The Spectral Envelope of Multiplicative Brownian Motion

QLAWS3: Random Matrices and Free Probability
IPAM

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Giving Credit where Credit is Due

- Biane, P.: *Free Brownian motion, free stochastic calculus and random matrices*. Fields Inst. Commun. vol. 12, Amer. Math. Soc., Providence, RI, 1-19 (1997)
- Biane, P.: *Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems*. J. Funct. Anal. 144, 1, 232-286 (1997)
- Driver; Hall; K: *The large-N limit of the Segal–Bargmann transform on $\mathbb{U}_N$*. J. Funct. Anal. 265, 2585-2644 (2013)
- K: *The Large-N Limits of Brownian Motions on $\mathbb{GL}_N$*. Int. Math. Res. Not. IMRN, no. 13, 4012-4057 (2016)
- Collins, Dahlqvist, K: *The Spectral Edge of Unitary Brownian Motion.* Probab. Theory Related Fields 170, no. 102, 49-93 (2018)
- Hall, K: *The Spectral Envelope of Multiplicative Brownian Motion.* (pre-)Preprint.

Prologue: Why Brownian Motion?

The GUE, Ginibre Ensemble, and Deformations

Our story starts with the Gaussian Unitary Ensemble X_N and the Ginibre Ensemble Z_N:

$$dX_N = \frac{1}{c_N} e^{-\frac{N}{2}\mathrm{Tr}(A^2)} L^N_{\mathrm{s.a.}}(dA) \qquad dZ_N = \frac{1}{c_N} e^{-\frac{N}{2}\mathrm{Tr}(A^*A)} L^N(dA).$$

(Equivalently: Gaussian “i.i.d.” entries.)

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The universality framework is about extending various eigenvalue statistics known to hold for the GUE to one of two different kinds of generalizations:

- Wigner matrices: all i.i.d. entries (subject to symmetry), sufficiently regular.
- Invariant ensembles: sampled from a law of the form

$$dL_N = \frac{1}{c_n} e^{-N\mathrm{Tr}(V(A))}\, dA$$

where $V : \mathbb{R} \to \mathbb{R}$ is a (sufficiently nice) potential function.

Brownian Motion in Lie algebras

If H is a finite-dimensional Hilbert space, the *Brownian motion* on H is the process

$$B^H(t) = \sum_{\xi \in \beta(H)} B_t^\xi \xi$$

where $\beta(H)$ is any o.n. basis for H and $\{B_t^\xi\}$ are i.i.d. standard Brownian motions on $\mathbb{R}$.

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The space of all complex matrices $M_N(\mathbb{C}) = \mathfrak{gl}(N, \mathbb{C})$ is a Lie algebra. The space of (skew-)Hermitian matrices $\mathfrak{u}(N)$ is a Lie algebra. Equip both with the inner product inner product

$$\langle A, B \rangle = N\mathrm{Tr}(B^* A).$$

- $B^{\mathfrak{gl}(N,\mathbb{C})}(1) = Z_N$ is a Ginibre ensemble.
- $B^{\mathfrak{u}(N)}(1) = iX_N$ is a GUE (after losing an i).

Lie Structure and Eigenvalues

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Conjecture. If $\mathfrak{g}$ is a Lie algebra, and π is a faithful representation, then the eigenvalues of $\pi(B_t^{\mathfrak{g}})$ form a determinantal point process for each $t \geq 0$.

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This talk is not about this question, but rather transporting these ideas from the Lie algebra to its Lie group. To connect the dots:

Theorem. [Dahlqvist, K, 2016 (unpublished)] Let K be a compact semisimple Lie *group*, equipped with a bi-invariant metric. Then the eigenvalues of the Brownian motion B_t^K form a determinantal point process for each $t \geq 0$.

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(Seems funny that we can prove something for Lie groups but it's harder for the Lie algebras. The proof uses an expansion of the heat kernel in terms of characters, Poisson resummation, and the Weyl integral formula.)

Brownian Motion on $\mathrm{U}(N)$, $\mathrm{GL}(N,\mathbb{C})$, and the Large-$N$ Limit

Brownian Motion on Lie Groups

On any Riemannian manifold M, there's a Laplace operator Δ_M. And where there's a Laplacian, there's a Brownian motion: the Markov process $(B_t^x)_{t\geq 0}$ on M with generator $\frac{1}{2}\Delta_M$, started at $B_0^x = x \in M$.

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Let Γ be a (matrix) Lie group. Any inner product on $\mathrm{Lie}(\Gamma) = T_I\Gamma$ gives rise to a unique left-invariant Riemannian metric, and corresponding Laplacian Δ_Γ. On Γ we canonically start the Brownian motion $(B_t)_{t\geq 0}$ at $I \in \Gamma$.

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There is a beautiful relationship between the Brownian motion W_t on the Lie algebra $\mathrm{Lie}(\Gamma)$ and the Brownian motion B_t: the *rolling map*

$$dB_t = B_t \circ dW_t, \qquad \text{i.e.} \qquad B_t = I + \int_0^t B_t \circ dW_t.$$

Here $\circ$ denotes the Stratonovich stochastic integral. This can always be converted into an Itô integral; but the answer depends on the structure of the group Γ (and the chosen inner product).

Brownian Motion on $\mathrm{U}(N)$ and $\mathrm{GL}(N,\mathbb{C})$

Fix the *reverse normalized* Hilbert–Schmidt inner product on $\mathbb{M}_N(\mathbb{C})$ for all matrix Lie algebras:

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Let $X_t = X_t^N$ and $Y_t = Y_t^N$ be independent Hermitian Brownian motions of variance t/N.

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The Brownian motion on $\mathfrak{gl}(N,\mathbb{C}) = \mathrm{Lie}(\mathrm{GL}(N,\mathbb{C})) = \mathbb{M}_N(\mathbb{C})$ is $Z_t = 2^{-1/2}i(X_t + iY_t)$; the Brownian motion G_t on $\mathrm{GL}(N,\mathbb{C})$ satisfies

$$\boxed{dG_t = G_t\,dZ_t.}$$

Free Additive Brownian Motion

If $X_t = X_t^N$ is a Hermitian Brownian motion process, then at each time $t > 0$ it is a GUE_N with entries of variance t/N. Wigner's law then shows that the empirical spectral distribution of X_t^N converges to the semicircle law $\varsigma_t = \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}\, dx$.

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A process $(x_t)_{t\geq 0}$ (in a W^*-probability space with trace τ) is a **free additive Brownian motion** if its increments are freely independent — $x_t - x_s$ is free from $\{x_r : r \leq s\}$ — and $x_t - x_s$ has the semicircular distribution ς_{t-s}, for all $t > s$.

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In 1991, Voiculescu showed that the Hermitian Brownian motion $(X_t^N)_{t\geq 0}$ converges to $(x_t)_{t\geq 0}$ in finite-dimensional non-commutative distributions:

$$\frac{1}{N}\mathrm{Tr}(P(X_{t_1}, \ldots, X_{t_n})) \to \tau(P(x_{t_1}, \ldots, x_{t_n})) \qquad \forall P.$$

Free Unitary and Free Multiplicative Brownian Motion

There is now a well-developed theory of free stochastic differential equations. Initially constructed in the free Fock space setting (by Kümmerer and Speicher in the early 1990s), it was used by Biane in 1997 to define “free versions” of U_t and G_t.

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Let x_t, y_t be freely independent free additive Brownian motions, and $z_t = 2^{-1/2}i(x_t + iy_t)$. The **free unitary Brownian motion** is the process started at $u_0 = 1$ defined by

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It is natural to expect that these processes should be the large-N limits of the $\mathrm{U}(N)$ and $\mathrm{GL}(N,\mathbb{C})$ Brownian motions.

Free Unitary Brownian Motion

Theorem. [Biane, 1997] For all non-commutative (Laurent) polynomials P in n variables and times $t_1, \ldots, t_n \geq 0$,

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Biane also computed the moments of u_t, and its spectral measure ν_t: it has a density (smooth on the interior of its support), supported on a compact arc for $t < 4$, and fully supported on $\mathbb{U}$ for $t \geq 4$.

