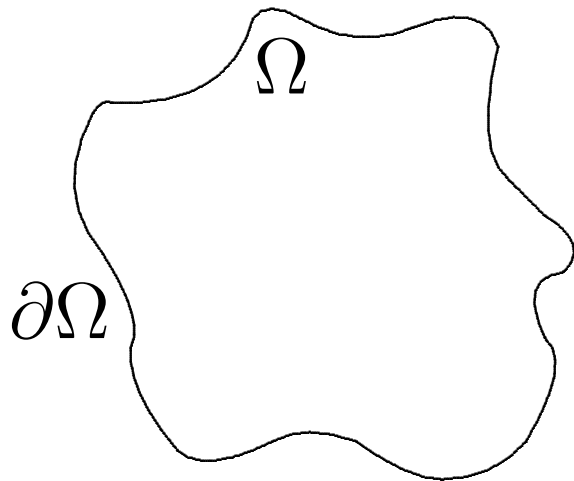


Theoretical description

- Laplace equation + mixed condition:
Description in average
- Partially reflected Brownian motion:
Random trajectory of each particle
- Dirichlet-to-Neumann operator:
Spectral analysis of the problem

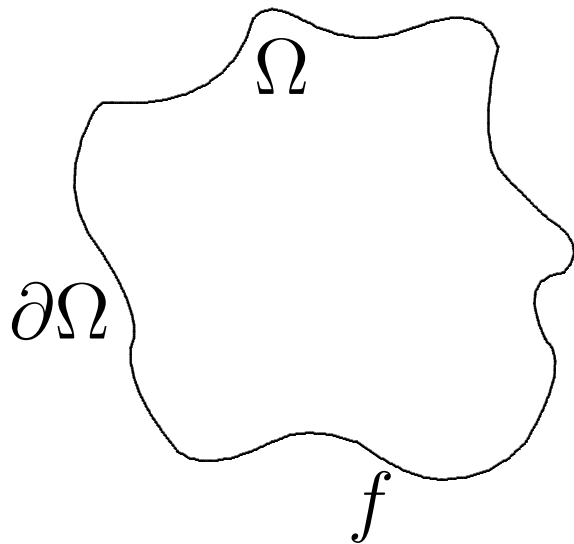
Definition of the Dirichlet-to-Neumann operator

For a given domain Ω ,



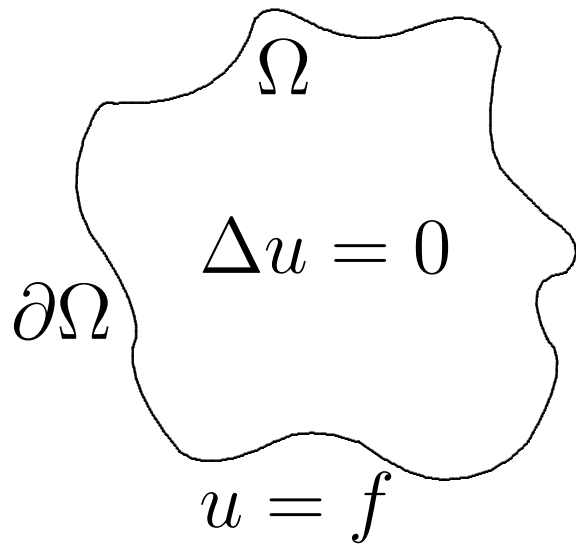
Definition of the Dirichlet-to-Neumann operator

For a given domain Ω , one takes **a function**
 $f \in H^1(\partial\Omega)$ on the boundary $\partial\Omega$,



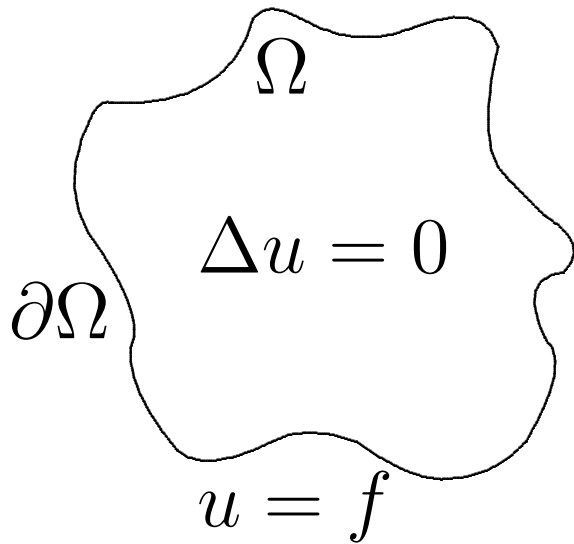
Definition of the Dirichlet-to-Neumann operator

For a given domain Ω , one takes a function $f \in H^1(\partial\Omega)$ on the boundary $\partial\Omega$, one solves **the Dirichlet problem** in this domain



Definition of the Dirichlet-to-Neumann operator

For a given domain Ω , one takes a function $f \in H^1(\partial\Omega)$ on the boundary $\partial\Omega$, one solves the Dirichlet problem in this domain, and one applies **the normal derivative** to the solution.



$$f \longrightarrow g = \frac{\partial u}{\partial n} \equiv [\mathcal{M}f]$$

Properties of the Dirichlet-to-Neumann operator




The Dirichlet-to-Neumann operator \mathcal{M} is

- dependent solely on the **geometry**;

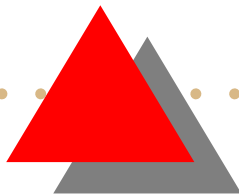


Properties of the Dirichlet-to-Neumann operator



The Dirichlet-to-Neumann operator \mathcal{M} is

- dependent solely on the **geometry**;
- **self-adjoint**;

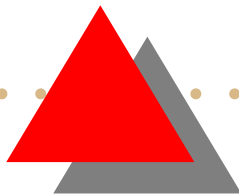


Properties of the Dirichlet-to-Neumann operator

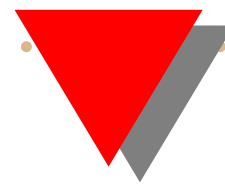


The Dirichlet-to-Neumann operator \mathcal{M} is

- dependent solely on the **geometry**;
- **self-adjoint**;
- with discrete spectrum $\{\mu_\alpha\}$ bounded below;

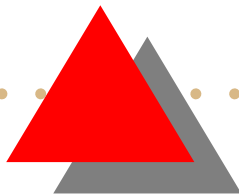


Properties of the Dirichlet-to-Neumann operator



The Dirichlet-to-Neumann operator \mathcal{M} is

- dependent solely on the **geometry**;
- **self-adjoint**;
- with discrete spectrum $\{\mu_\alpha\}$ bounded below;
- with smooth eigenfunctions $V_\alpha(s)$ (defined on the boundary $\partial\Omega$) which form a complete basis in $L^2(\partial\Omega)$.



Representation of the spectroscopic impedance

$$Z(\Lambda) = Z_{cell}(\Lambda) - Z_{cell}(0) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

Representation of the spectroscopic impedance

$$Z(\Lambda) = Z_{cell}(\Lambda) - Z_{cell}(0) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

$$Z(\Lambda) = \frac{\Lambda}{D} \sum_{\alpha=0}^{\infty} \frac{F_\alpha}{1 + \mu_\alpha \Lambda}$$

Representation of the spectroscopic impedance

$$Z(\Lambda) = Z_{cell}(\Lambda) - Z_{cell}(0) = \frac{C_0}{\Phi_\Lambda} - \frac{C_0}{\Phi_0}$$

$$Z(\Lambda) = \frac{\Lambda}{D} \sum_{\alpha=0}^{\infty} \frac{F_\alpha}{1 + \mu_\alpha \Lambda}$$

Physics

Λ, D

$$\Lambda = \frac{D}{W}, \frac{D}{K}, \frac{r}{\rho}$$

Geometry

μ_α, F_α

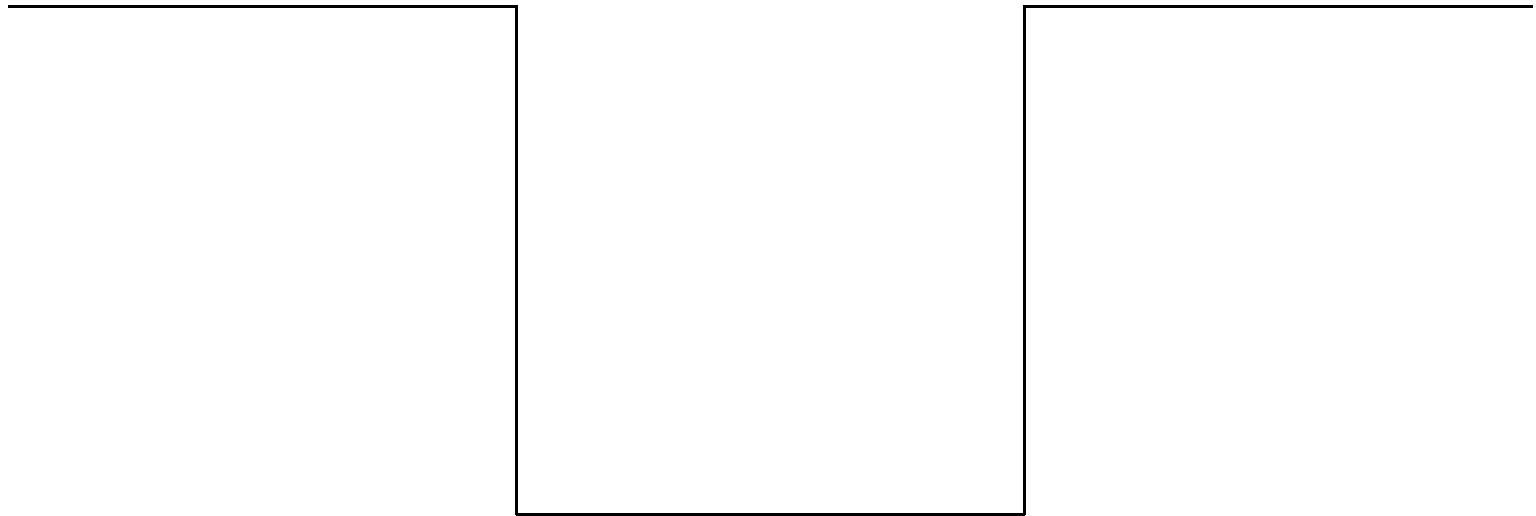
$$F_\alpha = (\mathbf{V}_\alpha \cdot \omega_0)^2$$



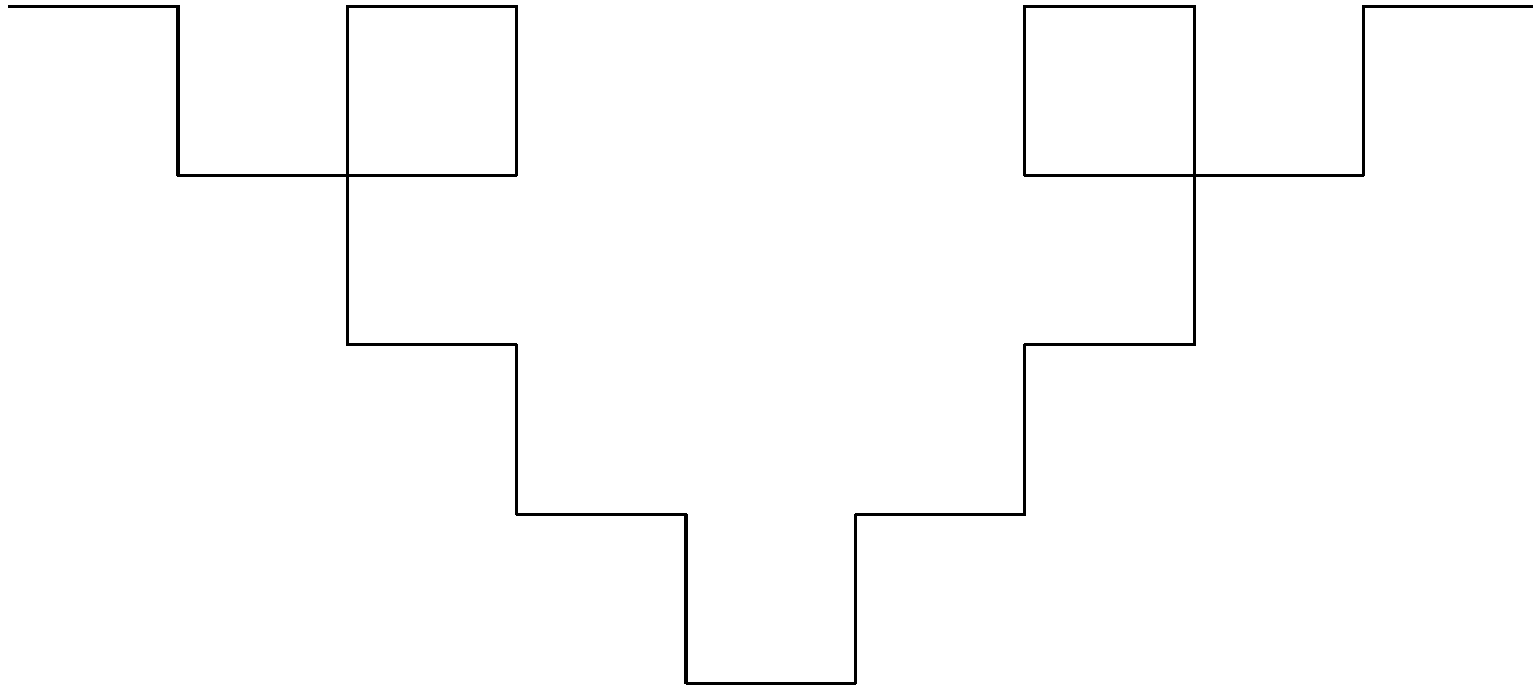
Plan

- Laplacian transport phenomena
- Theoretical approach
- **Numerical and experimental study**
- Conclusion, questions and perspectives

