Small perturbations of non-Hermitian matrices

Ofer Zeitouni
Based on joint work with Anirban Basak and Elliot Paquette

IPAM
May 2018

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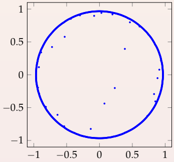

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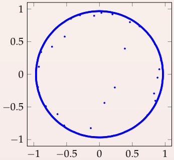

Goes back to Trefethen et als - pseudo-spectrum.

Background: Spectrum stability for symmetric matrices

A – matrix with singular values denoted $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$,

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No such control holds for eigenvalues of non-Hermitian matrices.

Background II: Ginibre matrices

Denote by G_N matrix with i.i.d. standard complex Gaussian entries, and set $g_N = N^{-1/2} G_N$.
λ_i - eigenvalues of g_N.
$L_N^g = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ - *empirical measure* of eigenvalues.

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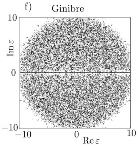
Theorem

L_N^g converges to the uniform measure on the unit disc.



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Thus, $\|N^{-\gamma} G_N\| \to 0$ if $\gamma > 1/2$.

Regularization by noise

Consider the nilpotent N-by-N matrix

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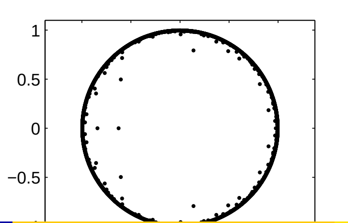

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Why is this particular perturbation picked up? (General criterion - Guionnet, Wood, Z.)

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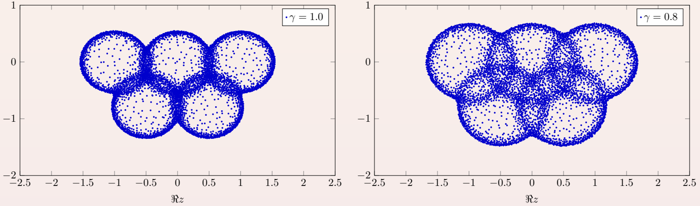
Simulations...



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Analogous result for $\gamma \in (1/2, 1]$ if collection of circles "does not spread too much" (e.g., olympics rings example OK).

More general models?

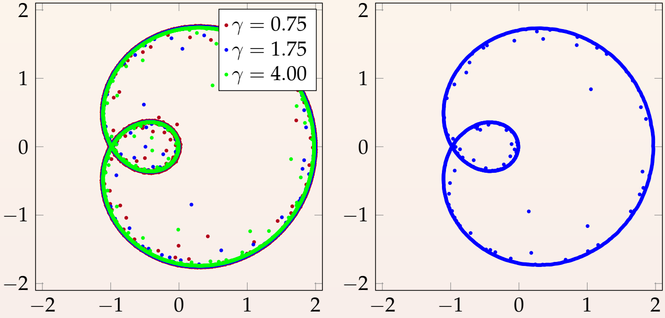

Figure: The eigenvalues of $J_N + J_N^2 + N^{-\gamma} G_N$, with $N = 4000$ and various γ. On left, actual matrix. On the right, $U_N(J_N + J_N^2)U_N^*$.

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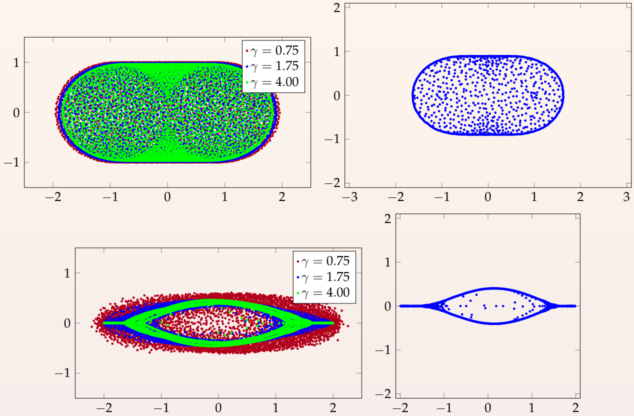

Figure: The eigenvalues of $D_N + J_N + N^{-\gamma} G_N$, with $N = 4000$ and various γ. Top: $D_N(i,i) = -1 + 2i/N$. Bottom: D_N i.i.d. uniform on $[-2,2]$. On left, actual matrix. On the right, $U_N(D_N + J_N)U_N^*$.

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Then $L_N \to \mu$, μ explicit: log-potential of μ at z is $(E \log |z - d_1|) \vee 0)$.

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$T_N = \sum_{i=0}^{k} a_i J_N^i$ (Toeplitz, finite symbol, upper triangular). Then,

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Extends to twisted Toeplitz $T_N(i,j) = a_i(j/N)$, $i = 1, \ldots, k$, a_i continuous:

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Confirms simulations and predictions (based on pseudo-spectrum) of Trefethen et als. Some two-diagonal Toeplitz cases studied by Sjöstrand and Vogel (2016)

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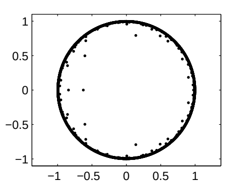

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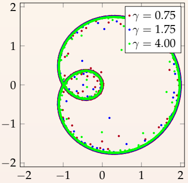

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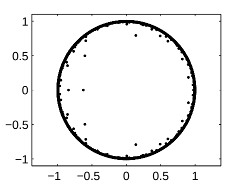

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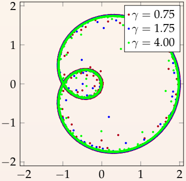

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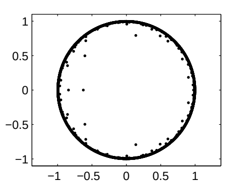

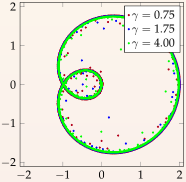

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- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.