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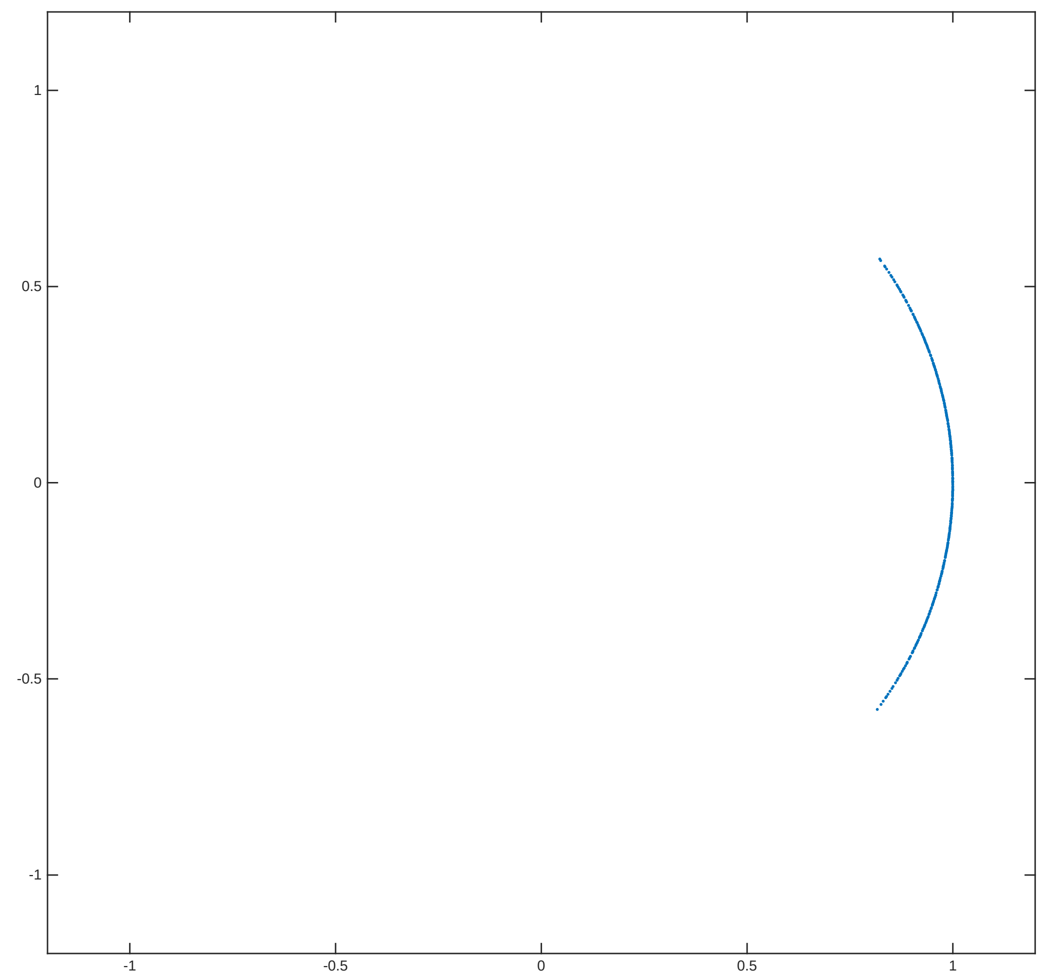

$$t = 0.1$$

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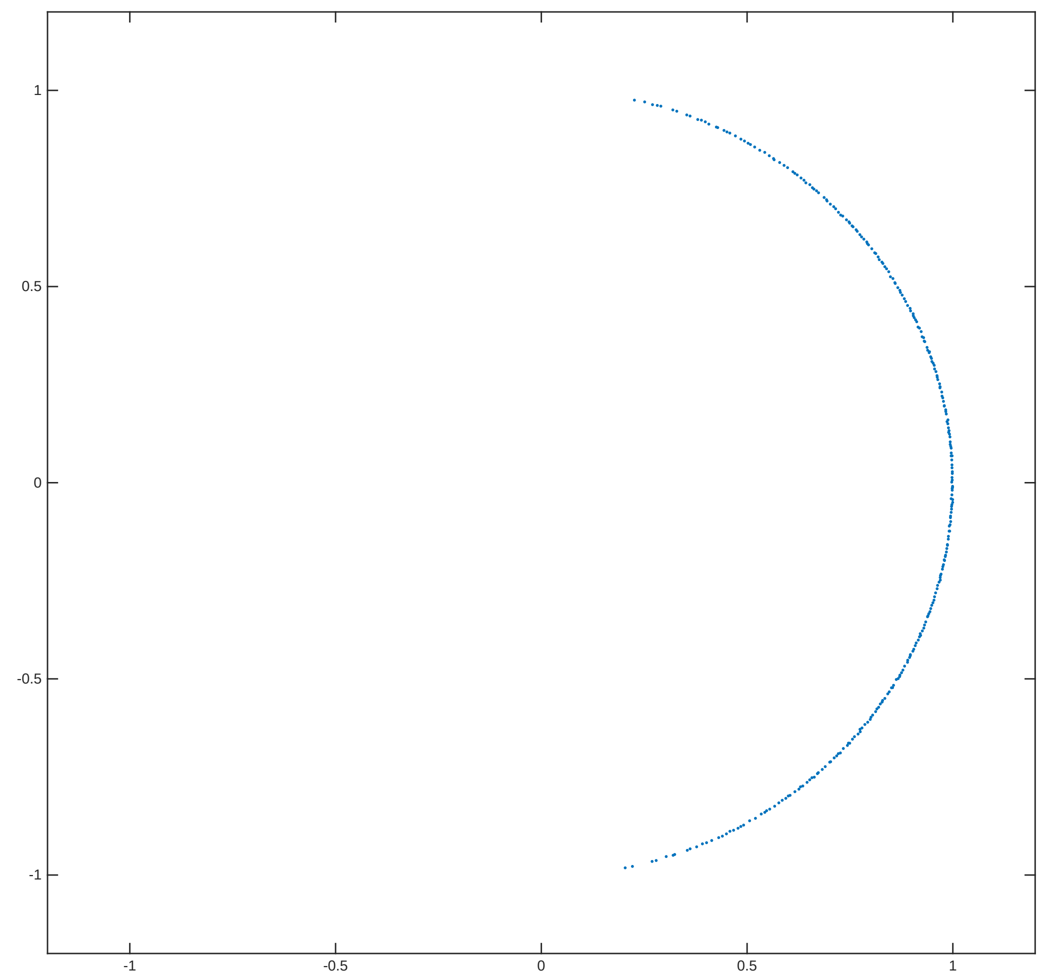

