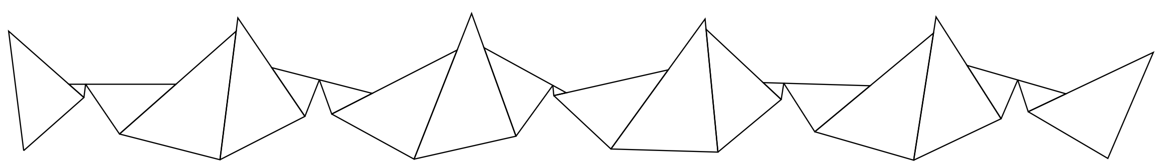
Discrete Geometry and Probability



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IPAM, march 2007

Given a planar graph, how to give it a geometric structure

so that "critical" probability models

have geometric significance?

e.g. Random walk

Spanning trees

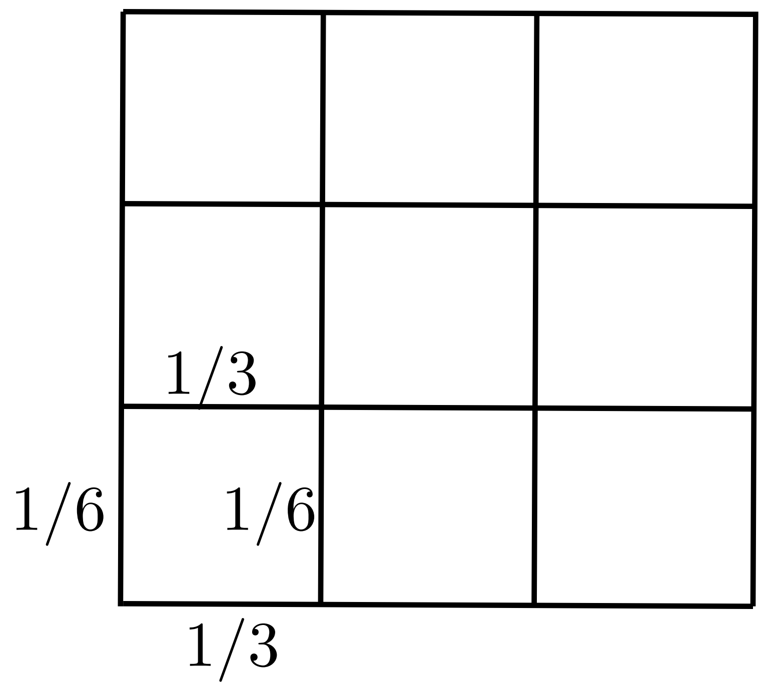
Loop-erased random walk

Percolation

Dimer model

Ising model

6-vertex model



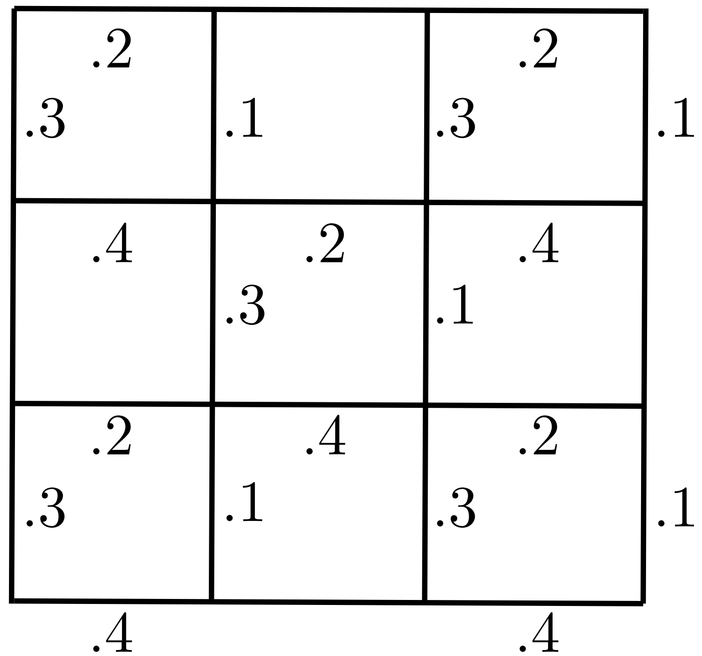
Example: random walk on the square grid

Horizontal/vertical step probability $1/3, 1/6$.

Exercise: How should we rescale vertical coordinate to get Brownian motion in scaling limit?

