Interface Problems arising in the Landau-de Gennes theory of Liquid Crystals

Jinhae Park

Department of Mathematics
Chungnam National University
Daejeon 34134, South Korea

IPAM Workshop on Partial Order:
Mathematics, Simulations and Applications,
UCLA, January 25-29, 2016

Outline

What is a Liquid Crystal?

A liquid crystal

- is an intermediate state of matter between isotropic and solid phases
- has properties of both isotropic liquid and solid crystalline
- is used in displays and optical switches etc

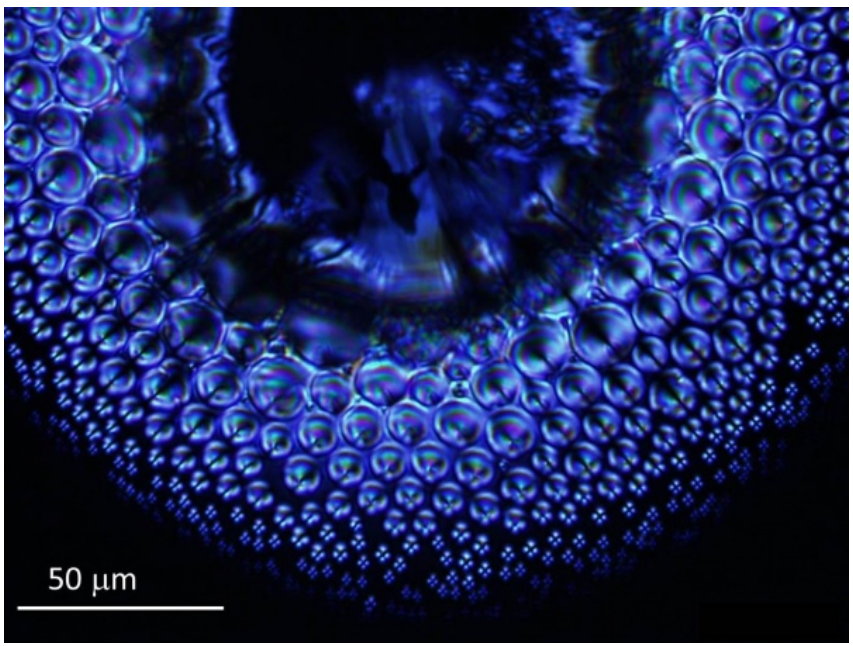


Figure : An array of microlenses(compound eyes) self assemble around a central pillar which could be used for three dimensional imaging (U Penn group, 2015)

Nematic Phases

There are many different types. The simplest one is a nematic phase of rod-like molecules whose length is 2-3nm.

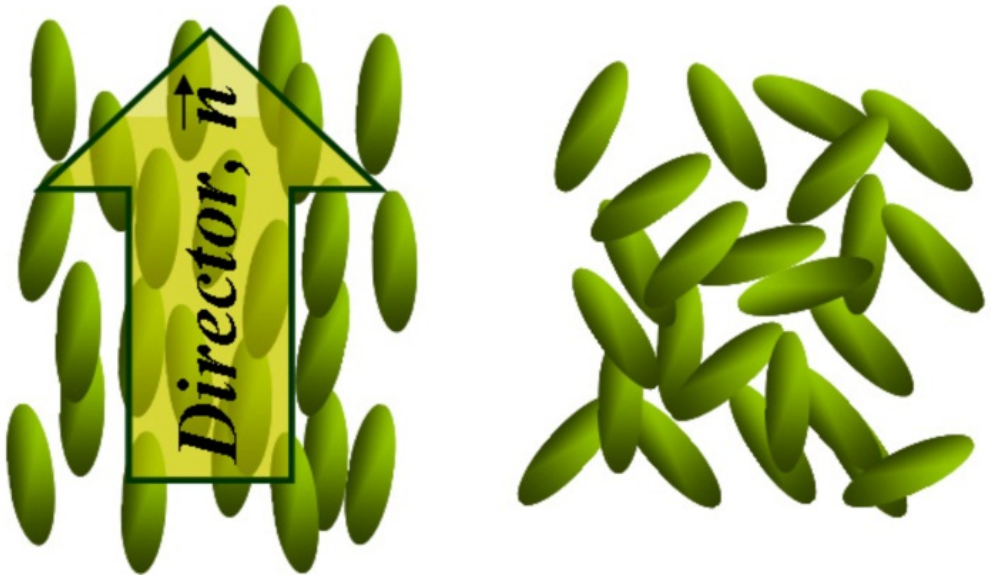


Figure : Nematic(left),Isotropic liquid(right)

Chiral Liquid Crystal

Figure : Chiral Phase

Smectic Phases

- Molecular centers of mass follow one-dimensional arrays and form layer structures
- Described by **n** and $\psi = \rho e^{i\omega}$: layers are level sets of ω, ρ represents centers of mass of the molecules

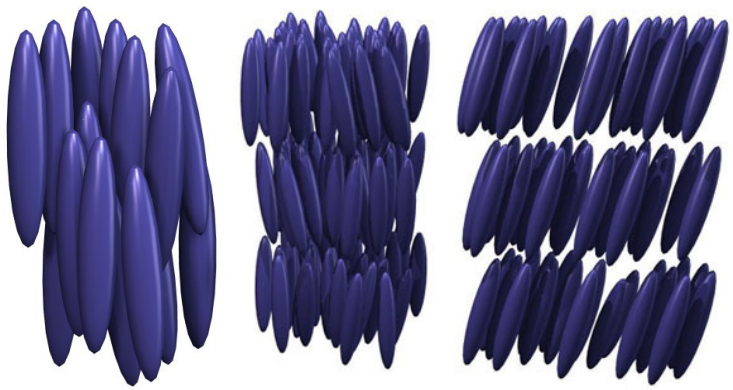


Figure : Nematic(left), Smectic A(middle), Smectic C(right)

- Smectic A: **n** is perpendicular to layers (left)
- Smectic C: **n** is tiled away from the layer normal $\nabla\omega$(right)

$$\tilde{\rho} = \rho_0 + \rho_1 \cos(qz + \phi) + \rho_2 \cos(2(qz + \phi)) + ...$$

where ρ_0 is the average of the density of center of mass, wave number $q = \frac{2\pi}{d_0}$, d_0 is a typical thickness of smectic layer, and ϕ is a fixed number.

$$\tilde{\rho} - \rho_0 = \rho Re(e^{i\omega}) = Re(\rho e^{i\omega}) = Re(\psi)$$

Chiral Smectic C Phase

- Smectic C + Chiral molecules
- $\mathbf{P} = P_0\mathbf{n} \times \mathbf{k}$: P_0 is a fixed constant depending on the material(after Bob Meyer 1976)