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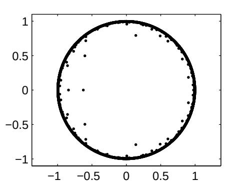

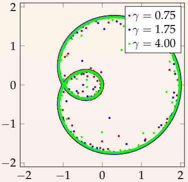

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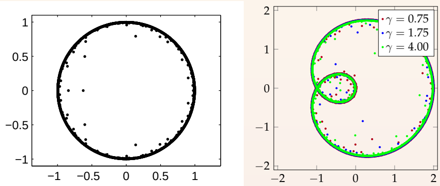

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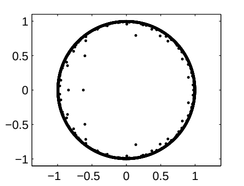

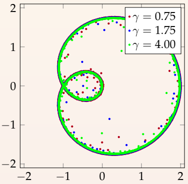

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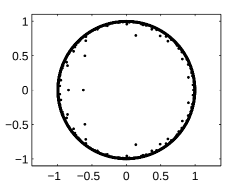

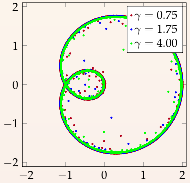

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Ongoing work - Outliers

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- Should depend on nature of noise.

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Even real Ginibre (instead of Ginibre) noise requires work (deterministic equivalence?)

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How can we take $t = t_N \to 0$?

Elements in proofs

Several methods, all use Logarithmic potential:
$U_\mu(z) = \int \log|z-x|\mu(dx)$ and Girko's Hermitization (Girko, Bai, Gotze-Tikhomirov, Tao-Vu, ...)

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One possibility (FPZ): expand determinant and identify dominant terms using concentration of measure.

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If $N^* = o(N/\log N)$ *then*

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So only need to understand small singular values of M_N.

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Reformulation: $M_N = \sum_{i=1}^{k} a_i J_N^i$, singular values?

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Theorem

$$U_{L_N}(z) \to \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta}) - z| dz$$

and therefore

$$U_{L_N}(z) \to \int_0^{2\pi} \log |P(e^{i\theta}) - z| dz = \log |a_k| + \sum_{i=1}^{k} [(\log |\mu_i(z)|) \vee 0].$$

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Lemma

Take $d_i = D_{ii}$ iid.
a) If $E \log|z - d_1| < 0$ then

$$N^{-1} \log \sigma_N(M_N) \to E \log|z - d_1|,$$

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Since

$$N^{-1} \sum_{i=1}^{N} \log \sigma_i(M_N) = N^{-1} \log \det(M_N),$$

get that log-potential converges to $(E \log |z - d_1|) \vee 0$.

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This reconstructs the GWZ result!

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Approximate singular vector construction

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Approximate singular vector construction Assume $|z| < 1$. Set $v_1 = 1$, $v_k = zv_{k-1}$. Then $(M_N v)_k = \begin{cases} 0, & k \leq N-1, \\ -z^N, & k = N. \end{cases}$

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$$x_k - av_k = (x_1 - av_1)z^{k-1} + \sum_{j=1}^{k-1} (M_N x)_j z^{k-j}$$

But if $\|x\|_2 = 1$ and $\langle x, v\rangle = 0$, we have that $1 \le \|x - av\|$.

BPZ - two diagonal case - $M_N = -zI + J_N$

Approximate singular vector construction Assume $|z| < 1$. Set $v_1 = 1$, $v_k = zv_{k-1}$. Then $(M_N v)_k = \begin{cases} 0, & k \le N-1, \\ -z^N, & k = N. \end{cases}$

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But if $\|x\|_2 = 1$ and $\langle x, v\rangle = 0$, we have that $1 \le \|x - av\|$. Choose $x_1 - av_1 = 0$, get a lower bound on $\|M_N x\|$ in terms of $\|x - av\|$ of the form

$$\|M_N x\| \ge C(z)/\log N \Rightarrow \sigma_{N-1}(M_N) \ge C(z)/\log N.$$

BPZ - two diagonal case - $M_N = -zI + D_N + J_N$

To control $\sigma_N(M_N)$, recall

$$\sigma_N(M_N) = \inf_x \frac{\|M_N x\|_2}{\|x\|_2}$$

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Write $x = av/\|v\|_2 + by$ with $\langle y, v\rangle = 0$, $\|y\|_2 = 1$. Then with π the projection to first $N-1$ coordinates,

$$\|M_N x\|_2^2 = \|\pi M_N y\|_2^2 |b|^2 + ((M_N x)_N)^2.$$

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Multi diagonal: choose appropriate basis according to composition of eigenvalues of transfer matrix.