Harder exercise:

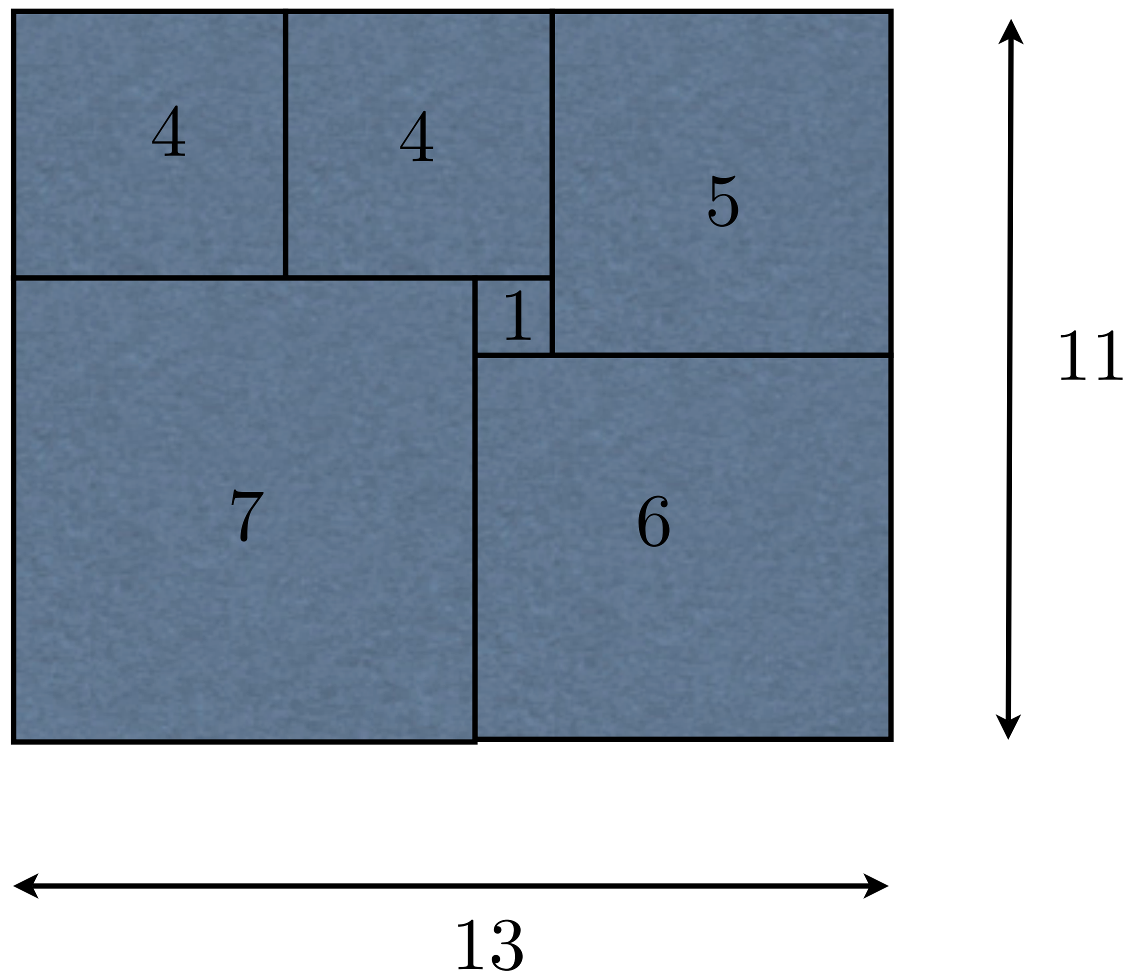
same question, more complicated (periodic) probabilities



