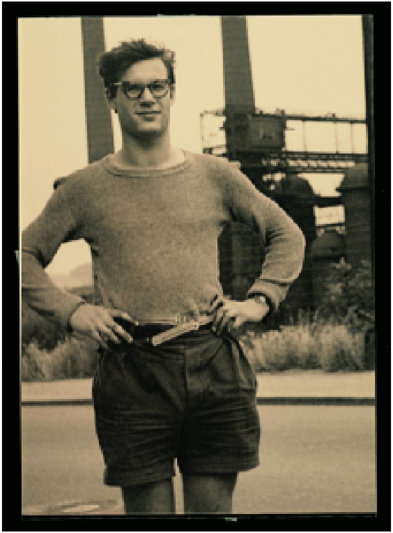
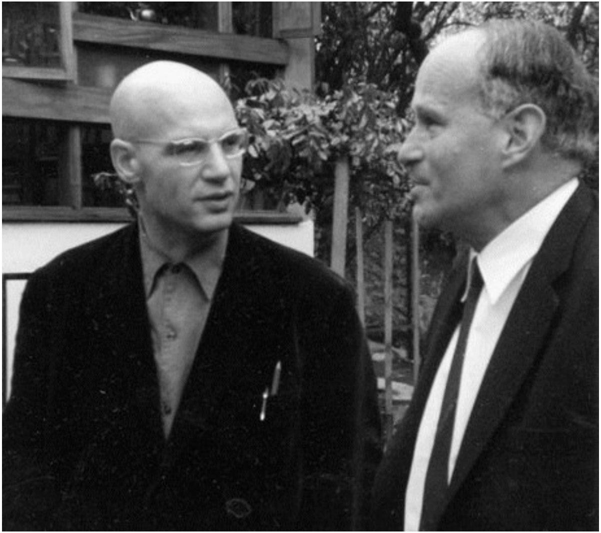
Grothendieck's works on Banach spaces and their surprising recent repercussions (parts 1 and 2)

Gilles Pisier

IPAM, April 2018





PLAN

- Classical GT
- Non-commutative and Operator space GT
- GT and Quantum mechanics : EPR and Bell's inequality
- GT in graph theory and computer science

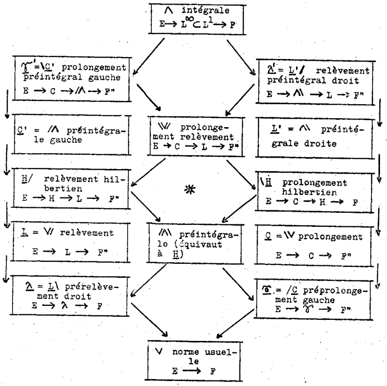
In 1953, Grothendieck published an extraordinary paper entitled "Résumé de la théorie métrique des produits tensoriels topologiques,"
now often jokingly referred to as "Grothendieck's résumé"(!).
Just like his thesis, this was devoted to tensor products of topological vector spaces, but in sharp contrast with the thesis devoted to the locally convex case, the "Résumé" was exclusively concerned with Banach spaces ("théorie métrique").

Boll.. Soc. Mat. São-Paulo 8 (1953), 1-79.
Reprinted in "Resenhas"

Initially ignored....
But after 1968 : huge impact on the development of “Geometry of Banach spaces"
starting with
Pietsch 1967 and Lindenstrauss-Pełczyński 1968
Kwapień 1972
Maurey 1974 and so on...

The “Résumé" is about the natural $\otimes$-norms

TABLEAU DES ⊗-NORMES NATURELLES.



Explications. - 1. Désignations et factorisations typiques. Nous avons inséré les diverses ⊗-normes usuelles par leur signe usuel ou leurs signes usuels (permettant d'en reconnaître la forme

The central result of this long paper
"Théorème fondamental de la théorie métrique des produits tensoriels topologiques"
is now called

Grothendieck's Theorem (or Grothendieck's inequality)

We will refer to it as

GT

Informally, one could describe GT as a surprising and non-trivial relation between Hilbert space, or say

$$L_2$$

and the two fundamental Banach spaces

$$L_\infty, L_1$$

(here L_∞ can be replaced by the space $C(\Omega)$ of continuous functions on a compact set S).

Why are L_∞, L_1 fundamental ?

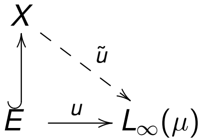
because they are UNIVERSAL !

Any Banach space is isometric to a **SUBSPACE** of L_∞
(ℓ_∞ in separable case)
Any Banach space is isometric to a **QUOTIENT** of L_1
(ℓ_1 in separable case)
(over suitable measure spaces)

— — — — — — — — — — — — — —

Moreover :
L_∞ is **injective**
L_1 is **projective**

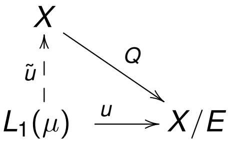
L_∞ is **injective**



Extension Pty :

$$\forall u \ \exists \tilde{u} \text{ with } \|\tilde{u}\| = \|u\|$$

L_1 is **projective**



Lifting Pty :

$$\forall u \text{ compact } \forall \varepsilon > 0 \ \exists \tilde{u} \text{ with } \|\tilde{u}\| \leq (1+\varepsilon)\|u\|$$

The relationship between

$$L_1, L_2, L_\infty$$

is expressed by an inequality involving

3 fundamental tensor norms :

Let X, Y be Banach spaces, let $X \otimes Y$ denote their algebraic tensor product. Then for any

$$T = \sum_1^n x_j \otimes y_j \in X \otimes Y \qquad (1)$$

$$T = \sum_1^n x_j \otimes y_j \tag{1}$$

(1."projective norm")

$$\|T\|_\wedge = \inf\left\{\sum \|x_j\|\|y_j\|\right\}$$

(2."injective norm")

$$\|T\|_\vee = \sup\left\{\left|\sum x^*(x_j)y^*(y_j)\right| \,\middle|\, x^* \in B_{X^*}, y^* \in B_{Y^*}\right\}$$

(3."Hilbert norm")

$$\|T\|_H = \inf\left\{\sup_{x^*\in B_{X^*}}\left(\sum |x^*(x_j)|^2\right)^{1/2} \sup_{y^*\in B_{Y^*}}\left(\sum |y^*(y_j)|^2\right)^{1/2}\right\}$$

where again the inf runs over all possible representations (1).