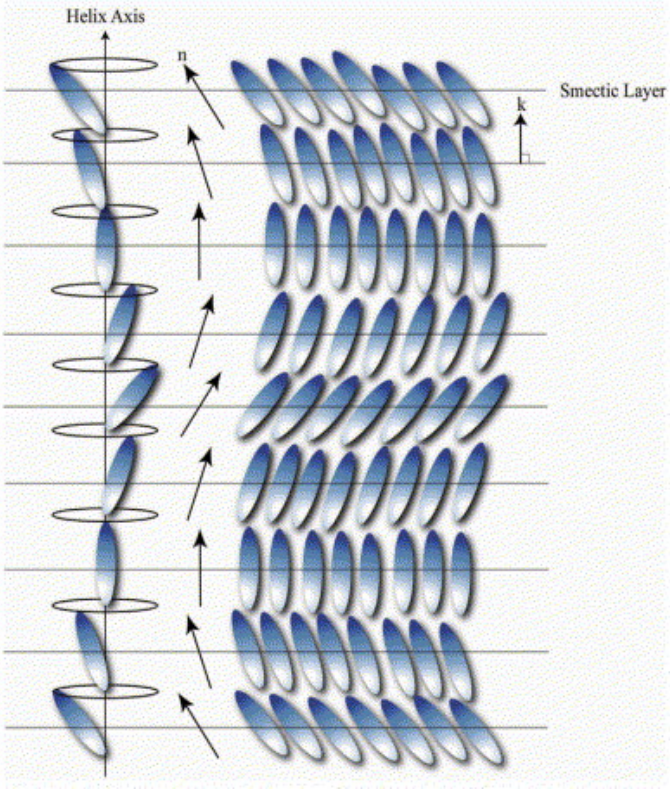
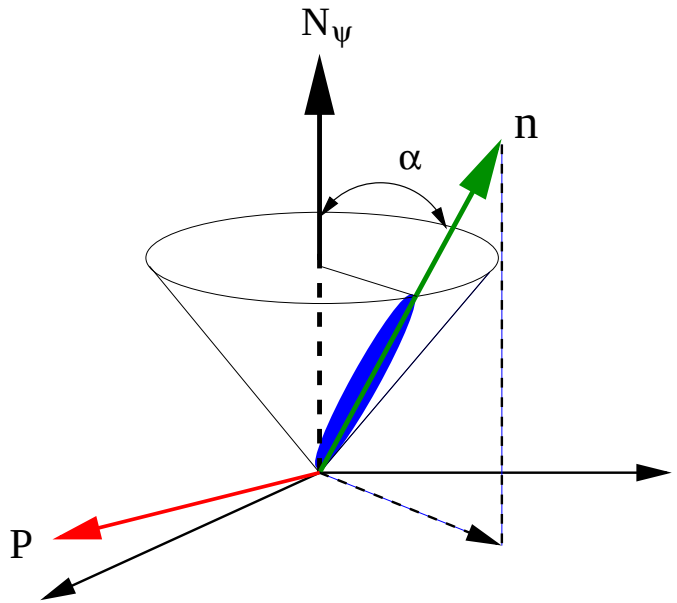


Figure :

Chiral Smectic C Phase

Variables: **n**, **P**, $\psi = \rho e^{i\omega}$ (level sets of ω are corresponding to smectic layers)

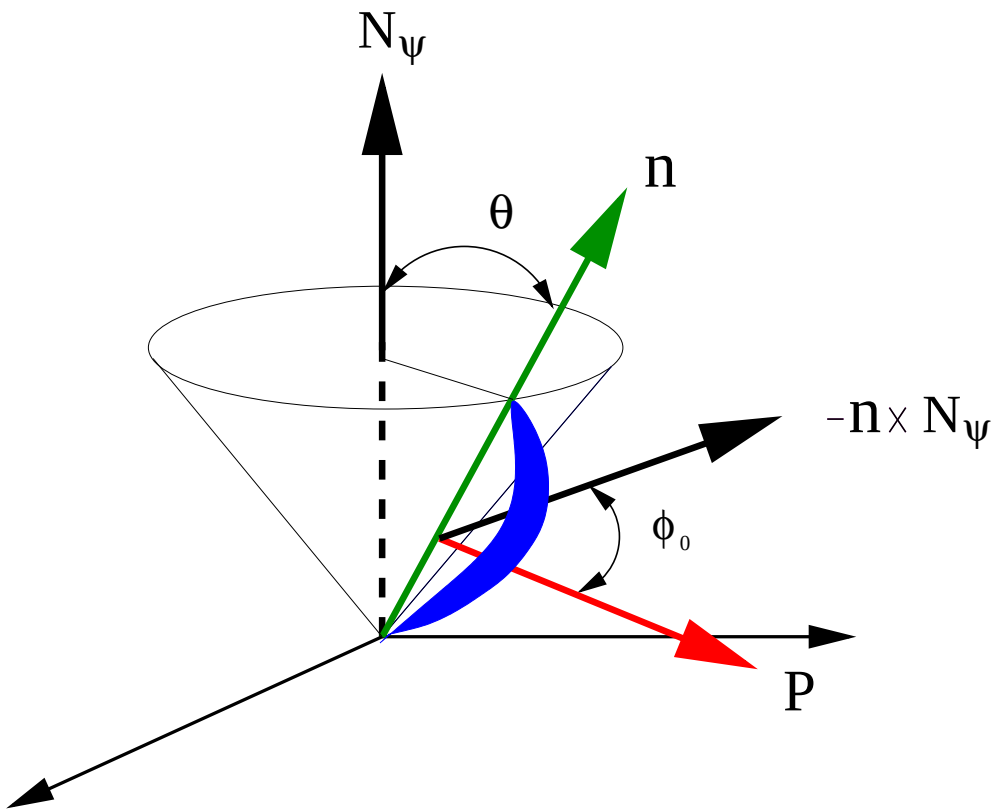
$$\mathbf{N}_\psi = -\frac{i}{2}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = \rho^2\nabla\omega(\text{layer normal}\psi^* = \rho e^{-i\omega})$$



Bent Core Molecules

$$\mathbf{N}_\psi = -\frac{i}{2}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = \rho^2\nabla\omega.$$

The angle ϕ_0 depends on the material.



How to describe liquid crystals?

Suppose that molecules occupy a domain Ω in $\mathbf{R}^3$,

- nematic liquid crystals : $\mathbf{n}$, a map from Ω to $\mathbb{S}^2$(sphere with radius 1)
- smectic liquid crystals: $\mathbf{n}$, $\psi = \rho e^{i\omega}$ (level sets of ω are corresponding to smectic layers)
- chiral smectic and bent core molecules: $\mathbf{n}, \psi, \mathbf{P}$(polarization field)

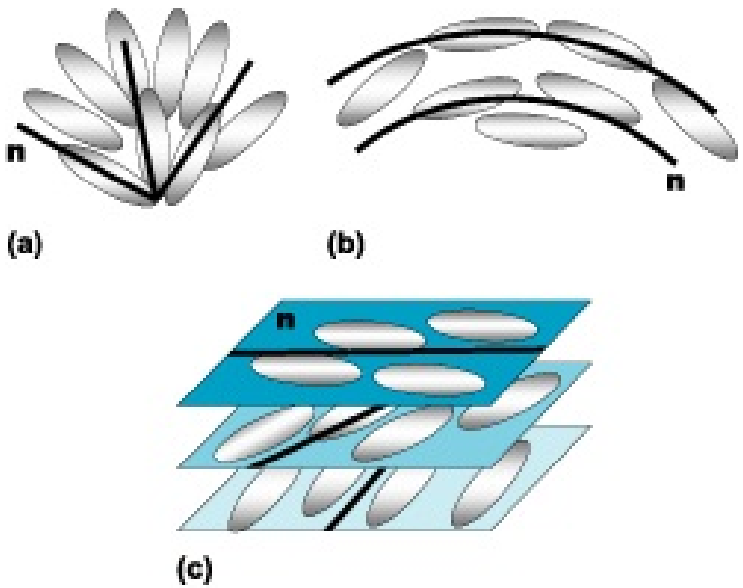
Nematic Liquid Crystals

Minimize the Oseen-Frank Energy of nematic liquid crystals

$$\mathcal{W} = \int_\Omega F_{OF}$$

$$F_{OF} = K_1(\nabla \cdot \mathbf{n})^2 + K_2(\mathbf{n} \cdot \nabla \times \mathbf{n} + \tau)^2 + K_3|\mathbf{n} \times \nabla \times \mathbf{n}|^2$$
$$+(K_2 + K_4)(\mathrm{tr}(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2), \qquad \mathbf{n} : \Omega \to \mathbb{S}^2$$

where $0 < K_2 + K_4 \leq \min\{K_1, K_3\}, K_4 \leq 0, \tau \in \mathbf{R}$.