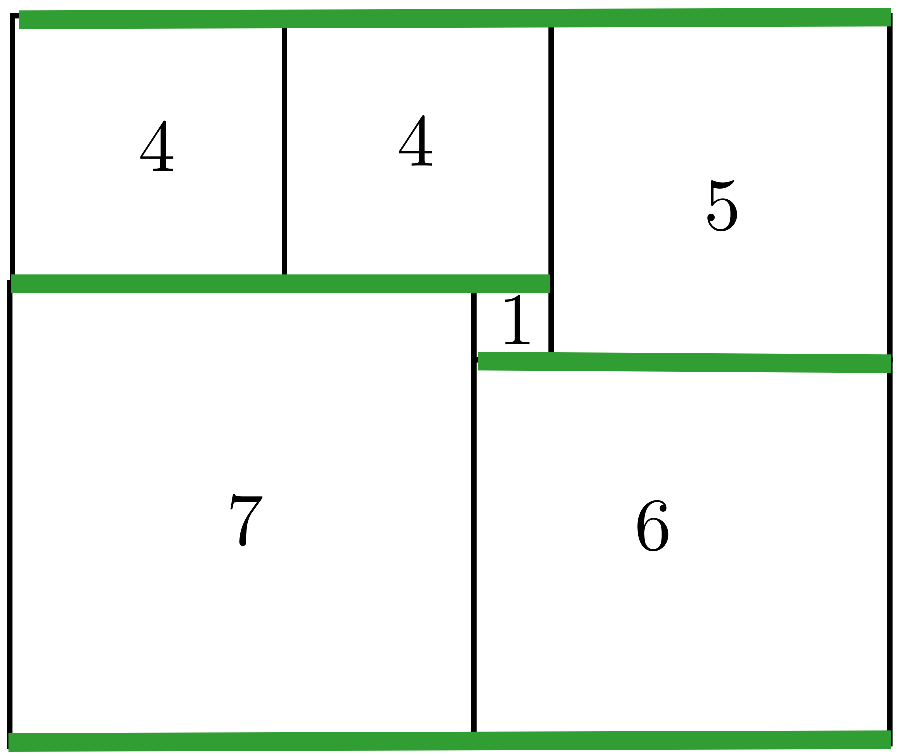
Geometry

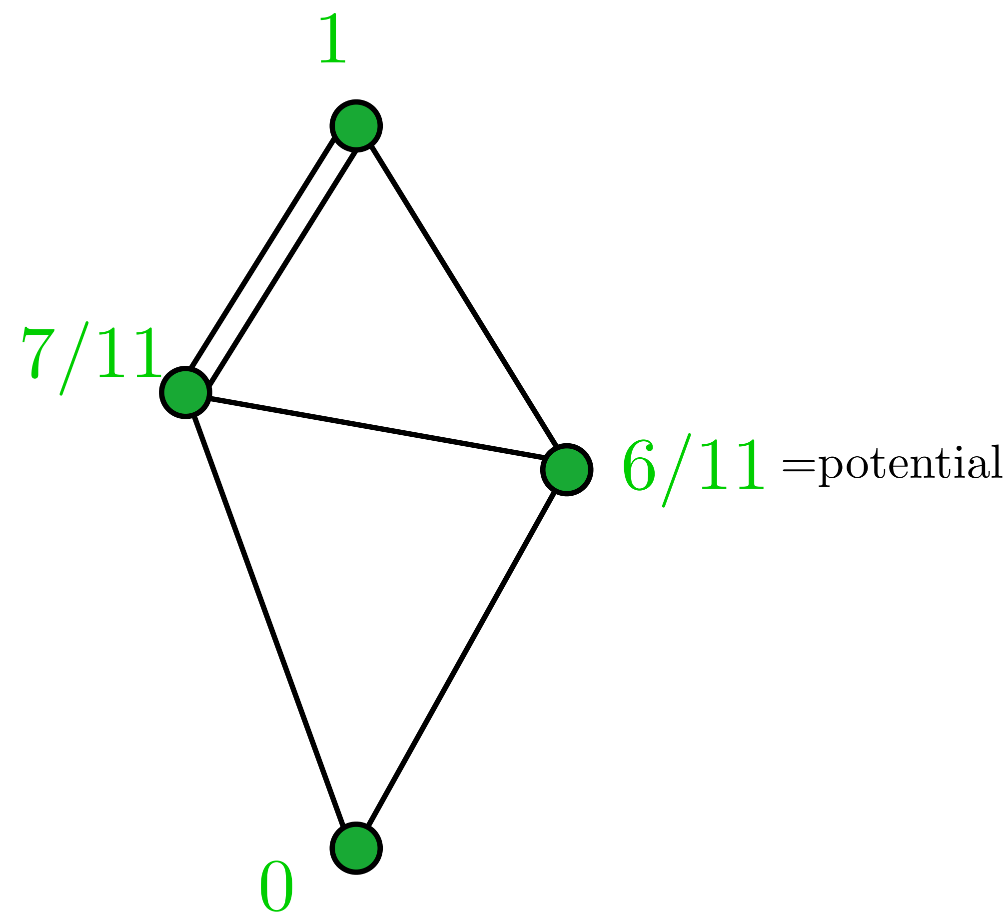
Theorem (Dehn, 1903)

An $a \times b$ rectangle can be tiled with squares if and only if a/b is rational.