Open Unit Ball of $X \otimes_\wedge Y$=convex hull of rank one tensors $x \otimes y$ with $\|x\| < 1$ $\|y\| < 1$

— — — — — — — — — — —

Note the obvious inequalities

$$\|T\|_\vee \leq \|T\|_H \leq \|T\|_\wedge$$

In fact $\| \ \|_\wedge$ (resp. $\| \ \|_\vee$) is the largest (resp. smallest) reasonable $\otimes$-norm

The γ_2-norm

Let $\widetilde{T}\colon\ X^* \to Y$ be the linear mapping associated to T,

$$\widetilde{T}(x^*) = \sum x^*(x_j)y_j$$

Then $\|T\|_\vee = \|\widetilde{T}\|_{B(X,Y)}$ and

$$\|T\|_H = \inf\{\|T_1\|\|T_2\|\} \tag{2}$$

where the infimum runs over all Hilbert spaces $\mathcal{H}$ and all possible factorizations of $\widetilde{T}$ through $\mathcal{H}$:

$$\widetilde{T}\colon\ X^* \xrightarrow{T_2} \mathcal{H} \xrightarrow{T_1} Y$$

with $T = T_1 T_2$.
More generally (with Z in place of X^*)

$$\gamma_2(V\colon\ Z \to Y) = \inf\{\|T_1\|\|T_2\| \mid V = T_1 T_2\}$$

called the norm of factorization through Hilbert space of $\widetilde{T}$

Important observations :
$\| \ \|_\vee$ is injective, meaning
$X \subset X_1$ and $Y \subset Y_1$(isometrically) implies

$$X \otimes_\vee Y \subset X_1 \otimes_\vee Y_1$$

$\| \ \|_\wedge$ is projective, meaning $X_1 \twoheadrightarrow X$ and $Y_1 \twoheadrightarrow Y$ implies

$$X_1 \otimes_\wedge Y_1 \twoheadrightarrow X \otimes_\wedge Y$$

(where $X_1 \twoheadrightarrow X$ means metric surjection onto X)
but $\| \ \|_\vee$ is NOT projective and $\| \ \|_\wedge$ is NOT injective
Note : $\| \ \|_H$ is injective but not projective

Natural question :
Consider $T \in X \otimes Y$ with $\|T\|_\vee = 1$
then let us enlarge $X \subset X_1$ and $Y \subset Y_1$ (isometrically)
obviously $\|T\|_{X_1 \otimes_\wedge Y_1} \leq |T\|_{X \otimes_\wedge Y}$
Question : What is the infimum over all possible enlargements X_1, Y_1

$$\|T\|_{/\!\wedge\!\backslash} = \inf\{\|T\|_{X_1 \otimes_\wedge Y_1}\}?$$

Answer using $X_1 = Y_1 = \ell_\infty$:

$$\|T\|_{/\!\wedge\!\backslash} = \|T\|_{\ell_\infty \otimes_\wedge \ell_\infty}$$

and (First form of GT) :

$$(\|T\|_H \leq) \quad \|T\|_{/\!\wedge\!\backslash} \leq K_G \|T\|_H$$

....was probably Grothendieck's favorite formulation

One of the great methodological innovations of "the Résumé" was the systematic use of duality of tensor norms : Given a norm α on $X \otimes Y$ one defines α^* on $X^* \otimes Y^*$ by setting

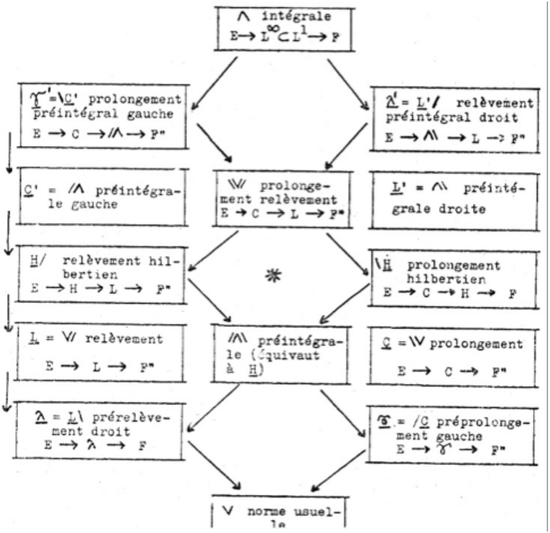
$$\alpha^*(T') = \sup\{|\langle T, T'\rangle| \mid T \in X \otimes Y, \alpha(T) \leq 1\}. \ \forall T' \in X^* \otimes Y^*$$

In the case

$$\alpha(T) = \|T\|_H,$$

Grothendieck studied the dual norm α^* and used the notation

$$\alpha^*(T) = \|T\|_{H'}.$$


∧ intégrale
prolongement préintégral gauche
relèvement préintégral droit
préintégrale gauche
prolongement relèvement
préintégrale droite
relèvement hilbertien
prolongement hilbertien
relèvement
préintégrale
prolongement
prérelèvement droit
préprolongement gauche
norme usuelle

GT can be stated as follows : there is a constant K such that for any T in $L_\infty \otimes L_\infty$ (or any T in $C(\Omega) \otimes C(\Omega)$) we have

$$GT_1 : \qquad \|T\|_\wedge \le K\|T\|_H \tag{3}$$

Equivalently by duality the theorem says that for any φ in $L_1 \otimes L_1$ we have

$$(GT_1)^* : \qquad \|\varphi\|_{H'} \le K\|\varphi\|_\vee. \tag{3$'$}$$

The best constant in either (3) or (3)$'$ is denoted by

K_G "the Grothendieck constant"(*actually* $K_G^{\mathbb{R}}$ *and* $K_G^{\mathbb{C}}$)

Exact values still unknown

although it is known that $1 < K_G^{\mathbb{C}} < K_G^{\mathbb{R}}$

$$1.676 < K_G^{\mathbb{R}} \le 1.782$$

Krivine 1979, Reeds (unpublished) more on this to come...

