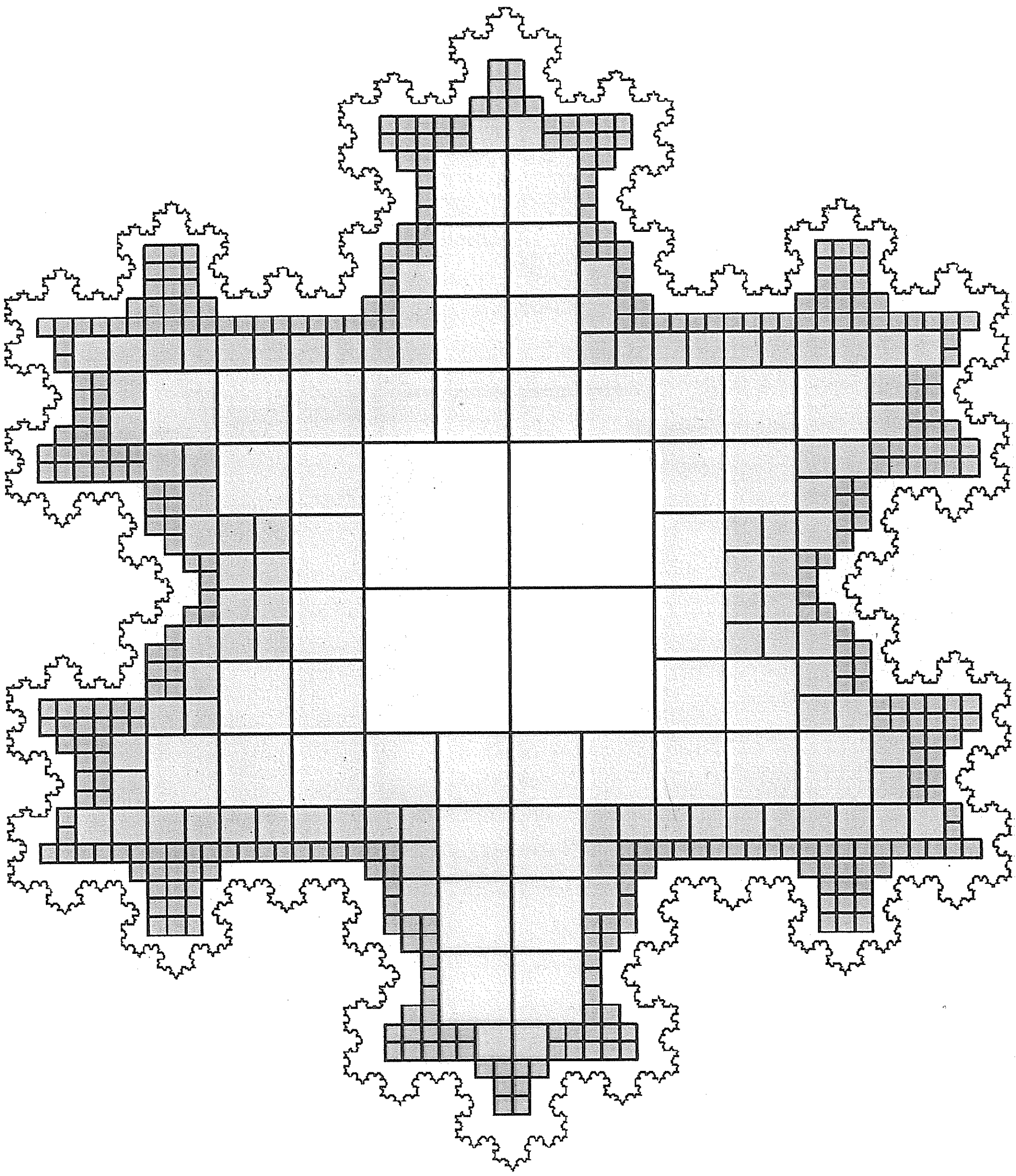
OUTLINE OF THE TALK

1. Multifractal Measures (Pictures)
2. A tour of the Universe (Pictures)
3. The Search for Filaments (Pictures)
4. The Gaussian Free Field (Pictures)
5. Lensing and Quasiconformal Mappings (Pictures)
6. Introduction to SLE and Random "Traces"
7. A Construction of Random Homeomorphisms (S^1)
8. Exponentiating the Gaussian Free Field
9. Ito Calculus for #8 and L^2 Estimates
10. Degenerate QC Mappings and Random Lehto Th.
11. Sobolev spaces and Uniqueness of Homeos
12. Construction of Random Jordan Curves

THE OBJECTS IN #7 AND 12 ARE "THE SAME"!



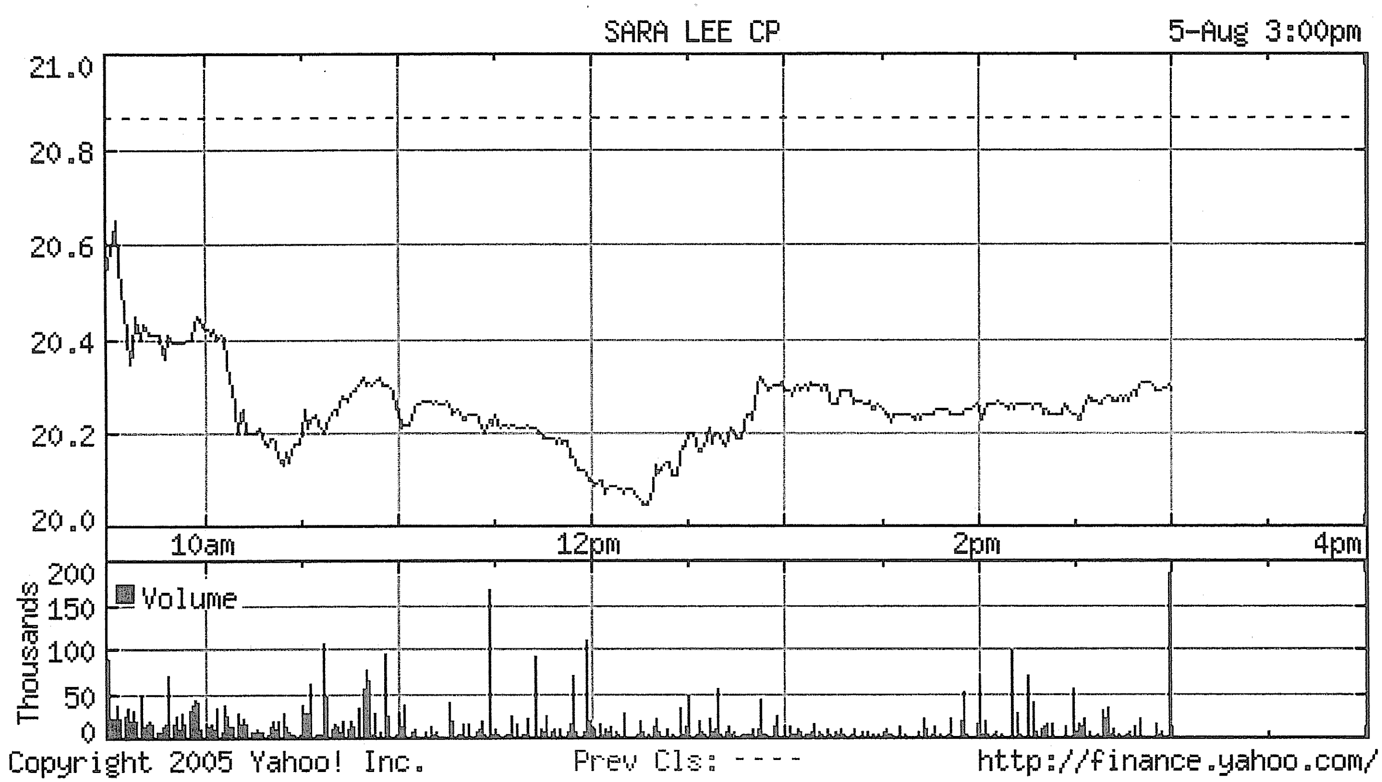
References to Related work:

1. O. Schramm and S. Sheffield have another approach using The Gaussian Free Field. They obtain SLE traces, but NOT Loops.