Proof via resistor networks:





replace each square with a unit resistor

vertical coordinate = voltage

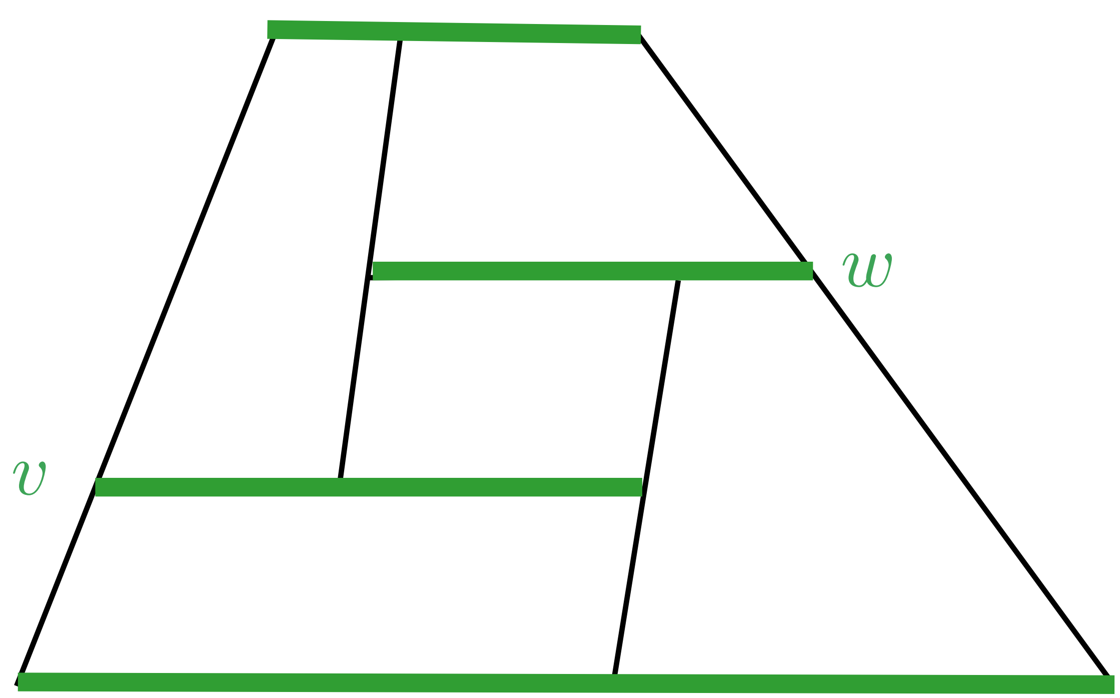
width of a square = current = voltage drop

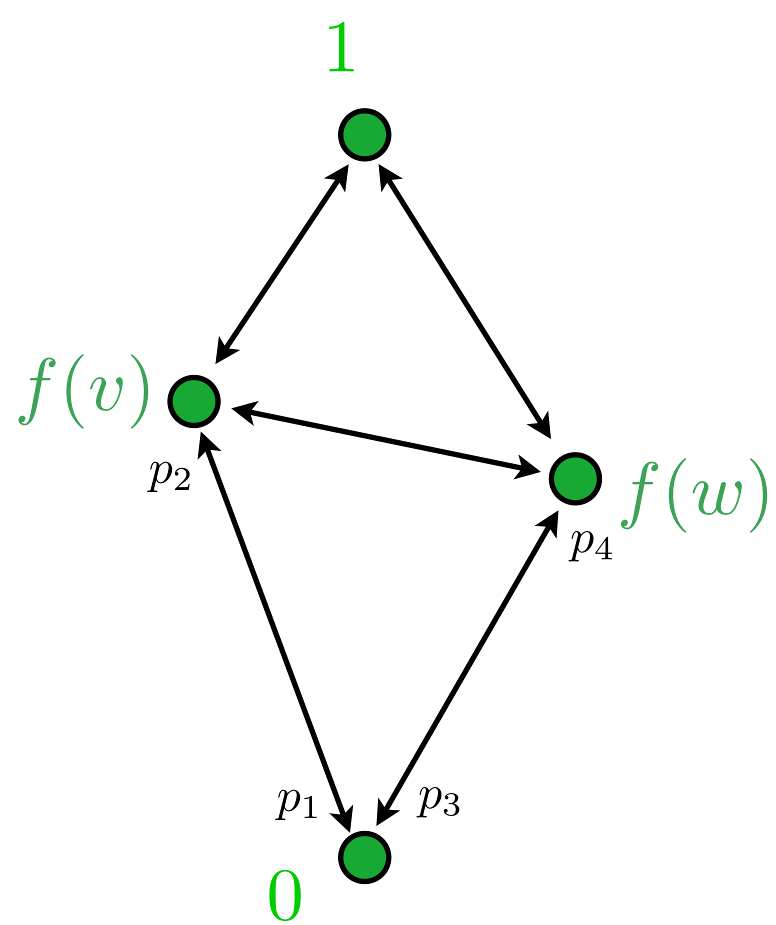
Kirchhoff's equations are rational, linear and have a unique solution. □

We can generalize this result to planar Markov chains

(graphs with transition probabilities) [K, 1998]

Now we'll get tilings with trapezoids with two horizontal boundaries:





heights are values of a harmonic function f

edge widths are given by the stationary measure on edges

times corresponding difference in f-values.