Smectic Liquid Crystals

Investigate minimum energy configurations of

$$\mathcal{E}_{Sm} = \int_\Omega \{F_{OF} + F_{Sm} + F_P + F_E\}$$

subject to

$$-\nabla \cdot ((I + \varepsilon_a \mathbf{n} \otimes \mathbf{n})\mathbf{E}) = \nabla \cdot \mathbf{P} \text{ in } \Omega,$$

$$-\nabla \cdot \varepsilon_0 E = 0 \text{ in } \mathbf{R}^3 - \Omega,$$

$$\nabla \times \mathbf{E} = 0 \text{ in } \mathbf{R}^3,$$

$$[(I + \varepsilon_a \mathbf{n} \otimes \mathbf{n})\mathbf{E} - \varepsilon_0 \mathbf{E}] \cdot \nu = (\mathbf{P} \cdot \nu) \text{ on } \partial\Omega,$$

where ν is outward normal vector to the boundary.

$$F_{Sm} = a_\perp |D_\perp \cdot D_\perp \psi|^2 - c_\perp |D_\perp \psi|^2 + a_\parallel |D_\parallel \cdot D_\parallel \psi|^2$$
$$+ c_\parallel |D_\parallel \psi|^2 + r|\psi|^2 + \frac{g}{2}|\psi|^4 \quad \text{(Chen-Lubensky energy)}$$
$$F_P = B(\nabla \mathbf{P})^2 + \frac{1}{\eta^2}(|\mathbf{P}|^2 - P_0^2)^2$$
$$\text{or} \quad \frac{1}{\eta^2}|\mathbf{P}|^2[(|\mathbf{P}|^2 - P_0^2)^2 - \alpha]^2$$
$$F_E = -\frac{1}{2}[(I + \varepsilon_a \mathbf{n} \otimes \mathbf{n})\mathbf{E}] \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{P}$$
$$\tilde{F}_p = B|\nabla \mathbf{P}|^2 + \frac{1}{\eta^2}(|\mathbf{P}|^2 - P_0^2)^2$$
$$+ \frac{1}{2}K_c \left| (\mathbf{n} \times \mathbb{N}_\psi \cdot \mathbf{P})^2 (\mathbf{n} \cdot \mathbb{N}_\psi)^2 - \chi_0^2 |\mathbf{P}|^4 |\mathbb{N}_\psi|^4 \right|^s \text{(Banana Shape Molecules)}$$

$$D = \nabla - iq\mathbf{n}, \quad D_\parallel = (\mathbf{n} \cdot \nabla - iq)\mathbf{n}, \quad D_\perp = D - D_\parallel, \psi = \rho e^{i\omega}, r = a(T - T^*), a > 0, g > 0, a_\perp > 0, a_\parallel > 0, B > 0.$$

Defects

If a minimizer $\mathbf{n}$ of the Oseen-Frank energy allows a singular point x_0, the director field $\mathbf{n}$ behaves like

$$\mathbf{n}(x) \sim \mathcal{R}\left(\frac{x - x_0}{|x - x_0|}\right)$$

near the singular point x_0 where $\mathcal{R} \in SO(3)$(R. Schoen and K. Uhlenbeck, F. Lin, D. Kinderlehrer, H. Brezis etc)

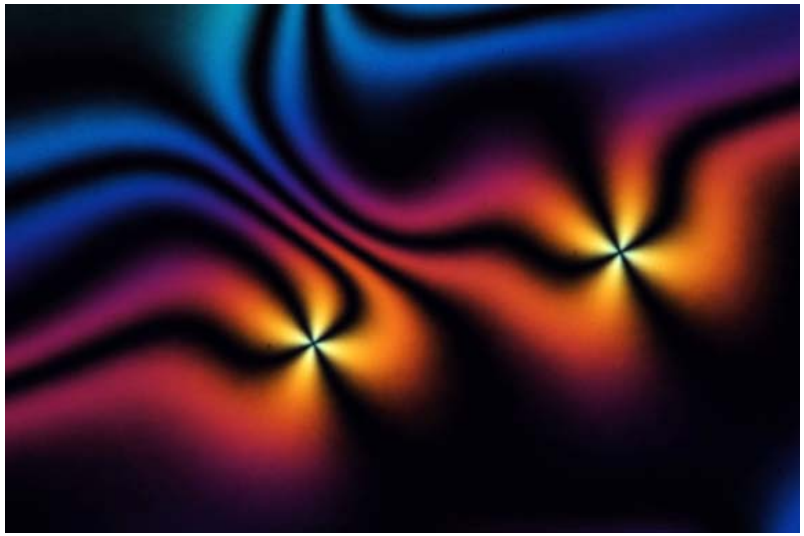


Figure : Defects for Oseen-Frank Energy

But it has been observed in the literature that there exists a singular point x_0 of $\pm\frac{1}{2}$ degree which cannot be a defect for a minimizer of the Oseen-Frank energy.