$t = 0.5$

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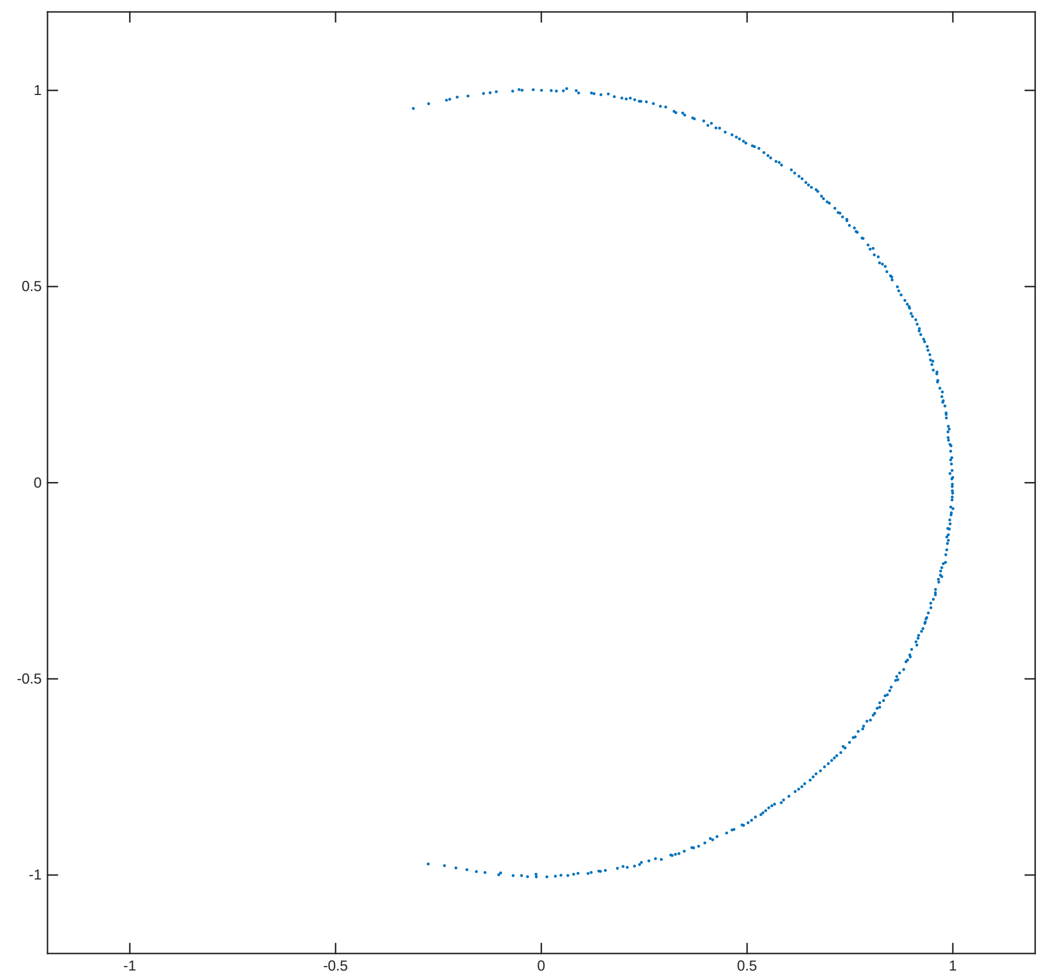
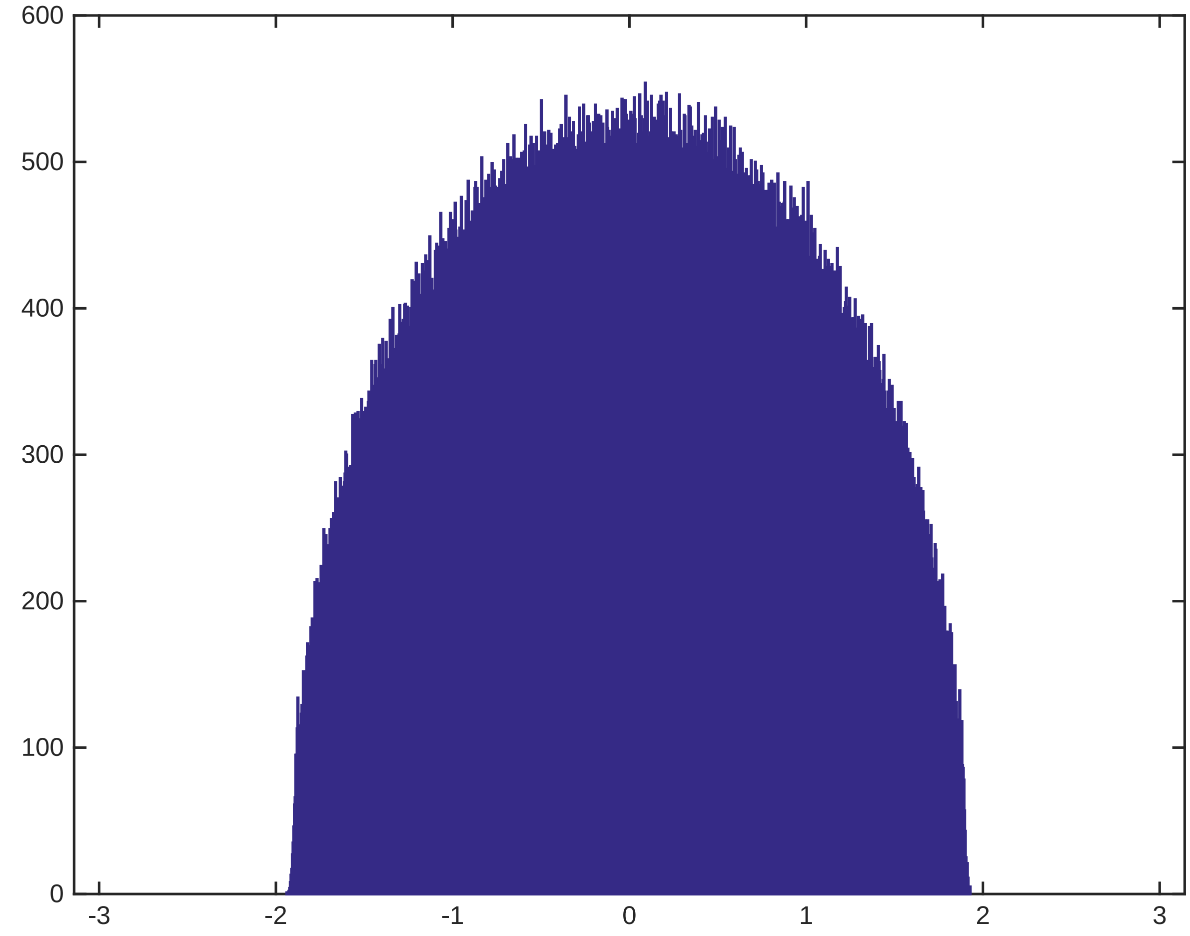


$$t = 1$$

Free Unitary Brownian Motion



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The Edge of The Spectrum of Unitary Brownian Motion

The measure ν_t is *not* the semicircle law wrapped around the circle. It is supported on the arc of angles $\theta \in (-\pi, \pi]$ satisfying

$$|\theta| \le \varepsilon(t) = \frac{1}{2}\sqrt{t(4-t)} + \cos^{-1}\left(1 - \frac{t}{2}\right).$$

In particular, when $t \ge 4$, supp ν_t is the whole circle.

- Citations

Prologue

Brownian Motion
- BM on Lie Groups
- U & GL
- Free$+$BM
- Free$\times$BM
- Free Unitary BM
- Edge
- Transforms
- Free Mult. BM
- GL Spectrum

Brown Measure

Segal–Bargmann

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Aside from this, little about the local behavior of the eigenvalues is known. In 2010, Lévy and Mäida proved that the linear statistics (fluctuations) for a single time $t > 0$ are Gaussian; Cébron and I extended this to sampling the process at many times. As to the fluctuations at the edge, it is natural to expect they should be Tracy-Widom before $t < 4$, and Pearcey at the collision time $t = 4$. I have been working on this with Liechty; not enough results to report just yet.

Analytic Transforms Related to u_t

Biane's approach to understanding the measure ν_t was through its moment-generating function

$$\psi_t(z) = \int_{\mathbb{U}} \frac{uz}{1-uz}\,\nu_t(du) = \sum_{n\geq 1} m_n(\nu_t)\,z^n$$

(the second $=$ holds for $|z|<1$; the integral converges for $1/z \notin \operatorname{supp}\nu_t$).

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(the second $=$ holds for $|z|<1$; the integral converges for $1/z \notin \operatorname{supp}\nu_t$). Then define

$$\chi_t(z) = \frac{\psi_t(z)}{1+\psi_t(z)}.$$

The function χ_t is injective on $\mathbb{D}$, and has a one-sided inverse f_t: $f_t(\chi_t(z)) = z$ for $z \in \mathbb{D}$ (but $\chi_t \circ f_t$ is only the identity on a certain region in $\mathbb{C}$; more on this later).

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Using the SDE for u_t and some clever complex analysis, Biane showed that

$$f_t(z) = ze^{\frac{t}{2}\frac{1+z}{1-z}}.$$

The Large-N Limit of G_t^N

In 1997 Biane conjectured a bulk large-N limit should hold for the Brownian motion on $\mathrm{GL}(N,\mathbb{C})$, but the ideas of his U_t^N proof (spectral theorem, representation theory of $\mathrm{U}(N)$) did not translate well to the a.s. non-normal process G_t^N.

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Theorem. [K, IMRN 2016] For all non-commutative Laurent polynomials P in $2n$ variables, and times $t_1,\ldots,t_n \ge 0$,

$$\frac{1}{N}\mathrm{Tr}\left(P(G_{t_1}^N,(G_{t_1}^N)^*,\ldots,G_{t_n}^N,(G_{t_n}^N)^*)\right) \to \tau\left(P(g_{t_1},g_{t_1}^*,\ldots,g_{t_n},g_{t_n}^*)\right) \text{ a.s.}$$

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This is convergence of the (multi-time) $*$-distribution, of a *non-normal* matrix process. What about the eigenvalues?