Can we generalize this further, to tilings with arbitrary convex polygons?

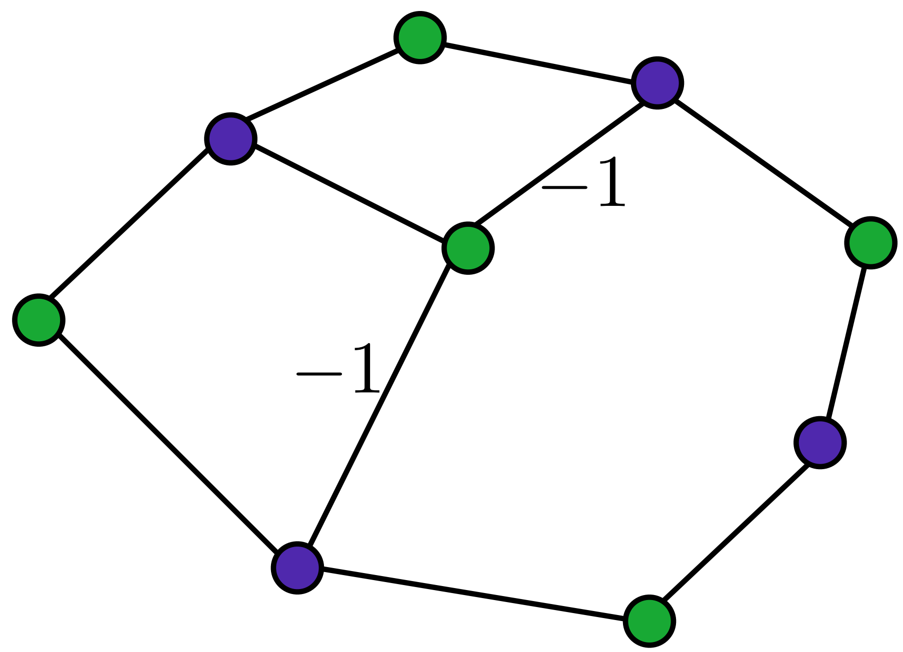
Yes, [K, Sheffield 2003]

For this we need to discuss

Discrete analytic functions and Kasteleyn theory

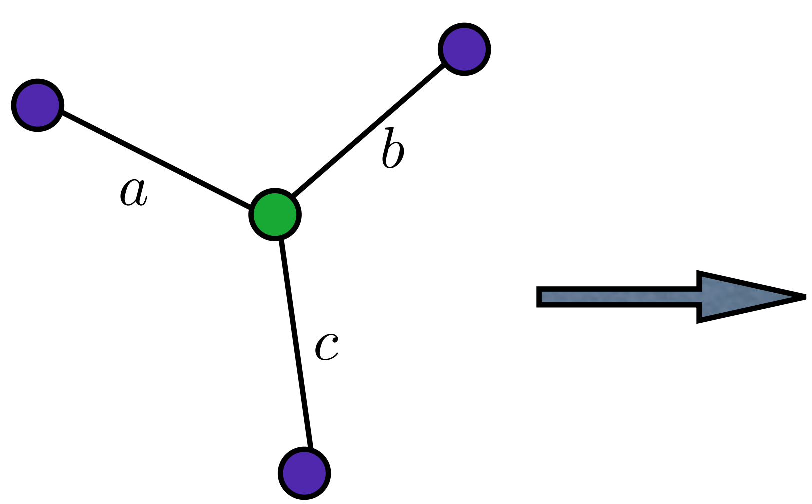
Discrete analytic functions and Kasteleyn theory

On an edge-weighted bipartite planar graph G, choose signs so that a face with $2k$ edges has $k+1 \bmod 2$ minus signs.



Complex version: choose $e^{i\theta}$ per edge so that alternating product (first, divided by second, times third, etc) is $(-1)^{k+1}$

Lemma Any two such choices are related by operations of the type:



"gauge transformations"

The Kasteleyn matrix

Let K be the associated signed adjacency matrix

$$K\colon \mathbb{C}^W \to \mathbb{C}^B.$$

Theorem [Kasteleyn 1965]

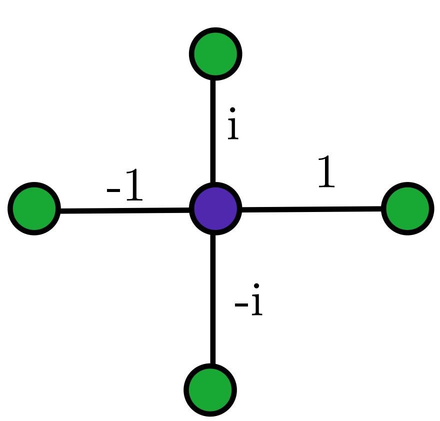
$$|\det K| = \sum_{\text{perfect matchings } M} \text{weight}(M)$$

where weight(M) is the product of the (positive) edge weights.