2. Barral, Mandelbrot et al have a very useful set of papers on mutifractal measures. In particular they have useful methods to show when the measures are (a.s.) 0.

3. Malliavin et al have defined "Brownian Motion on the Space Diff(S1)" It is a very different process, but philosophically similar. That process also has long time existence, unlike ours.

4. S. Sheffield has a beautiful introduction to analysis and the GFF. (On the web)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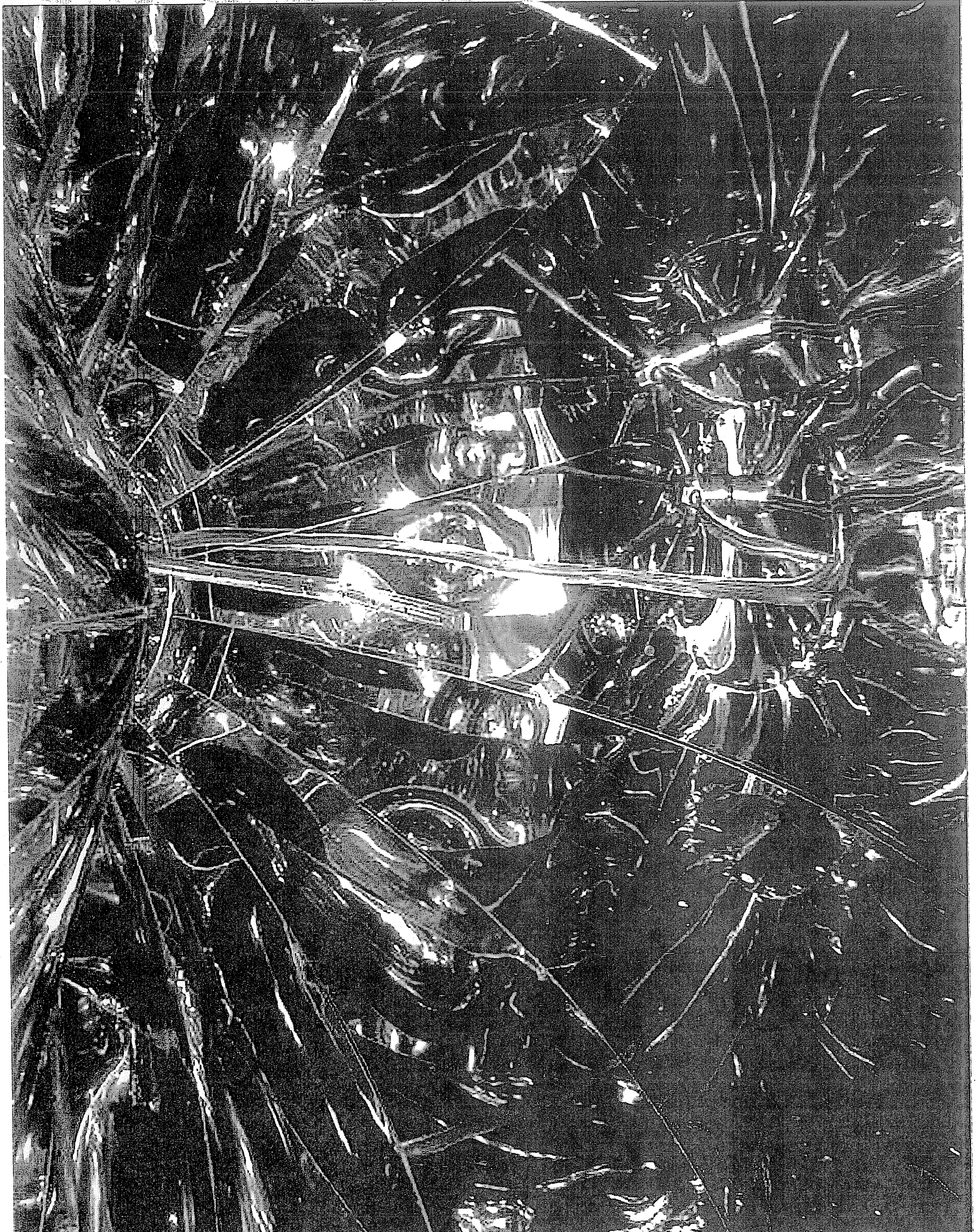
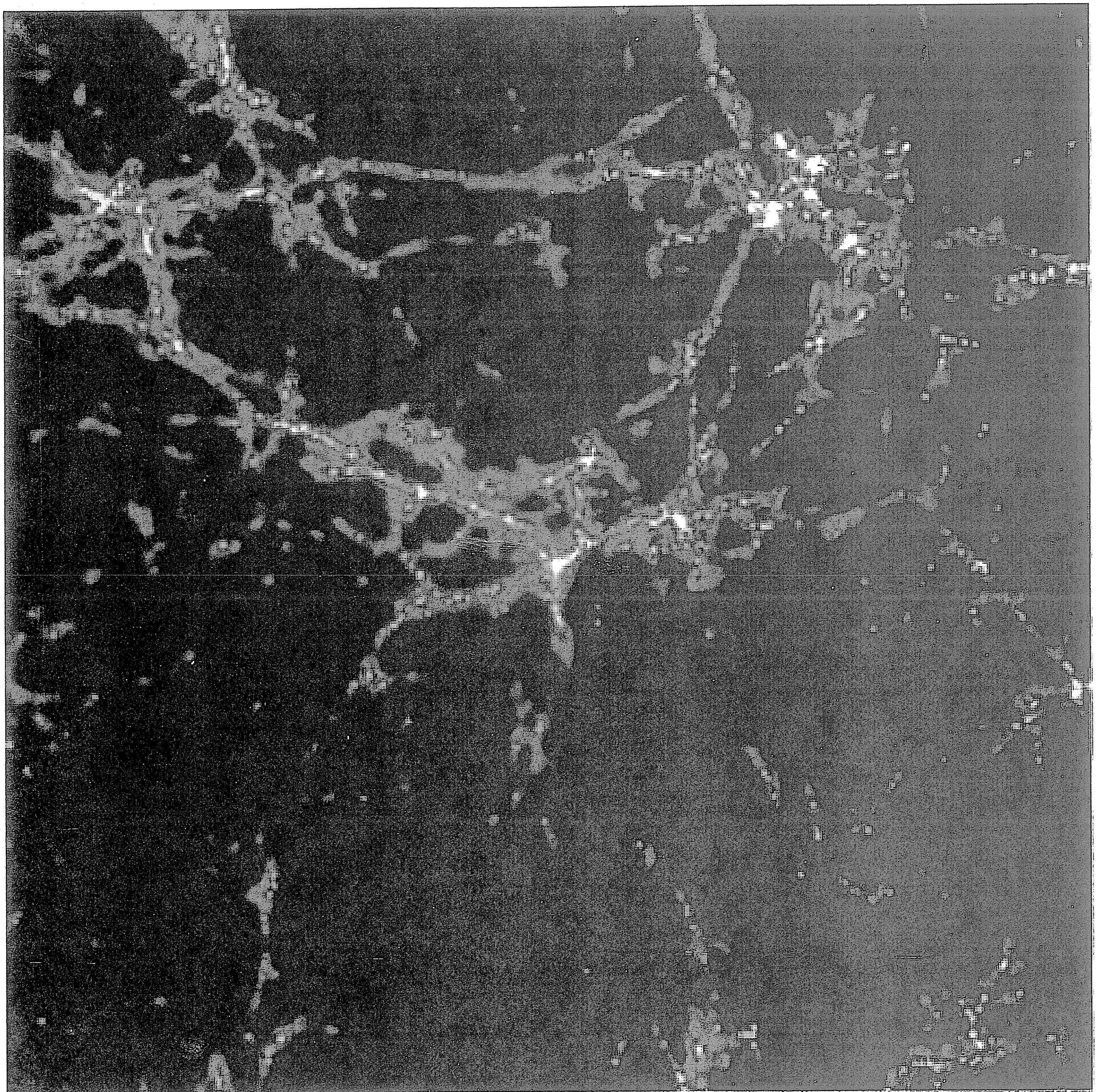
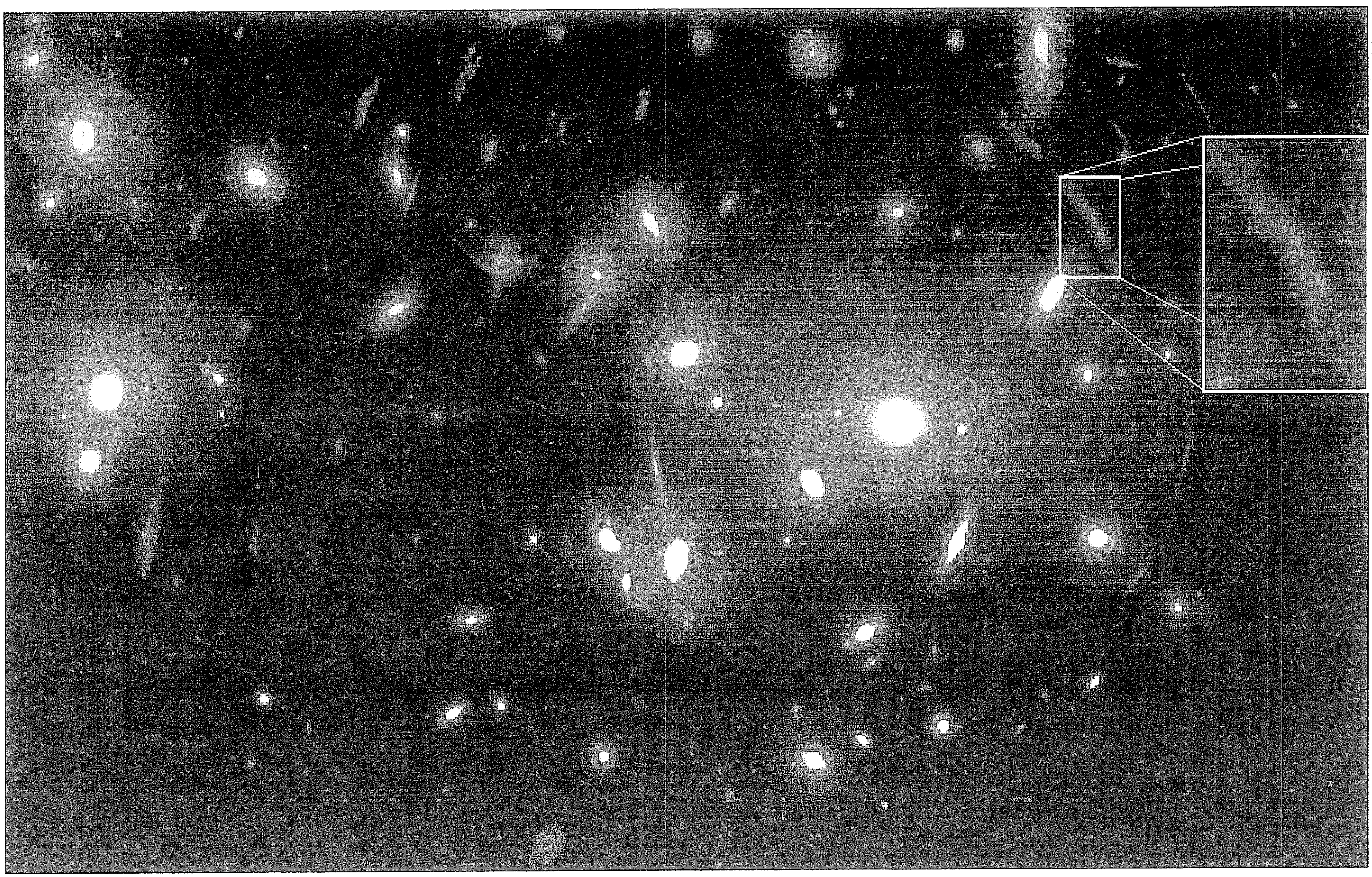
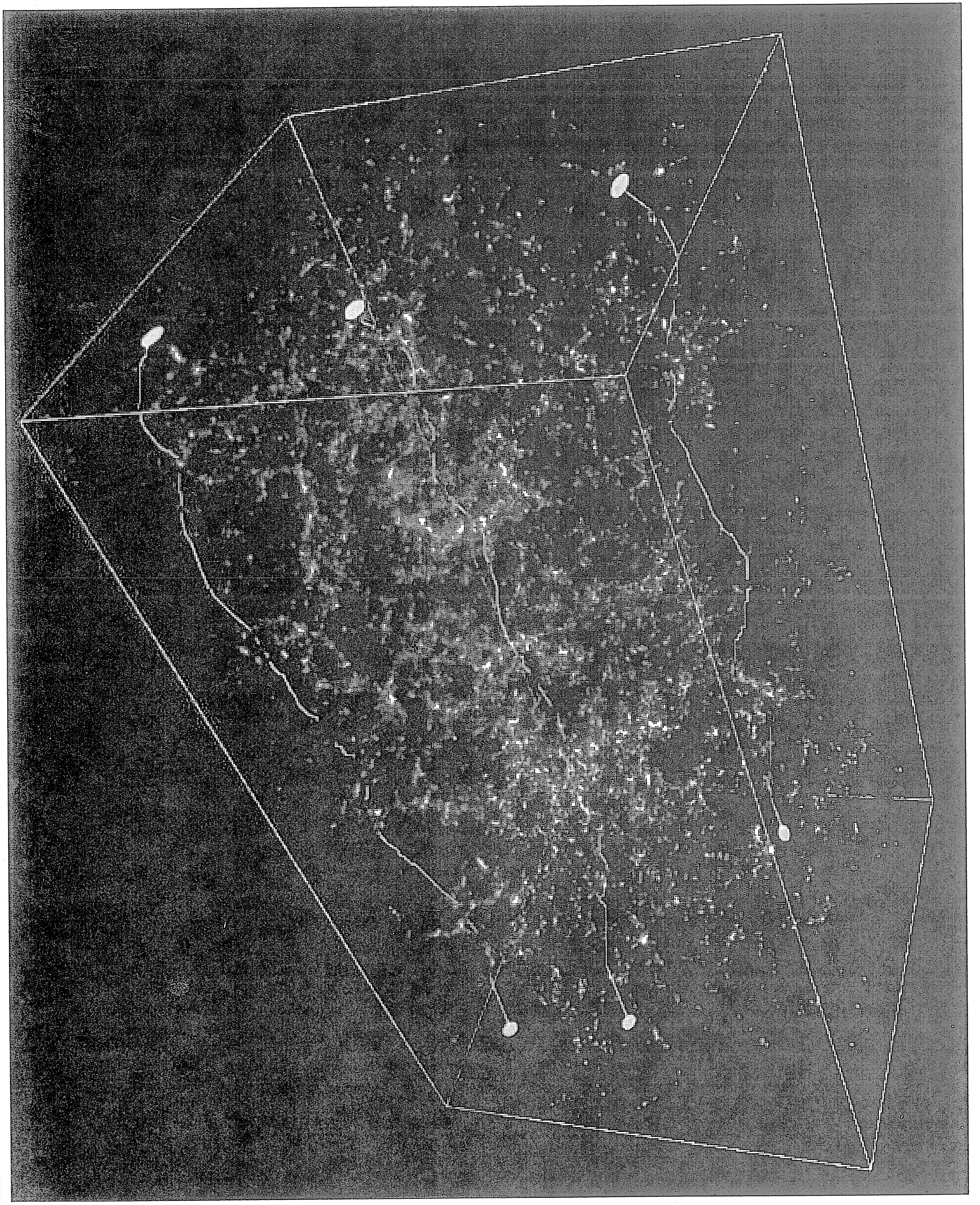