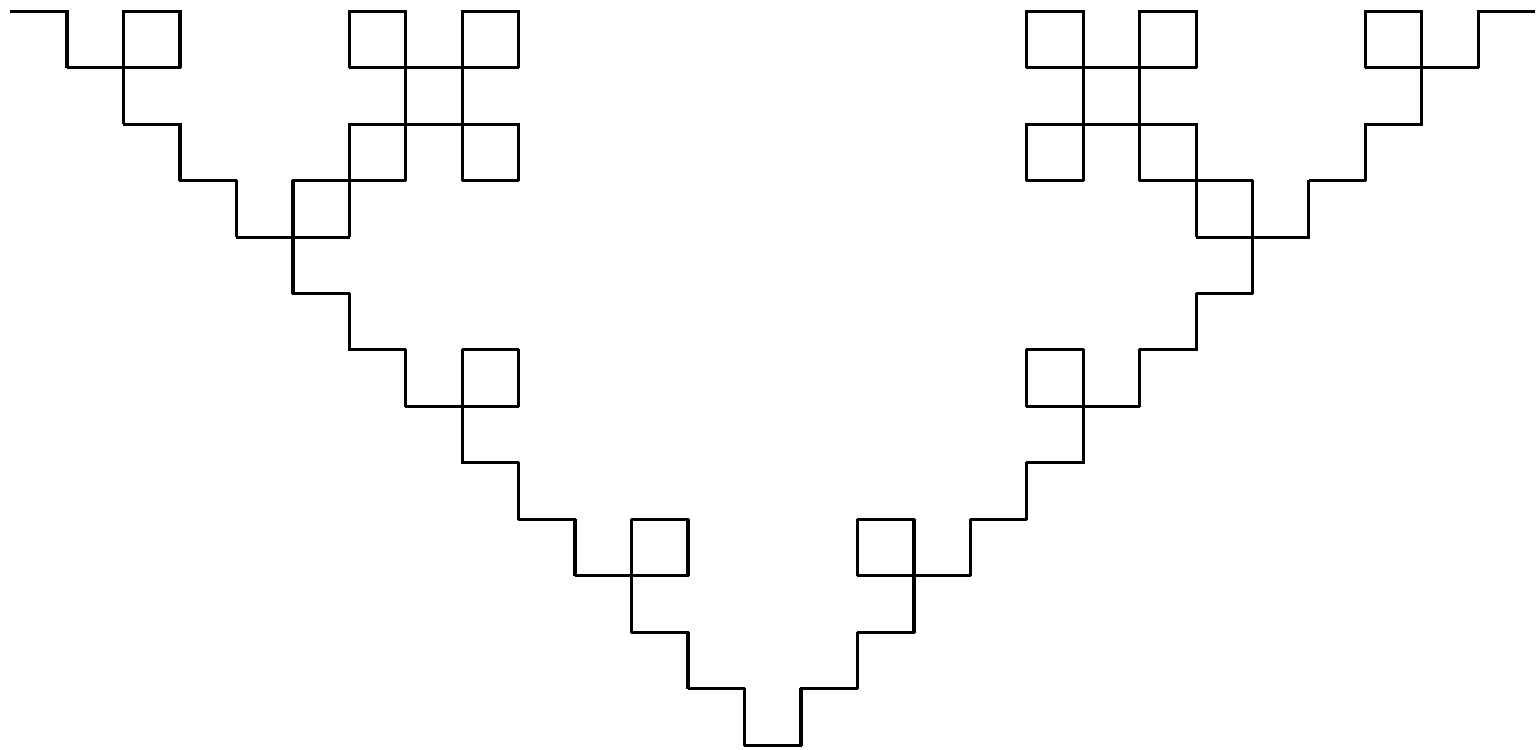
Koch interface



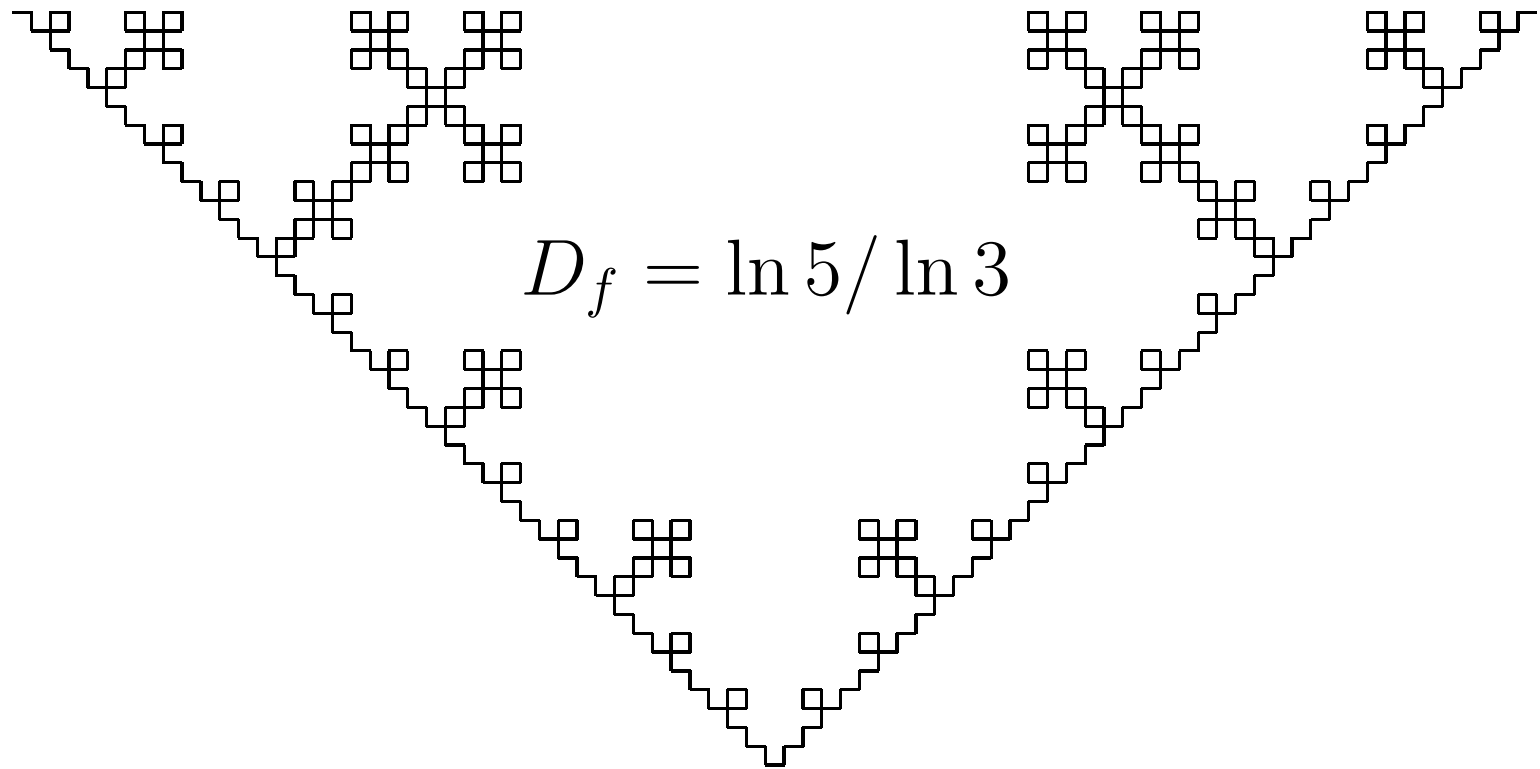
Koch interface



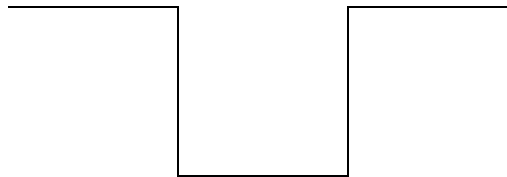
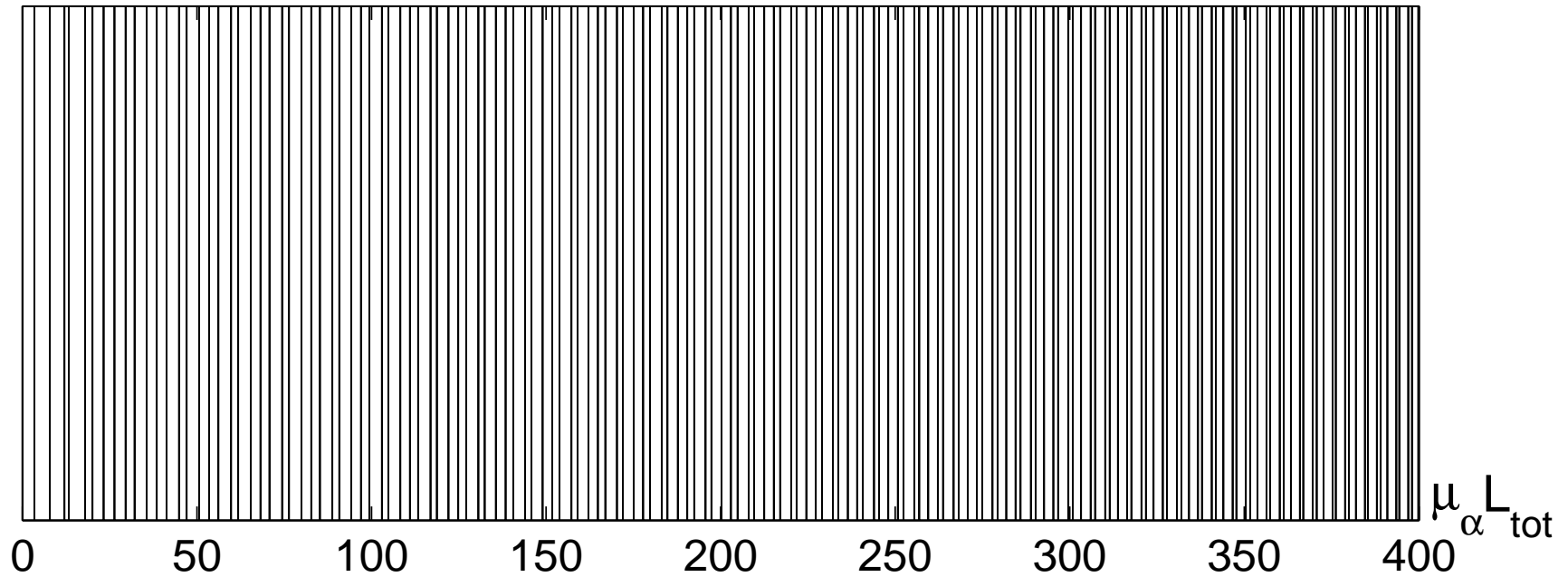
Koch interface



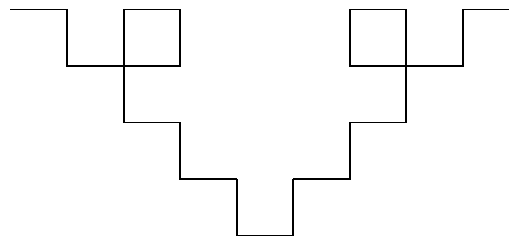
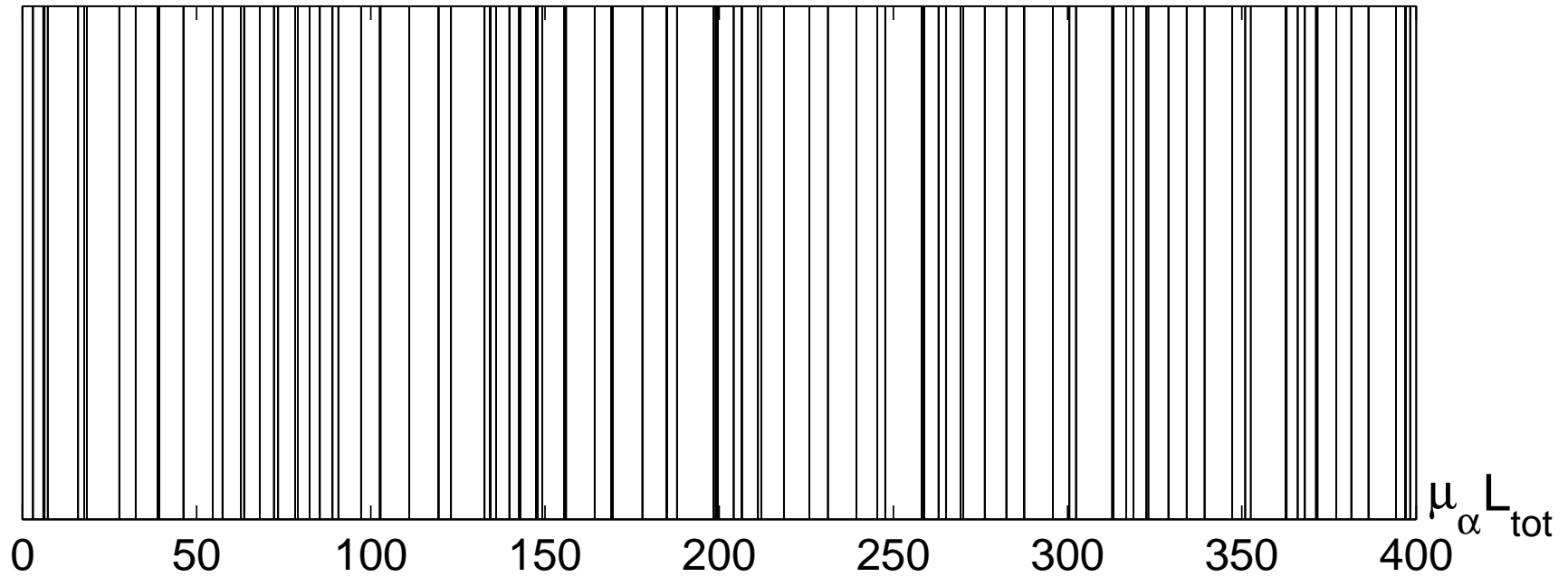
Koch interface



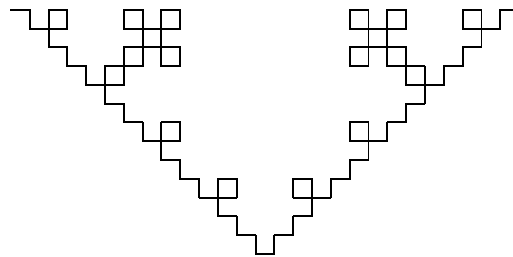
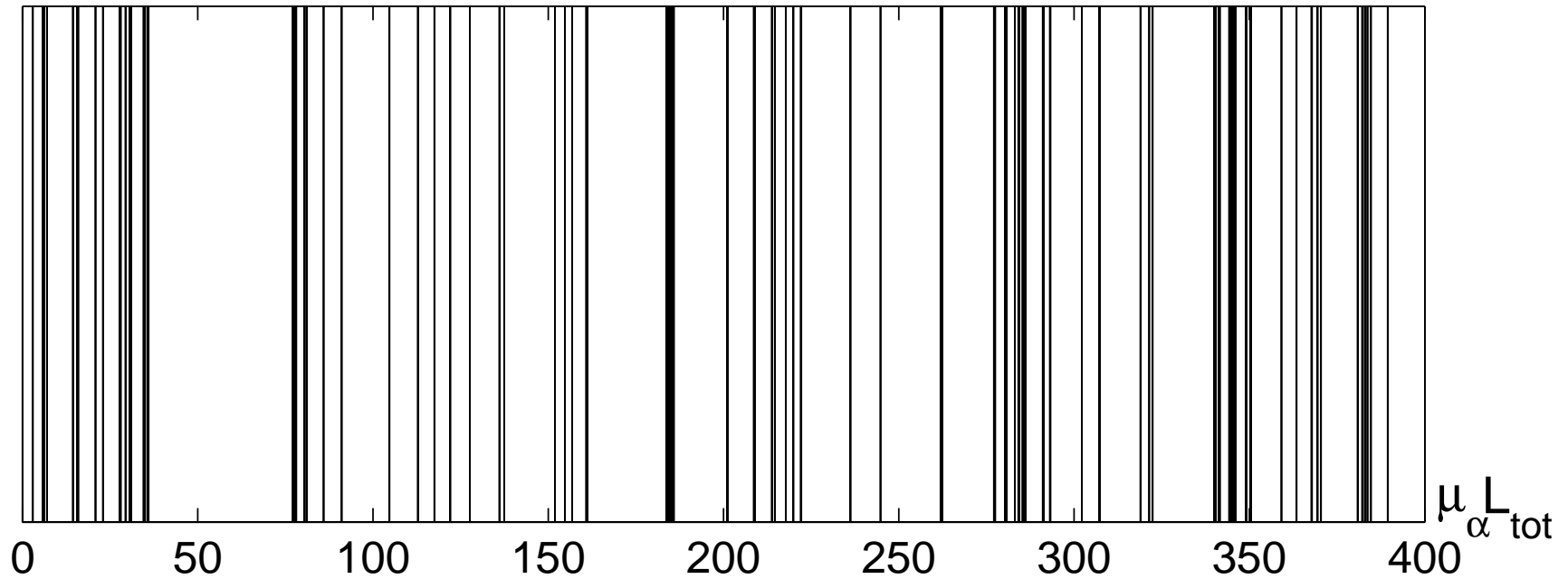
“Localization” of the spectrum



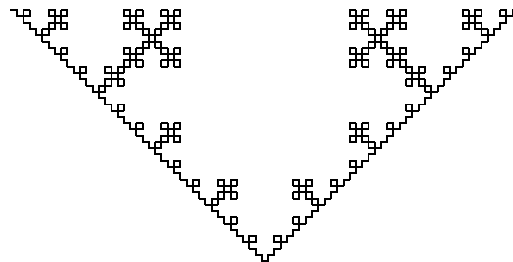
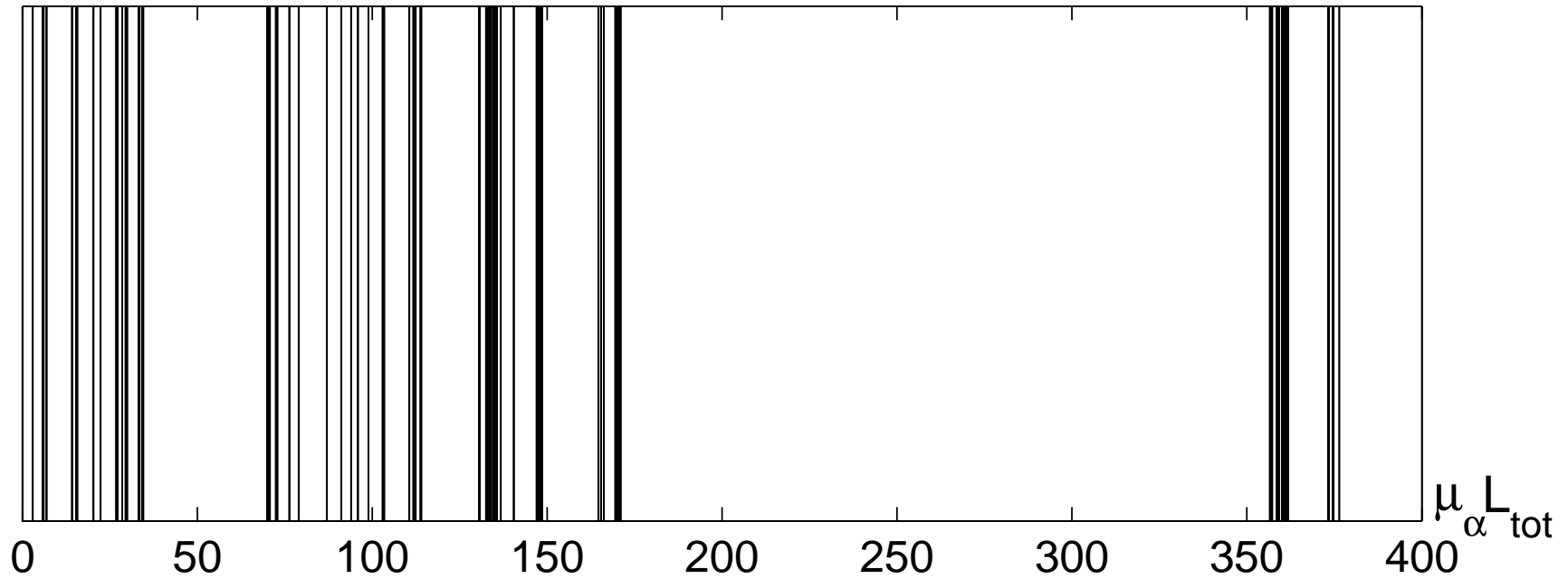
“Localization” of the spectrum