More “concrete" functional version of GT

$$GT_2$$

Let $B_H = \{x \in H \mid \|x\| \leq 1\}$

$$\forall n \quad \forall x_i, y_j \in B_H \quad (i, j = 1, \cdots, n)$$
$$\exists \phi_i, \psi_j \in L_\infty([0,1])$$

such that

$$\forall i, j \quad \langle x_i, y_j \rangle = \langle \phi_i, \psi_j \rangle_{L_2}$$
$$\sup_i \|\phi_i\|_\infty \sup_j \|\psi_j\|_\infty \leq K$$

Remark.

We may assume w.l.o.g. that

$$x_i = y_i$$

but nevertheless we cannot (in general) take

$$\phi_i = \psi_i!!$$

... more on this later

GT$_2$ implies GT$_1$ in the form $\forall T \in \ell_\infty^n \otimes \ell_\infty^n \quad \|T\|_\wedge \le K\|T\|_H$

$T \in \ell_\infty^n \otimes \ell_\infty^n$ is a matrix $T = [T_{i,j}]$

$$\text{Then} \quad \|T\|_H \le 1 \quad \textit{iff} \quad \exists x_i, y_j \in B_H \quad T_{i,j} = \langle x_i, y_j \rangle$$

Let

$$C = \{[\varepsilon_i' \varepsilon_j''] \mid |\varepsilon_i'| \le 1 |\varepsilon_j''| \le 1]$$

then $\{T \in \ell_\infty^n \otimes \ell_\infty^n \mid \|T\|_\wedge \le 1\} =$ convex-hull(C)=$C^{\circ\circ}$
But now if $\|T\|_H \le 1$ for any $b \in C^\circ$

$$|\langle T, b\rangle| = |\sum T_{i,j} b_{i,j}| = |\sum \langle x_i, y_j \rangle b_{i,j}| = |\int \sum \varphi_i \psi_j b_{i,j}|$$

$$\le \sup_i \|\phi_i\|_\infty \sup_j \|\psi_j\|_\infty \le K$$

Conclusion :

$$\|T\|_\wedge = \sup_{b \in C^\circ} |\langle T, b\rangle| \le K$$

and the top line is proved !

But now how do we show :

$$\textit{Given} \quad x_i, y_j \in B_H$$

$$\textit{there are } \phi_i, \psi_j \in L_\infty([0,1])$$

such that

$$\forall i,j \quad \langle x_i, y_j \rangle = \langle \phi_i, \psi_j \rangle_{L_2}$$

$$\sup_i \|\phi_i\|_\infty \sup_j \|\psi_j\|_\infty \leq K$$

???

Let $H = \ell_2$. Let $\{g_j \mid j \in \mathbb{N}\}$ be an i.i.d. sequence of standard Gaussian random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.
For any $x = \sum x_j e_j$ in ℓ_2 we denote $G(x) = \sum x_j g_j$.

$$\langle G(x), G(y)\rangle_{L_2(\Omega,\mathbb{P})} = \langle x, y\rangle_H .$$

Assume $\mathbb{K} = \mathbb{R}$. The following formula is crucial both to Grothendieck's original proof and to Krivine's :

$$\langle x, y\rangle = \sin\left(\frac{\pi}{2}\langle \text{sign}(G(x)),\ \text{sign}(G(y))\rangle\right). \tag{4}$$

Krivine's proof of GT with $K = \pi(2\mathbf{Log}(1+\sqrt{2}))^{-1}$
Here $K = \pi/2a$ where $a > 0$ is chosen so that

$$\sinh(a) = 1 \quad i.e. \quad a = \text{Log}(1+\sqrt{2}).$$

Krivine's proof of GT with $K = \pi(2\mathrm{Log}(1+\sqrt{2}))^{-1}$

We view $T = [T_{i,j}]$. Assume $\|T\|_H < 1$ i.e.
$T_{ij} = \langle x_i, y_j \rangle,\ x_i y_j \in B_H$
We will prove that $\|T\|_\wedge \leq K$.
Since $\| \quad \|_H$ is a Banach algebra norm we have

$\|\sin(aT)\|_H \leq \sinh(a\|T\|_H) < \sinh(a) = 1.$ (*here* $\sin(aT) = [\sin(aT_{i,j})]$)

$$\Rightarrow \quad \sin(aT_{i,j}) = \langle x_i', y_j' \rangle \quad \|x_i'\| \leq 1 \ \|y_j'\| \leq 1$$

By (4) we have

$$\sin(aT_{i,j}) = \sin\left(\frac{\pi}{2}\int \xi_i \eta_j \, d\mathbb{P}\right)$$

where $\xi_i = \mathrm{sign}(G(x_i'))$ and $\eta_j = \mathrm{sign}(G(y_j'))$. We obtain

$$aT_{i,j} = \frac{\pi}{2}\int \xi_i \eta_j \, d\mathbb{P}$$

and hence $\|aT\|_\wedge \leq \pi/2$, so that we conclude $\|T\|_\wedge \leq \pi/2a$.