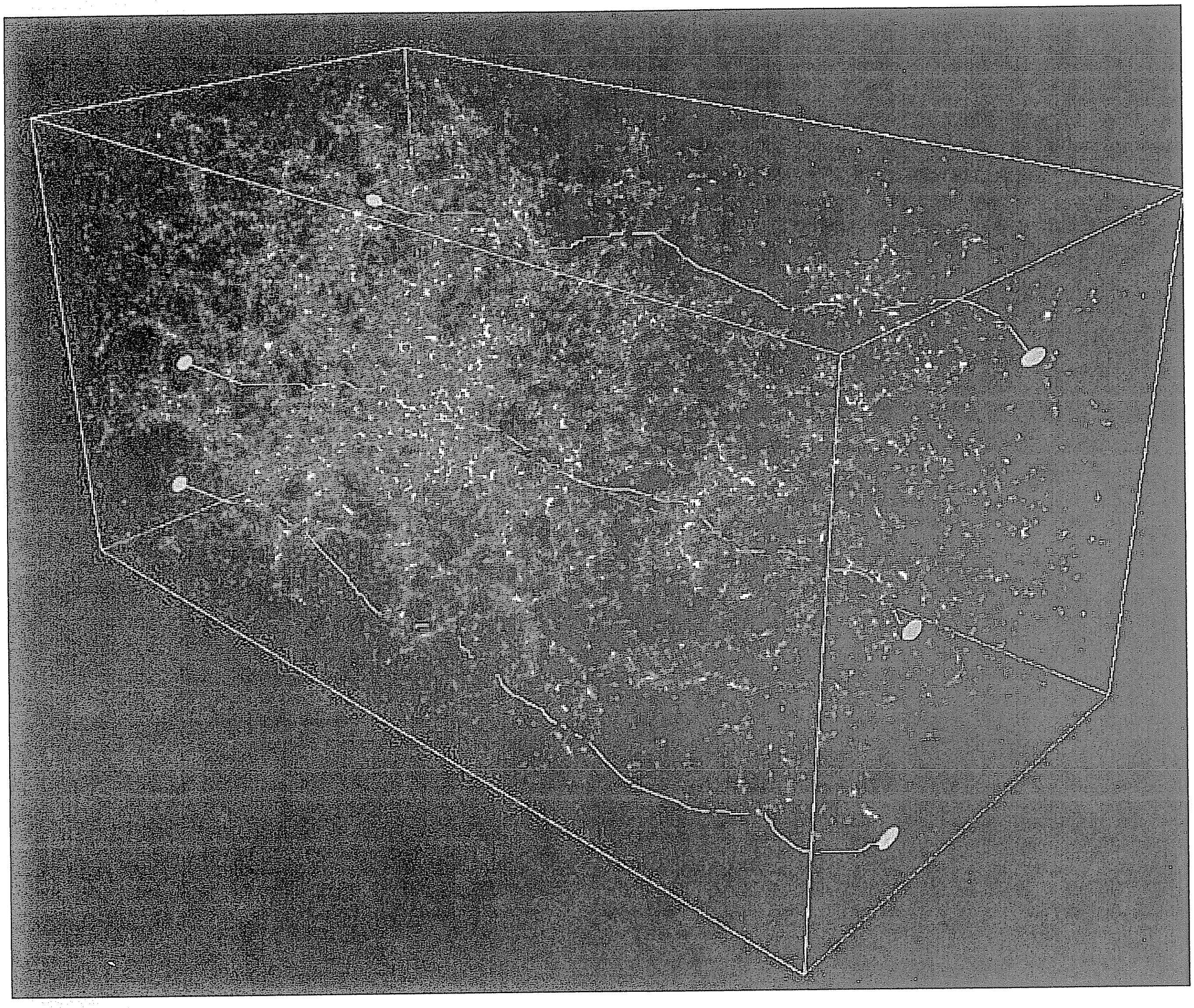


Tom Theuns/Max-Planck-Institute for Astrophysics, Germany

Strong Gravitational Lensing







DO ALIENS LIKE QUASICIRCLES?