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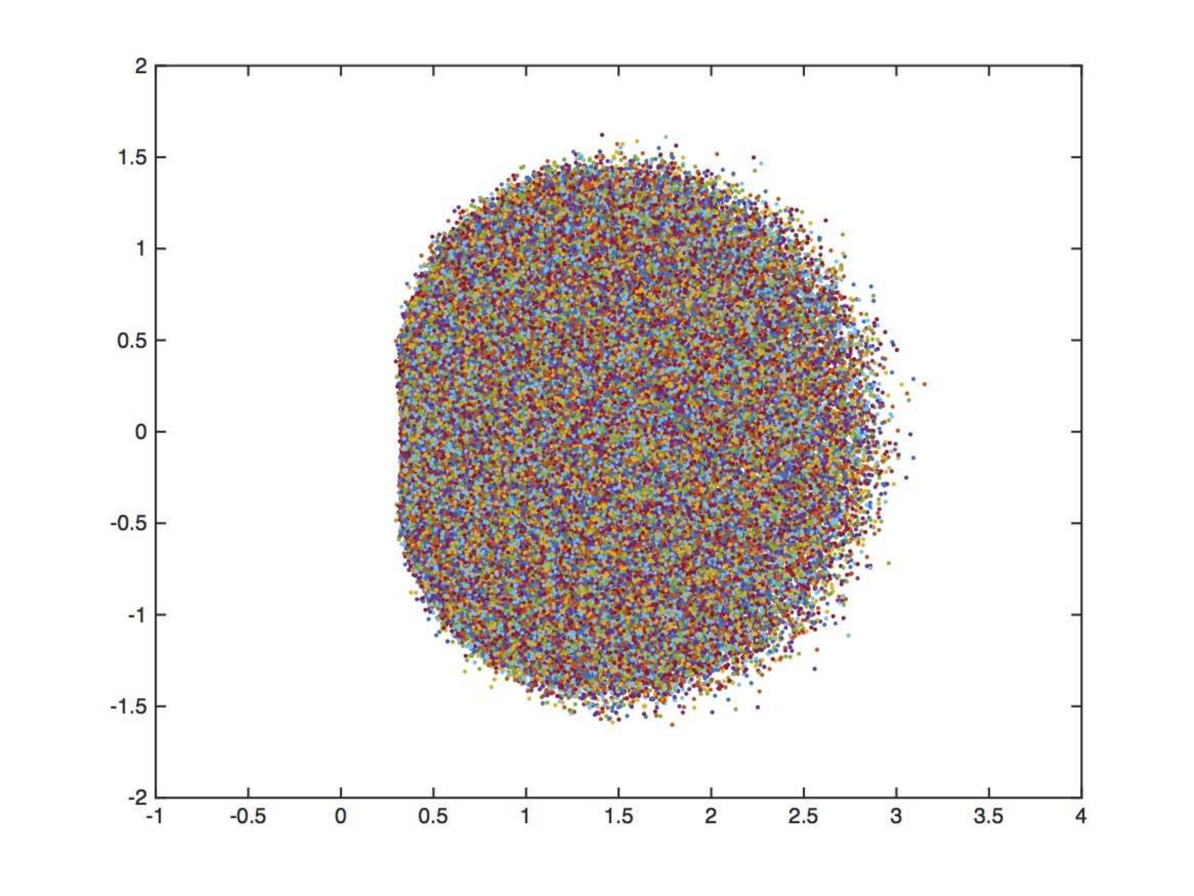

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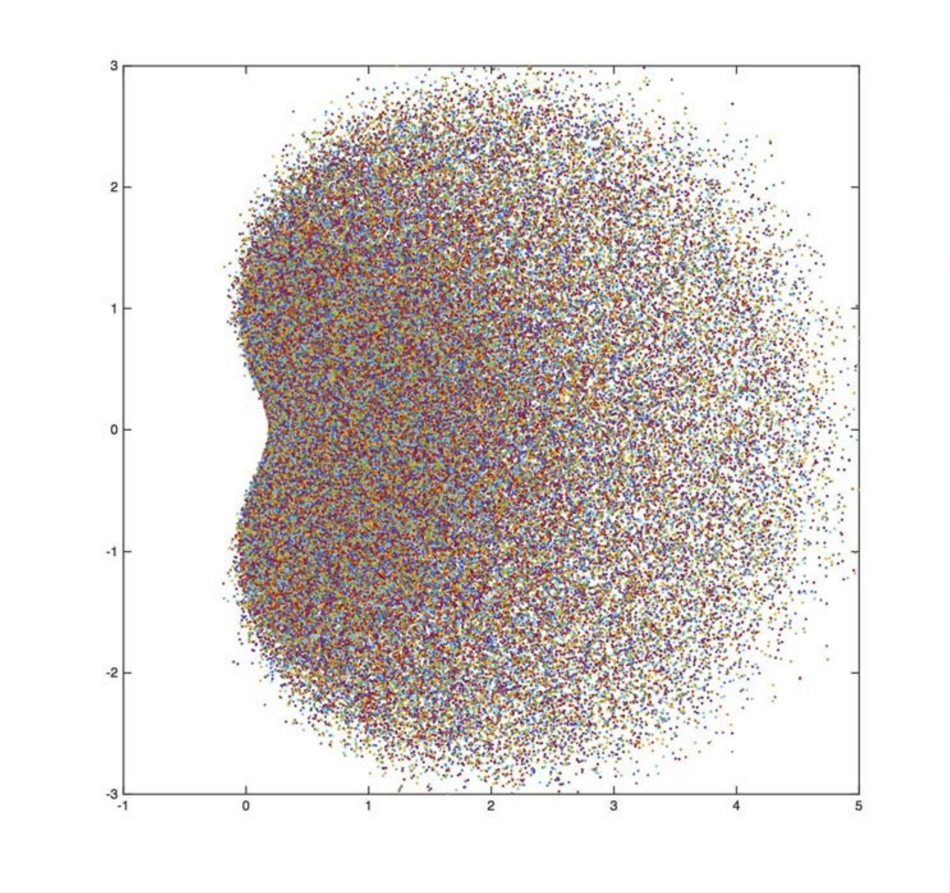

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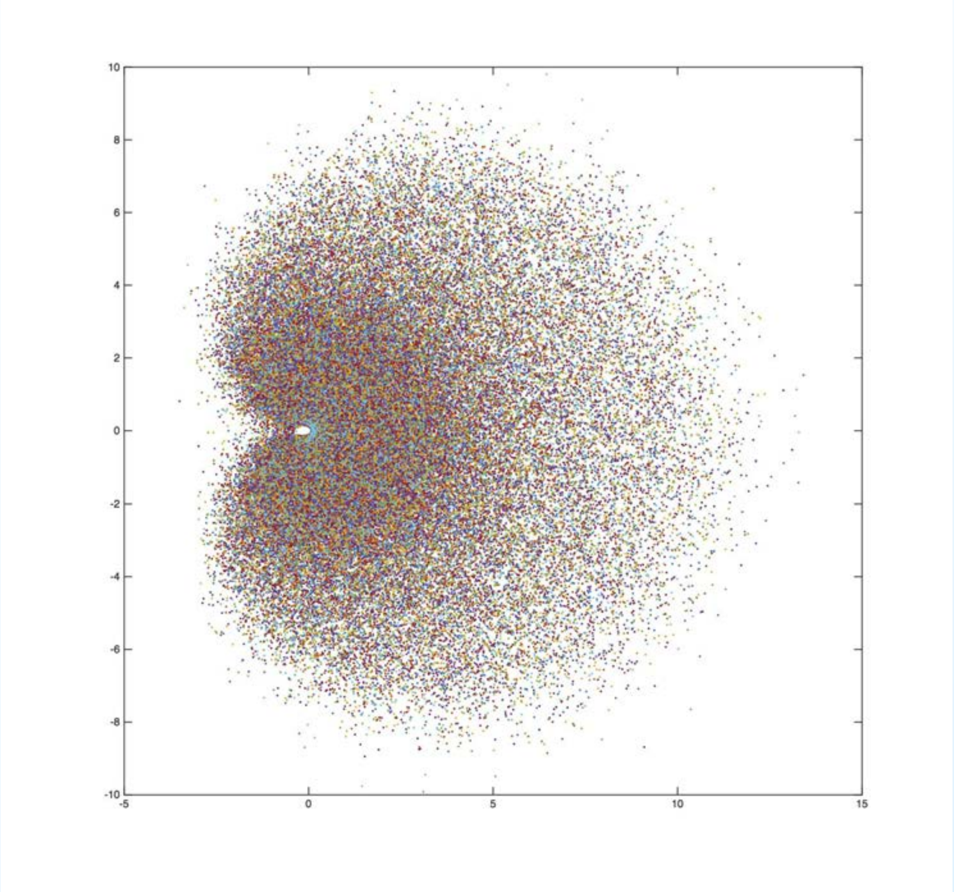

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Brown's Spectral Measure

Brown's Spectral Measure in Tracial von Neumann Algebras

If $(\mathcal{A}, \tau)$ is a W^*-probability space, then any normal operator $a \in \mathcal{A}$ has a spectral measure $\mu_a = \tau \circ E^a$. If A is a normal matrix, μ_A is its ESD. It is characterized (nicely) by the $*$-distribution of a:

$$\int_{\mathbb{C}} z^k \bar{z}^\ell \, \mu_a(dz d\bar{z}) = \tau(a^k a^{*\ell}).$$

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If a is not normal, there is no such measure. But there is a substitute: Brown's spectral measure. Let $L(a)$ denote the (log) Kadison–Fuglede determinant:

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$$\mu_a = \frac{1}{2\pi} \nabla_\lambda^2 L(a-\lambda)$$

is a probability measure on $\mathbb{C}$. If A is *any* matrix, μ_A is its ESD.

Properties of Brown Measure

The Brown measure has some nice properties analogous to the spectral measure, but not all:

- $\tau(a^k) = \int_{\mathbb{C}} z^k \, \mu_a(dzd\bar{z})$ and $\tau(a^{*k}) = \int_{\mathbb{C}} \bar{z}^k \, \mu_a(dzd\bar{z})$ but you *cannot max and match*.