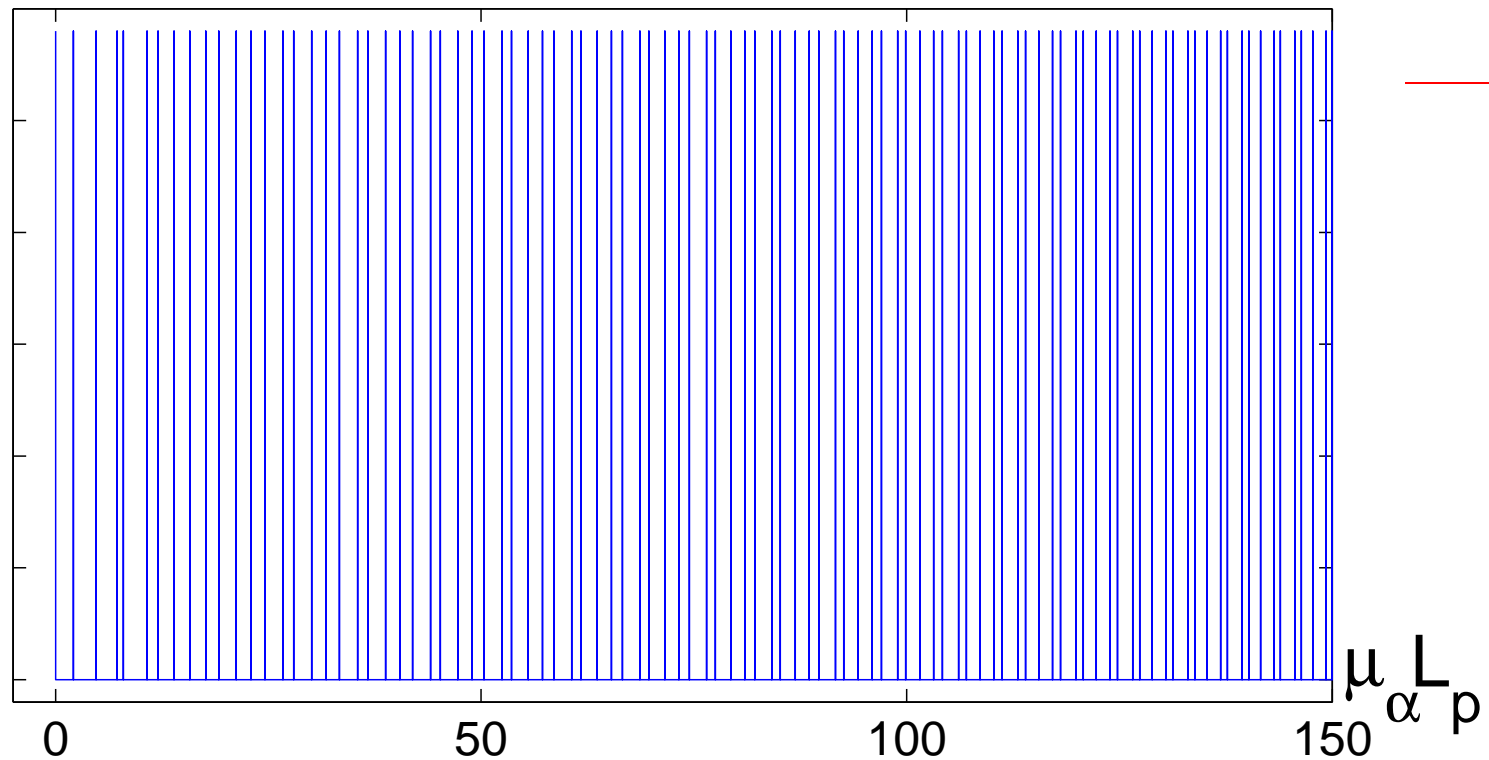
“Localization” of the spectrum



“Localization” of the spectrum

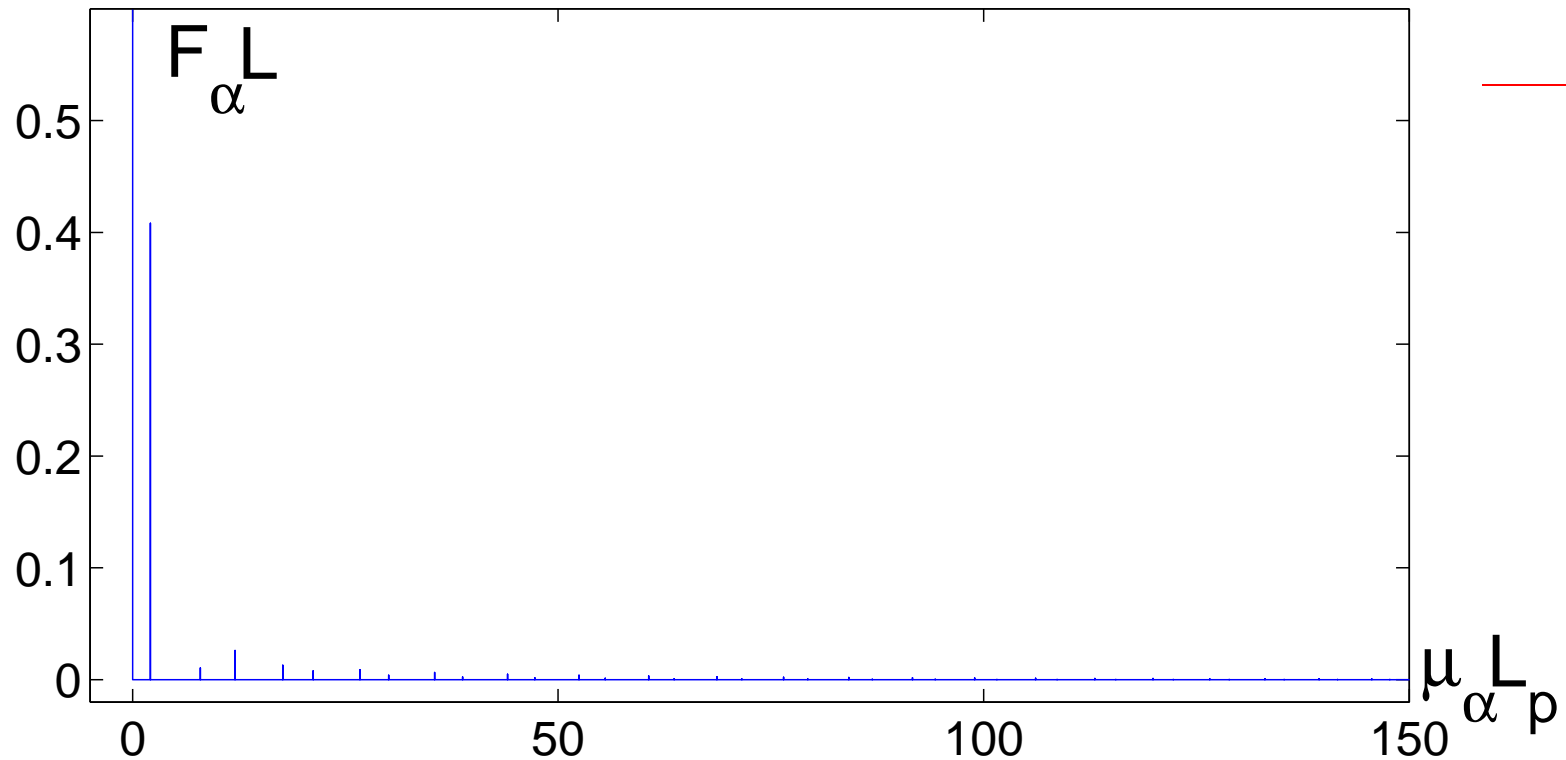


Reduction of eigenmodes



$$Z(\Lambda) = \frac{\Lambda}{D} \sum_{\alpha=0}^{\infty} \frac{F_{\alpha}}{1 + \mu_{\alpha} \Lambda}$$

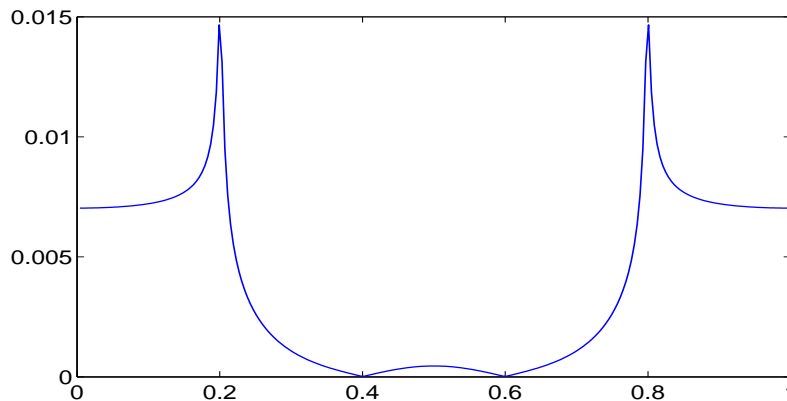
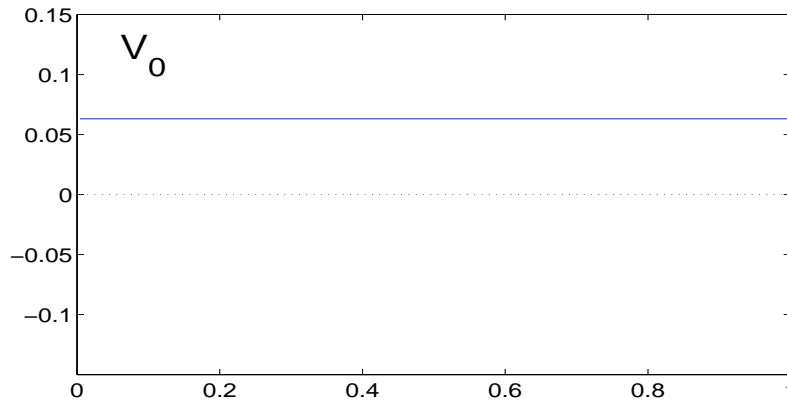
Reduction of eigenmodes



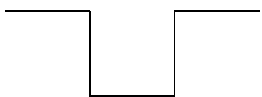
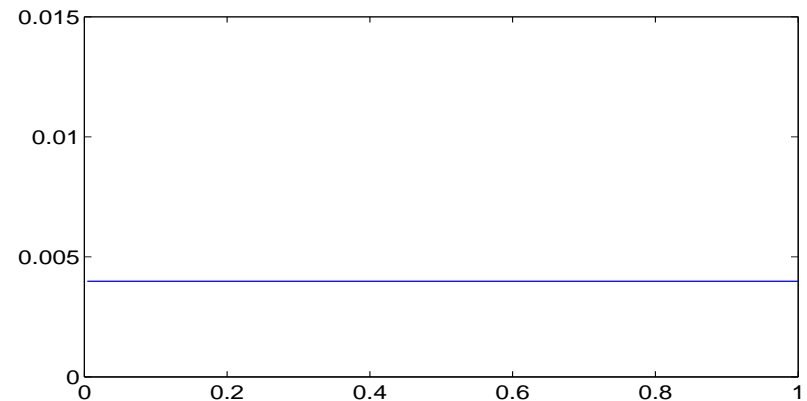
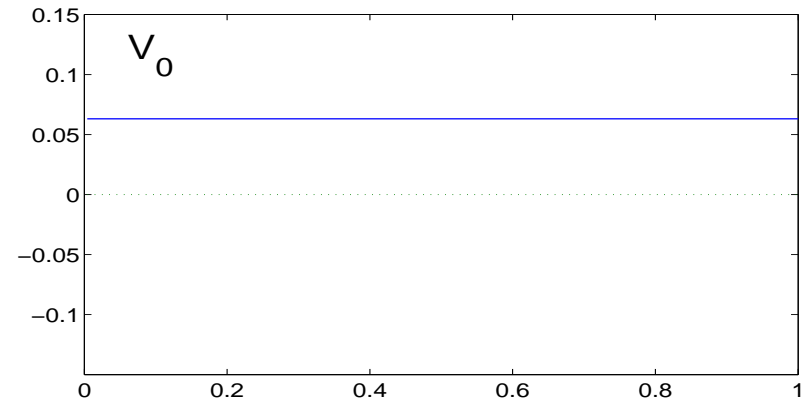
$$Z(\Lambda) \simeq \frac{\Lambda}{D} \left(\frac{F_0}{1 + \mu_0 \Lambda} + \frac{F_1}{1 + \mu_1 \Lambda} \right)$$

Eigenvectors

First generation



Plane surface (line)

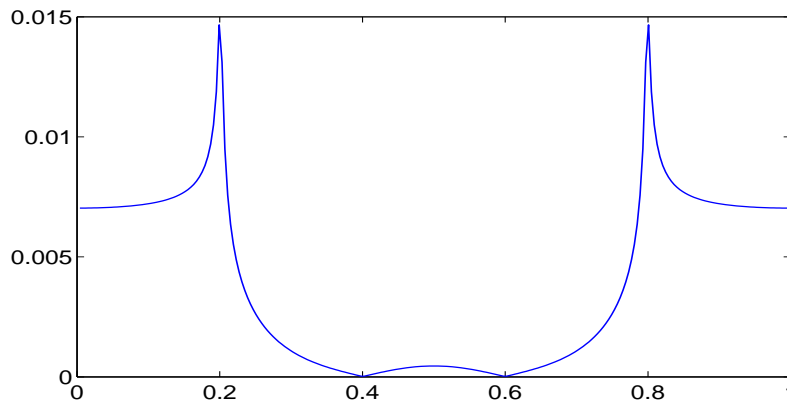
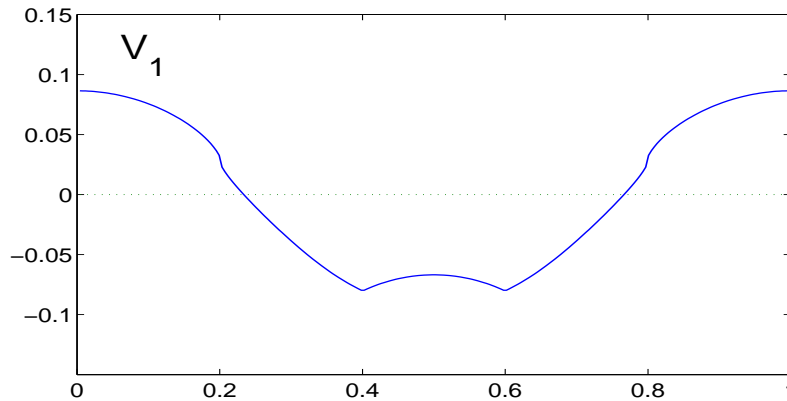


$$F_0 L_{tot} = 1$$

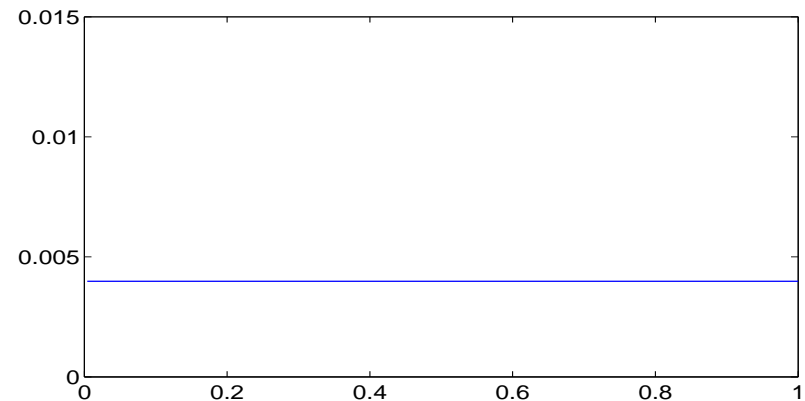
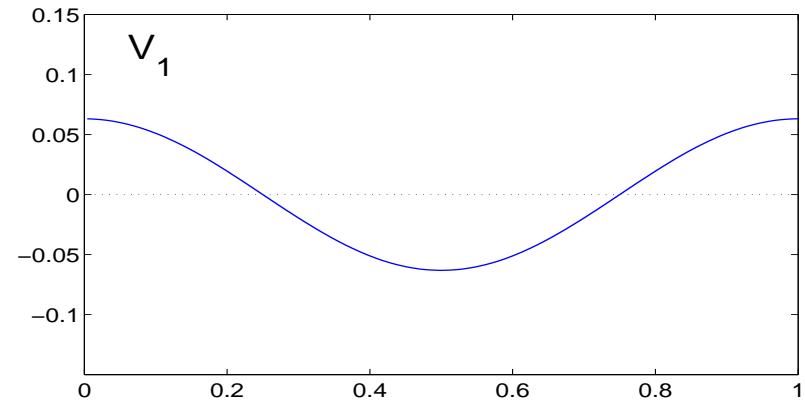
$$F_0 L_{tot} = 1$$

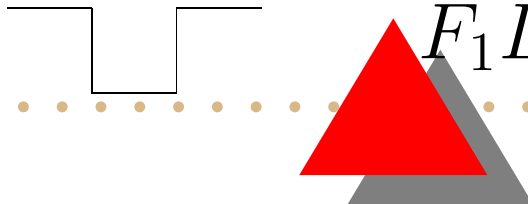
Eigenvectors

First generation



Plane surface (line)