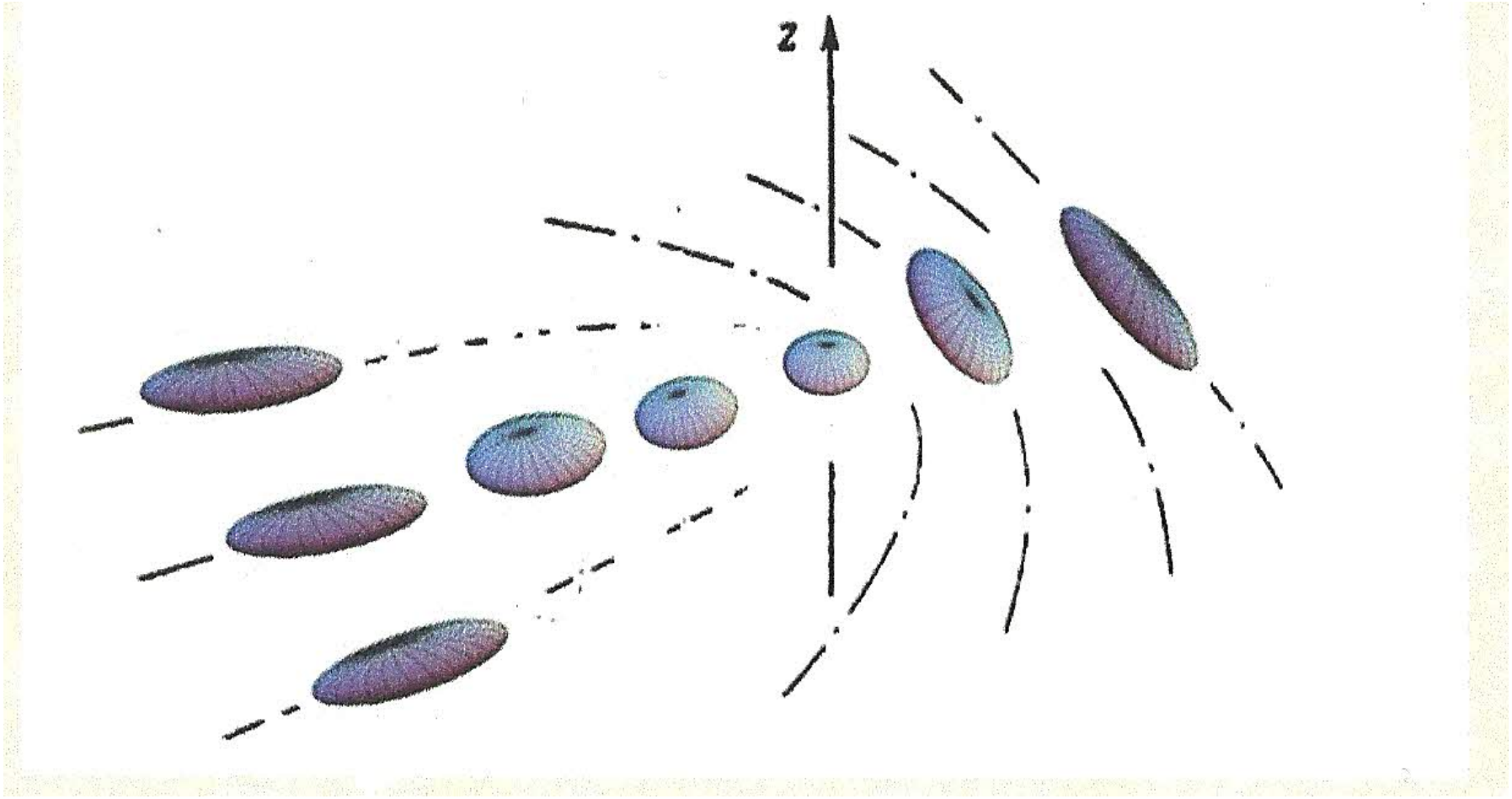


Figure : Singularity of $\frac{1}{2}$ degree

The Order Tensor Q

Let Ω be a domain in $\mathbf{R}^3$ and fix $x \in \Omega$. Let f be a probability density of molecular orientations satisfying $f(-\ell) = f(\ell)$ for all $\ell \in \mathbb{S}^2$ and μ be the corresponding probability measure.

- The first moments of f is $\int_{\mathbb{S}^2} \ell f(\ell) d\ell = 0$
- The second moments M is $\int_{\mathbb{S}^2} \ell \otimes \ell f(\ell) d\ell$: $\mathrm{tr} M = 1, M^T = M$, where $\mathbf{u} \otimes \mathbf{v} = (u_i v_j)_{i,j=1,2,3}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$
- If $f = \frac{1}{4\pi}$ (molecules are equally distributed in all directions), then the second moments M_0 is $\frac{1}{3} I$.
- $Q = M - M_0$ is called the second order tensor: measures the second moments tensor associated with a given probability density deviates from its isotropic value (a measure of local degree of orientational order in liquid crystals)
- $\mathrm{tr}\, Q = 0$

The Order Tensor Q

- Q is a 3×3 symmetric matrix
  $\Longrightarrow \exists$ an orthogonal matrix $\mathcal{O}$ such that $\mathcal{O}Q\mathcal{O}^T$ is diagonal.
- $\text{tr}\, Q = 0$
  $\Longrightarrow Q = S_1\left(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}I\right) + S_2\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}I\right)$
  where $\{\mathbf{m}, \mathbf{n}, \mathbf{m}\times\mathbf{n}\}$ is an orthonormal basis(eigenvectors of Q) for $\mathbf{R}^3$.
- Three eigenvalues are
  $\frac{1}{3}(2S_1 - S_2), \quad -\frac{1}{3}(S_1 + S_2), \quad \frac{1}{3}(2S_2 - S_1)$.

Liquid crystal is uniaxial
$\Longleftrightarrow$ two eigenvalues are equal

Liquid Crystal is biaxial
$\Longleftrightarrow$ Q has three distinct eigenvalues

The Governing Energy

Let B_Q be a symmetric traceless 3×3 matrix satisfying $\frac{1}{\int_{\mathbb{S}^2} \exp(B_Q(x):\mathbf{m}\otimes\mathbf{m})\, d\mathbf{m}} \int_{\mathbb{S}^2} \left(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}I\right) \exp(B_Q(x) : \mathbf{m}\otimes\mathbf{m})\, d\mathbf{m} = Q(\mathbf{x})$.

The generalized Landau-de Gennes energy(P.Zhang et al, 2015) is given by

$$\tilde{\mathcal{F}}(Q, \nabla Q) = \int_\Omega \{F_e(Q, \nabla Q) + \tilde{F}_b(Q)\}\, dx,$$

$$F_e = \frac{1}{2}\left(L_1|\nabla Q|^2 + L_2 Q_{ij,j} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} + L_4 Q_{ij} Q_{kl,i} Q_{kl,j}\right),$$

$$\tilde{F}_b = k_B T c \left(Q : B_Q - \ln Z_Q - \gamma |Q|^2\right)$$

Here $\gamma > 0$, $c > 0$, and $Z_Q = \int_{\mathbb{S}^2} \exp(B_Q : \mathbf{m}\otimes\mathbf{m})\, d\mathbf{m}$.

Remark.

J. Ball and A. Majumdar(2009): consider the mean field Maier-Saupe free energy in the total energy

$$\Psi(\mathbf{Q}) = T \min_{\rho \in \mathcal{A}_{\mathbf{Q}}} \int_{\mathbb{S}^2} \rho(\mathbf{m}) \ln \rho(\mathbf{m})\, d\mathbf{m} - \kappa |\mathbf{Q}|^2$$

$\mathcal{A}_{\mathbf{Q}}$ is the set of all $\rho : \mathbb{S}^2 \to \mathbf{R}$ satisfying

$$\rho \geq 0, \int_{\mathbb{S}^2} \rho(\mathbf{m})\, d\mathbf{m} = 1, \mathbf{Q} = \int_{\mathbb{S}^2} \left(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3} \right) \rho(\mathbf{m})\, d\mathbf{m}$$

In the case that $L_1 > 0$, $L_1 + L_2 + L_3 > 0$ and $L_4 = 0$, the direct method of the calculus of variations guarantees the existence of minimizers for $\tilde{\mathcal{F}}$ in the space

$$\mathcal{A} = \left\{ Q \in W^{1,2}(\Omega, \mathcal{S}_0) : Q = Q_0 \text{ on } \partial\Omega \right\},$$

where Q_0 is a smooth boundary data on $\partial\Omega$ and $\mathcal{S}_0$ denotes the set of all symmetric traceless 3×3 matrices.

The total energy $\tilde{F}$ may not be bounded from below if $L_4 \neq 0$(Ball and Majumdar, 2009).

The global minimizer for the bulk energy $\tilde{F}_b$ is uniaxial(Ball and Majumdar 2009, P. Zhang et al, 2015) and
near the isotropic-nematic transition the bulk energy $\tilde{F}_b$ is approximated by

$$\frac{a}{2}trQ^2 - \frac{b}{3}trQ^3 + \frac{c}{4}(trQ^2)^2 + \text{ higher order terms.}$$

For isotropic-Nematic Interface problems, we focus on a special form of the energy (Landau-de Gennes energy)

$$\mathcal{F}(Q,\nabla Q)=\int_\Omega \underbrace{\frac{a}{2}trQ^2-\frac{b}{3}trQ^3+\frac{c}{4}(trQ^2)^2}_{f_{bulk}:\text{bulk energy}}$$

$$+\underbrace{\frac{1}{2}\left(L_1|\nabla Q|^2+L_2Q_{ij,j}Q_{ik,k}+L_3Q_{ij,k}Q_{ik,j}+L_4Q_{ij}Q_{kl,i}Q_{kl,j}\right)}_{\mathcal{F}_{el}:\text{elastic energy}}dx.$$

Here a, b, c are material-dependent and temperature-dependent nonnegative constants and $L_i (i = 1, 2, 3, 4)$ are material dependent elastic constants.