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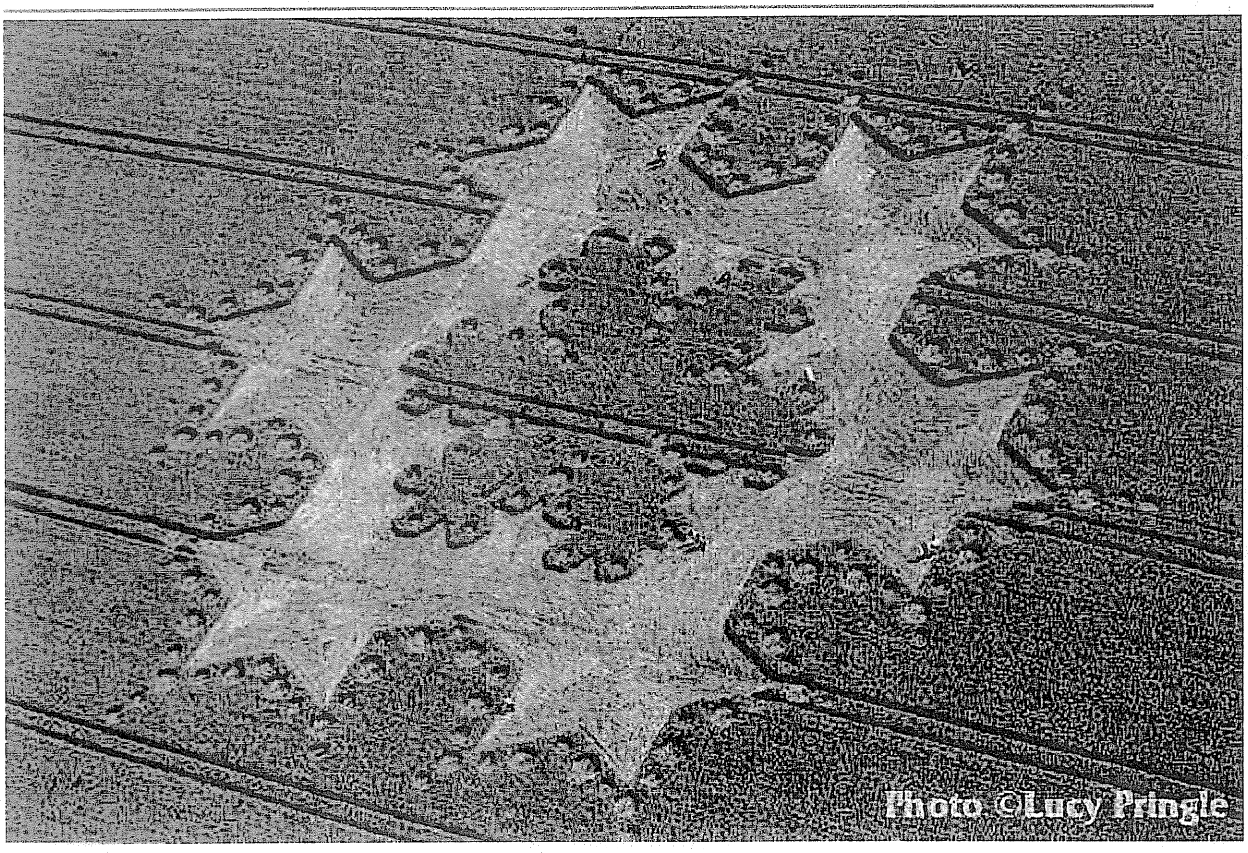