Def: A function $f : B \to \mathbb{C}$ is discrete analytic if $Kf = 0$.

Similarly for functions $f : W \to \mathbb{C}$.

Ex. square grid: use this gauge:



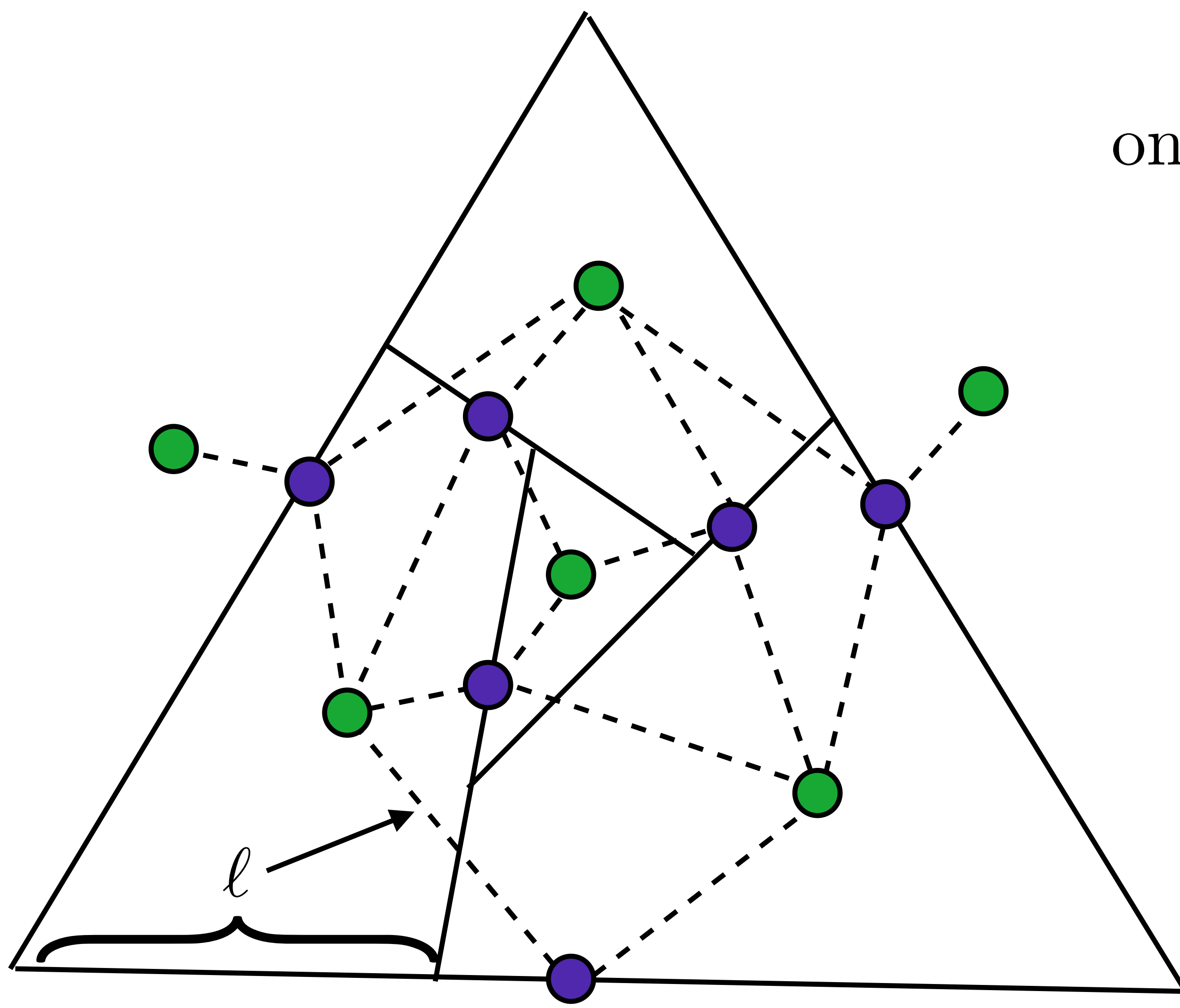
$$Kf(w) = f(w+1) - f(w-1) + if(w+i) - if(w-i)$$