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Convergence of the Brown Measure

Let $\{a, a_n\}_{n\in\mathbb{N}}$ be a uniformly bounded set of operators in some W^*-probability spaces, with $a_n \to a$ in $*$-distribution. We would hope that $\mu_{a_n} \to \mu_a$. Without some very fine information about the spectral measure of $|a_n - \lambda|$ near the edge of $\mathrm{Spec}(a_n)$, the best that can be said in general is the following.

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Proposition. Suppose that $\mu_{a_n} \to \mu$ weakly for some probability measure μ on $\mathbb{C}$. Then

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Corollary. Let V_a be the unbounded connected component of $\mathbb{C} \setminus$ supp μ_a. Then supp $\mu \subseteq \mathbb{C} \setminus V_a$. (In particular, if supp μ_a is simply-connected, then supp $\mu \subseteq$ supp μ_a.)

Brown Measure via Regularization

The function $L(a-\lambda) = \int_{\mathbb{R}} \log t \, \mu_{|a|}(dt)$ is essentially impossible to compute with. But we can use regularity properties of the spectral resolution to approach it in a different way. Define

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$$h_a^{\epsilon}(\lambda) = \frac{1}{2\pi}\nabla_{\lambda}^2 L_{\epsilon}(a-\lambda).$$

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It is not difficult to explicitly calculate the density h_a^ϵ for fixed $\epsilon > 0$.

The Density h_a^ϵ and the Spectrum of a

Lemma. Let $\lambda \in \mathbb{C}$, and denote $a_\lambda = a - \lambda$. Then

$$h_a^\epsilon(\lambda) = \frac{1}{\pi}\epsilon\tau\left((a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\right).$$

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This is locally uniformly bounded in λ; so taking $\epsilon \downarrow 0$, the factor of ϵ in $h_a^\epsilon(\lambda)$ kills the term; we find $\mu_a = 0$ in a neighborhood of λ.

Invertibility in $L^p(\mathcal{A},\tau)$

Recall that $L^p(\mathcal{A},\tau)$ is the closure of $\mathcal{A}$ in the norm

$$\|a\|_p^p = \tau(|a|^p) = \tau\Big((a^*a)^{p/2}\Big).$$

(It can be realized as a set of densely-defined unbounded operators, acting on the same Hilbert space as $\mathcal{A}$). The non-commutative L^p-norms satisfy the same Hölder inequality as the classical ones.

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Definition. The $L^p(\mathcal{A},\tau)$ *resolvent* $\mathrm{Res}_{p,\tau}(a)$ is the interior of the set of $\lambda\in\mathbb{C}$ for which $a-\lambda$ has an inverse in $L^p(\mathcal{A},\tau)$. The $L^p(\mathcal{A},\tau)$ *spectrum* $\mathrm{Spec}_{p,\tau}(a)$ is $\mathbb{C}\setminus\mathrm{Res}_{p,\tau}(a)$.

The $L^p(\mathcal{A}, \tau)$ Spectrum

From Hölder's inequality, we have the inclusions

$$\mathrm{Spec}_{p,\tau}(a) \subseteq \mathrm{Spec}_{q,\tau}(a) \subseteq \mathrm{Spec}(a)$$

for $1 \leq p \leq q < \infty$. Without including the closure in the definition, these inclusions can be strict; with the closure, my (wild) conjecture is that $\mathrm{Spec}_{1,\tau}(a) = \mathrm{Spec}(a)$ for all a.

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Note, this is *not* equal to $\|a_\lambda^{-1}\|_4^4$ when a_λ is not normal.

The $L^2_{2,\tau}$ Spectrum

Proposition. Let $a \in \mathcal{A}$, and suppose a^2 is invertible in $L^2(\mathcal{A}, \tau)$. Then for all $\epsilon > 0$,

$$\tau\Big((a^*a + \epsilon)^{-1}(aa^* + \epsilon)^{-1}\Big) \le \|a^{-2}\|_2^2.$$

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Another wild conjecture: this is actually equality. (That depends on showing that, if a^2 is *not* invertible in $L^2(\mathcal{A}, \tau)$, the above quantity blows up at rate $\Omega(1/\epsilon)$. This appears to be what happens in the case that a is normal, which would imply $\mathrm{Spec}^2_{2,\tau}(a) = \mathrm{Spec}_{4,\tau}(a)$ $= \mathrm{Spec}(a)$ in that case.)

The Segal–Bargmann Transform

The Unitary Segal–Bargmann Transform

The **Segal–Bargmann (Hall) Transform** is a map from functions on $\mathrm{U}(N)$ to holomorphic functions on $\mathrm{GL}(N,\mathbb{C})$. It is defined by the analytic continuation of the action of the heat operator:

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Writing out what this integral formula means in probabilistic terms, here is a nice way to express it: let F already be a holomorphic function on $\mathrm{GL}(N), \mathbb{C})$, and let $f = F|_{\mathrm{U}(N)}$. Let U_t and G_t be independent Brownian motions on $\mathrm{U}(N)$ and $\mathrm{GL}(N, \mathbb{C})$. Then

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This extends beyond f that already possess an analytic continuation; it defines an *isometric isomorphism*

$$\mathbf{B}_t^N : L^2(\mathrm{U}(N), U_t) \to \mathcal{H}L^2(\mathrm{GL}(N,\mathbb{C}), G_t).$$

The Free Unitary Segal–Bargmann Transform

In 1997, Biane introduced a free version of the Unitary SBT, which can be described in similar terms: acting on, say, polynomials f in a single variable, $\mathscr{G}_t f$ is defined by

$$(\mathscr{G}_t f)(g_t) = \tau[f(g_t u_t)|g_t].$$

He conjectured that $\mathscr{G}_t$ is the large-N limit of $\mathbf{B}_t^N$ in an appropriate sense; this was proven by Driver, Hall, and me in 2013. (It was for this work that we invented trace polynomial concentration.)

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Biane proved directly (and it follows from the large-N limit) that $\mathscr{G}_t$ extends to an isometric isomorphism

$$\mathscr{G}_t \colon L^2(\mathbb{U}, \nu_t) \to \mathscr{A}_t$$

where $\mathscr{A}_t$ is a certain reproducing-kernel Hilbert space of holomorphic functions. The norm on $\mathscr{A}_t$ is given by

$$\|F\|^2_{\mathscr{A}_t} = \tau(|F(g_t)|^2) = \tau(F(g_t)^* F(g_t)) = \|F(g_t)\|_2^2.$$

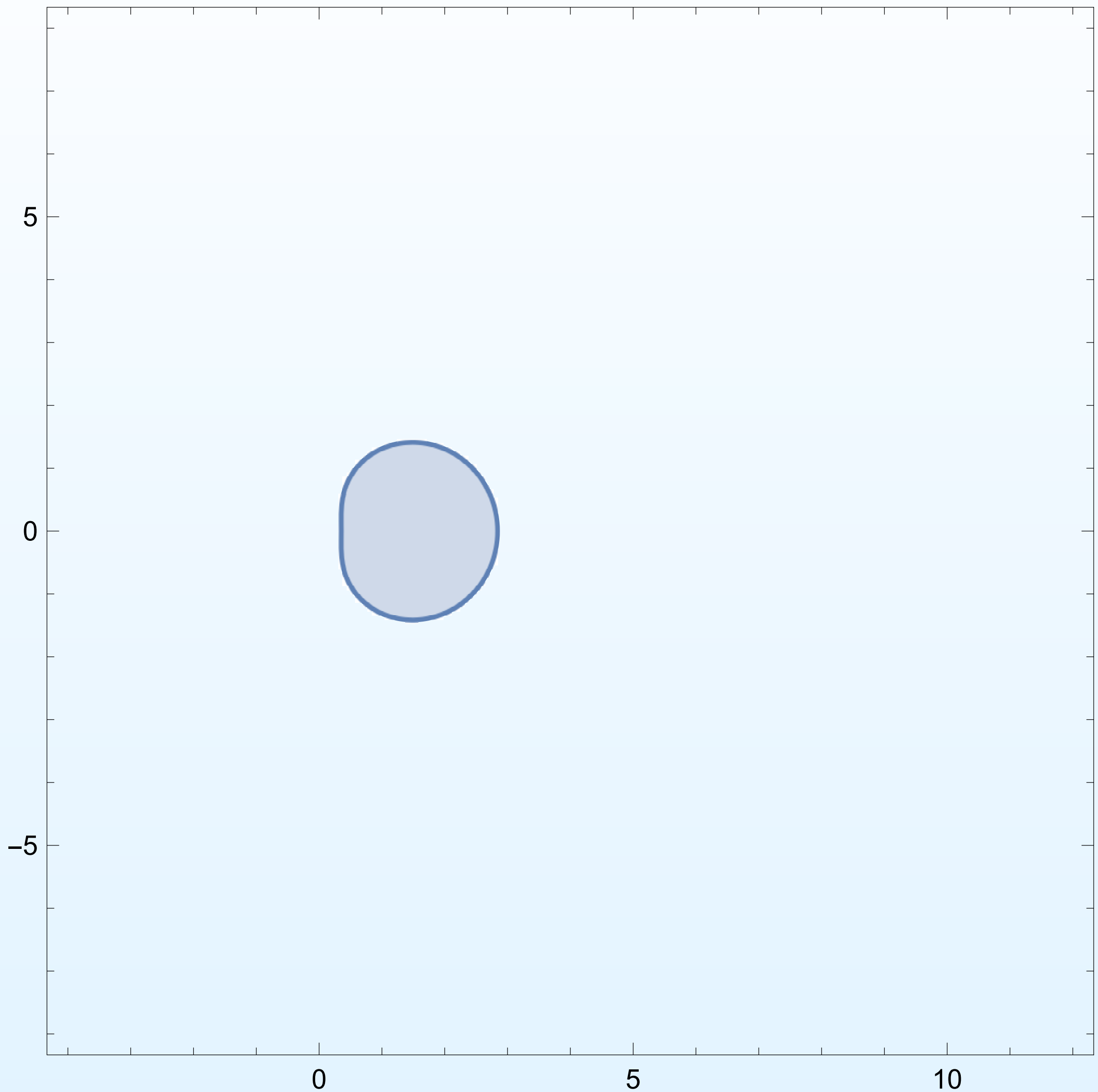
The Range of the Free Segal–Bargmann Transform