Best Constants

The constant K_G is "the Grothendieck constant." Grothendieck proved that

$$\pi/2 \leq K_G^{\mathbb{R}} \leq \sinh(\pi/2)$$

Actually (here g is a standard $N(0,1)$ Gaussian variable)

$$\|g\|_1^{-2} \leq K_G$$

$$\mathbb{R}: \ \|g\|_1 = \mathbb{E}|g| = (2/\pi)^{1/2} \qquad \mathbb{C}: \ \|g\|_1 = (\pi/4)^{1/2}$$

and hence $K_G^{\mathbb{C}} \geq 4/\pi$. Note $K_G^{\mathbb{C}} < K_G^{\mathbb{R}}$.
Krivine (1979) proved that

$$1.66 \leq K_G^{\mathbb{R}} \leq \pi/(2\,\mathrm{Log}(1+\sqrt{2})) = 1.78\ldots$$

and conjectured $K_G^{\mathbb{R}} = \pi/(2\,\mathrm{Log}(1+\sqrt{2}))$.
$\mathbb{C}$: Haagerup and Davie $1.338 < K_G^{\mathbb{C}} < 1.405$
The best value ℓ_{best} of the constant in Corollary 0.4 seems also unknown in both the real and complex case. Note that in the real case we have obviously $\ell_{best} \geq \sqrt{2}$ because the 2-dimensional L_1 and L_∞ are isometric.

Disproving Krivine's 1979 conjecture
Braverman, Naor, Makarychev and Makarychev proved in 2011 that :
The Grothendieck constant is strictly smaller than krivine's bound
i.e.

$$K_G^{\mathbb{R}} < \pi/(2\,\mathrm{Log}(1+\sqrt{2}))$$

The Approximation Property (AP)

Def : X has AP if for any Y

$$X\widehat{\otimes}Y \to X\check{\otimes}Y \quad \textit{is injective}$$

Answering Grothendieck's main question
ENFLO (1972) gave the first example of Banach FAILING AP
SZANKOWSKI (1980) proved that $B(H)$ fails AP
also proved that for any $p \neq 2$ ℓ_p has a subspace failing AP....

Nuclearity

A Locally convex space X is NUCLEAR if

$$\forall Y \quad X\widehat{\otimes}Y = X\check{\otimes}Y$$

Grothendieck asked whether it suffices to take $Y = X$, i.e.

$$X\widehat{\otimes}X = X\check{\otimes}X$$

but I gave a counterexample (1981) even among Banach spaces
also $\quad X\widehat{\otimes}X^* \to X\check{\otimes}X^*$ is onto, this X also fails AP.

Other questions

[2] Solved by Gordon-Lewis Acta Math. 1974. (related to the notion of Banach lattice and the so-called “local unconditional structure")

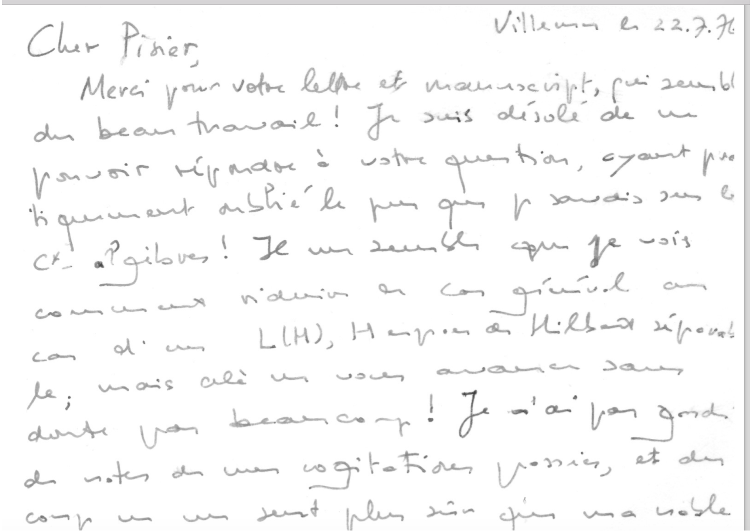
[3] Best constant ? Still open !

[5] Solved negatively in 1978 (P. Annales de Fourier) and Kisliakov independently : The Quotients L_1/R for $R \subset L_1$ reflexive satisfy GT.

[4] non-commutative GT

Is there a version of the fundamental Th. (GT) for bounded bilinear forms on non-commutative C^*-algebras ?
On this I have a small story to tell
and a letter from Grothendieck...

4. Propriétés algébrico-topologiques des C^*-algèbres. Soit A une C^*-algèbre. Le théorème 3 du N°2 suggère la conjecture suivante: Soit u une forme sesquilinéaire continue sur $A \times A$, peut on trouver une forme positive φ sur A telle que $u \ll u_\varphi$ (où on pose, comme au n°5, $u_\varphi(x,y) = \varphi(y^*x)$)? S'il on était toujours ainsi, on pourrait trouver une constante universelle λ (peut on prendre même $\lambda = h$?) telle que l'on puisse choisir cette φ de norme $\leq \lambda \|u\|$. Il suffirait de prouver alors l'énoncé sous cette forme pour le cas où A est du type $L(H)$, H étant un espace de Hilbert de dimension finie. Cette conjecture peut s'énoncer de diverses autres façons équivalentes dignes d'intérêt. Signalons qu'elle impliquerait que toute forme bilinéaire continue sur le produit de deux C^*-algèbres est hilbertienne. Quand l'une des deux C^*-algèbres est prise égale à c_0, on obtient facilement la conséquence suivante: toute suite sommable dans le dual A' d'une C^*-algèbre a une suite de normes qui est de carré sommable. Cela permettrait par exemple de prouver la proposition 6 du N°4 sans supposer le groupe G abélien.



Villecun le 22.7.76

Cher Pisier,

Merci pour votre lettre et manuscript, qui semble du beau travail ! Je suis désolé de ne pouvoir répondre à votre question, ayant pratiquement oublié le peu que je savais sur les C*-algèbres ! Il me semble que je vois comment réduire le cas général au cas d'un L(H), H espace de Hilbert séparable ; mais cela ne vous avance sans doute pas beaucoup ! Je n'ai pas gardé de notes de mes cogitations passées, et du coup ne me sent plus sûr qu'une noble

UNIVERSITÉ DES SCIENCES
ET TECHNIQUES DU LANGUEDOC

INSTITUT DE MATHEMATIQUES
Place Eugène Bataillon
34060 MONTPELLIER CEDEX

Montpellier, le

assertion de la p. 73 soit prouvée – ou vraie! Ça serait bien rigolo que L(H) n'ait pas la propriété d'approximation!