Assume that $L_4 = 0$. Then the Landau-de Gennes energy is given by

$$\int_\Omega (F_{el} + f_{bulk})$$

$$\mathcal{F}_{el} = \frac{1}{2}\left(L_1 Q_{ij,k} Q_{ij,k} + L_2 Q_{ij,i} Q_{kj,k} + L_3 Q_{ij,k} Q_{ik,j}\right)$$

$$f_{bulk} = \frac{A}{2}\mathrm{tr}\, Q^2 - \frac{B}{3}\mathrm{tr}\, Q^3 + \frac{C}{4}\left(\mathrm{tr}\, Q^2\right)^2$$

$$= \frac{A}{2}\sum_{i=1}^{3}\lambda_i^2 - \frac{B}{3}\sum_{i=1}^{3}\lambda_i^3 + \frac{C}{4}\left(\sum_{i=1}^{3}\lambda_i^2\right)^2$$

where $\lambda_i (i = 1, 2, 3)$ are three eigenvalues of Q

Remark

If $Q = \frac{3}{2}S(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}I)$ with S being fixed, then

$$\mathcal{F}_{el} = \frac{9}{8}S^2\Big[(2L_1 + L_2 + L_3)(\nabla \cdot \mathbf{n})^2 + 2L_1(\mathbf{n} \cdot \nabla \times \mathbf{n})^2$$

$$+(2L_1 + L_2 + L_3)|\mathbf{n} \times \nabla \times \mathbf{n}|^2 + (2L_1 + L_3)(\mathrm{tr}(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2)\Big]$$

Minimizers Q of the Landau-de Gennes allows a singular point x_0 of $\pm\frac{1}{2}$ degree

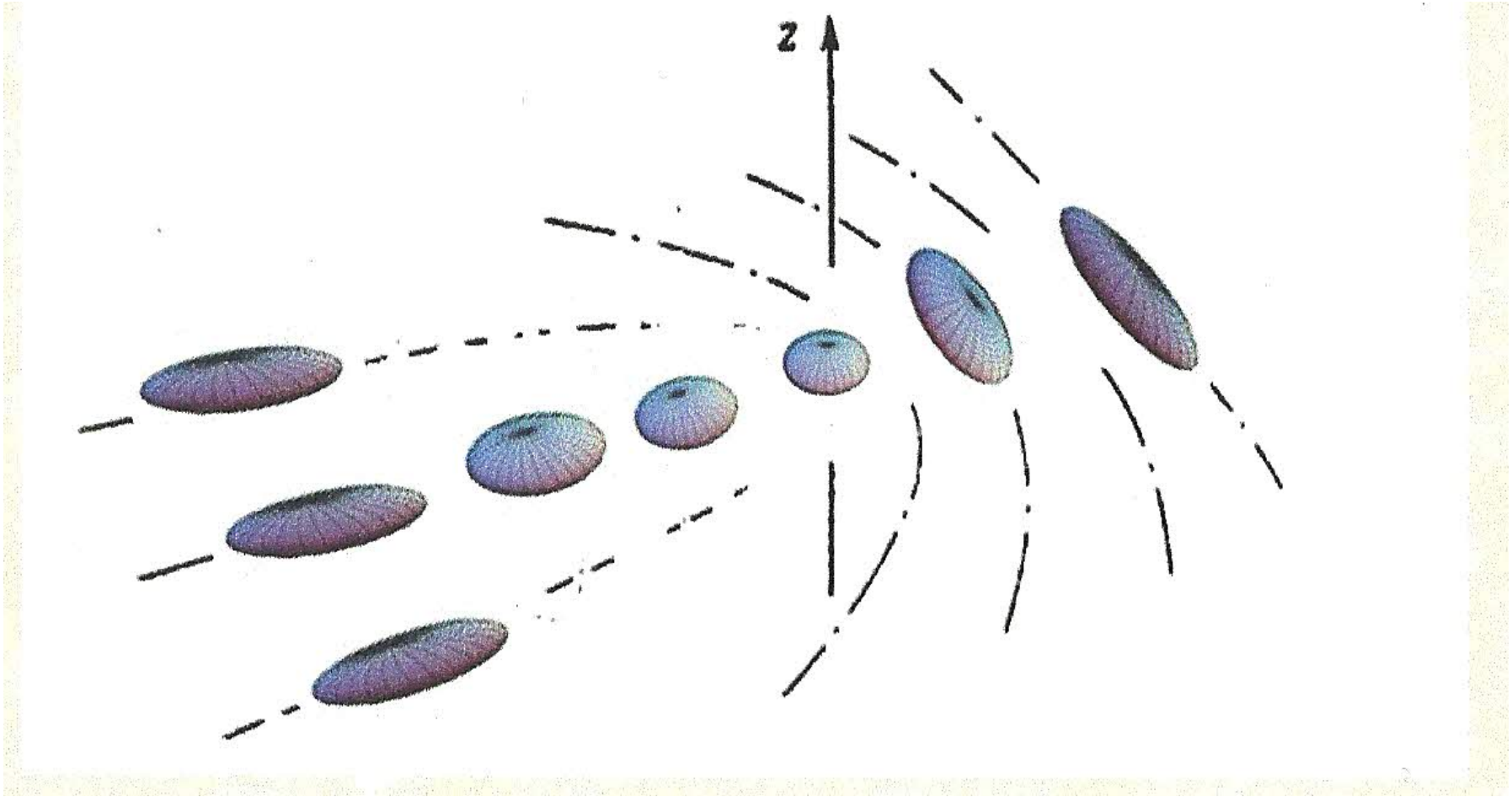
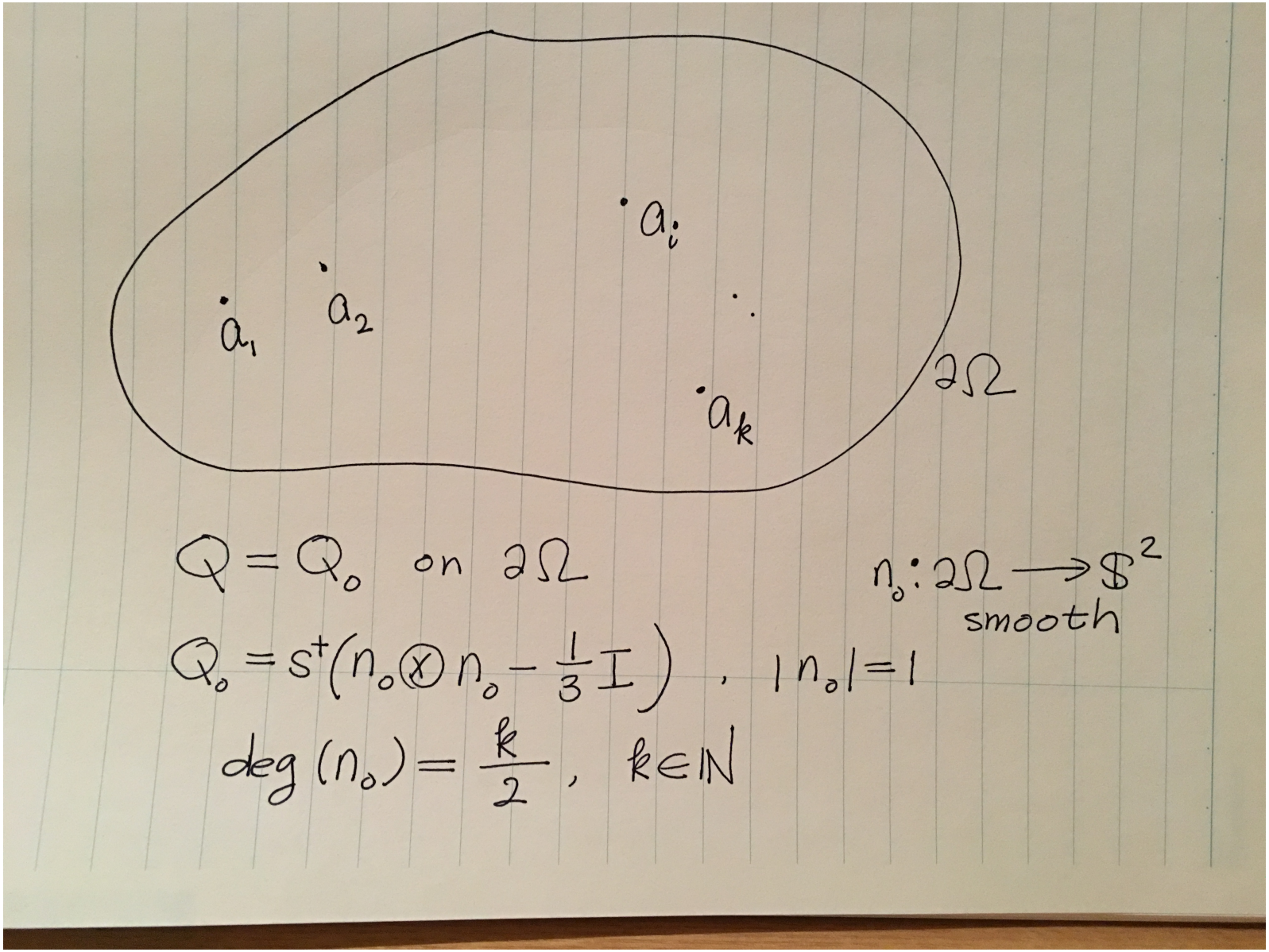