$$= \text{``}\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\text{''} f$$

Back to tilings

From a tiling with convex polygons,

we construct an (edge-weighted) bipartite planar graph

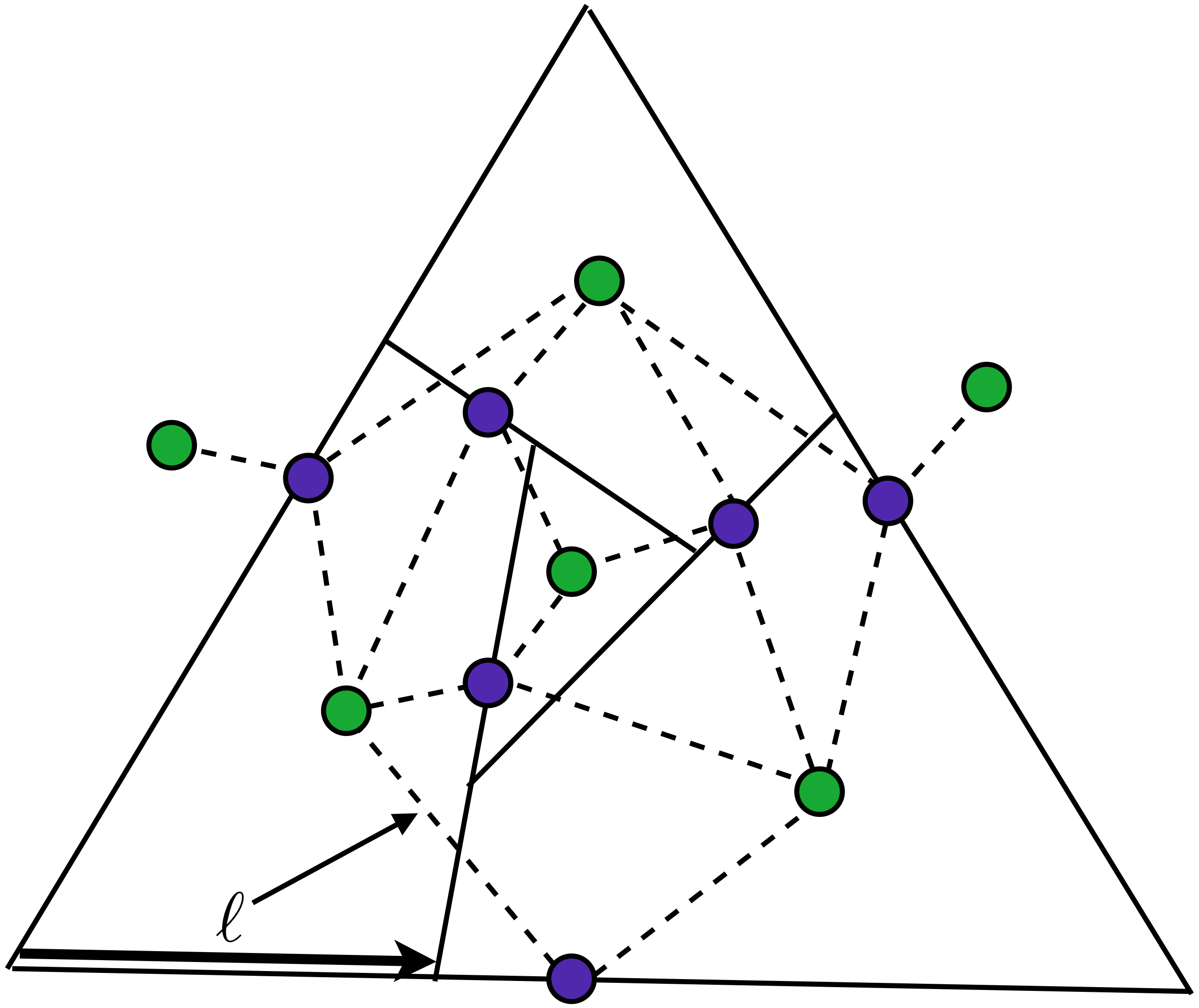


one ● per tile,

plus some on boundary

one ● per maximal segment

Edge weight = length of corresponding segment



Kasteleyn matrix:

complex edge weight= corresponding complex vector

exercise 2. check sign condition around a face.

exercise 3. Show that $f(w) \equiv 1$ is discrete analytic except at the boundary

Constructing a tiling from a graph.

Given a weighted bipartite planar graph G, with Kasteleyn matrix K,

find a function $g : W \to \mathbb{R}$

so that $\sum_w g(w)K(w,b) = 0$ unless b is a boundary vertex

Then define f so that $\sum_b g(w)K(w,b)f(b) = 0$
unless w is a boundary vertex.