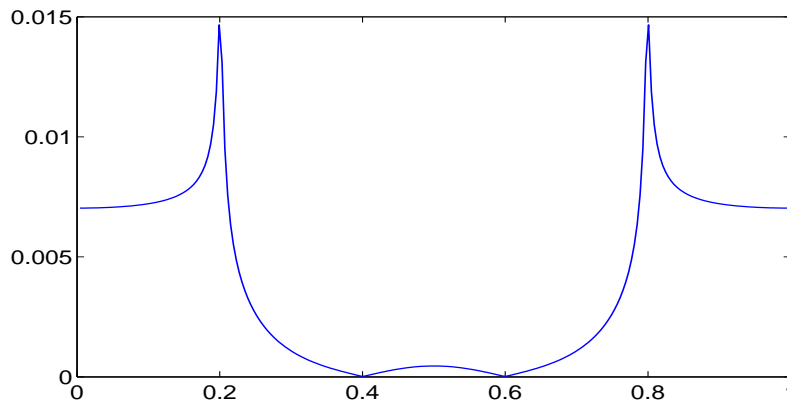
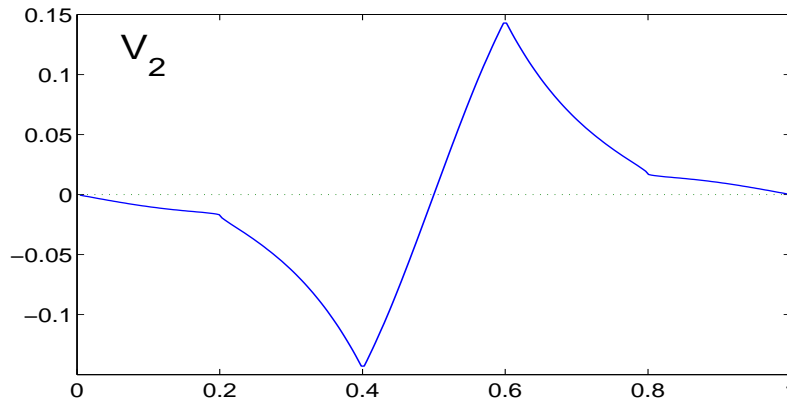



$$F_1 L_{tot} = 0.67$$

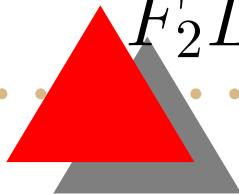
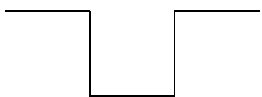
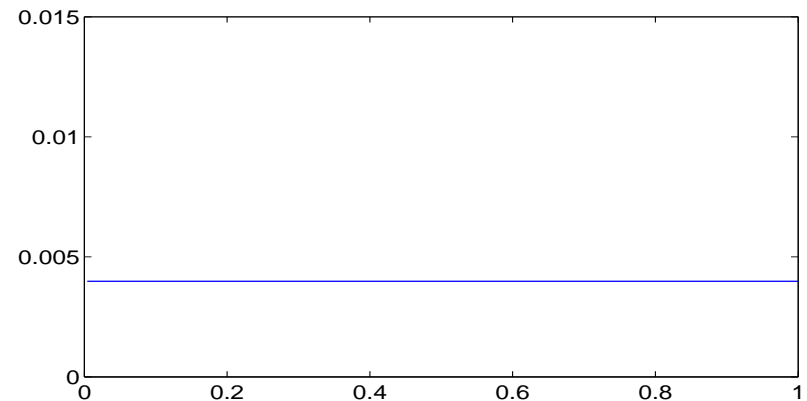
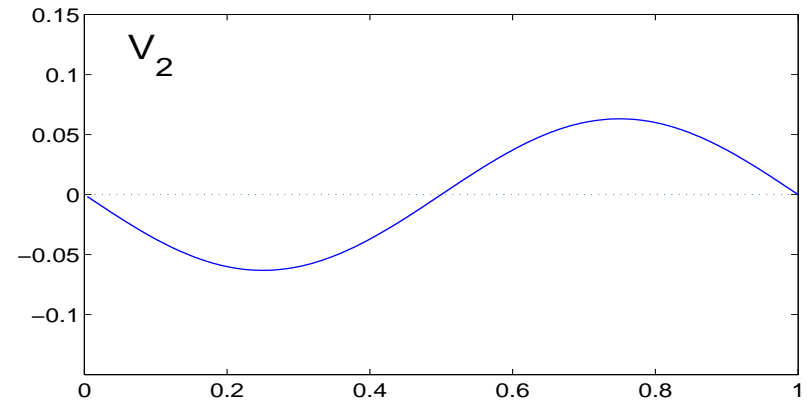
$$F_1 L_{tot} = 0$$

Eigenvectors

First generation



Plane surface (line)

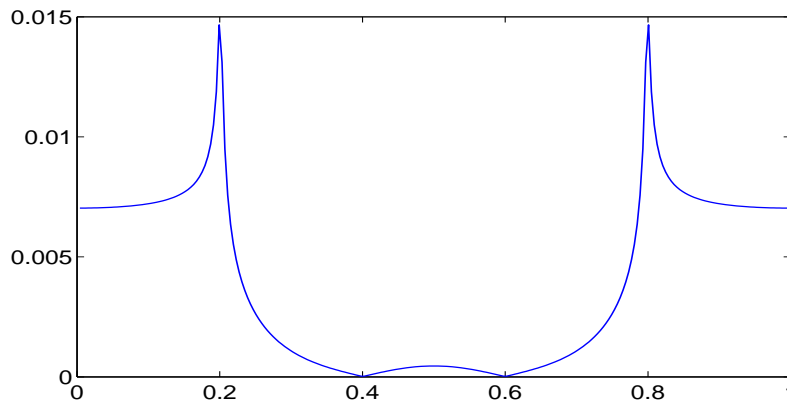
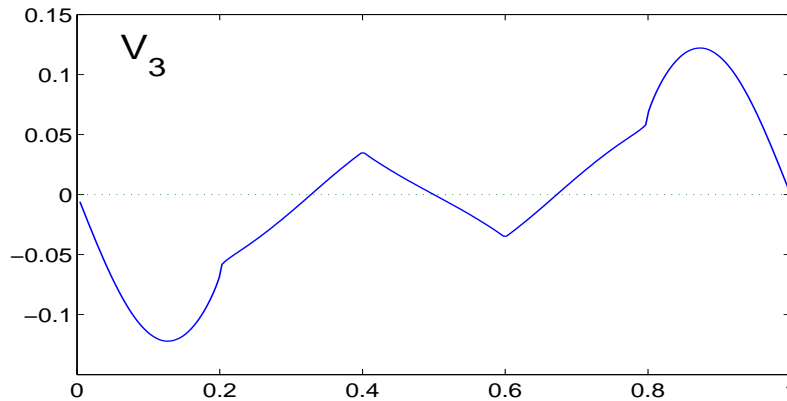


$$F_2 L_{tot} = 0$$

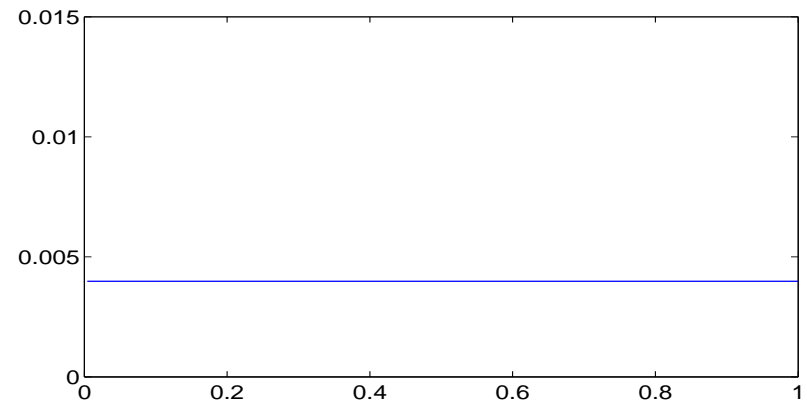
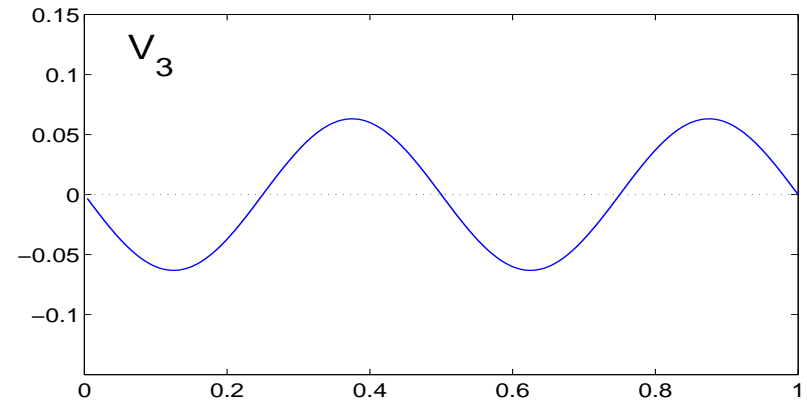
$$F_2 L_{tot} = 0$$

Eigenvectors

First generation



Plane surface (line)

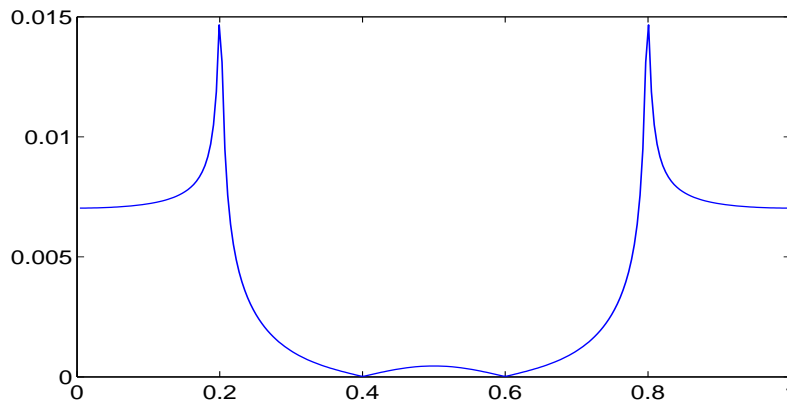
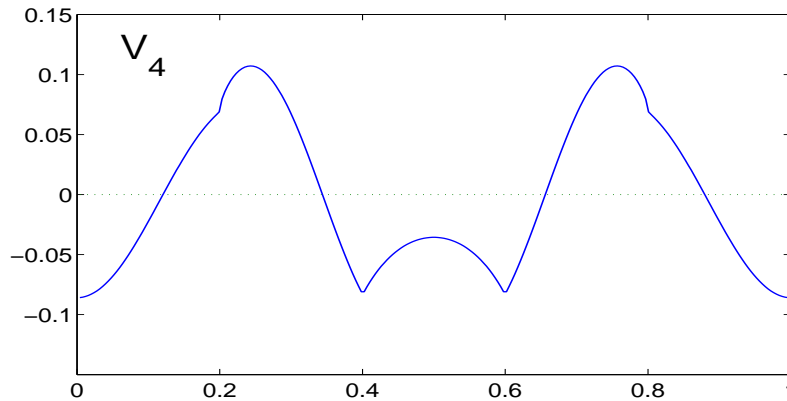


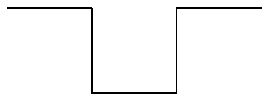

$$F_3 L_{tot} = 0$$

$$F_3 L_{tot} = 0$$

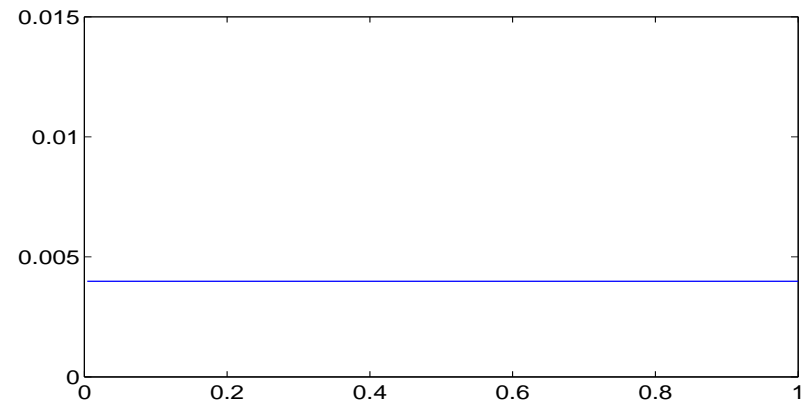
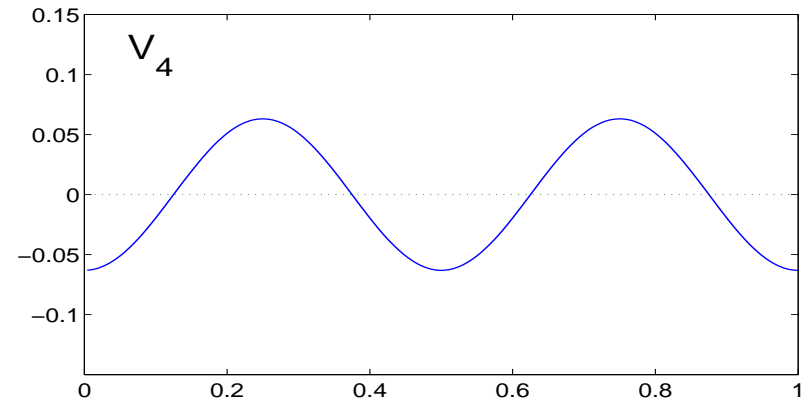
Eigenvectors

First generation




$$F_4 L_{tot} = 0.017$$

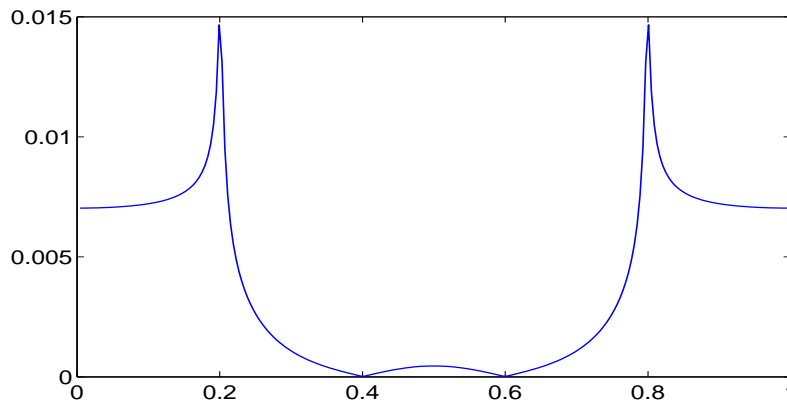
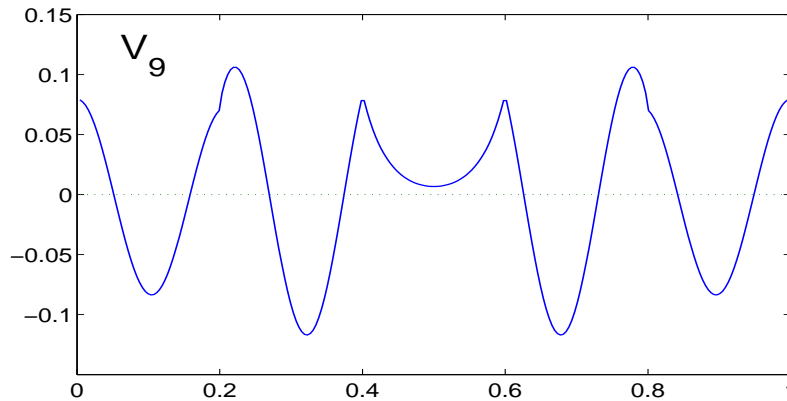
Plane surface (line)



$$F_4 L_{tot} = 0$$

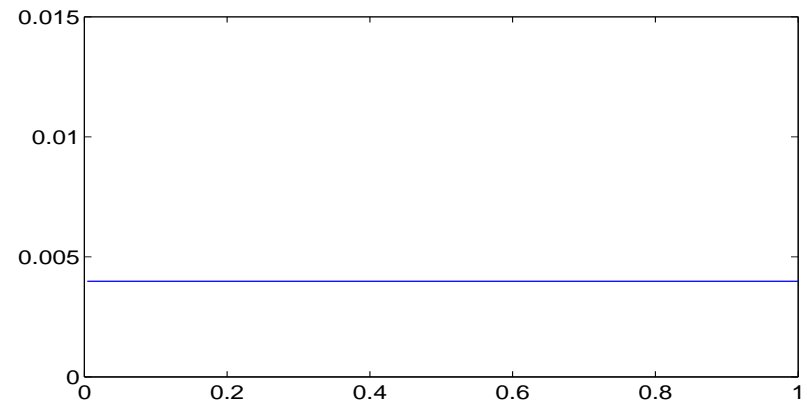
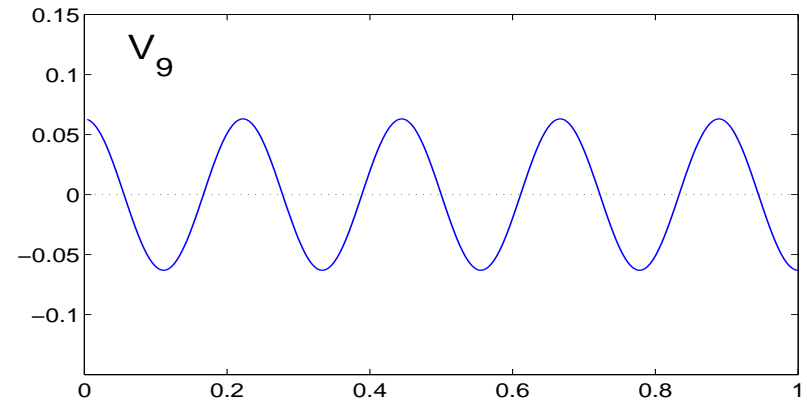
Eigenvectors

First generation



$$F_9 L_{tot} = 0.021$$

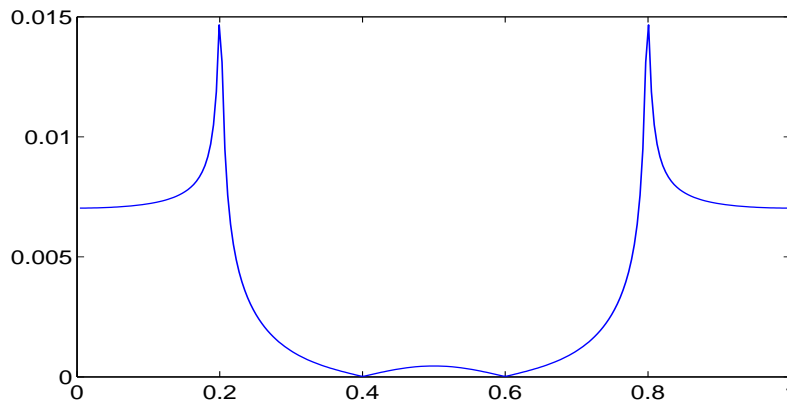
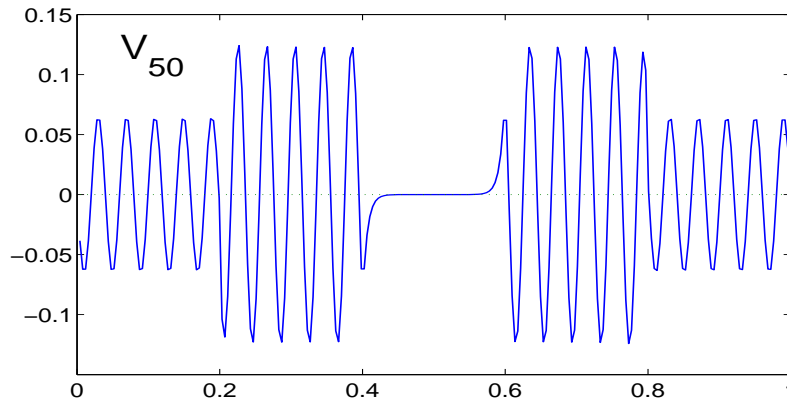
Plane surface (line)