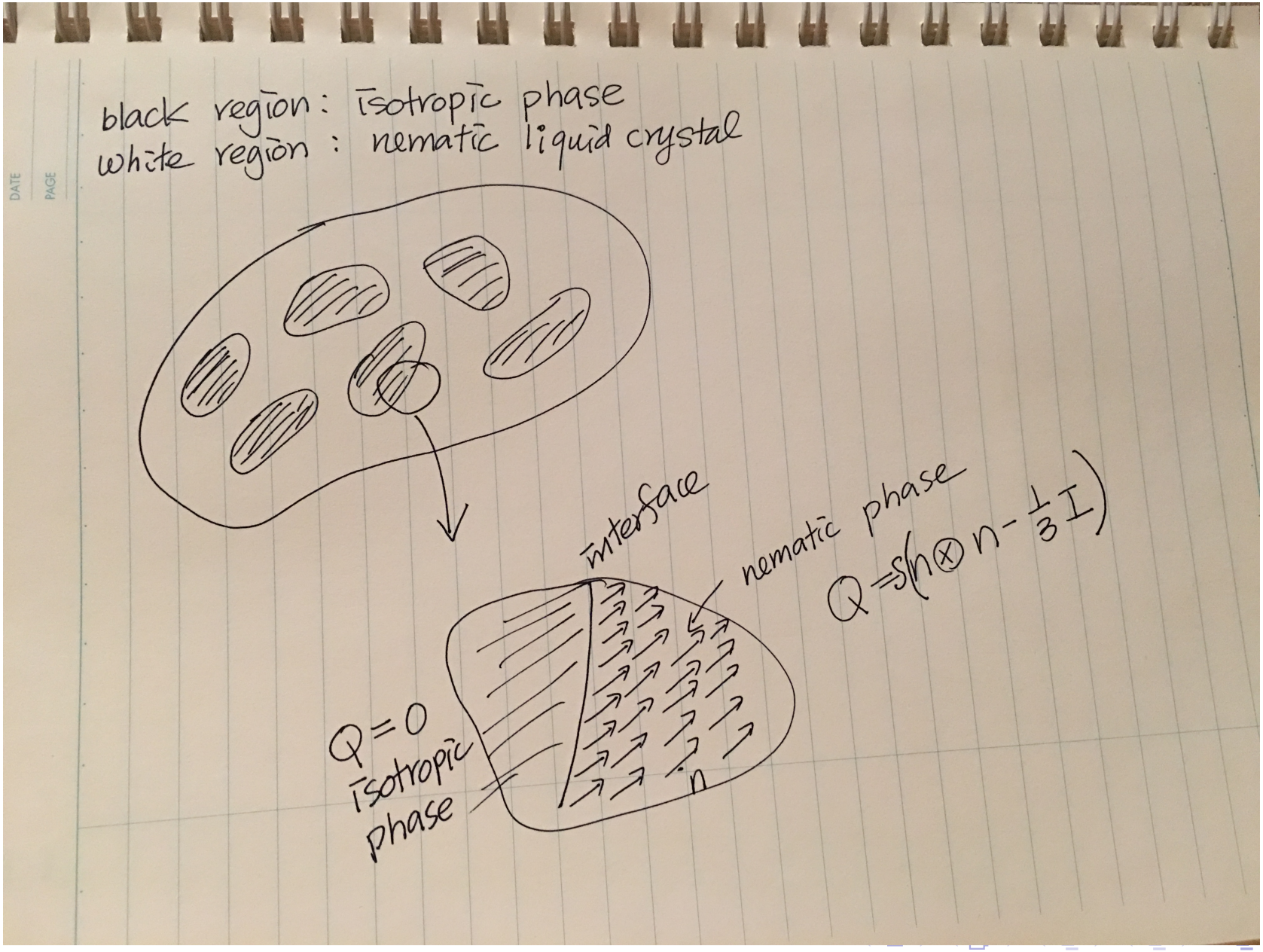
Figure : Singularity of $\frac{1}{2}$ degree

In fact, P. Bauman, J.P and D. Phillips, 2012) proved existence of half degree singularities in 2D.



- G. Di Fratta, A. Zarnescu, et al: Half-Integer Point Defects in the Q-Tensor Theory of Nematic Liquid Crystals(J. of Nonlinear Sci., 2015)
- D. Golovaty, J.A. Montero and P. Sternberg: Dimension Reduction for the Landau-de Gennes Model in Planar Nematic Thin Films (J. of Nonlinear Sci., 2015)
- Y. Hu, Y. Qu, P. Zhang: On the Disclination Lines of Nematic Liquid Crystals(2015)
- G. Canevari:Biaxiality in the asymptotic analysis of a 2-D Landau-de Gennes model for liquid crystals(2014)
- S. Alama, L. Bronsard, and X. Lamy: Minimizers of the Landau-de Gennes energy around a spherical colloid particle(2015)
- D. Golovaty and J. A. Montero: On minimizers of a Landau-de Gennes energy functional on planar domains(ARMA, 2014)

Isotropic-Nematic Interface



Remark

1. Based on the framework of the Ginzburg-Landau, de Gennes studied the interface between the isotropic and nematic phases(de Gennes 1971). With a special ansatz that de Gennes made on the variation of the order tensor, the biaxiality does not appear in the isotropic-nematic interface.

2.In the absence of the anisotropic energy($L_2 = 0$), the de Gennes' ansatz predicts that both the homeotropic and planar anchorings on the interface are possibly stable.

3.Popa-Nita et al(1997) investigated the isotropic-nematic interface by numerical and asymptotic analysis and showed that the de Gennes' ansatz is valid when the bend and splay elastic energies dominate over the twist energy.

4. Numerical simulations(Kamil et al, 2010) give positive answer with the de Gennes ansatz in the absence of anisotropic elastic energy corresponding to L_2-term.

5.When the anisotropic energy presents ($L_2 \neq 0$), de Gennes argued energetically that the homeotropic anchoring is stable when $L_2 < 0$ while the planar anchoring is stable when $L_2 > 0$. It turns out that uniaxiality may lose in the interfacial profile(Kamil et al, 2010, Popa-Nita 1997).