The functions $F \in \mathscr{A}_t$ are not all entire functions. They are holomorphic on a bounded region Σ_t

$$\Sigma_t = \mathbb{C} \setminus \overline{\chi_t(\mathbb{C} \setminus \text{supp}\, \nu_t)}$$

where (recall) χ_t is the (right-)inverse of $f_t(z) = ze^{\frac{t}{2}\frac{1+z}{1-z}}$.

The Range of the Free Segal–Bargmann Transform



$t = 1$

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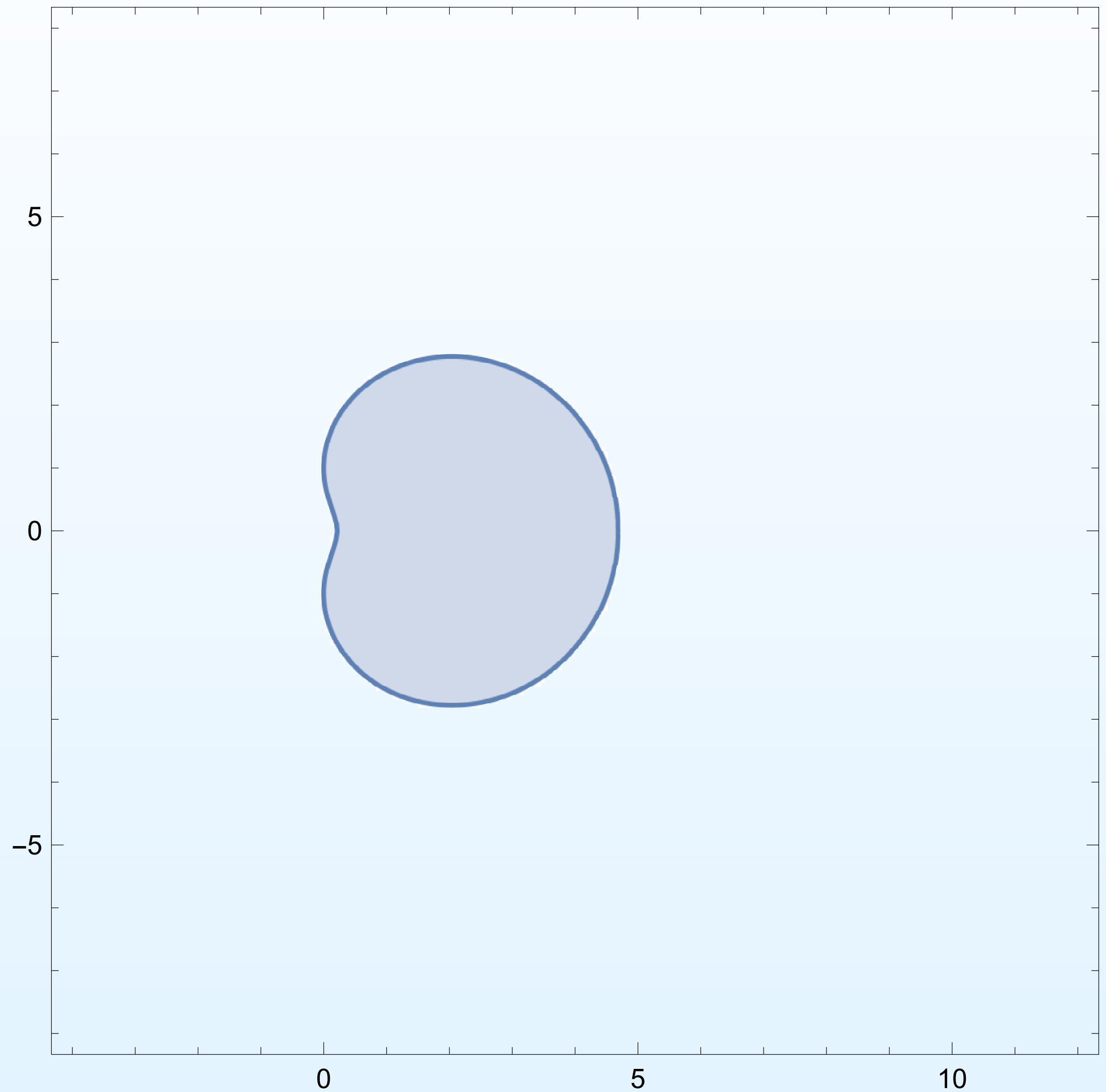
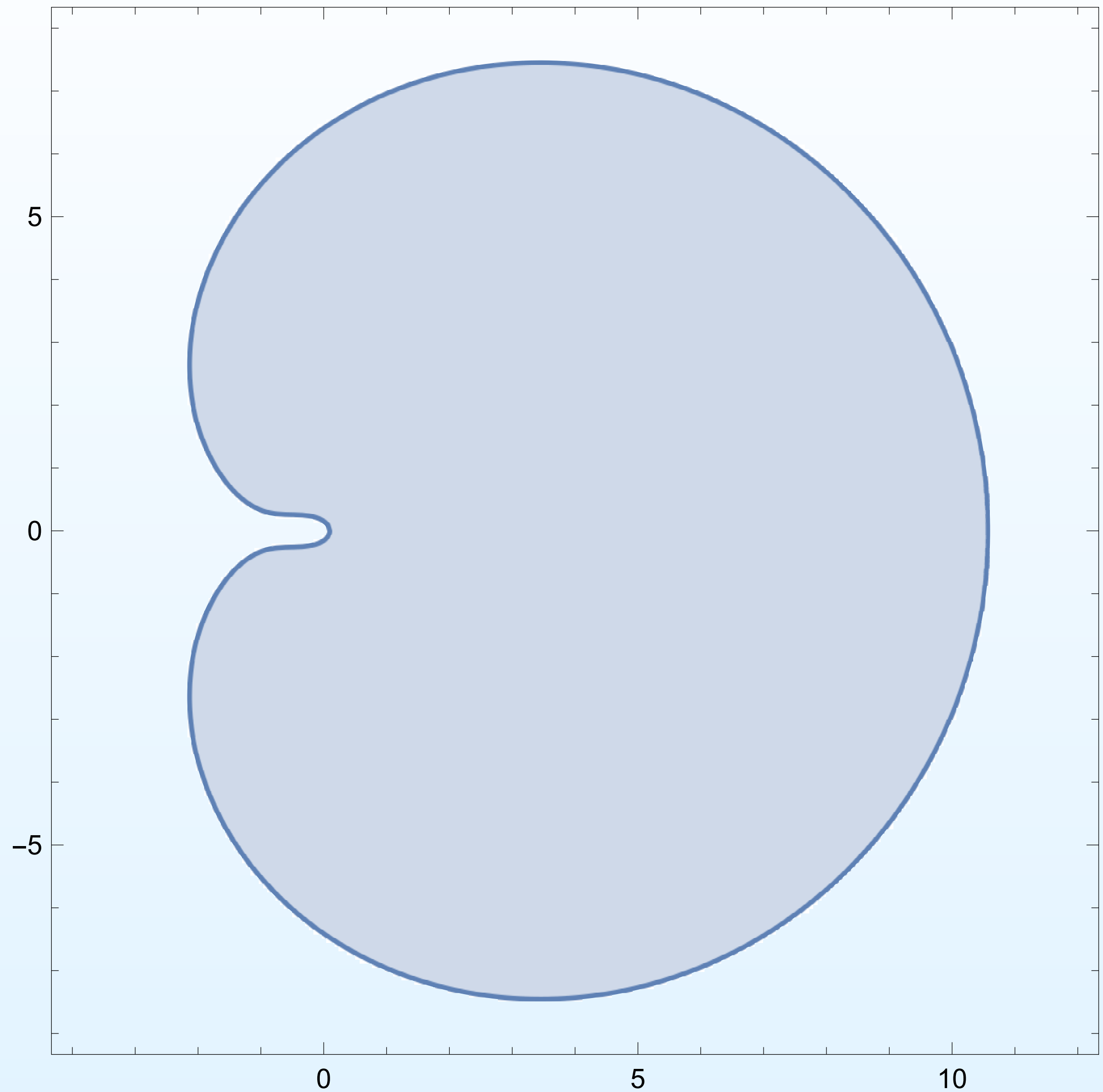