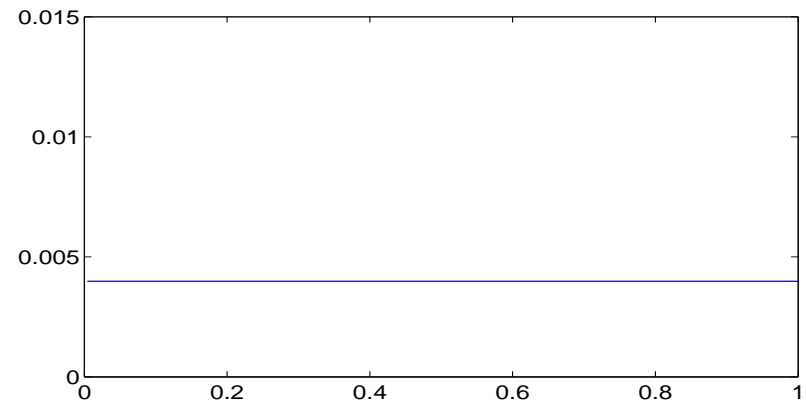
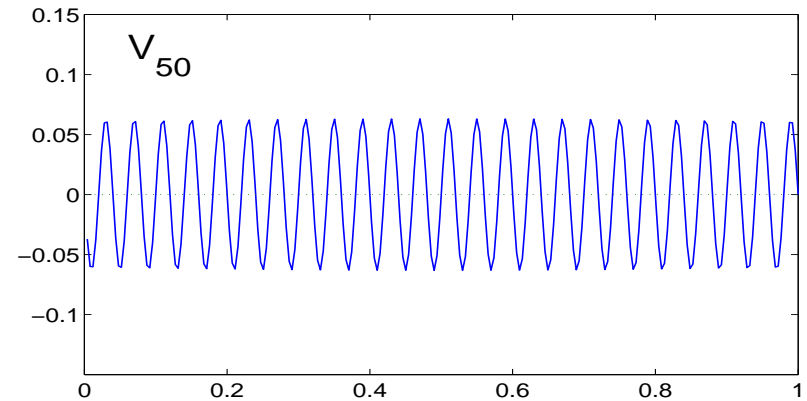
$$F_9 L_{tot} = 0$$

Eigenvectors

First generation



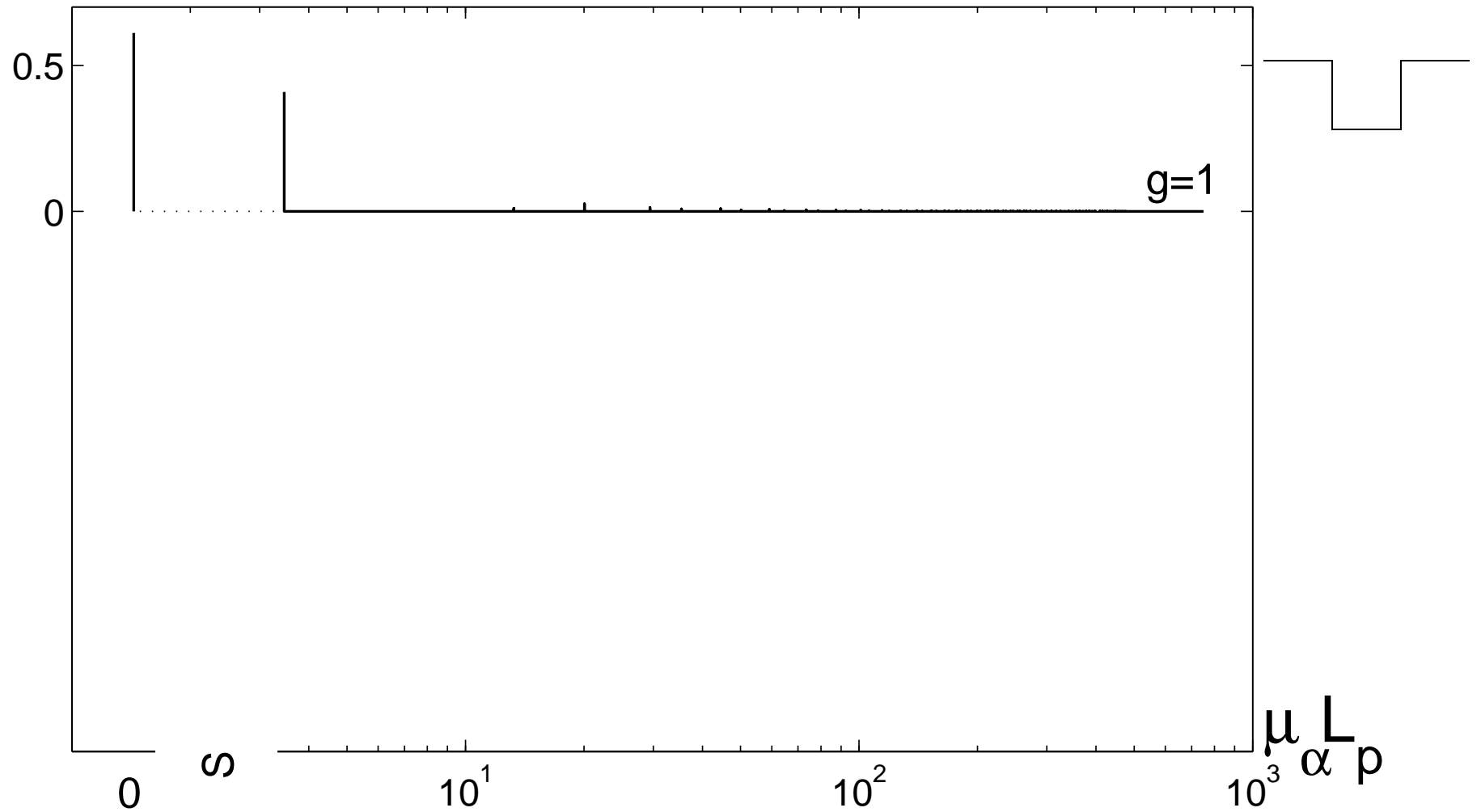
Plane surface (line)