En vous souhaitant un bon amusement pour finir de tirer au clair ces questions

bien cordialement votre

Alexander Grothendieck

Dual Form and factorization :

Since $\|\varphi\|_{\ell_1^n \otimes_\vee \ell_1^n} = \|\varphi\|_{[\ell_\infty^n \otimes_\wedge \ell_\infty^n]^*}$

$$(GT_1)^* \quad \forall \varphi \in \ell_1^n \otimes \ell_1^n \quad \|\varphi\|_{H'} \leq K\|\varphi\|_\vee$$

is the formulation put forward by Lindenstrauss and Pełczyński ("Grothendieck's inequality") :

Theorem

Let $[a_{ij}]$ be an $N \times N$ scalar matrix $(N \geq 1)$ such that

$$\left|\sum a_{ij}\alpha_i\beta_j\right| \leq \sup_i |\alpha_i| \sup_j |\beta_j|. \qquad \forall \alpha, \beta \in \mathbb{K}^n$$

Then for Hilbert space H and any N-tuples $(x_j),(y_j)$ in H we have

$$\left|\sum a_{ij}\langle x_i, y_j\rangle\right| \leq K \sup \|x_i\| \sup \|y_j\|. \tag{5}$$

Moreover the best K (valid for all H and all N) is equal to K_G.

We can replace $\ell_\infty^n \times \ell_\infty^n$ by $C(\Omega) \times C(\Pi)$ (Ω, Π compact sets)

Theorem (Classical GT/inequality)

For any $\varphi\colon\ C(\Omega) \times C(\Pi) \to \mathbb{K}$ *and for any finite sequences* (x_j, y_j) *in* $C(\Omega) \times C(\Pi)$ *we have*

$$\left|\sum \varphi(x_j, y_j)\right| \le K\|\varphi\| \left\|\left(\sum |x_j|^2\right)^{1/2}\right\|_\infty \left\|\left(\sum |y_j|^2\right)^{1/2}\right\|_\infty . \quad (6)$$

(We denote $\|f\|_\infty = \sup_\Omega |f(.)|$ *for* $f \in C(\Omega)$*) Here again*

$$K_{\text{best}} = K_G.$$

For later reference observe that here φ is a bounded bilinear form on $A \times B$ with A, B **commutative C^*-algebras**

By a Hahn–Banach type argument, the preceding theorem is equivalent to the following one :

Theorem (Classical GT/factorization)

Let Ω, Π be compact sets. (here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)
$\forall \varphi: \ C(\Omega) \times C(\Pi) \to \mathbb{K}$ bounded bilinear form $\exists\ \lambda, \mu$ probabilities resp. on Ω and Π, such that $\forall (x, y) \in C(\Omega) \times C(\Pi)$

$$|\varphi(x,y)| \le K \|\varphi\| \left(\int |x|^2 d\lambda \right)^{1/2} \left(\int |y|^2 d\mu \right)^{1/2} \tag{7}$$

where constant $K_{best} = K_G^{\mathbb{R}}$ or $K_G^{\mathbb{C}}$

$$\begin{array}{ccc} C(\Omega) & \xrightarrow{\ \widetilde{\varphi}\ } & C(\Pi)^* \\ J_\lambda \downarrow & & \uparrow J_\mu^* \\ L_2(\lambda) & \xrightarrow{\ u\ } & L_2(\mu) \end{array}$$

Note that any L_∞-space is isometric to $C(\Omega)$ for some Ω, and any L_1-space embeds isometrically into its bidual, and hence embeds into a space of the form $C(\Omega)^*$.

Corollary

Any bounded linear map $v\colon\ C(\Omega) \to C(\Pi)^*$ or any bounded linear map $v\colon\ L_\infty \to L_1$ (over arbitrary measure spaces) factors through a Hilbert space. More precisely, we have

$$\gamma_2(v) \le \ell\|v\|$$

where ℓ is a numerical constant with $\ell \le K_G$.

Analogous C^*-algebra tensor products

$$A \otimes_{\min} B \quad \textit{and} \quad A \otimes_{\max} B$$

Guichardet, Turumaru 1958, (later on Lance)
Def : A C^*-algebra A is called NUCLEAR (abusively...) if

$$\forall B \quad A \otimes_{\min} B = A \otimes_{\max} B$$

Example : all commutative C^*-algebras,
$K(H) = \{\textit{compact operators on } H\}$,
$C^*(G)$ for G amenable discrete group

For C^*-algebras :

nuclear $\simeq$ amenable

Connes 1978, Haagerup 1983

KIRCHBERG (1993) gave the first example of a C^*-algebra A such that

$$A \otimes_{\min} A^{op} = A \otimes_{\max} A^{op}$$

but

$$A \textit{ is NOT nuclear}$$

He then conjectured that this equality holds for the two **fundamental** examples

$$A = B(H)$$

and

$$A = C^*(\mathbb{F}_\infty)$$

Why are $B(H)$ and $C^*(\mathbb{F}_\infty)$ fundamental C^*-algebras ?
because they are UNIVERSAL

Any separable C^*-algebra **EMBEDS** in $B(\ell_2)$
Any separable C^*-algebra is a **QUOTIENT** of $C^*(\mathbb{F}_\infty)$

Moreover, $B(H)$ is injective (i.e. extension property)
and $C^*(\mathbb{F}_\infty)$ has a certain form of lifting property
called (by Kirchberg) Local Lifting Property (LLP)

With JUNGE (1994) we proved that if $A = B(H)$
(well known to be non nuclear, by S. Wassermann 1974)

$$A \otimes_{\min} A^{op} \neq A \otimes_{\max} A^{op}$$

which gave a counterexample to the first Kirchberg conjecture

The other Kirchberg conjecture has now become the most important OPEN problem on operator algebras :
(here $\mathbb{F}_\infty$ is the free group)

$$\text{If } A = C^*(\mathbb{F}_\infty), \quad A \otimes_{\min} A^{op} \stackrel{?}{=} A \otimes_{\max} A^{op}?$$