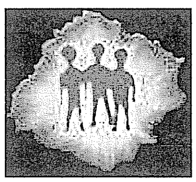

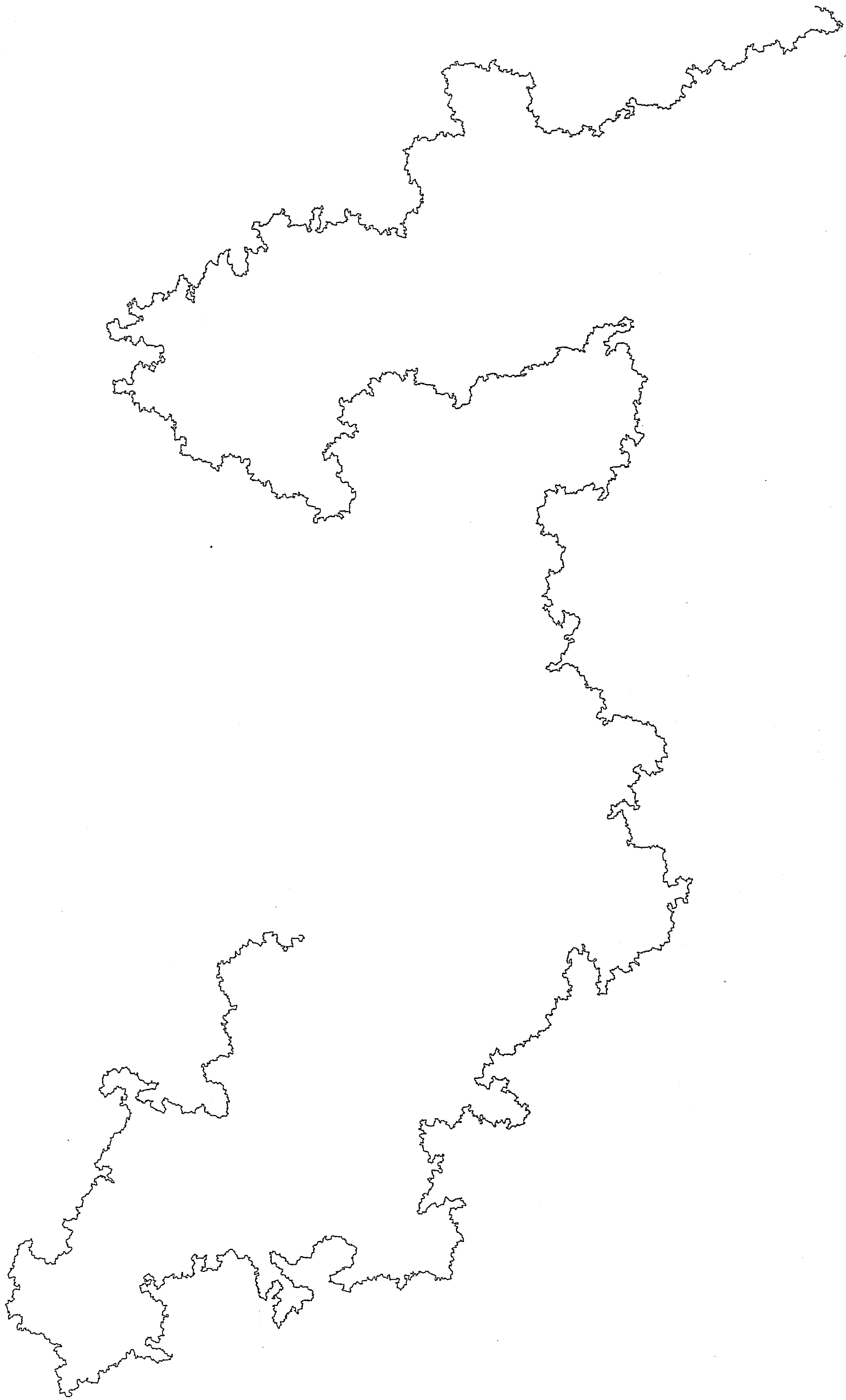

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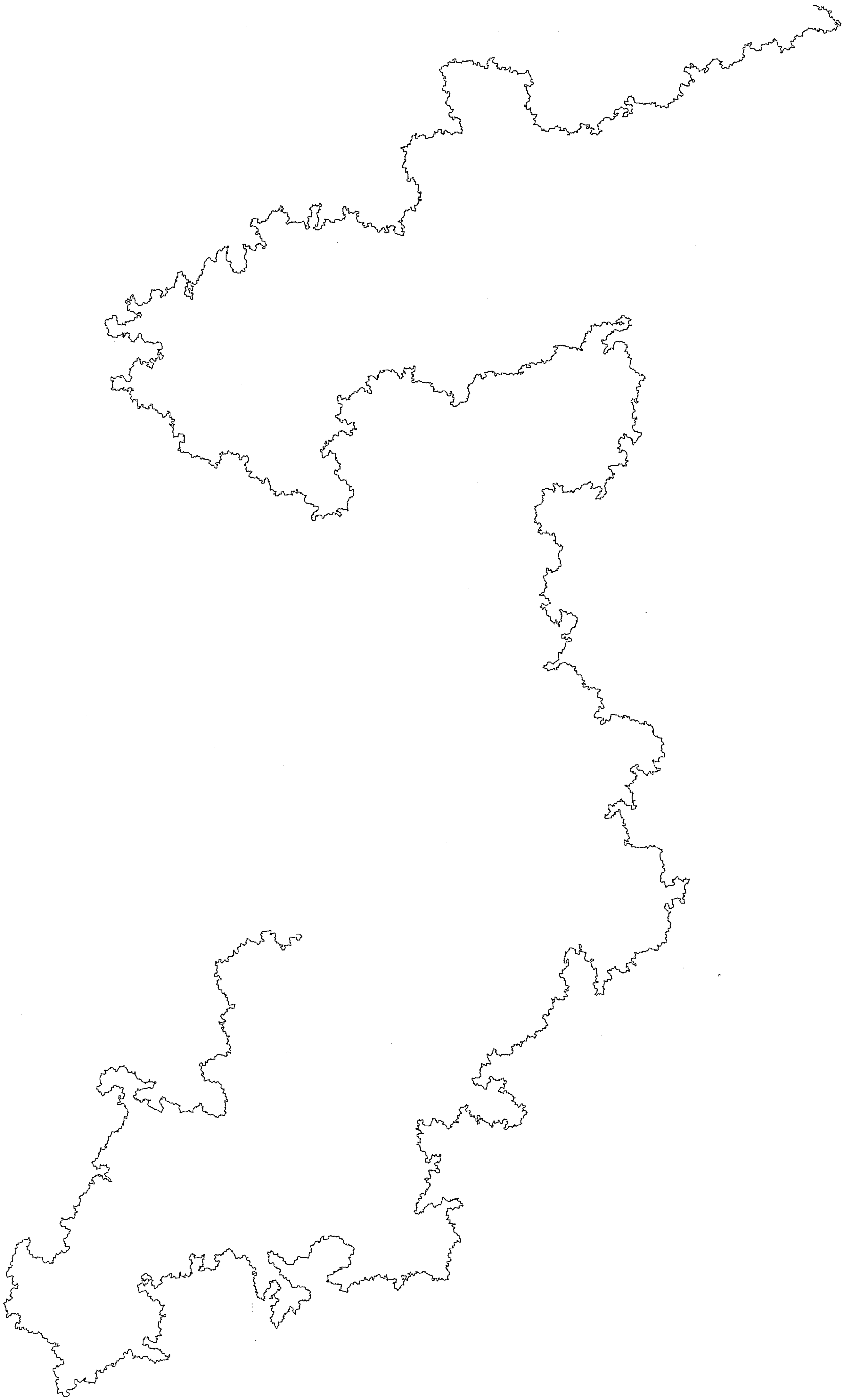


Gaussian Free Field

GFF

Gaussian Hilbert Space

$\mathcal{H}$ = Hilbert Space

$\{f_n\}$ = o.n. basis

$$\sum a_n(\omega) f_n$$

↑ i.i.d. $N(0,1)$ or $N(0,t)$

Base Change: $\{g_n\}$ = o.n. basis

$$\sum a_n(\omega) f_n = \sum \tilde{a}_n(\tilde{\omega})\, g_n$$

"Nothing" Changes

Gaussian Free Field on a Surface M
(Essentially same for Manifold)

Let M be a smooth, compact surface having (positive) Laplace-Beltrami operator

$$\Delta \geq 0$$

By the SPECTRAL THEOREM there is an o.n. basis for L^2 Consisting of eigenfunctions φ_k satisfying

$$\Delta \varphi_k = \lambda_k \varphi_k$$

Let H be the HILBERT space with inner product

$$< F, G >_H = \iint_M \nabla(F) \bullet \overline{\nabla(G)}$$

An o.n. basis for the Hilbert Space H is $\{ (\lambda_k)^{-1/2} \varphi_k \}$.

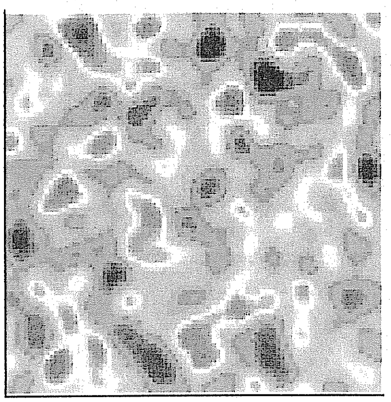
Example: The Torus An o.n. basis is given by

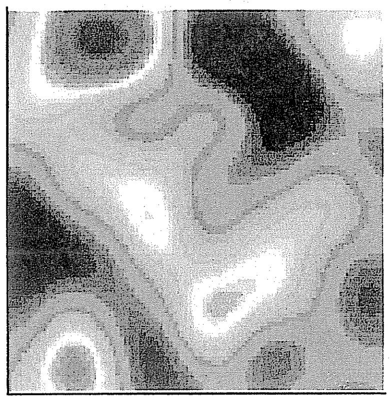
$$\{ (2\pi)^{-1} (m^2 + n^2)^{-1/2} \exp(i2\pi mx + i2\pi ny) \}$$

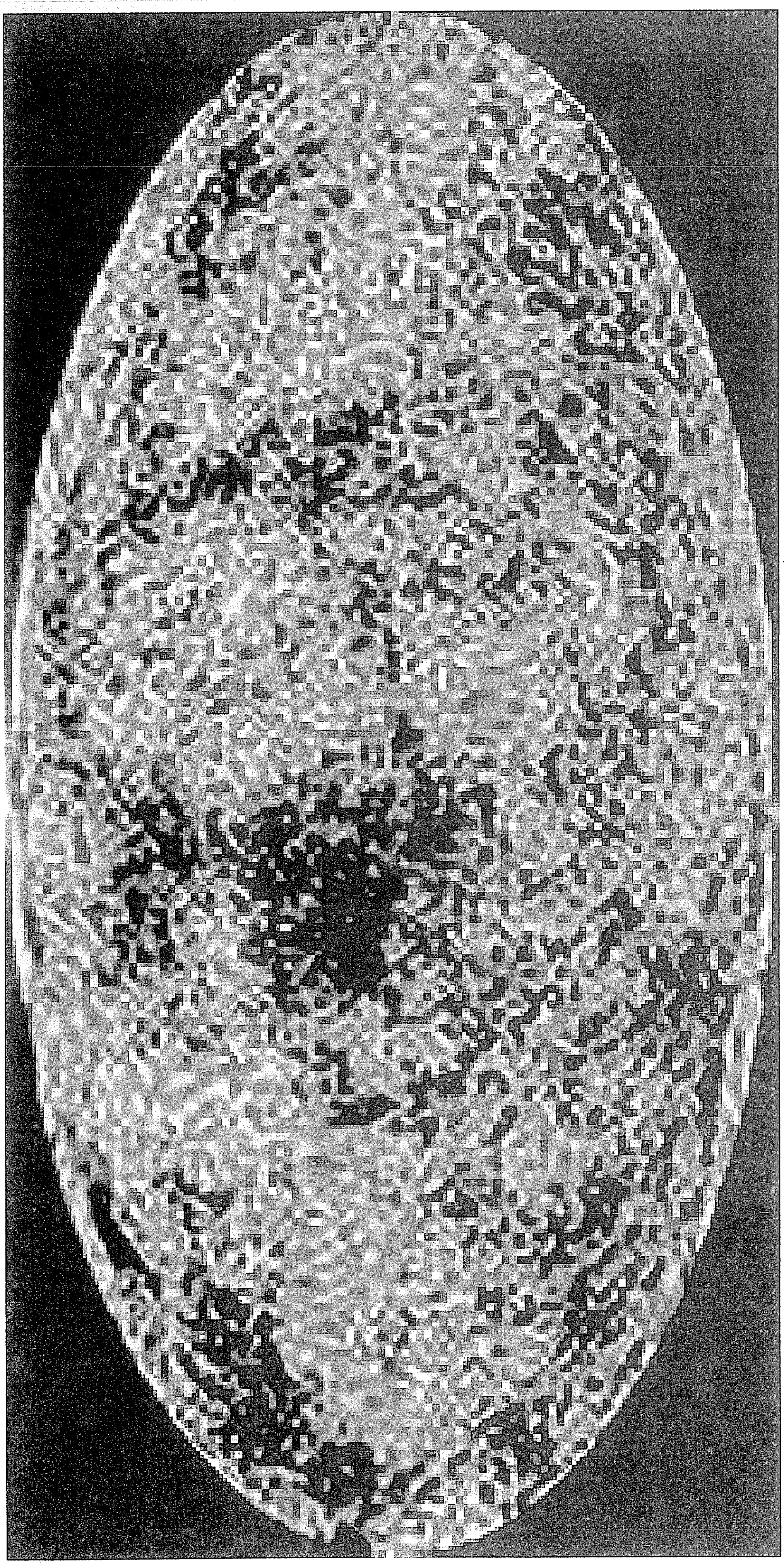
Gaussian Hilbert Space $\leftrightarrow$ $\sum a_k F_k$,