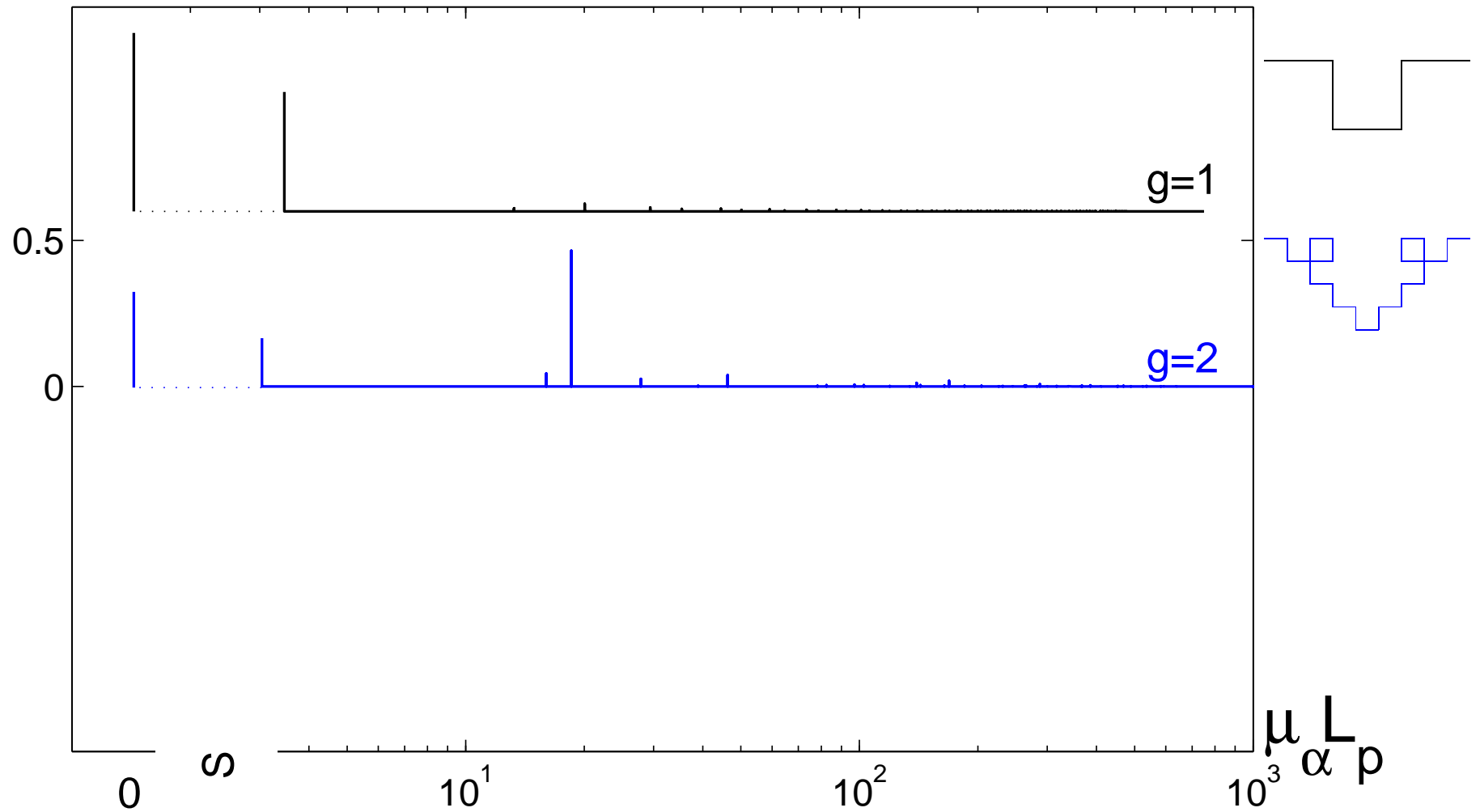

$$F_{50} L_{tot} = 0$$

$$F_{50} L_{tot} = 0$$

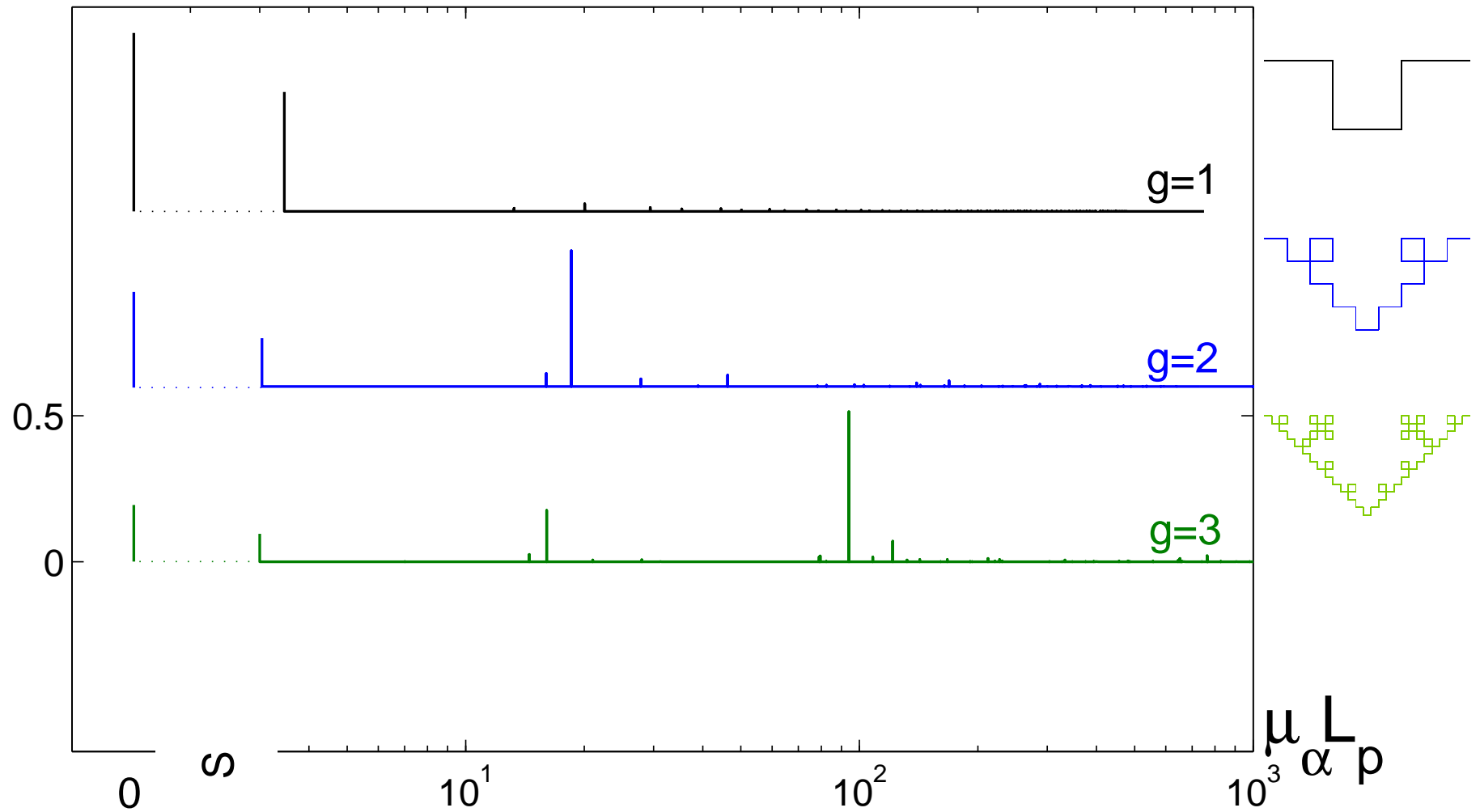
Reduction of eigenmodes



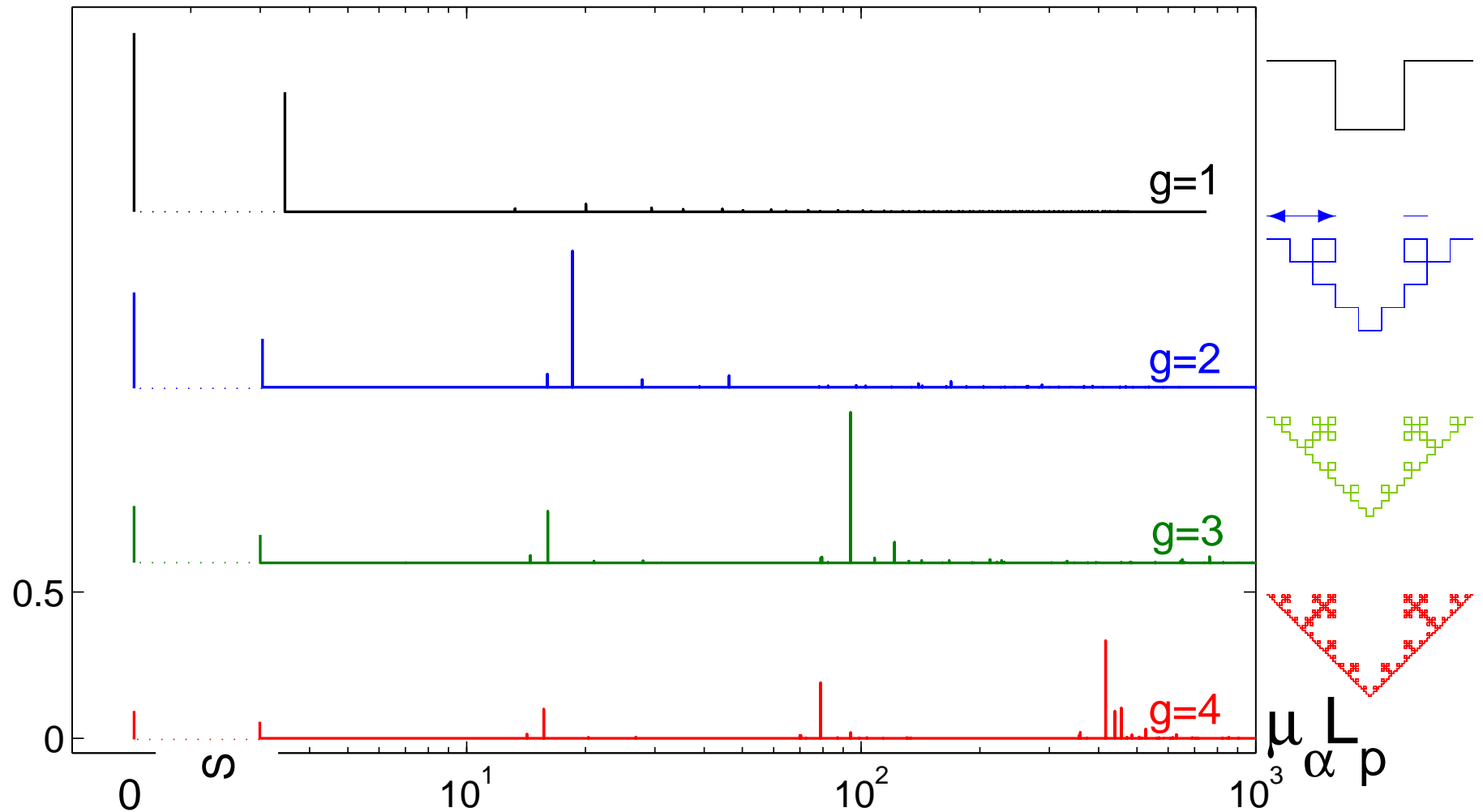
Reduction of eigenmodes



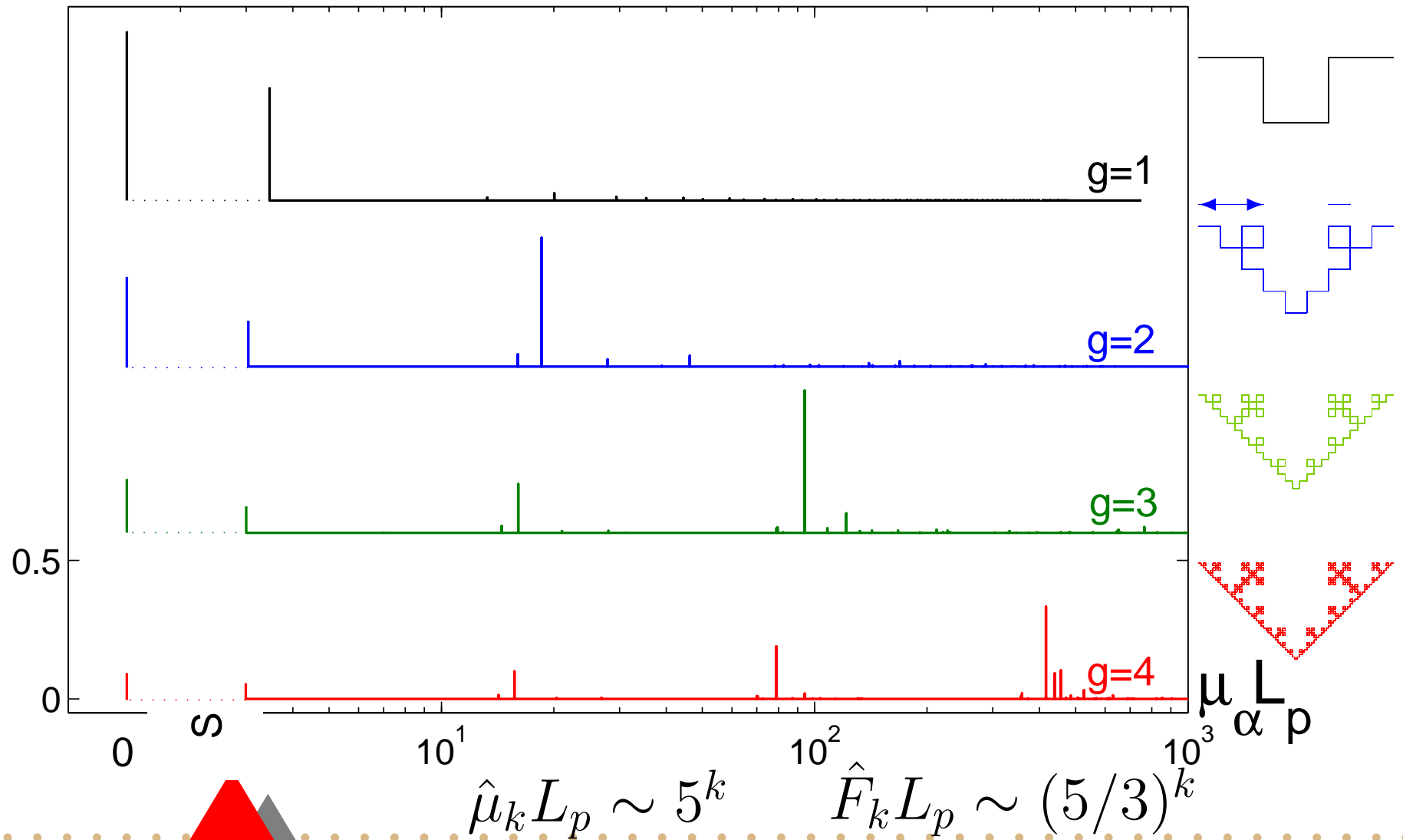
Reduction of eigenmodes



Reduction of eigenmodes



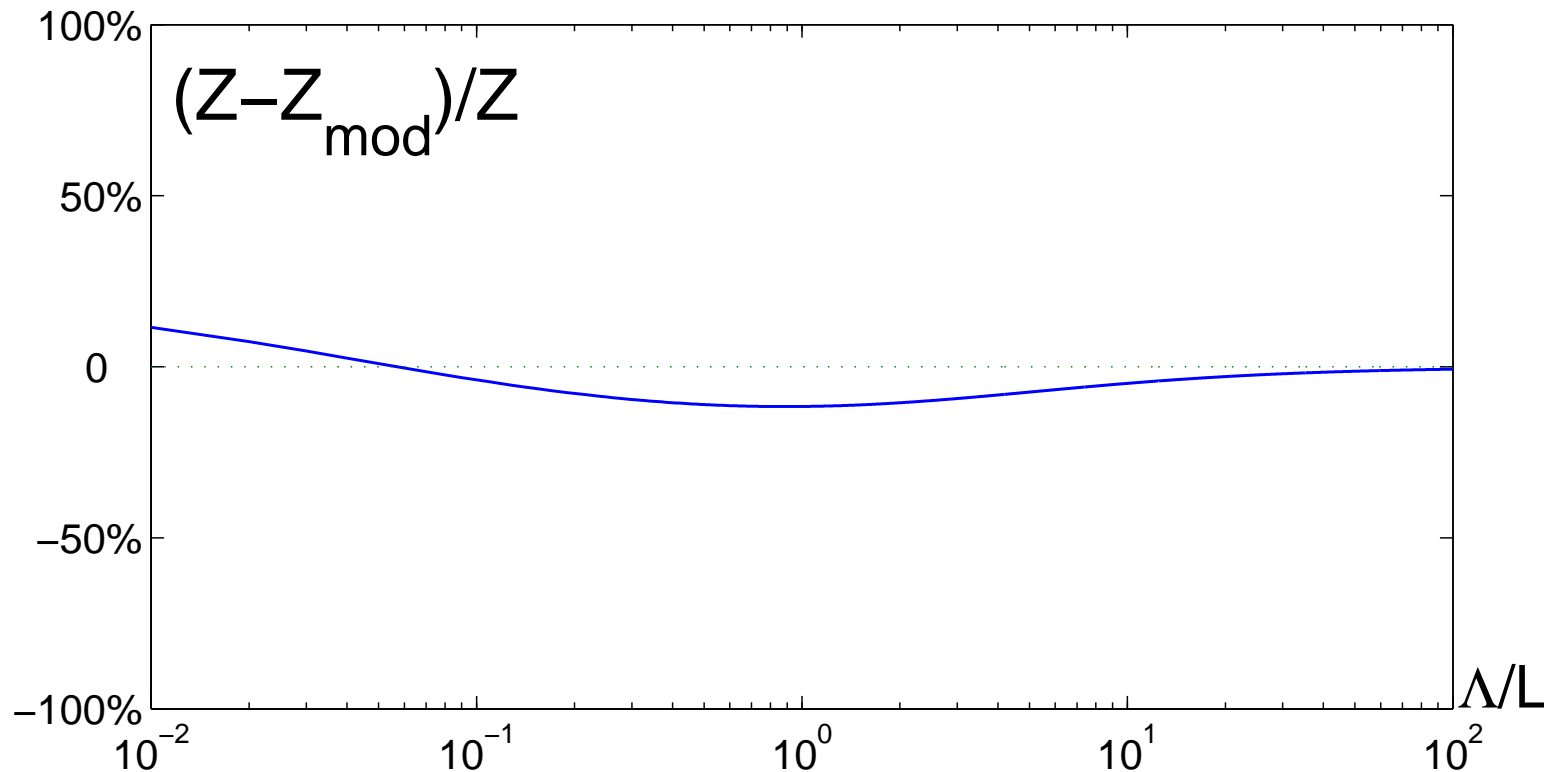
Reduction of eigenmodes



Spectroscopic impedance: model and calculations

Fourth generation:

$$Z_{mod}(\Lambda) = \frac{\Lambda}{D} \sum_{k=0}^4 \frac{\hat{F}_k}{1 + \Lambda \hat{\mu}_k}$$



Analytical model: anomalous behavior

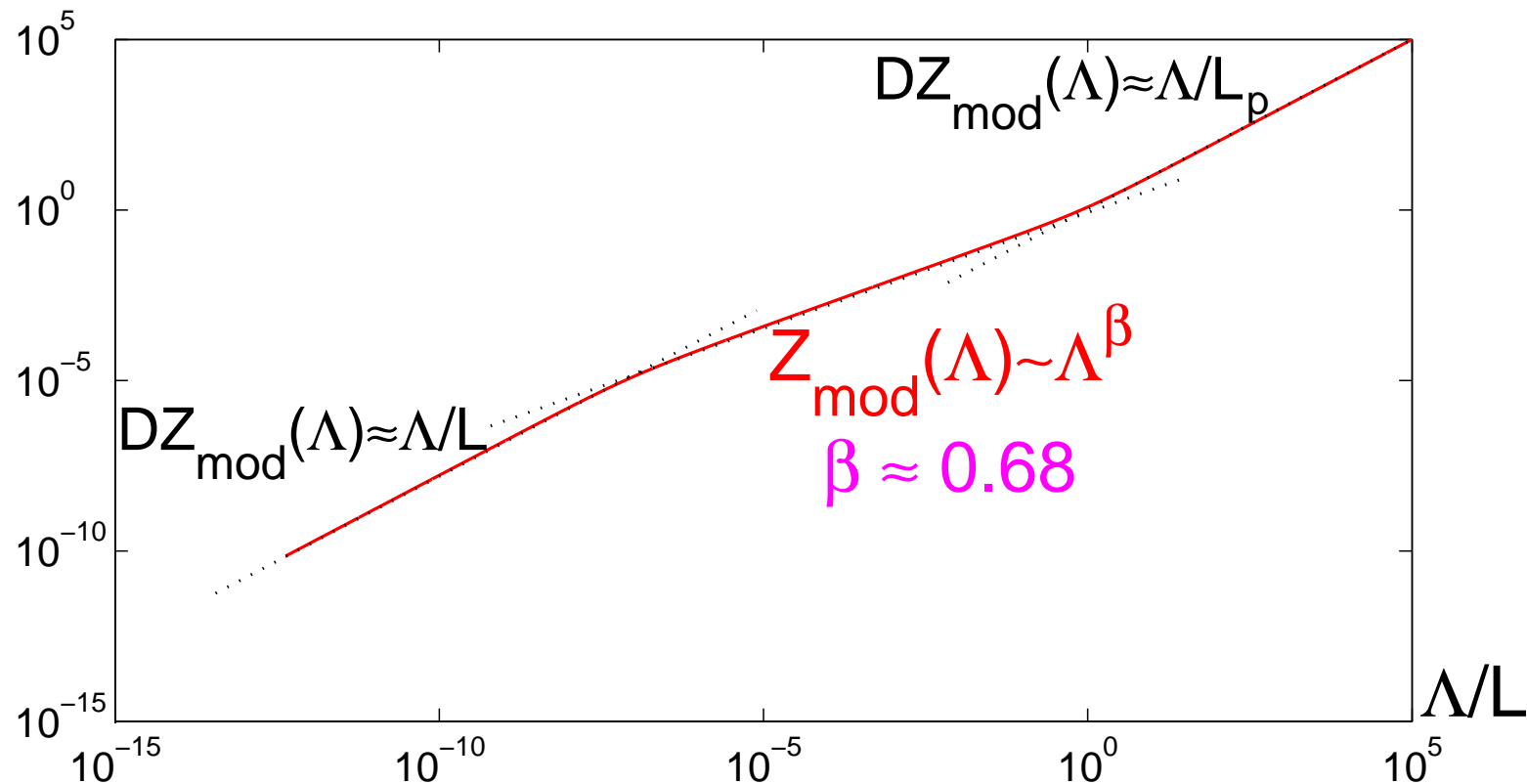
Tenth generation:

$$Z_{mod}(\Lambda) = \frac{\Lambda}{D} \sum_{k=0}^{10} \frac{\hat{F}_k}{1 + \Lambda \hat{\mu}_k}$$

Analytical model: anomalous behavior

Tenth generation:

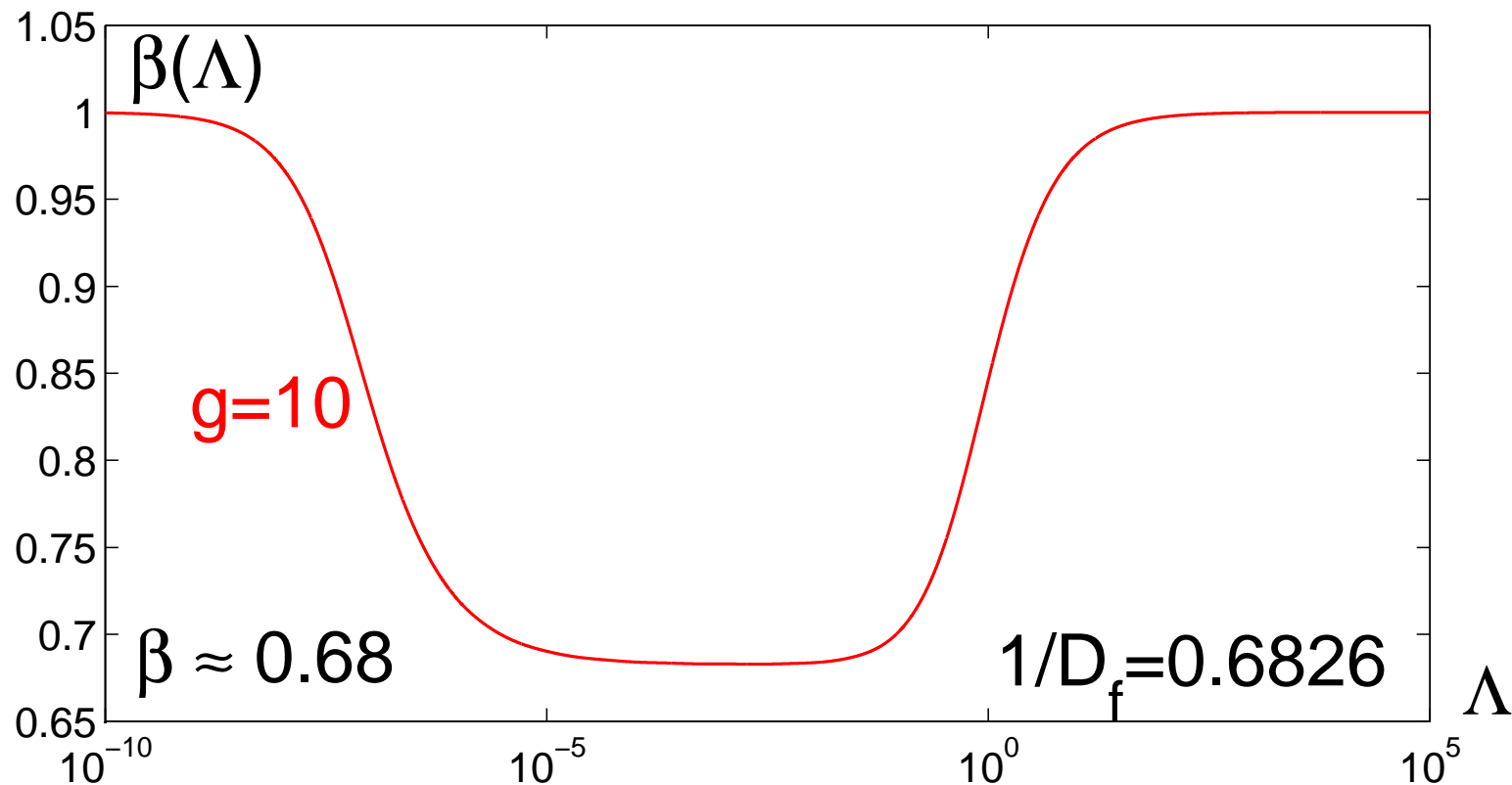
$$Z_{mod}(\Lambda) = \frac{\Lambda}{D} \sum_{k=0}^{10} \frac{\hat{F}_k}{1 + \Lambda \hat{\mu}_k}$$



Analytical model: anomalous behavior

Tenth generation:

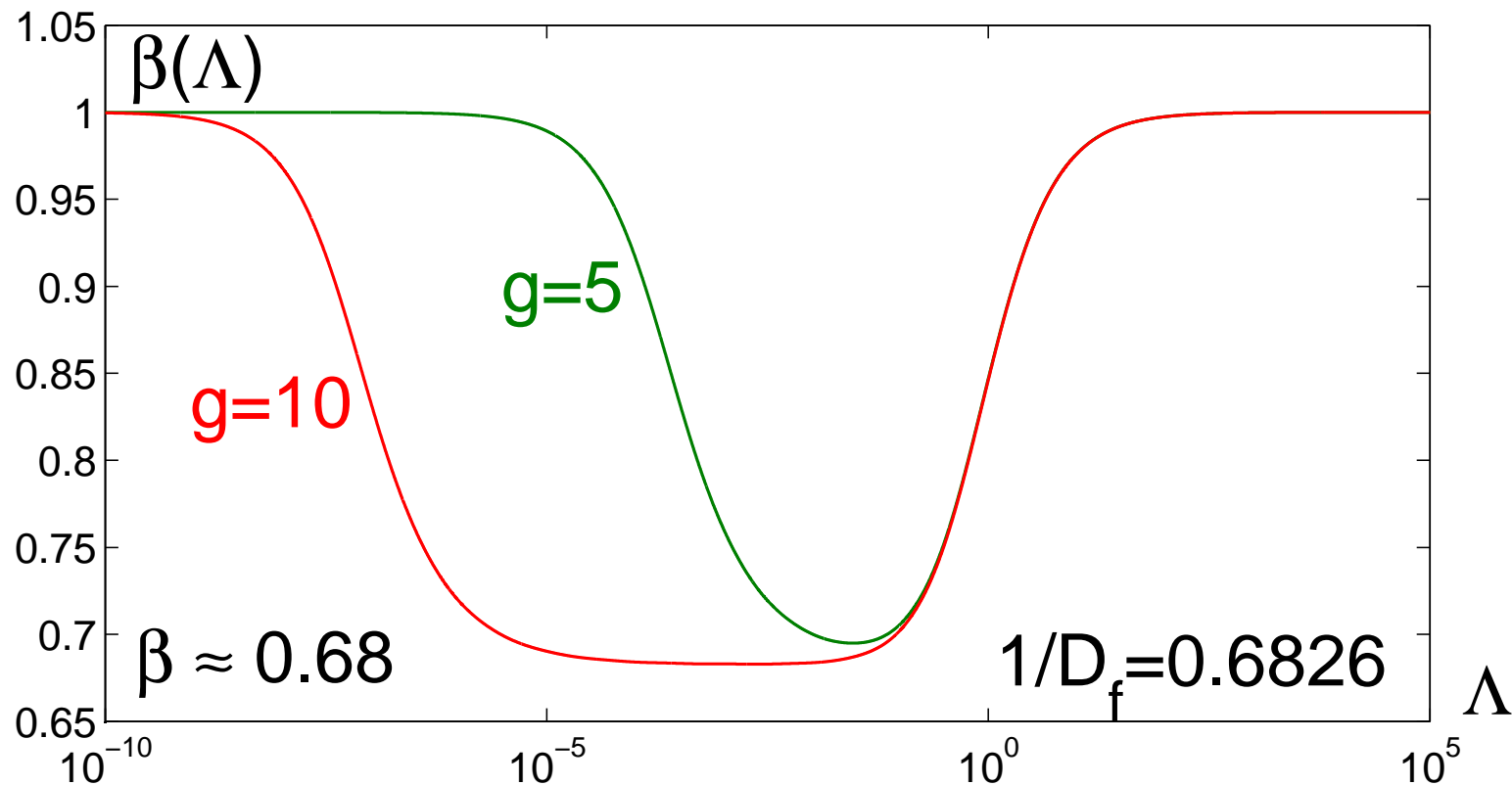
$$Z_{mod}(\Lambda) = \frac{\Lambda}{D} \sum_{k=0}^{10} \frac{\hat{F}_k}{1 + \Lambda \hat{\mu}_k}$$



Analytical model: anomalous behavior

Tenth generation:

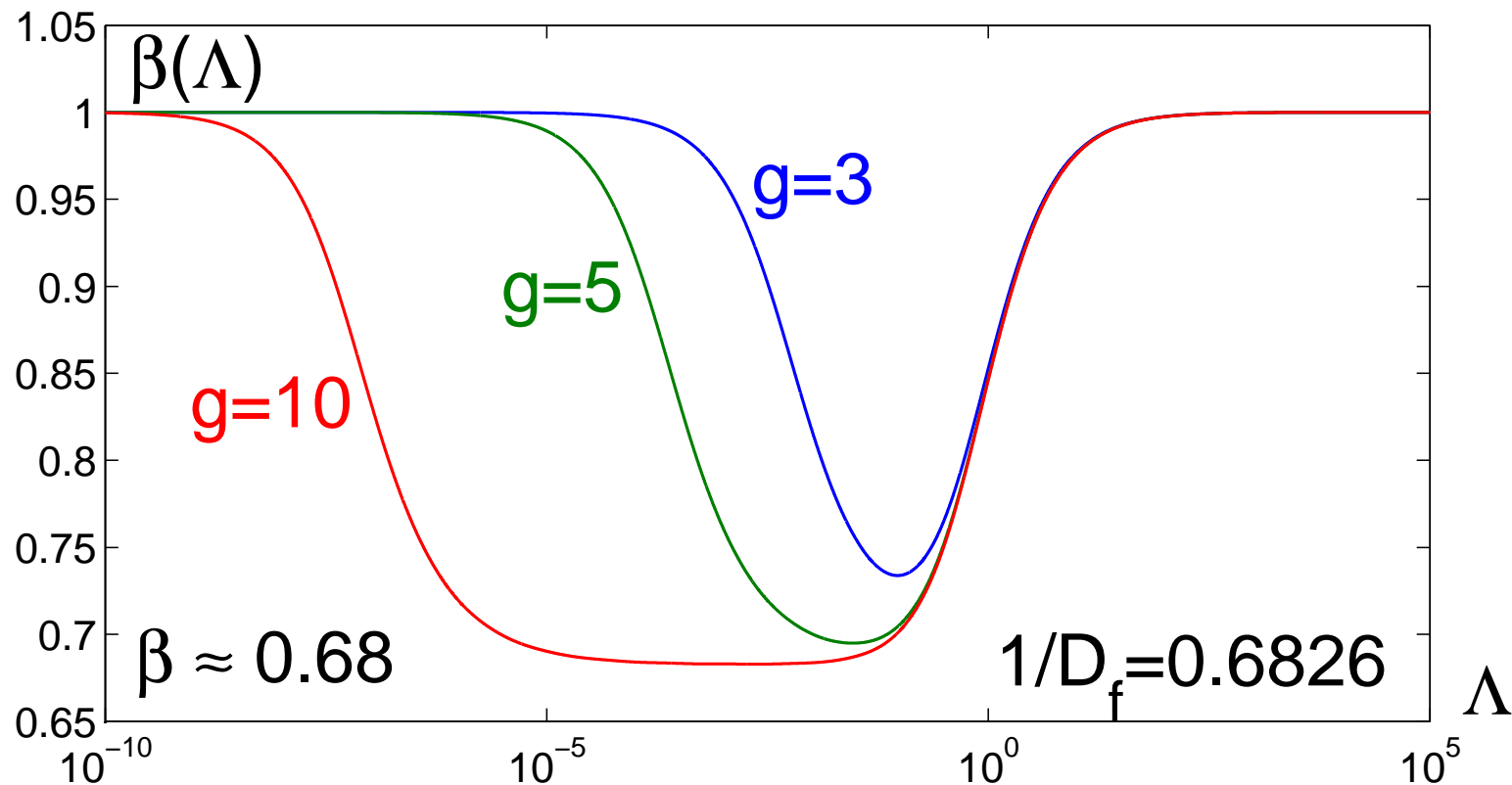
$$Z_{mod}(\Lambda) = \frac{\Lambda}{D} \sum_{k=0}^{10} \frac{\hat{F}_k}{1 + \Lambda \hat{\mu}_k}$$



Analytical model: anomalous behavior

Tenth generation:

$$Z_{mod}(\Lambda) = \frac{\Lambda}{D} \sum_{k=0}^{10} \frac{\hat{F}_k}{1 + \Lambda \hat{\mu}_k}$$



Irregular boundaries

- *Random Koch curves*

Irregular boundaries

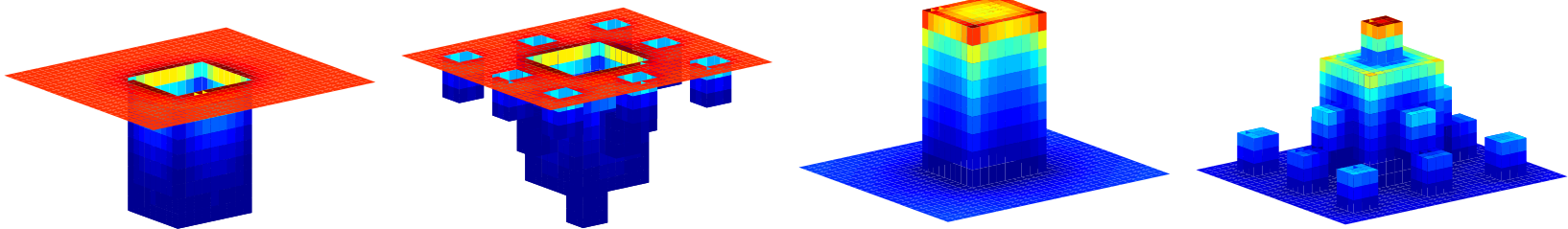
- *Random Koch curves*
- *Koch curves of different dimensions*

Irregular boundaries

- *Random Koch curves*
- *Koch curves of different dimensions*
- *Koch surfaces*

Irregular boundaries

- *Random Koch curves*
- *Koch curves of different dimensions*
- *Koch surfaces*



Irregular boundaries

- *Random Koch curves*
- *Koch curves of different dimensions*
- *Koch surfaces*
- Effect of eigenmodes reduction

Irregular boundaries

- *Random Koch curves*
- *Koch curves of different dimensions*
- *Koch surfaces*

- Effect of eigenmodes reduction
- Hierarchy of eigenvalues

Irregular boundaries

- *Random Koch curves*
- *Koch curves of different dimensions*
- *Koch surfaces*

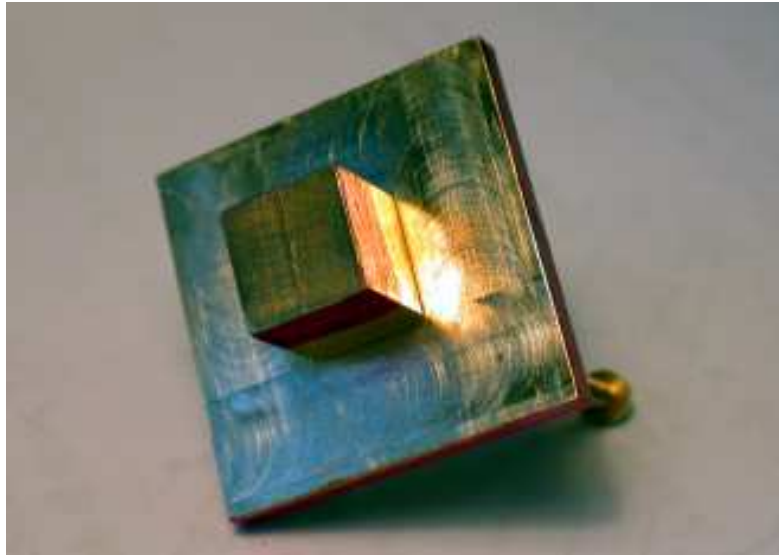
- Effect of eigenmodes reduction
- Hierarchy of eigenvalues
- Analytical model of the impedance

Experimental study

How does the geometrical irregularity influence impedance behavior of real electrodes?

Experimental study

How does the geometrical irregularity influence impedance behavior of real electrodes?



Classical theory of double layer

Electric transport:

$\Delta V = 0$ in the volume

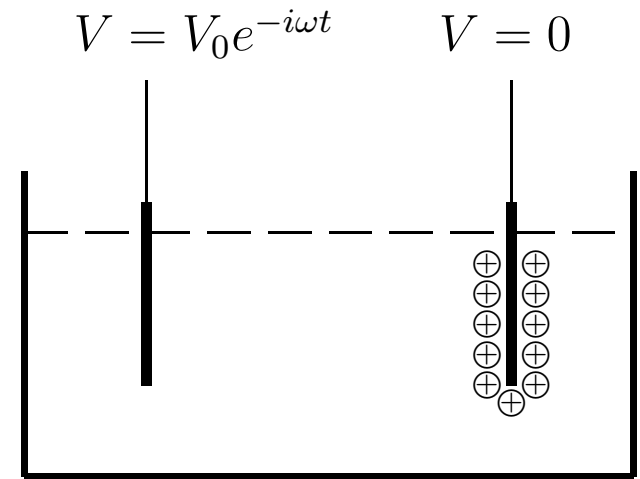
$V = V_0$ on the counter-electrode

$\frac{\partial V}{\partial n} = \frac{1}{\Lambda} V$ on the working electrode

$$\Lambda = \frac{\zeta(\omega)}{\rho}$$

where $\zeta(\omega)$ is the surface impedance

$$Z(\Lambda) = \rho \Lambda \sum_{\alpha=0}^{\infty} \frac{F_{\alpha}}{1 + \mu_{\alpha} \Lambda}$$



Practical implementation

- One measures the **physical** characteristics, ρ and $\zeta(\omega)$, to obtain $\Lambda = \zeta(\omega)/\rho$;

Practical implementation

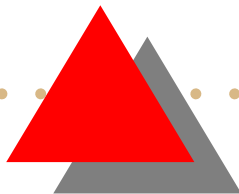
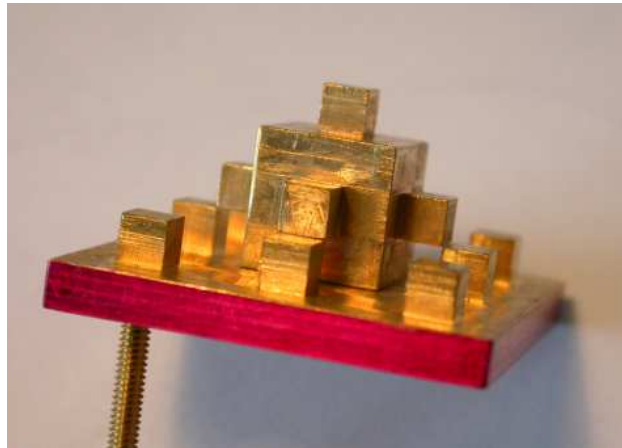
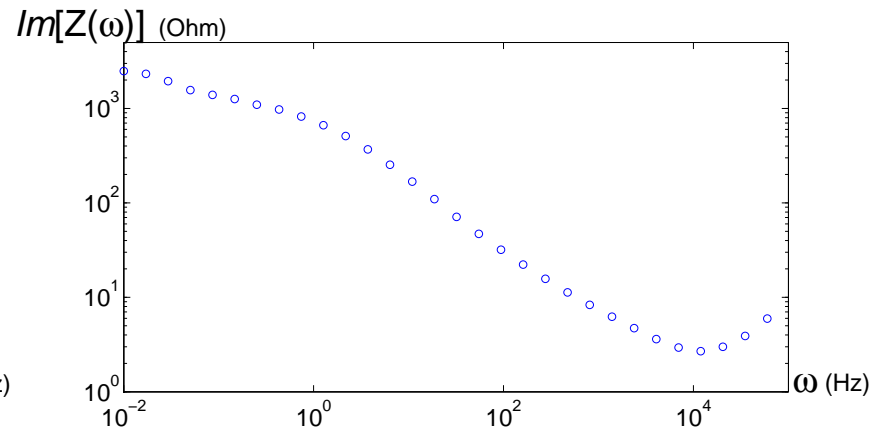
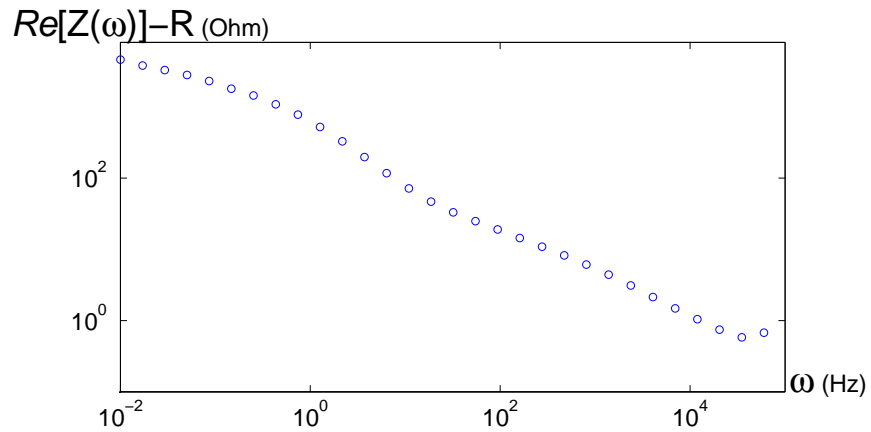
- One measures the **physical** characteristics, ρ and $\zeta(\omega)$, to obtain $\Lambda = \zeta(\omega)/\rho$;
- One finds the **geometrical** characteristics, μ_α and F_α ;