$\Leftrightarrow$ CONNES embedding problem

Let (U_j) be the free unitary generators of $C^*(\mathbb{F}_\infty)$
Ozawa (2013) proved

Theorem

The Connes-Kirchberg conjecture is equivalent to

$$\forall n \geq 1 \forall a_{ij} \in \mathbb{C} \quad \left\| \sum_{i,j=1}^{n} a_{ij} U_i \otimes U_j \right\|_{\max} = \left\| \sum_{i,j=1}^{n} a_{ij} U_i \otimes U_j \right\|_{\min}$$

Grothendieck's inequality implies

$$\left\| \sum_{i,j=1}^{n} a_{ij} U_i \otimes U_j \right\|_{\max} \leq K_G^{\mathbb{C}} \left\| \sum_{i,j=1}^{n} a_{ij} U_i \otimes U_j \right\|_{\min}$$

Indeed,

$$\left\|\sum_{i,j=1}^{n} a_{ij} U_i \otimes U_j\right\|_{\max} = \sup\{|\langle \eta, \sum_{i,j=1}^{n} a_{ij} u_i v_j \xi\rangle|, \xi, \eta \in B_H\}$$

$$\leq \sup\{|\sum_{i,j=1}^{n} a_{ij} \langle u_i^* \eta, v_j \xi\rangle|, \xi, \eta \in B_H\}$$

$$\leq \sup\{|\sum_{i,j=1}^{n} a_{ij} \langle x_i, y_j\rangle|, x_i, y_j \in B_H\}$$

$$\leq K_G^{\mathbb{C}} \sup\{|\sum_{i,j=1}^{n} a_{ij} \langle x_i, y_j\rangle|, x_i, y_j \in B_{\mathbb{C}}\}$$

$$\leq K_G^{\mathbb{C}} \left\|\sum_{i,j=1}^{n} a_{ij} U_i \otimes U_j\right\|_{\min}$$

Theorem (Tsirelson 1980)

If $a_{ij} \in \mathbb{R}$ for all $1 \leq i, j \leq n$. Then

$$\|\sum\nolimits_{i,j} a_{ij} U_i \otimes U_j\|_{\max} = \|\sum\nolimits_{i,j} a_{ij} U_i \otimes U_j\|_{\min} = \|a\|_{\ell_1^n \otimes_{H'} \ell_1^n}.$$

Moreover, these norms are all equal to

$$\sup \|\sum a_{ij} u_i v_j\| \tag{8}$$

where the sup runs over all $n \geq 1$ and all self-adjoint unitary $n \times n$ matrices u_i, v_j such that $u_i v_j = v_j u_i$ for all i, j.

Non-commutative and Operator space GT

Theorem (C^*-algebra version of GT, P-1978, Haagerup-1985)

Let A, B be C^*-algebras. Then for any bounded bilinear form $\varphi\colon\ A \times B \to \mathbb{C}$ there are states f_1, f_2 on A, g_1, g_2 on B such that $\forall (x, y) \in A \times B$

$$|\varphi(x, y)| \leq \|\varphi\|(f_1(x^*x) + f_2(xx^*))^{1/2}(g_1(yy^*) + g_2(y^*y))^{1/2}.$$

Many applications to amenability, similarity problems, multilinear cohomology of operator algebras (cf. Sinclair-Smith books)

Non-commutative Banach spaces (sometimes called "quantum Banach spaces"...)

Definition

An operator space E is a closed subspace of a C^*-algebra, i.e.

$$E \subset A \subset B(H)$$

Any Banach space can appear, but
In category of operator spaces, the **morphisms** are *different*

$$u : E \to F \quad \|u\|_{cb} = \sup_n \|[a_{ij}] \to [u(a_{ij})]\|_{B(M_n(E) \to M_n(F))}$$

$$B(E,F) \quad \textit{is replaced} \quad \textit{by } CB(E,F) \quad (\textit{Note} : \|u\| \le \|u\|_{cb})$$

bounded maps *are replaced* *by completely bounded maps*

isomorphisms *are replaced* *by complete isomorphisms*

If A is commutative : recover usual Banach space theory

L_∞ is replaced by
Non-commutative L_∞ : any von Neumann algebra
Operator space theory :
developed roughly in the 1990's by
EFFROS-RUAN BLECHER-PAULSEN and others
admits Constructions Parallel to Banach space case
SUBSPACE, QUOTIENT, DUAL, INTERPOLATION,
$\exists$ ANALOGUE OF HILBERT SPACE ("OH")...
Analogues of projective and injective Tensor products

$$E_1 \subset B(H_1) \quad E_2 \subset B(H_2)$$

$$\textit{Injective} \qquad E_1 \otimes_{\min} E_2 \subset B(H_1 \otimes_2 H_2)$$

Again Non-commutative L_∞ and Non-commutative L_1
are UNIVERSAL objects

Theorem (Operator space version of GT)

Let A, B be C^*-algebras. Then for any CB bilinear form $\varphi\colon\ A \times B \to \mathbb{C}$ with $\|\varphi\|_{cb} \le 1$ there are states f_1, f_2 on A, g_1, g_2 on B such that $\forall (x, y) \in A \times B$

$$|\langle \varphi(x, y)\rangle| \le 2\left(f_1(xx^*)g_1(y^*y))^{1/2} + (f_2(x^*x)g_2(yy^*))^{1/2}\right).$$

Conversely if this holds then $\|\varphi\|_{cb} \le 4$.

With some restriction : SHLYAKHTENKO-P (Invent. Math. 2002)
Full generality : HAAGERUP-MUSAT (Invent. Math. 2008) and 2 is optimal !
Also valid for “exact" operator spaces A, B (no Banach space analogue !)

In 1935, Einstein, Podolsky and Rosen [EPR] published a famous article vigorously criticizing the foundations of quantum mechanics

They pushed forward the alternative idea that there are, in reality, "hidden variables" and that the statistical aspects of quantum mechanics can be replaced by this concept.