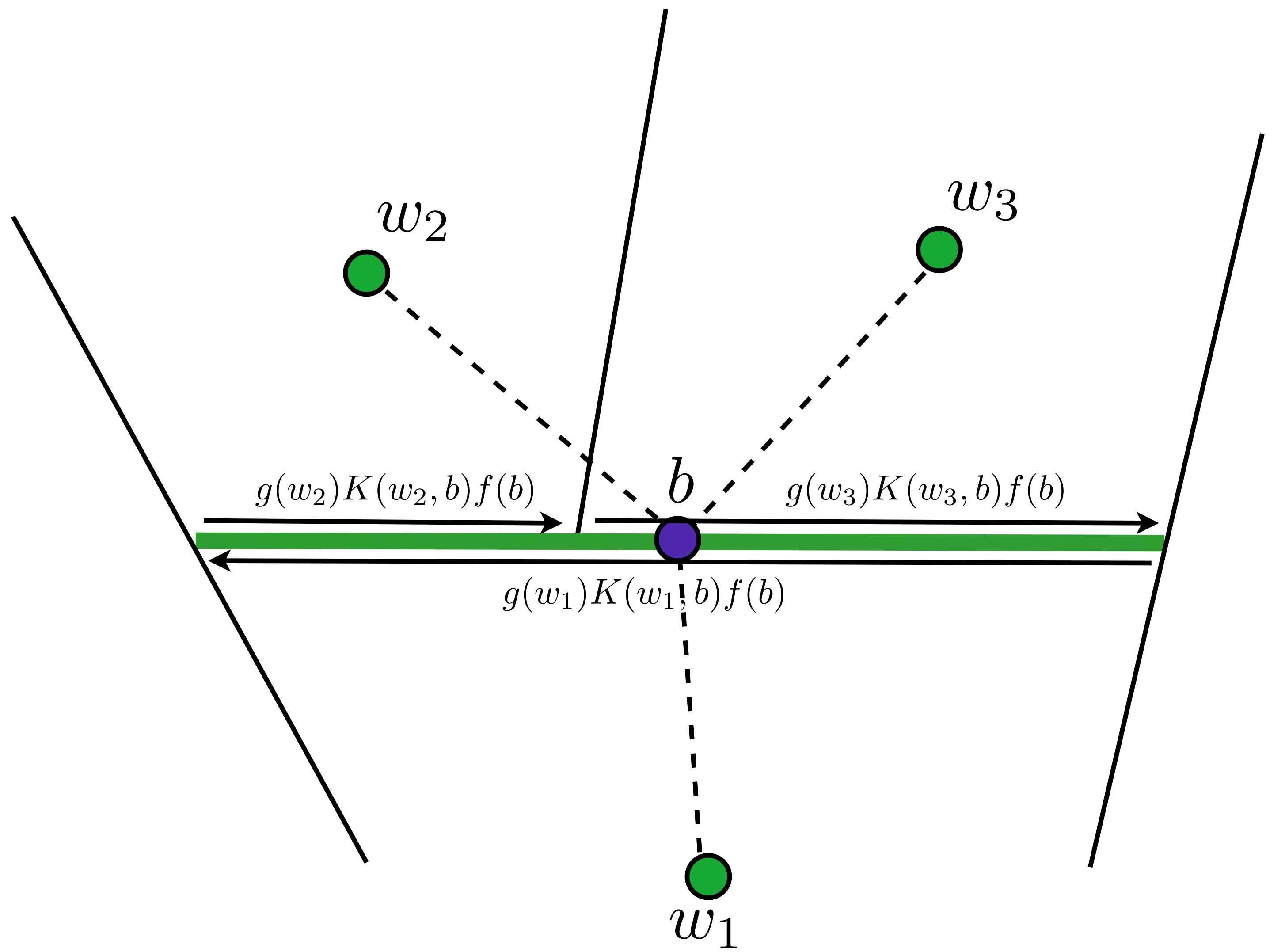
The function $g(w)K(w,b)f(b)$ on edges is a flow

(divergence free) on interior vertices and so defines a map $G^* \to \mathbb{C}$

(G^*=planar dual).

Theorem [K-Sheffield] If G has a perfect matching,
for appropriate choices of boundary conditions this gives a tiling.

Proof idea



$$\sum_{i=1}^{3} g(w_i)K(w_i,b)f(b) = 0$$ by discrete analyticity of g

so we know what the map looks like near a black vertex. Similarly, vectors around a white (green) vertex close up by discrete analyticity of f.

Convexity (harder) follows from the Kasteleyn property and a maximum principle. □

Another motivation

Theorem: There is a weight-preserving mapping from perfect matchings to (indirected) spanning trees on a related Markov chain