Where $\{F_k\}$ is an o.n. basis for H and the a_k are i.i.d. N(0,1). These Sums (a.s.) ARE NOT FUNCTIONS. (They are Distributions that "just miss" being functions.) THEY PUT EQUAL ENERGY AT EVERY SCALE. (As in the very early universe!)

GAUSSIAN FREE FIELD in BRAIN IMAGING







GFF $(\mathbb{D})$

$$\mathcal{H} \to \left(\iint_{\mathbb{D}} |\nabla f|^2 \, dx\, dy\right)^{1/2} < +\infty$$

is conformally invariant

$$\left\{ \frac{1}{j^{1/2}} z^j , \frac{1}{k^{1/2}} \bar{z}^k \right\} \underset{\uparrow}{=} \text{o.n.}$$

const!

π : GFF$(\mathbb{D}) \longrightarrow$ Harm. GFF$(\mathbb{D})$

These <u>are</u> functions

R.O.C. $= 1$ (a.s.)

If we take REAL FCNS,
take trace on S^1,
get GFF(S^1) (Same Norm)

$\Rightarrow$ Conformal Invariance for GFF(S^1)

GFF(S^1)

$$\sum_{n=1}^{\infty} a_n(\omega)\,\frac{\cos(n\theta)}{n^{1/2}} \quad (+\ \sin\text{'s})$$

$$= \left(\frac{d}{d\theta}\right)^{1/2} \underbrace{\sum_{n=1}^{\infty} a_n(\omega)\,\frac{\cos(n\theta)}{n} \quad (+\ \sin\text{'s})}_{\text{Brownian Motion}}$$

(Actually Brownian Bridge)

B.M. $= \Lambda_{1/2-\varepsilon}$

G.F.F = Does NOT Converge,

but is almost a function

$\longleftrightarrow$ "Random Bloch" B.V.'s

non !

$$\sum_{n=0}^{\infty}\left(a_n(t)\,\frac{\cos(n\theta)}{n^{1/2}} + \right.$$

$$\left. + \ \tilde{a}_n(t)\frac{\sin(n\theta)}{n^{1/2}} - \frac{t}{2n}\right)$$

$a_n, \tilde{a}_n$ i.i.d. $N(0,t)$ $\quad t \nearrow$

$$\sum = \Sigma_{\omega,t}$$

$\omega \in [0,1] =$ Wiener Space
$0 \leq t$

$$\varphi'(\theta) = c_\omega \exp\{\Sigma_{\omega,t}\}$$

$$\updownarrow$$

$$\int_0^{2\pi}\varphi'\,d\theta = 2\pi$$

Theorem (to be written at IPAM!) 19

$0 \leq t < 2$

1. $\varphi'(\theta) = c_\omega \exp\{\sum_{\omega,t}(\theta)\}$ is (a.s.) a homeo of S^1.

2. This class of homeos is conformally invariant under $\tau: \mathbb{D} \to \mathbb{D}$ Möbius, and is measure preserving. (Wiener measure on $\Omega = [0,1] \ni \omega$)

3. φ is (a.s.) a welding map for a Hölder domain. (Lehto Thm.)

4. The welding curve is (a.s.) unique. (J, Smirnov Thm.)

CONJECTURE:

This gives us

SLE(κ) $\quad 0 \leq \kappa < 4$

$= ?$

"Everything" degenerates at $\ell = 2$ (and $\kappa = 4$).
But it probably still makes sense.

MBA's

$$\omega \to \exp\left\{ B_\omega(t) - \frac{t}{2} \right\}$$

Take Average $\equiv E$

$$E(f_\omega) = \int_0^1 f_\omega \, d\omega$$

$\leftarrow$ ω. Space

Feynman-Kac Formula:

$$E\left(\exp\left\{B_\omega(t) - \frac{t}{2}\right\}\right) \equiv 1$$

$\uparrow$ (all t)

$$\Rightarrow E\left(\exp\left\{ \frac{a_n(\omega, t)\cos}{n^{1/2}} - \frac{t}{2n}\right\}\right) = 1$$

etc.

$$\cos^2 + \sin^2 = 1 \quad !$$

$\Longrightarrow$ (Formally)

$$E\left(\exp\left\{\Sigma_{\omega,t}(\theta)\right\}\right) = 1$$

$$\Rightarrow E\left(\int_0^{2\pi} \exp\left\{\Sigma_{\omega,t}(\theta)\right\} d\theta\right) = 2\pi$$