In 1964, J.S. BELL observed that the hidden variables theory could be put to the test. He proposed an inequality (now called "Bell's inequality") that is a CONSEQUENCE of the hidden variables assumption.

After Many Experiments initially proposed by Clauser, Holt, Shimony and Holt (CHSH, 1969), the consensus is :

The Bell-CHSH inequality is VIOLATED, and in fact the measures tend to agree with the predictions of QM.

Ref : Alain ASPECT, Bell's theorem : the naive view of an experimentalist (2002)

In 1980 TSIRELSON observed that GT could be interpreted as giving AN UPPER BOUND for the violation of a (general) Bell inequality,
and that the VIOLATION of Bell's inequailty is related to the assertion that

$$K_G > 1!!$$

He also found a variant of the CHSH inequality (now called "Tsirelson's bound")

The experiment

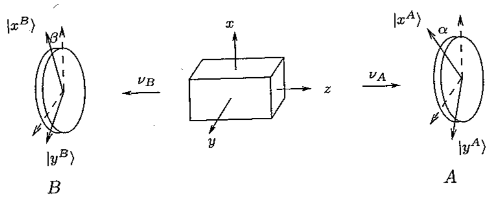


Figure 10.1: A polarisation measurement on pairs of photons. The dashed lines indicate the x- and y-axes. The solid lines are the rotated axes.

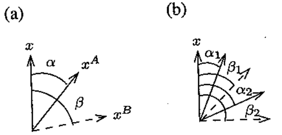


Figure 10.2: The orientations of the analysers.

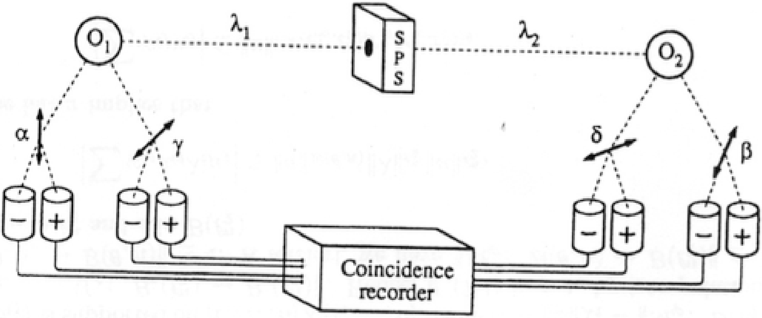


Fig. 6.8. Aspect's experiment: Pairs of photons are emitted in *SPS* cascades. Optical switches O_1 and O_2 randomly redirect these photons toward four polarization analyzers, symbolized by thick arrows. Each analyzer tests the linear polarization along one of the directions indicated in Fig. 6.7(b). The detector outputs are checked for coincidences in order to find correlations between them.

Outline of Bell's argument :

Hidden Variable Theory :
If A has spin detector in position i
and B has spin detector in position i
Covariance of their observation is

$$\xi_{ij} = \int A_i(\lambda) B_j(\lambda) \rho(\lambda) d\lambda$$

where ρ is a probability density over the "hidden variables"
Now if $a \in \ell_1^n \otimes \ell_1^n$, viewed as a matrix $[a_{ij}]$, for ANY ρ we have

$$|\sum a_{ij}\xi_{ij}| \leq HV(a)_{\max} = \sup_{\phi_i=\pm1 \psi_j=\pm1} |\sum a_{ij}\phi_i\psi_j| = \|a\|_\vee$$

But Quantum Mechanics predicts

$$\xi_{ij} = \mathrm{tr}(\rho A_i B_j)$$

where A_i, B_j are self-adjoint unitary operators on H ($\dim(H) < \infty$) with spectrum in $\{\pm 1\}$ such that $A_i B_j = B_j A_i$ and ρ is a non-commutative probability density,
i.e. $\rho \geq 0$ trace class operator with $\mathrm{tr}(\rho) = 1$. This yields

$$|\sum a_{ij}\xi_{ij}| \leq {\color{red}QM(a)_{\max}} = \sup_{x \in B_H} |\sum a_{ij}\langle A_i B_j x, x\rangle| = \|a\|_{\min}$$

with $\|a\|_{\min}$ relative to embedding (here g_j = free generators)

$$\ell_1^n \otimes \ell_1^n \subset C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$$

$$e_i \otimes e_j \mapsto g_i \otimes g_j$$

Easy to show $\|a\|_{\min} \leq \|a\|_{H'}$, so GT implies :

$$\|a\|_{\vee} \leq \|a\|_{\min} \leq K_G \, \|a\|_{\vee}$$

$$\Rightarrow HV(a)_{\max} \leq QM(a)_{\max} \leq K_G \, HV(a)_{\max}$$

But the covariance ξ_{ij} can be physically measured, and hence also $|\sum a_{ij}\xi_{ij}|$ for a fixed suitable choice of a, so we obtain an experimental answer

$$EXP(a)_{\max}$$

and (for well chosen a) it DEVIATES from the HV value
In fact the experimental data strongly confirms the QM predictions :

$$HV(a)_{\max} < EXP(a)_{\max} \simeq QM(a)_{\max}$$

GT then appears as giving a bound for the deviation :

$$HV(a)_{\max} < QM(a)_{\max} \quad \text{but} \quad QM(a)_{\max} \le K_G \, HV(a)_{\max}$$

JUNGE (with Perez-Garcia, Wolf, Palazuelos, Villanueva, Comm.Math.Phys.2008) considered the same problem for three separated observers A, B, C

The analogous question becomes : If

$$a = \sum a_{ijk} e_i \otimes e_j \otimes e_k \in \ell_1^n \otimes \ell_1^n \otimes \ell_1^n \subset C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$$

Is there a constant K such that

$$\|a\|_{\min} \leq K \|a\|_{\vee} ?$$

Answer is

NO

One can get on $\ell_1^n \otimes \ell_1 \otimes \ell_1$

$$K \geq c n^{1/8}$$

and in some variant a sharp result

JUNGE (with Perez-Garcia, Wolf, Palazuelos, Villanueva, Comm.Math.Phys.2008) considered the same problem for three separated observers A, B, C
The analogous question becomes : If

$$a = \sum a_{ijk} e_i \otimes e_j \otimes e_k \in \ell_1^n \otimes \ell_1^n \otimes \ell_1^n \subset C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$$