$t = 2$

The Range of the Free Segal–Bargmann Transform



$t = 3.9$

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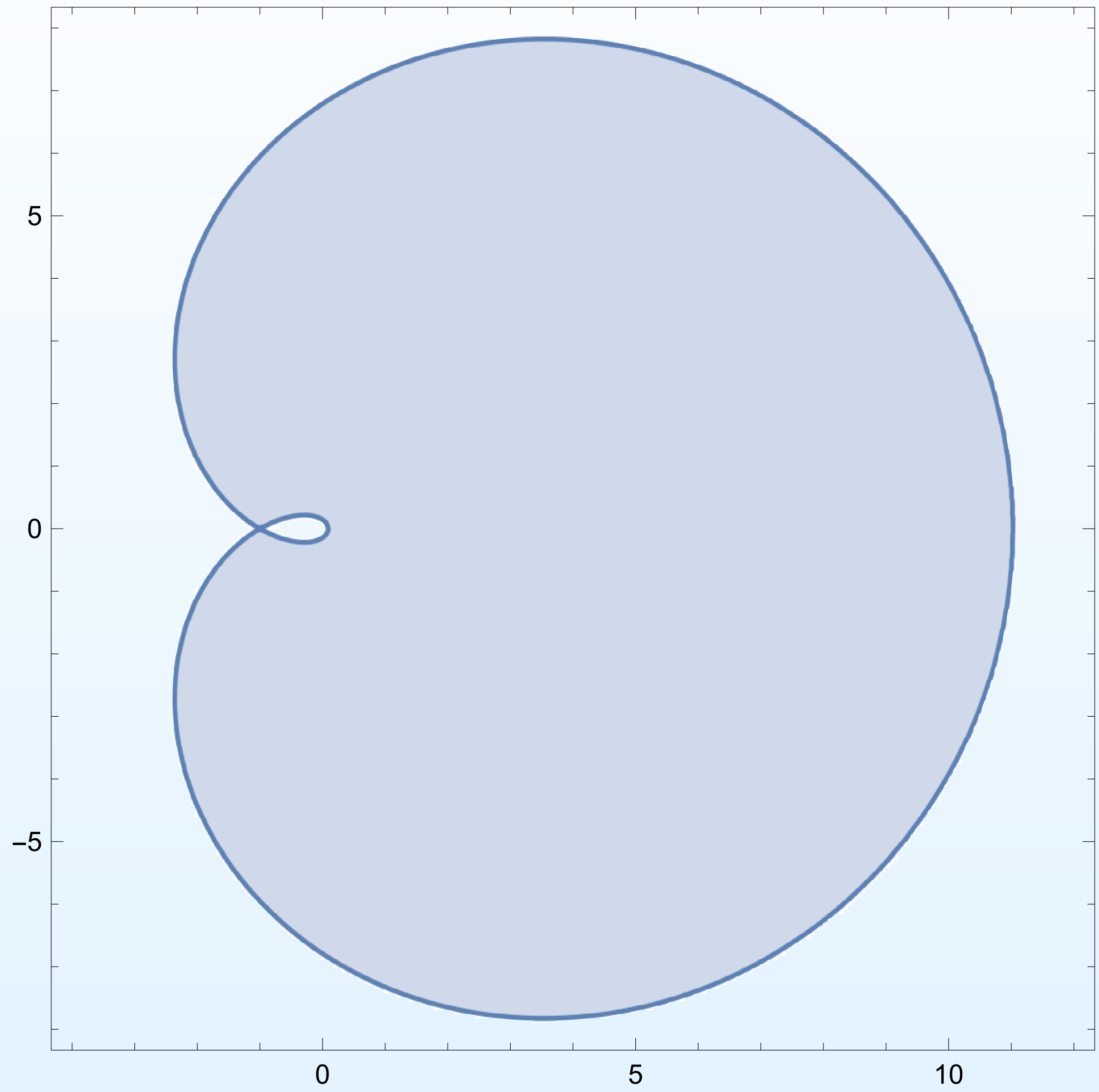

$t = 4$

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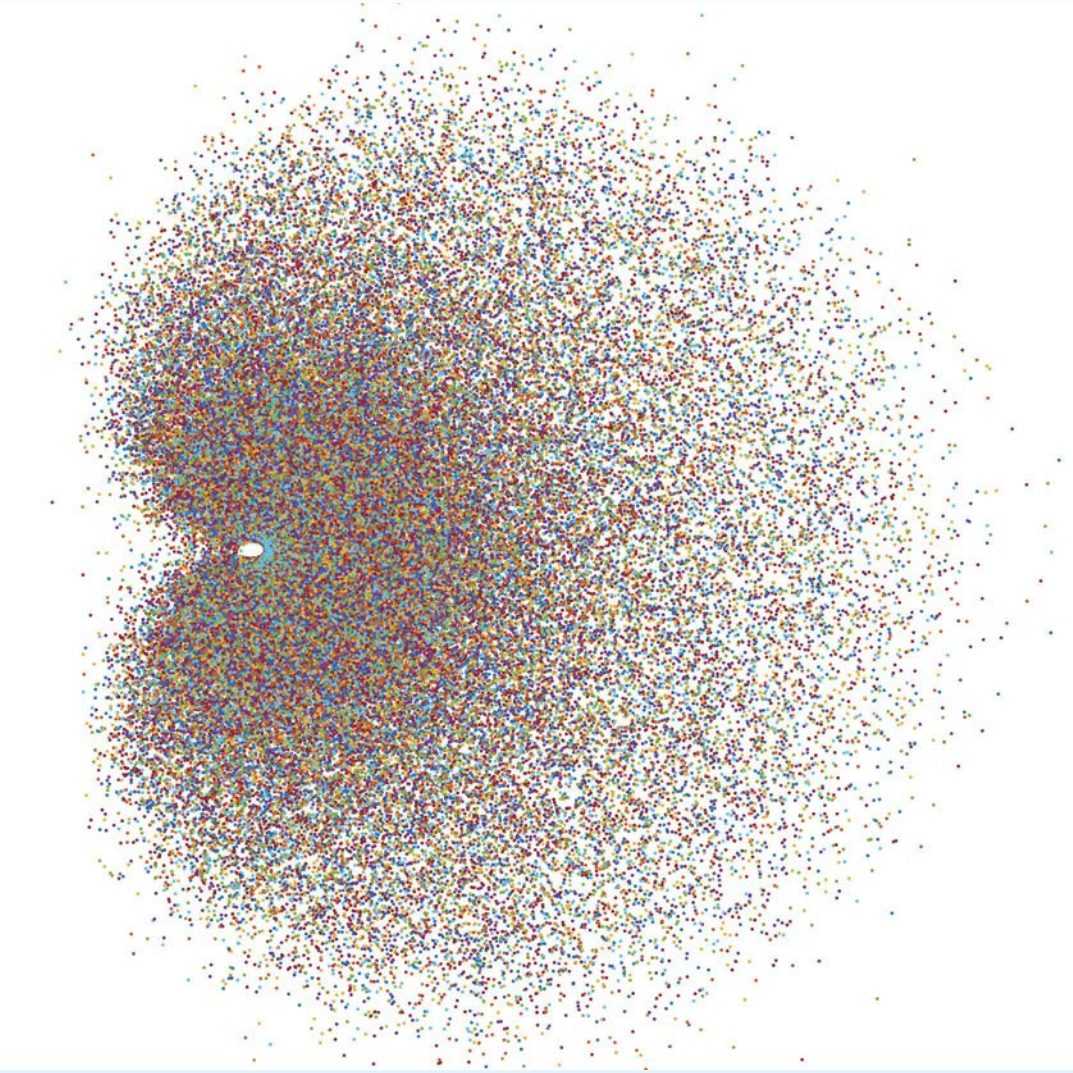