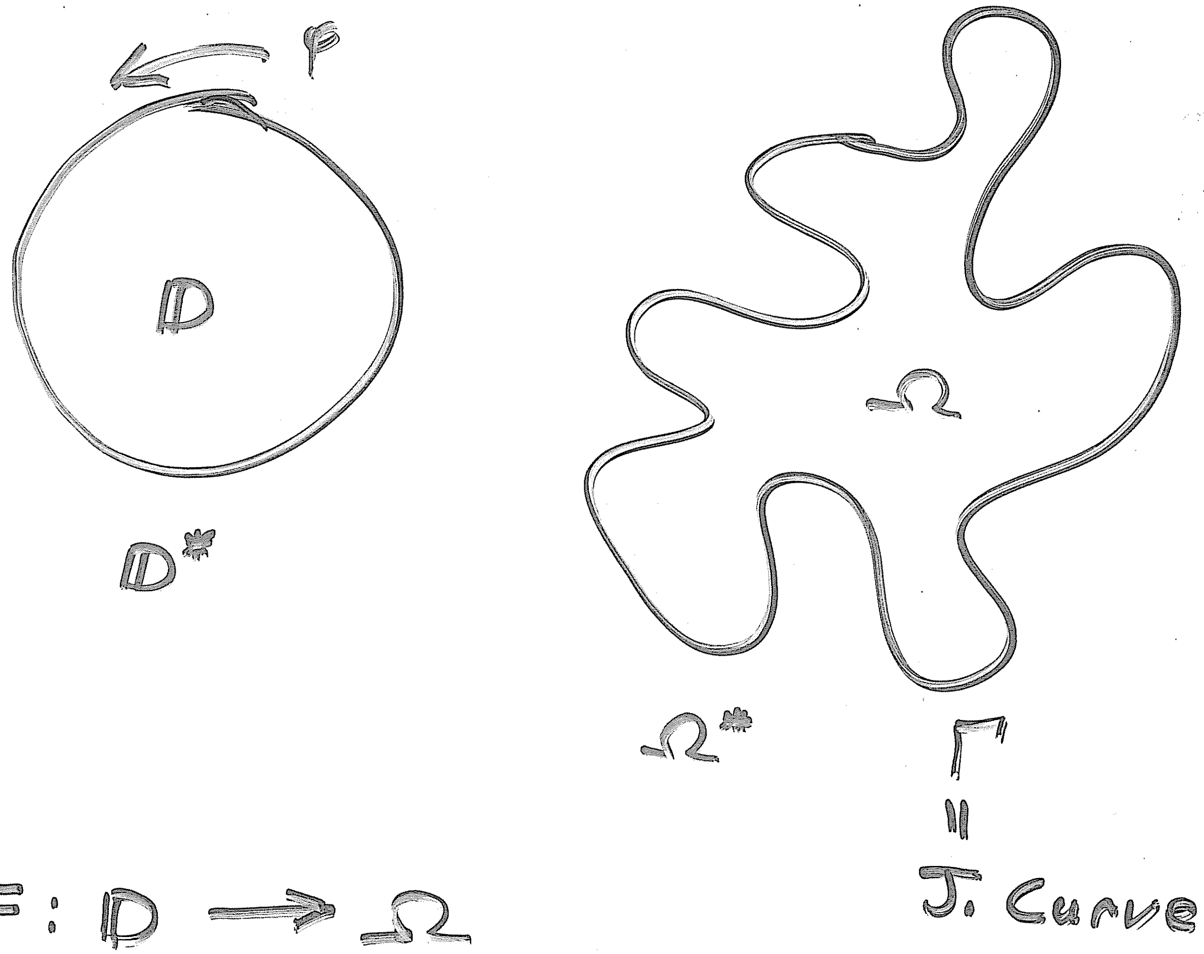
Careful L^2 analysis:

$$0 \leq t < 1 \Longrightarrow \text{(a.s.)}$$

$$0 < \int_0^{2\pi} \exp\left\{\Sigma_{\omega,t}(\theta)\right\} d\theta < +\infty$$

(non-degenerate)

Welding Maps



$$F : \mathbb{D} \longrightarrow \Omega$$

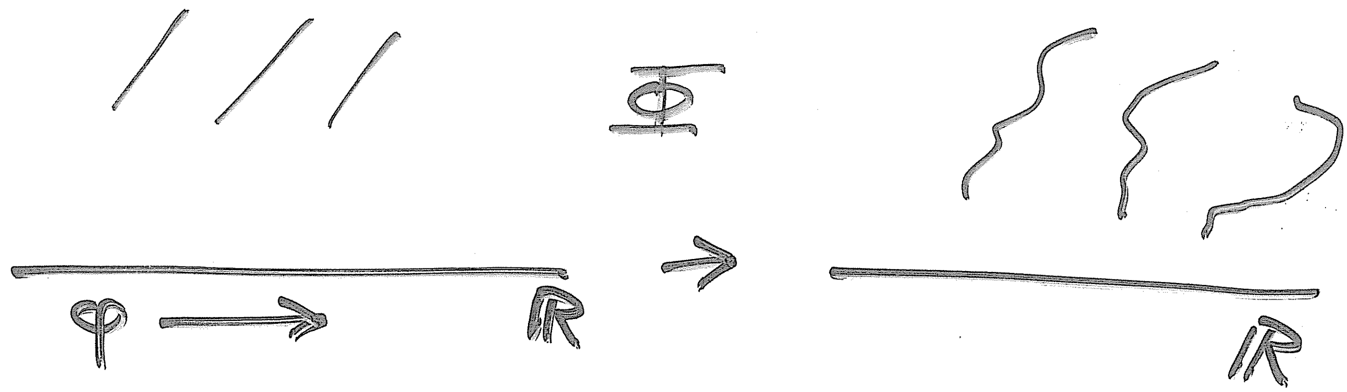
$$F_* : \mathbb{D}_* \longrightarrow \Omega^*$$

$$\varphi = (F^*)^{-1} \circ F$$

φ is a WELDING MAP

Every Γ gives a φ,
BUT NOT EVERY φ
IS WELDING.

How to Weld



extension to UHP

$$\bar{\partial}\Phi = \mu\,\partial\Phi$$

μ ← Beltrami Data

$$\bar{\partial} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right), \quad \partial = \tfrac{1}{2}\left(\text{''} - \text{''}\right)$$

TRY TO SOLVE

$$\bar{\partial}F = \mu\,\partial F \qquad \text{global}$$

$(\mu \equiv 0$ on LHP$)$

Can Replace UHP by $\mathbb{D}$

Quasiconformal Mappings

F is a Homeomorphism of The Complex Plane

BELTRAMI DATA:

EQ # 1 $\bar{\partial} F(z) = \mu(z)\, \partial F(z)$

Where $\bar{\partial} = \frac{1}{2}(\partial_x + i\, \partial_y)$ (Annihilates holomorphic functions)

$\partial = \frac{1}{2}(\partial_x - i\, \partial_y)$ (H'(z) for H holomorphic)

Conditions on $\mu(z)$:

F diffeo $\rightarrow |\mu(z)| < 1$ ($\mu(z) = 0$ iff $F = (az + b)/(cz + d)$)

THERE IS A CONVERSE!

MEASURABLE RIEMANN MAPPING THEOREM:

1. If $\|\mu(z)\|_\infty < 1$ and is measurable, then there exists a homeo F such that EQ #1 holds a.e.

2. F fixes $0, 1, \infty$ implies F is unique.

We will need a stronger theorem later. (LEHTO)

LEHTO's Thm (Brutal Form)

$$\psi(z,r) = \left(\int_0^{2\pi} \frac{1}{1-|\mu(z+re^{i\theta})|} \frac{d\theta}{2\pi} \right)^{-1}$$

$$\int_0^1 \psi(z,r) \frac{dr}{r} = +\infty$$

$\Longrightarrow$ $\bar\partial F = \mu \partial F$

$\mu = 0$ on $\mathbb{D}^*$

has a <u>homeo</u> solution F.

We must evaluate

$\psi(e^{i\theta}, r)$

Use Large Deviations

"Mostly $\psi \geq \varepsilon > 0$"

SOBOLEV SPACES AND UNIQUENESS

Def. A closed set $K \subseteq \mathbb{R}^d$ is said to be removable for a Sobolev space S if $f \in S(\mathbb{R}^d \setminus K)$, and f globally continuous $\Rightarrow$ f = restriction to $\mathbb{R}^d \setminus K$ of $F \in S(\mathbb{R}^d)$. (f extends to the "same" Sobolev Space.)

There is now a theory (almost best possible in any dimension) about such problems due to J and J + S. Smirnov. In particular one obtains the following special case:

Theorem (J, Smirnov) If $D \subseteq \mathbb{R}^2$ is a Hölder domain, then its boundary is removable for the Sobolev space

$$W^{1,2} \text{ (One Derivative in } L^2)$$

Corollary: The Jordan Curves constructed via the GFF Are unique up to Möbius transformations.

Comment: The particular case listed above as the theorem has proved to be very useful for conformal dynamics.