Is there a constant K such that

$$\|a\|_{\min} \leq K \|a\|_{\vee} ?$$

Answer is

NO

One can get on $\ell_1^n \otimes \ell_1 \otimes \ell_1$

$$K \geq c n^{1/8}$$

and in some variant a sharp result

Alon-Naor-Makarychev^2 [ANMM] introduced the Grothendieck constant of a graph $\mathcal{G} = (V, E)$: the smallest constant K such that, for every $a\colon\ E \to \mathbb{R}$, we have

$$\sup_{f\colon\ V\to S} \sum_{(s,t)\in E} a(s,t)\langle f(s), f(t)\rangle \le K \sup_{f\colon\ V\to\{-1,1\}} \sum_{(s,t)\in E} a(s,t)f(s)f(t) \tag{9}$$

where S is the unit sphere of $H = \ell_2$ (may always assume $\dim(H) \le |V|$). We will denote by

$$K(\mathcal{G})$$

the smallest such K.

Consider for instance the complete bipartite graph $\mathcal{CB}_n$ on vertices $V = I_n \cup J_n$ with $I_n = \{1, \ldots, n\}$, $J_n = \{n+1, \ldots, 2n\}$ with

$$(i, j) \in E \Leftrightarrow i \in I_n, \; j \in J_n$$

In that case (9) reduces to GT and we have

$$K(\mathcal{CB}_n) = K_G^{\mathbb{R}}(n)$$
$$\sup_{n \geq 1} K(\mathcal{CB}_n) = K_G^{\mathbb{R}}.$$

If $\mathcal{G} = (V', E')$ is a subgraph of $\mathcal{G}$ (i.e. $V' \subset V$ and $E' \subset E$) then obviously

$$K(\mathcal{G}') \leq K(\mathcal{G}).$$

Therefore, for any bipartite graph $\mathcal{G}$ we have

$$K(\mathcal{G}) \leq K_G^{\mathbb{R}}.$$

However, this constant does not remain bounded for general (non-bipartite) graphs. In fact, it is known (cf. Megretski 2000 and independently Nemirovski-Roos-Terlaky 1999) that there is an absolute constant C such that for any $\mathcal{G}$ with no selfloops (i.e. $(s, t) \notin E$ when $s = t$)

$$K(\mathcal{G}) \leq C(\log(|V|) + 1). \tag{10}$$

Moreover by Kashin-Szarek and [AMMN] this logarithmic growth is asymptotically optimal.

Flashback :

$$\forall n \quad \forall x_i, x_j \in B_H \quad (i, j = 1, \cdots, n)$$

$$\exists \phi_i, \psi_j \in L_\infty([0, 1])$$

such that

$$\forall i, j \quad \langle x_i, x_j \rangle = \langle \phi_i, \psi_j \rangle_{L_2}$$

$$\sup_i \|\phi_i\|_\infty \sup_j \|\psi_j\|_\infty \leq K$$

but nevertheless we cannot (in general) take

$$\phi_i = \psi_i !!$$

If $\phi_i = \psi_i$, then $K \geq c \log(n)$!

WHAT IS THE POINT ?

In computer science the CUT NORM problem is of interest : We are given a real matrix $(a_{ij})_{\substack{i\in R\\ j\in S}}$ we want to compute efficiently

$$Q = \max_{\substack{I\subset R\\ J\subset S}} \left| \sum_{\substack{i\in I\\ j\in J}} a_{ij} \right| .$$

Of course the connection to GT is that this quantity Q is such that

$$4Q \geq Q' \geq Q$$

where

$$Q' = \sup_{x_i, y_j \in \{-1,1\}} \sum a_{ij} x_i y_j .$$

So roughly computing Q is reduced to computing Q'.
In fact if we assume $\sum_j a_{ij} = \sum_i a_{ij} = 0$ for any i and any j then

$$4Q = Q'$$

Then precisely Grothendieck's Inequality tells us that

$$Q'' \geq Q' \geq \frac{1}{K_G} Q''$$

where

$$Q'' = \sup_{x_i, y_j \in S} \sum a_{ij} \langle x_i, y_j \rangle.$$

The point is that computing Q' in polynomial time is not known (in fact it would imply $P = NP$) while the problem of computing Q'' falls into the category of *"semi-definite programming"* problems and these are known to be solvable in polynomial time.
cf. Grötschel-Lovasz-Schriver 1981 : "The ellipsoid method"
Goemans-Williamson 1995 : These authors introduced the idea of "relaxing" a problem such as Q' into the corresponding problem Q''.
Known : $\exists \rho < 1$ *such that* even computing Q' up to a factor ρ in polynomial time would imply $P = NP$. So the Grothendieck constant seems to play a role here!

Alon and Naor (Approximating the cut norm via Grothendieck's inequality, 2004) rewrite several known proofs of GT (including Krivine's) as (polynomial time) algorithms for solving Q'' and producing a cut I, J such that

$$\left|\sum_{\substack{i\in I\\ j\in J}} a_{ij}\right| \geq \rho Q = \rho \max_{\substack{I\subset R\\ J\subset S}} \left|\sum_{\substack{i\in I\\ j\in J}} a_{ij}\right| .$$

According to work by P. Raghavendra and D. Steurer, for any $0 < K < K_G$, assuming a strengthening of $P \neq NP$ called the "unique games conjecture", it is NP-hard to compute any quantity q such that $K^{-1}q \leq Q'$. While, for $K > K_G$, we can take $q = Q''$ and then compute a solution in polynomial time by semi-definite programming. So in this framework K_G seems connected to the $P = NP$ problem !
Reference : S. Khot and A. Naor , Grothendieck-type inequalities in combinatorial optimization, 2012.

THANK YOU !