$$t = 4$$

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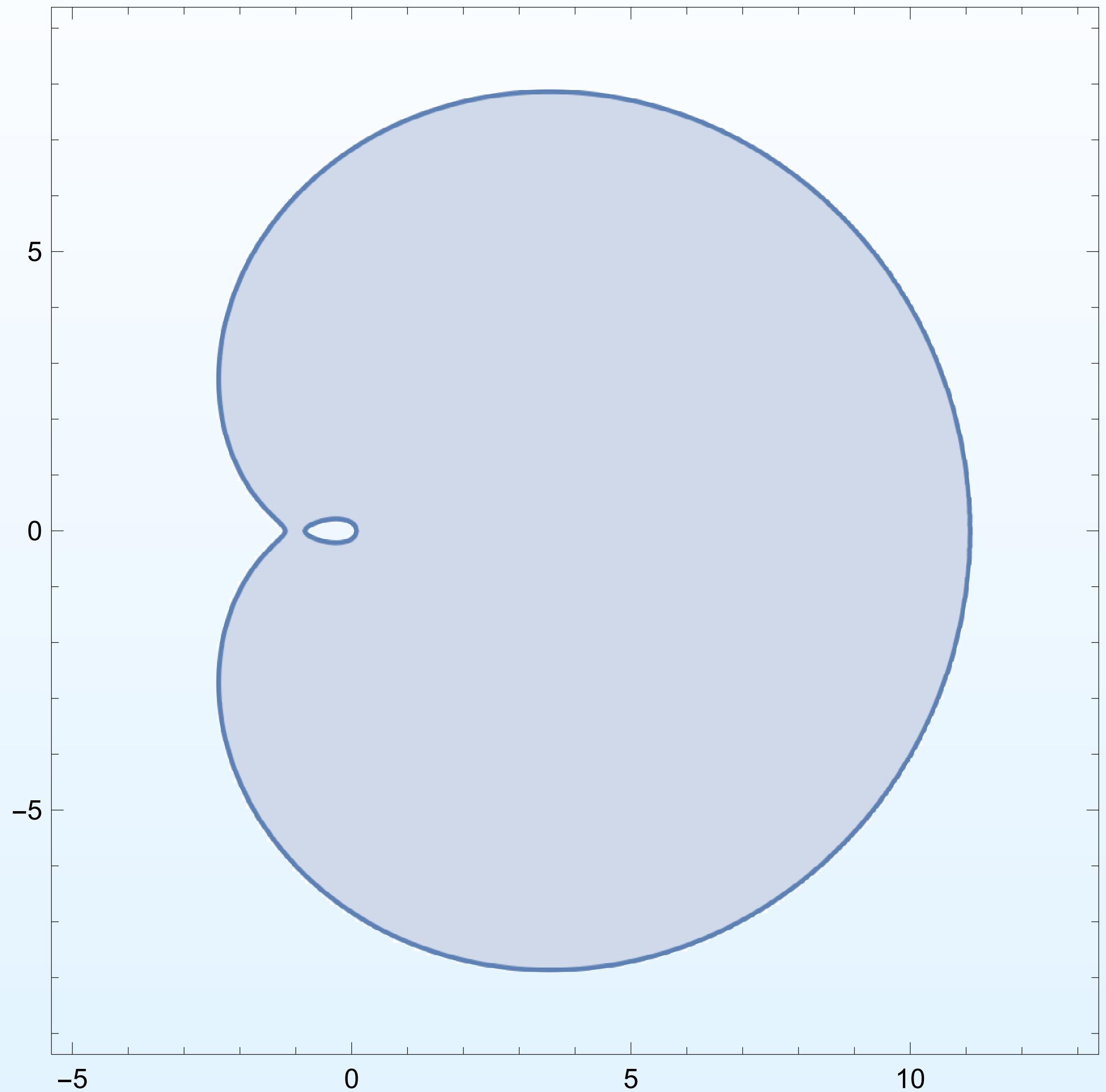

$t = 4.01$

The Support of The Brown Measure of g_t

Theorem. (Hall, K, late 2017)

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Using PDE techniques, we can compute that

$$\mathscr{G}_t^{-1}((z-\lambda)^{-1}) = \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda) - u}.$$

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$$\mathscr{G}_t : \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda)-u} \mapsto \frac{1}{z-\lambda}.$$

Since $\frac{1}{(z-\lambda)^2} = \frac{d}{d\lambda}\frac{1}{z-\lambda}$, using regularity properties of $\mathscr{G}_t$ we have

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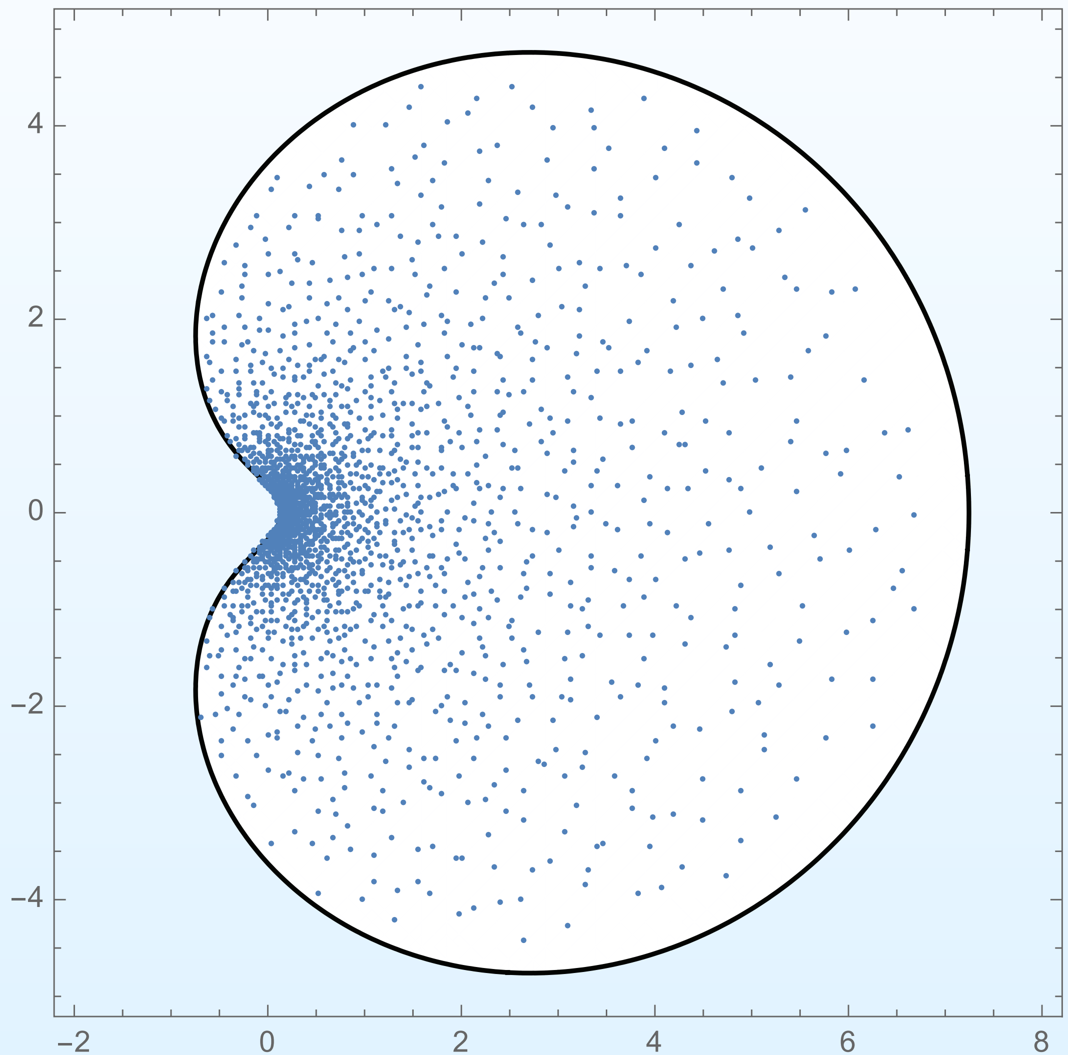
The answer is: precisely when $f_t(\lambda) \notin \operatorname{supp}\nu_t$. I.e.

$$\operatorname{Res}^2_{2,\tau}(g_t) = f_t^{-1}(\mathbb{C}\setminus \operatorname{supp}\nu_t) = \chi_t(\mathbb{C}\setminus\operatorname{supp}\nu_t).$$

□

The Empirical Spectrum and Σ_t

Here is a simulation of eigenvalues of $G_t^{(N)}$ for $N = 2000$, together with the boundary of Σ_t, at $t = 3$ (produced in Mathematica).



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I'll let you know what more I know next time we meet. ○