Practical implementation

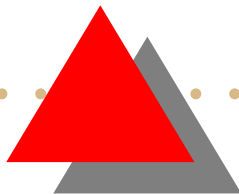
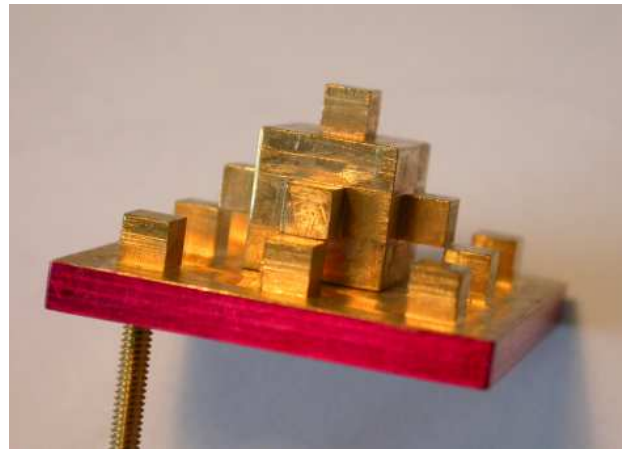
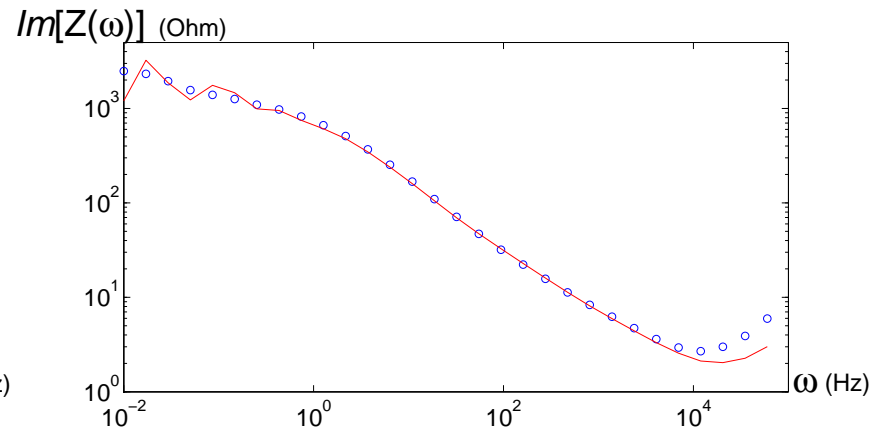
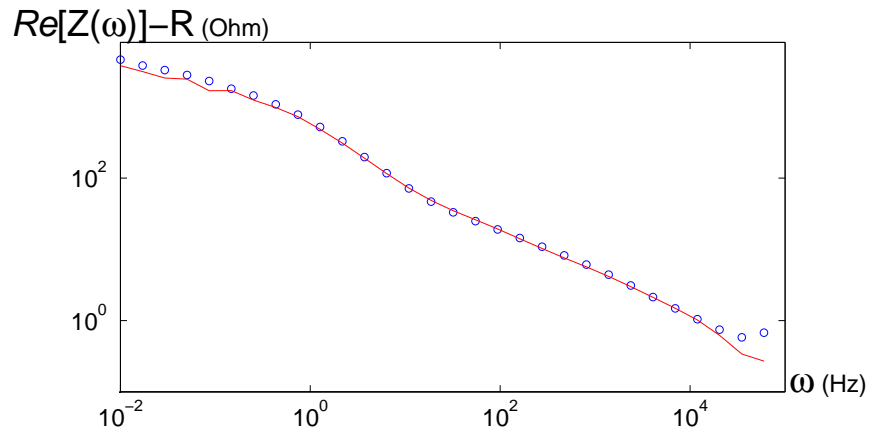
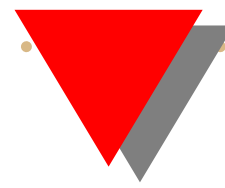
- One measures the **physical** characteristics, ρ and $\zeta(\omega)$, to obtain $\Lambda = \zeta(\omega)/\rho$;
- One finds the **geometrical** characteristics, μ_α and F_α ;
- One substitutes them into the spectral decomposition

$$Z(\Lambda) = \rho\Lambda \sum_{\alpha=0}^{\infty} \frac{F_\alpha}{1 + \mu_\alpha\Lambda}$$

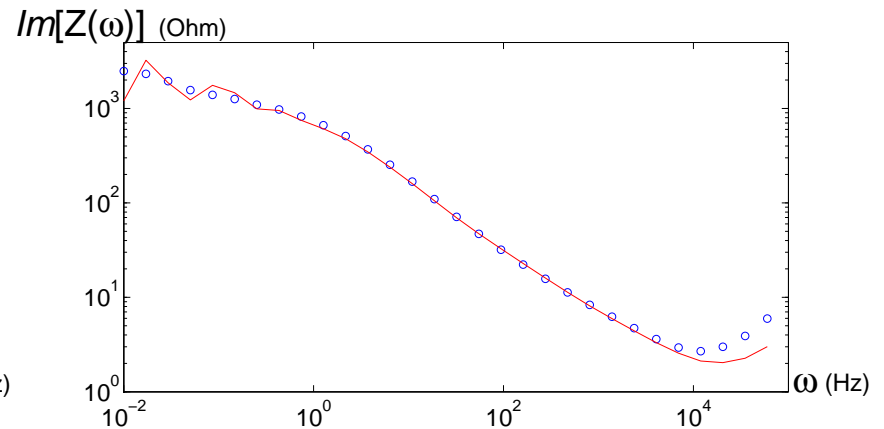
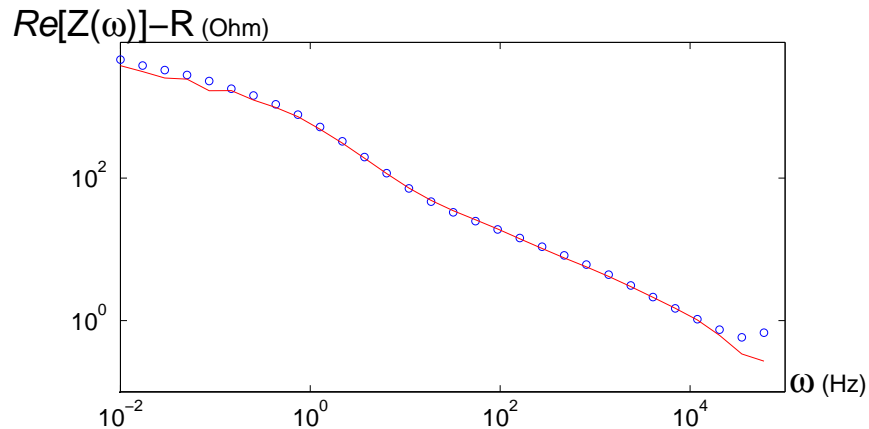
Second generation: experiment and theory



Second generation: experiment and theory



Second generation: experiment and theory

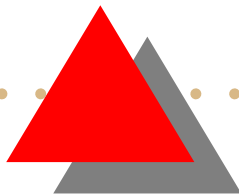


In conclusion, without knowledge of the microscopic transport mechanism, the geometrical irregularity can be taken into account by our theoretical approach



Theoretical results

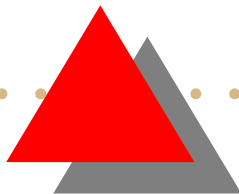
- Theoretical description of Laplacian transport towards irregular interfaces based on the Dirichlet-to-Neumann operator and study of the partially reflected Brownian motion
- Implementation of the (discrete) boundary element method for construction of the Dirichlet-to-Neumann operator





Numerical results

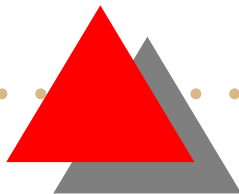
- Effective contribution of a small number of eigenmodes to linear response of the system
- Hierarchy of these significant eigenmodes (characteristic scales of the boundary)
- Analytical model of the impedance





Experimental results

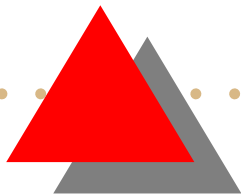
- Measurement of the impedance of brass and nickel electrodes with boundaries reproducing two generations of the convex Von Koch surface
- Observation of anomalous behavior of the impedance in the case of planar electrodes
- Accounting for geometrical irregularity by theoretical approach





Construction of the Dirichlet-to-Neumann operator

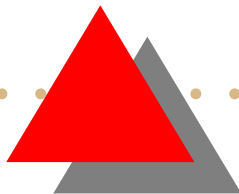
- **Present approach:** A kind of the boundary element method based on potential theory





Construction of the Dirichlet-to-Neumann operator

- **Present approach:** A kind of the boundary element method based on potential theory
- **Main problem:** one has to calculate all the elements of the matrix representing this operator in a discrete form



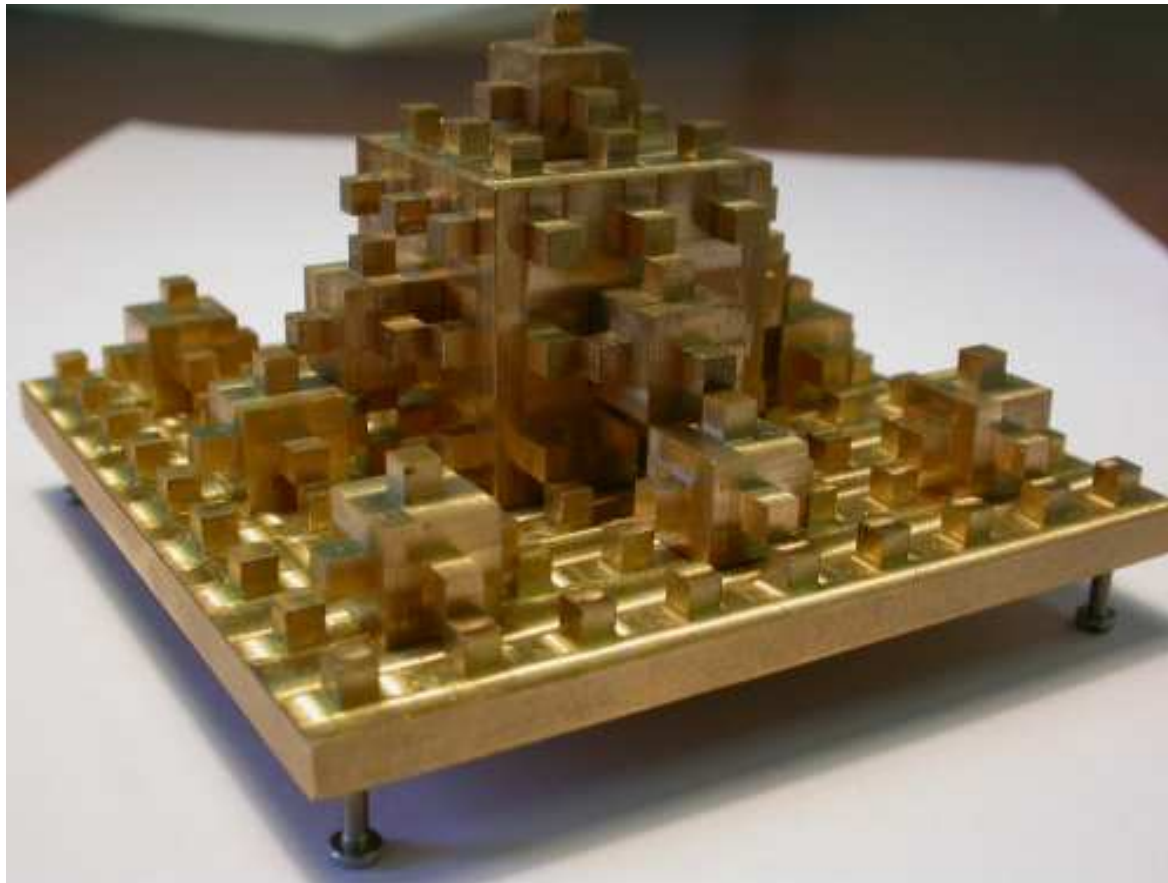
Construction of the Dirichlet-to-Neumann operator

- **Present approach:** A kind of the boundary element method based on potential theory
- **Main problem:** one has to calculate all the elements of the matrix representing this operator in a discrete form
- **Reality:** Only a small fraction of the eigenmodes is in fact needed

Construction of the Dirichlet-to-Neumann operator

We are looking for a numerical method allowing to compute the eigenmodes of the operator \mathcal{M} successively, without constructing the whole operator

Construction of the Dirichlet-to-Neumann operator



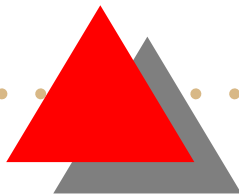
Perspectives

- Extensions of the theoretical approach:



Perspectives

- Extensions of the theoretical approach:
 - other equations (time-dependent diffusion)



Perspectives

- Extensions of the theoretical approach:
 - other equations (time-dependent diffusion)
 - other conditions (non linear)

Perspectives

- Extensions of the theoretical approach
- Numerical study of higher orders generations, especially in 3 dimensions

Perspectives

- Extensions of the theoretical approach
- Numerical study of higher orders generations, especially in 3 dimensions
- Analytical model of the impedance for random interfaces?

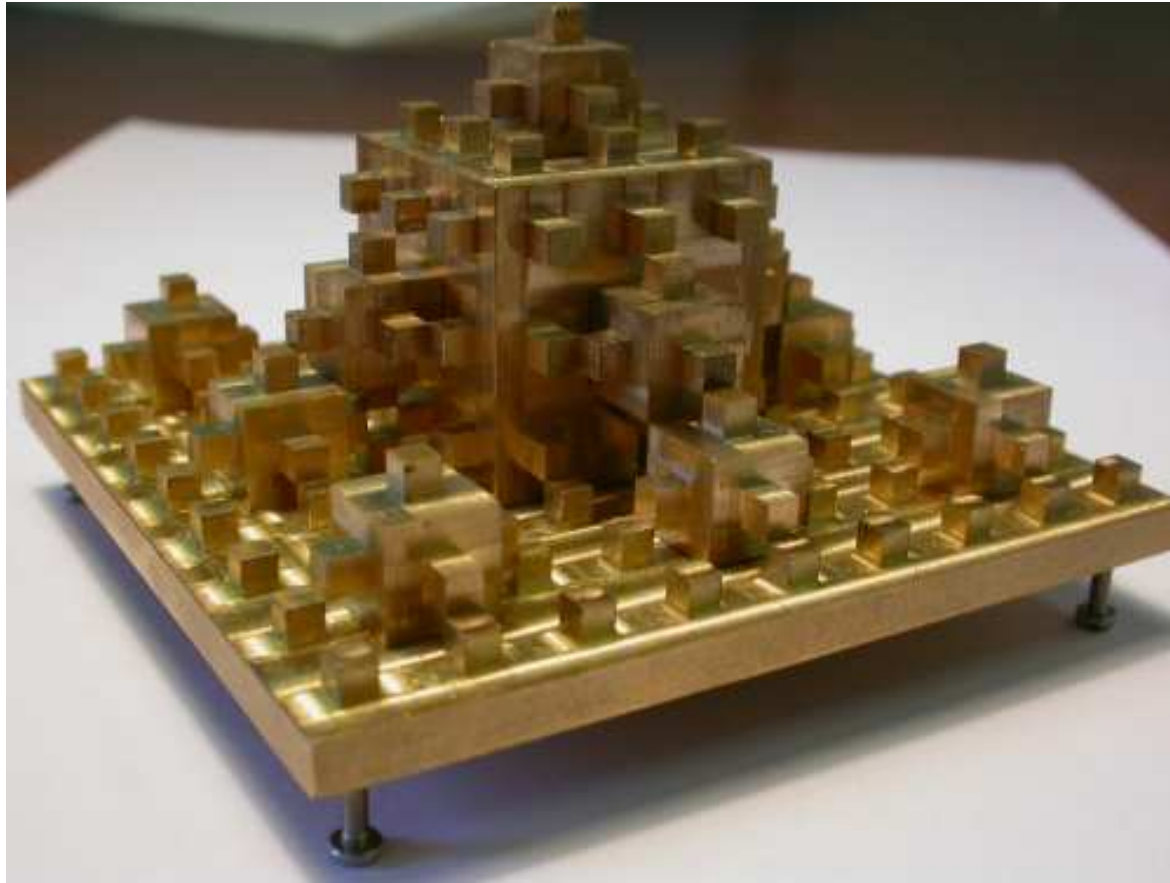
Perspectives

- Extensions of the theoretical approach
- Numerical study of higher orders generations, especially in 3 dimensions
- Analytical model of the impedance for random interfaces?
- Inverse problem: knowing the spectroscopic impedance, what can one say about the geometry?

Perspectives

- Extensions of the theoretical approach
- Numerical study of higher orders generations, especially in 3 dimensions
- Analytical model of the impedance for random interfaces?
- Inverse problem: knowing the spectroscopic impedance, what can one say about the geometry?
- Experimental study of irregular electrodes

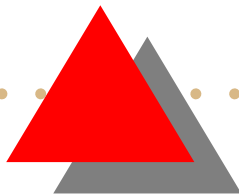
Perspectives





Open problems

- Justification of the approach for really irregular interfaces



Open problems

- Justification of the approach for really irregular interfaces
- Reduction of the number of contributing eigenmodes

Open problems

- Justification of the approach for really irregular interfaces
- Reduction of the number of contributing eigenmodes
- Relation between the eigenvalues and the characteristic length scales of the boundary

Open problems

- Justification of the approach for really irregular interfaces
- Reduction of the number of contributing eigenmodes
- Relation between the eigenvalues and the characteristic length scales of the boundary
- Correction, generalization and mathematical justification for the simplified impedance model

$$\beta = \frac{1}{D_f} \implies \beta = \frac{\tau(2)}{D_f}$$



Related papers

1. B. Sapoval, *Transport Across Irregular Interfaces: Fractal Electrodes, Membranes and Catalysts*, in “Fractals and Disordered Systems”, Eds. A. Bunde, S. Havlin, pp. 233-261 (Springer, 1996).
2. B. Sapoval, *General Formulation of Laplacian Transfer Across Irregular Surfaces*, Phys. Rev. Lett. 73, 3314 (1994).
3. M. Filoche and B. Sapoval, *Can One Hear the Shape of an Electrode? II. Theoretical Study of the Laplacian Transfer*, Eur. Phys. J. B 9, 755 (1999).
4. D. S. Grebenkov, M. Filoche, B. Sapoval, *Spectral Properties of the Brownian Self-Transport Operator*, Eur. Phys. J. B 36, 221-231 (2003).
5. D. S. Grebenkov, M. Filoche, B. Sapoval, *Mathematical Basis for a General Theory of Laplacian Transport towards Irregular Interfaces*, Phys. Rev. E 73, 021103 (2006).
6. D. S. Grebenkov, M. Filoche, B. Sapoval, *A Simplified Analytical Model for Laplacian Transfer Across Deterministic Prefractal Interfaces*, Fractals (2007, in press)
7. D. S. Grebenkov, *Partially Reflected Brownian Motion: A Stochastic Approach to Transport Phenomena*, in “Focus on Probability Theory”, Ed. L. R. Velle, pp. 135-169 (Nova Science Publishers, 2006) [Electronic version is available on <http://arxiv.org/abs/math/0610080>]