Isotropic-Nematic Interface Problems

Assume that $L_3 = 0$.

$$\mathcal{F}[Q] = \int_\Omega \{\frac{L_1}{2}|\nabla Q|^2 + \frac{L_2}{2} Q_{ij,j} Q_{ik,k} + F_b(Q)\}\, dx,$$

$$F_b(Q) = \frac{a}{2} tr(Q) - \frac{b}{3} tr(Q^2) + \frac{c}{4}[tr(Q^2)]^2.$$

where Ω is a bounded domain in $\mathbf{R}^3$.

Assume that $b^2 = 27ac$. Then the stable two constant states 0 and $s^+\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\right)$ $(s^+ = \frac{b}{4c})$ could coexist in the global minimizer

If $0 < a < \frac{b^2}{24c}$, then $\mathbf{Q} = 0$ and

$Q = s\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} I\right), s^+ = \frac{b \pm \sqrt{b^2 - 24ac}}{4c}$ are local minimizers of F_b.

Introduce new variables

$\tilde{\mathbf{x}} = \frac{1}{\sqrt{L_1}}\mathbf{x}, \quad \tilde{\mathbf{Q}}(\tilde{\mathbf{x}}) = \mathbf{Q}(\sqrt{L_1}\tilde{\mathbf{x}})$ and consider the scaled energy

$$\frac{1}{3L_1\sqrt{L_1}}\mathcal{F}(\mathbf{Q}, \nabla\mathbf{Q}) = \int_{\frac{1}{\sqrt{L_1}}\Omega} \frac{1}{6}\left(|\nabla\tilde{\mathbf{Q}}|^2 + \frac{L_2}{L_1}\tilde{Q}_{ij,j}\tilde{Q}_{ik,k}\right) + F_b(\tilde{\mathbf{Q}})\right) d\tilde{\mathbf{x}}.$$

we assume that the following limits exist

$$\lim_{L_1 \to 0} \frac{2L_2}{3L_1} = L, \quad \lim_{L_1 \to 0} \frac{a}{L_1} = \tilde{a},$$

$$\lim_{L_1 \to 0} \frac{b}{L_1} = \tilde{b} \quad \lim_{L_1 \to 0} \frac{c}{L_1} = \tilde{c}$$

Passing to the limit as $L_1 \to 0$, we obtain the following limiting energy functional (not relabelled) after removing the tilde

$$\mathcal{F}(\mathbf{Q}, \nabla\mathbf{Q}) = \int_{\mathbf{R}^3} \left\{ \frac{1}{6}|\nabla\mathbf{Q}|^2 + \frac{L}{4} Q_{ij,j} Q_{ik,k} + F_b(\mathbf{Q}) \right\} d\mathbf{x}.$$

After scaling if necessary, we take $a = \frac{1}{3}, b = 3, c = 1$ so that $f(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} I) = f(0)$. Then energy functional takes the form

$$\mathcal{F}(\mathbf{Q}, \nabla\mathbf{Q}) = \int_{\mathbf{R}^3} \left\{ \frac{1}{6}|\nabla\mathbf{Q}|^2 + \frac{L}{4} Q_{ij,j} Q_{ik,k} + \frac{1}{6}\mathrm{tr}\mathbf{Q}^2 - \mathrm{tr}\mathbf{Q}^3 + \frac{1}{4}(\mathrm{tr}\mathbf{Q}^2)^2 \right\} d\mathbf{x}$$

The global minimizer for the case $L = 0$

We first investigate the global minimizer of the energy

$$\mathcal{F}_0(\mathbf{Q}, \nabla\mathbf{Q}) = \int_{\mathbf{R}} \left\{ \frac{1}{6}|\mathbf{Q}'|^2 + \frac{1}{6}\mathrm{tr}\mathbf{Q}^2 - \mathrm{tr}\mathbf{Q}^3 + \frac{1}{4}(\mathrm{tr}\mathbf{Q}^2)^2 \right\} ds$$

with the boundary condition

$$\mathbf{Q}(+\infty) = \mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}, \quad \mathbf{Q}(-\infty) = 0.$$

THEOREM(J. P., W. Wang, P.Zhang, Z. Zhang, 2015). The global minimizer of $\mathcal{F}_0(\mathbf{Q}, \nabla\mathbf{Q})$ must take the form

$$\mathbf{Q}(s) = \frac{1}{2}(1 + \tanh(s - t))(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I})$$

which is an uniaxial interfacial profile where t is an arbitrary constant due to the translation.

The global minimizer for the case $L \neq 0$

In the case of $L \neq 0$, the one-dimensional Landau-de Gennes energy functional reads

$$\mathcal{F}_L(\mathbf{Q}, \nabla \mathbf{Q}) = \int_{\mathbf{R}} \left\{ \frac{1}{6}|\mathbf{Q}'|^2 + \frac{L}{4} \sum_{i=1}^{3} (Q'_{i3})^2 + \frac{1}{6}\mathrm{tr}\mathbf{Q}^2 - \mathrm{tr}\mathbf{Q}^3 + \frac{1}{4}\mathrm{tr}\mathbf{Q}^2)^2 \right\} ds$$

with the boundary condition

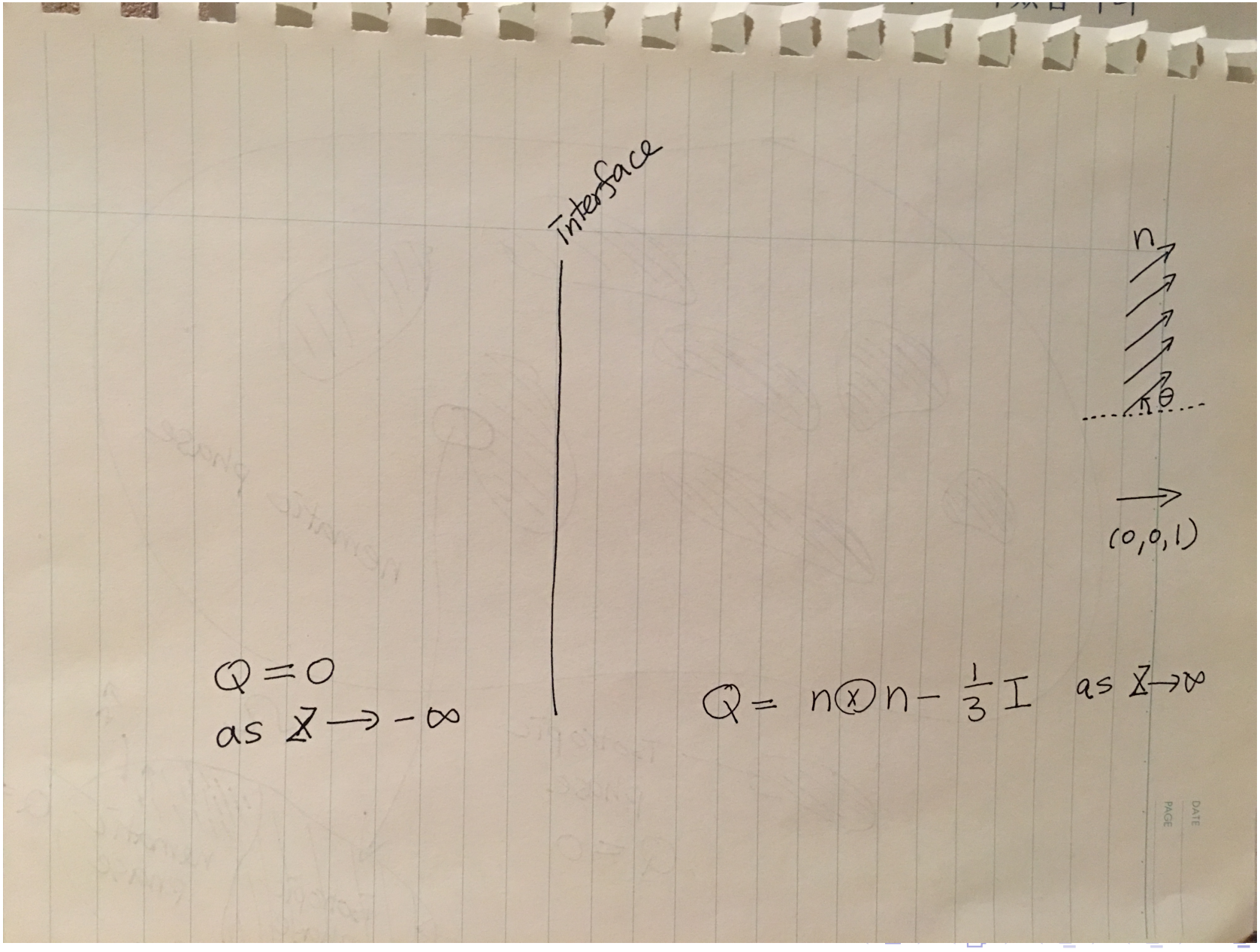
$$\mathbf{Q}(+\infty) = \mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}, \quad \mathbf{Q}(-\infty) = 0.$$

Boundary Conditions

Unlike in the case of $L = 0$ the direction vector $\mathbf{n}$ on the anchoring condition at $+\infty$ makes a significant effect on the behavior of the global minimizers. There are three different types of the alignment director $\mathbf{n}$ on the boundary

1. Homeotropic anchoring: $\mathbf{n} \cdot (0, 0, 1) = 1$;
2. Planar anchoring: $\mathbf{n} \cdot (0, 0, 1) = 0$;
3. Oblique(Tilt) anchoring: $0 < \mathbf{n} \cdot (0, 0, 1) < 1$.

Boundary Conditions



Without loss of generality, we look for minimizers of the form

$$\mathbf{Q} = \begin{pmatrix} -\frac{1}{3}(S+T) & 0 & 0 \\ 0 & -\frac{1}{3}(S-T) & 0 \\ 0 & 0 & \frac{2}{3}S \end{pmatrix}$$

with $S(+\infty) = 1$, $T(+\infty) = S(-\infty) = T(-\infty) = 0$. Then the energy functional becomes

$$\mathcal{F}_L(S,T) = \frac{2}{9}\int_{\mathbf{R}} \Big(\frac{1+L}{2}(S')^2 + \frac{1}{6}(T')^2 + \frac{1}{6}(3S^2+T^2) - S(S^2-T^2) + \frac{1}{18}(3S^2+T^2)^2 \Big) ds$$

Euler-Lagrange Equations

The corrresponding Euler-Lagrange equations are

$$-\frac{1+L}{2}S'' + \frac{S}{2} - \frac{3S^2}{2} + \frac{T^2}{2} + \frac{S(3S^2+T^2)}{3} = 0, -\infty < s < \infty,$$

$$-\frac{1}{6}T'' + \frac{T}{6} + ST + \frac{T(3S^2+T^2)}{9} = 0, -\infty < s < \infty$$

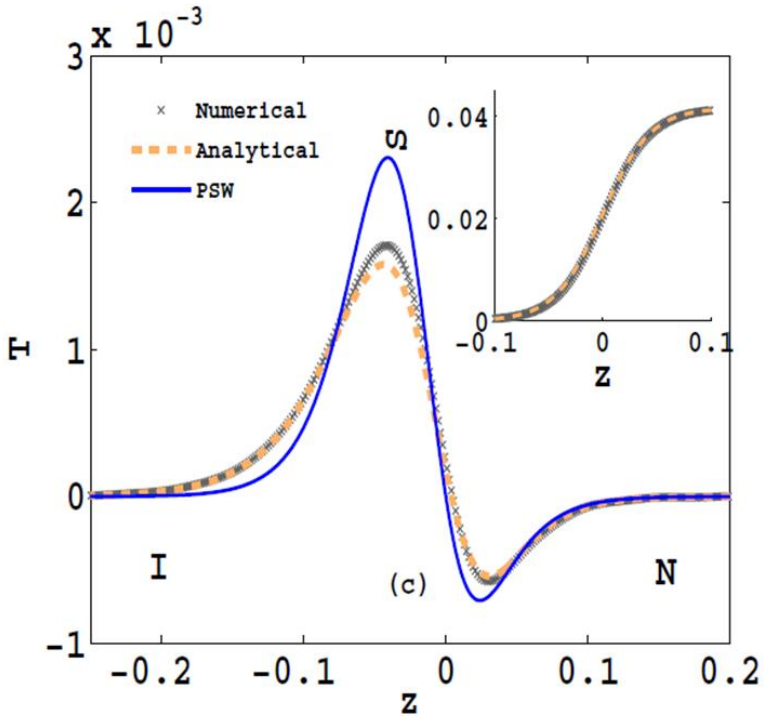
Note that a uniaxial state with $T = 0$ and $S(z) = S^*(z/\sqrt{1+L})$ solves

$$-(1+L)S'' + S - 3S^2 + 2S^3 = 0$$

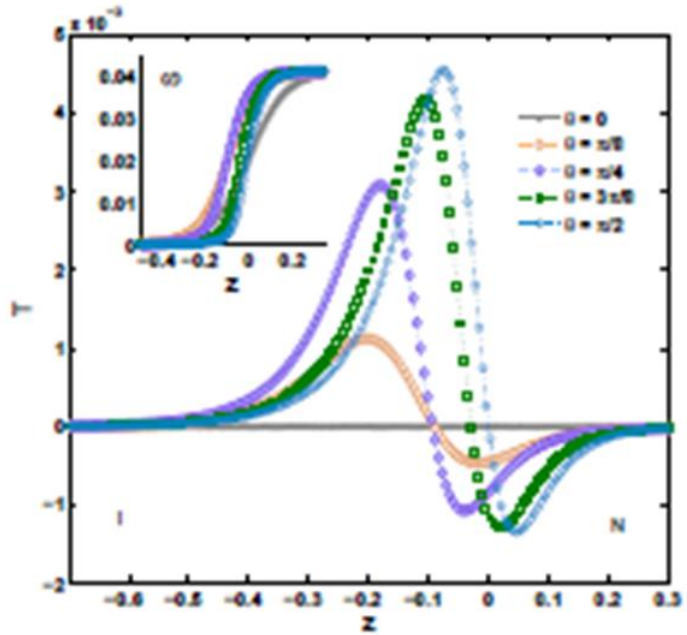
where

$$S^*(\tau) = \frac{exp(\tau - t)}{1 + exp(\tau - t)}.$$

THEOREM(J.P., W. Wang, P.Zhang, Z. Zhang, 2015). There exists a uniaxial equilibrium state of the energy functional $\mathcal{F}_L$ which isstable(with homeotropic anchoring) when $L \leq 0$ and unstable(with homeotropic anchoring) when $L > 0$(Biaxial interfacial profile for Planar boundary condition is shown in the picture(A. Bhattacharjee,2010, $L \approx 1$)



Biaxial Interfacial Profiles for Oblique Boundary Conditions(A. Bhattacharjee,2010,$L > 1$)



For this, let us consider a special order tensor depending only on z of the form

$$Q=\begin{pmatrix} -\frac{(S+T)}{2}\cos^2\theta + S\cos^2\theta & 0 & -\frac{(3S+T)}{4}\sin 2\theta \\ 0 & -\frac{(S-T)}{2} & 0 \\ -\frac{3(3S+T)}{4}\sin 2\theta & 0 & -\frac{(S+T)}{2}\sin^2\theta + S\cos^2\theta \end{pmatrix}$$

with the boundary condition

$$S(-\infty) = T(\pm\infty) = 0,\ S(+\infty) = 1, \theta(+\infty) = \theta_0$$

Thank you for your attention!