# Geometry and topology of turbulence in active nematics

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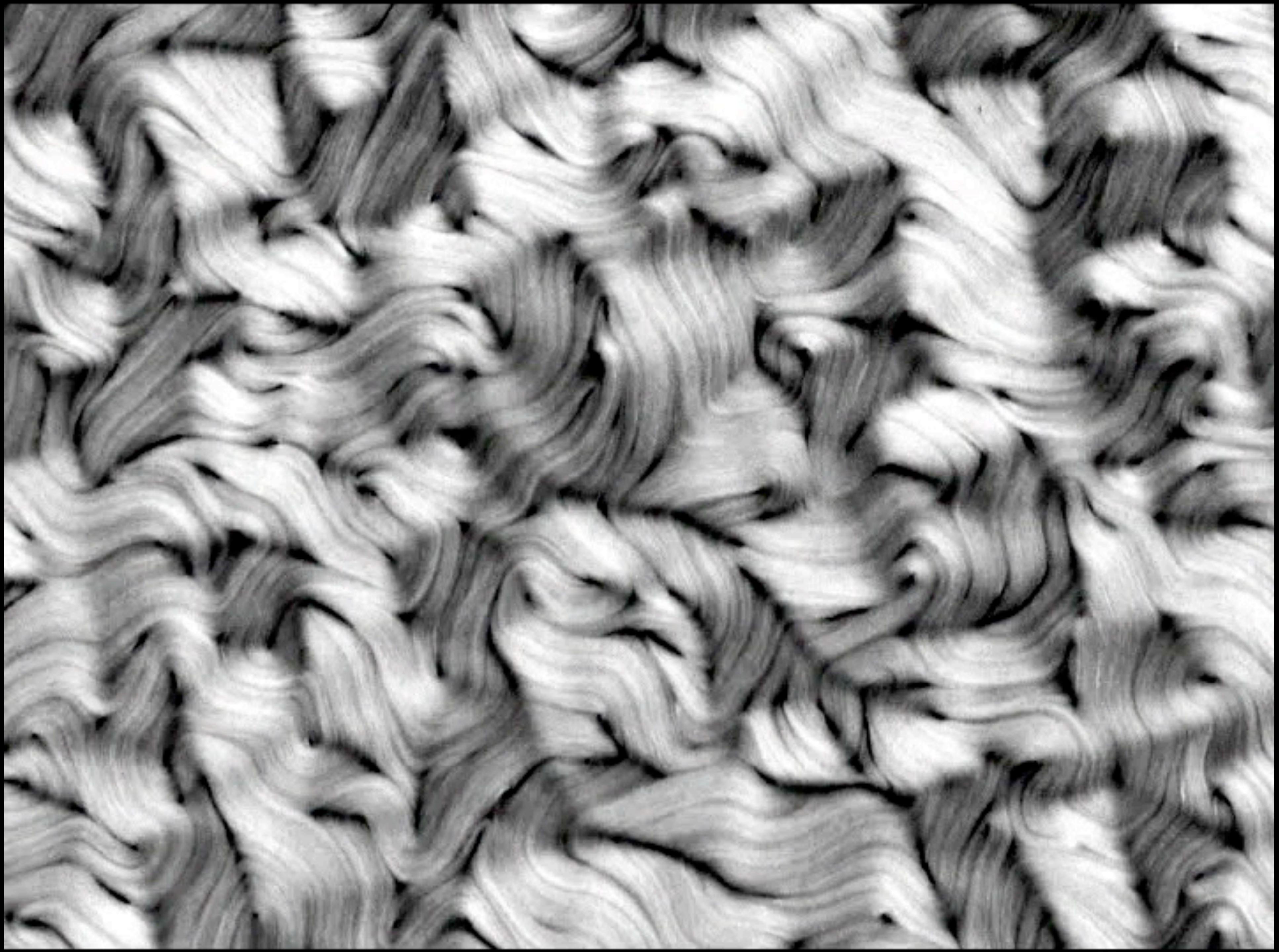
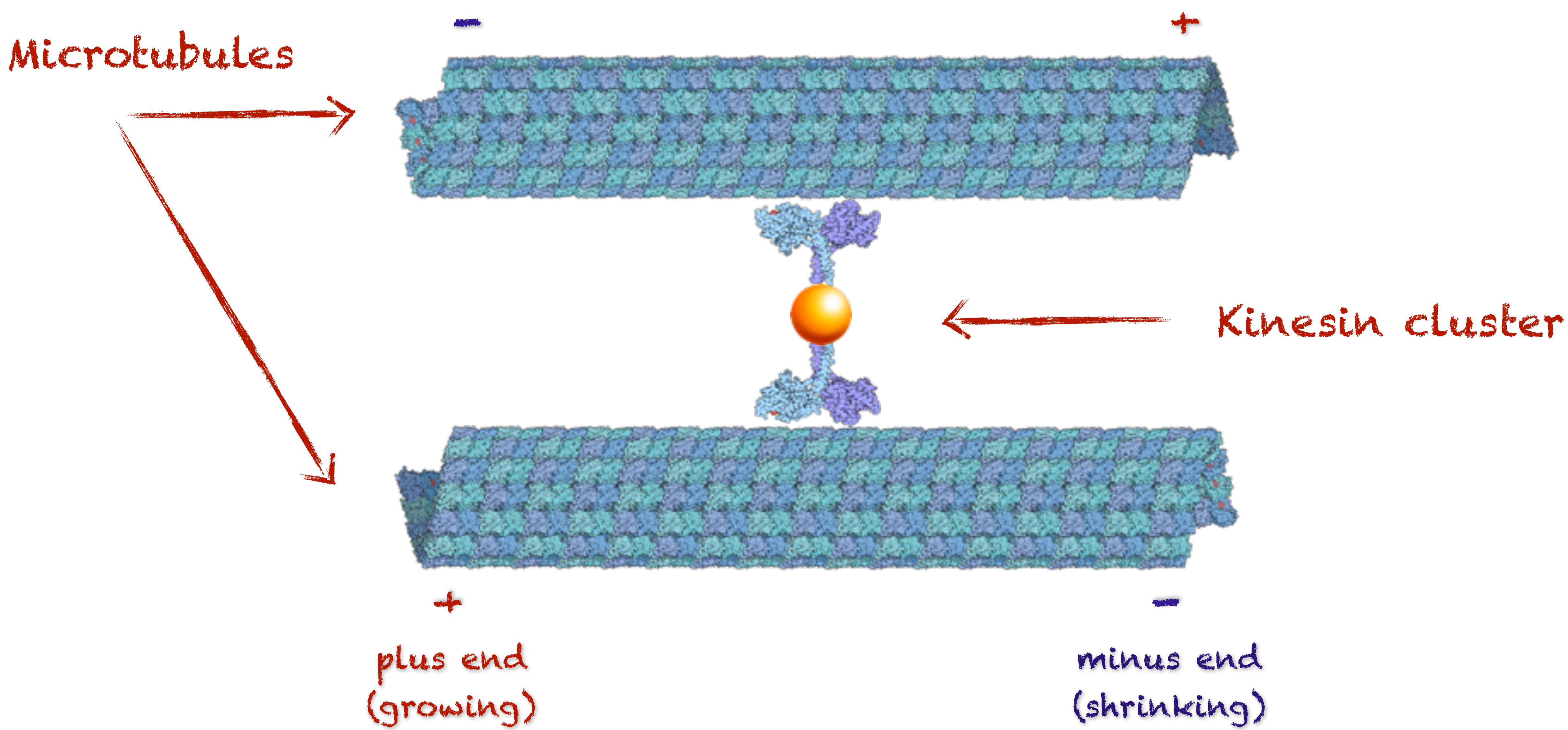
Courtesy of Z. Dogic



# Cytoskeletal fluids

Dense mixtures of microtubules and kinesin behaves as fluids of mutually propelled rods: *active liquid crystals*.



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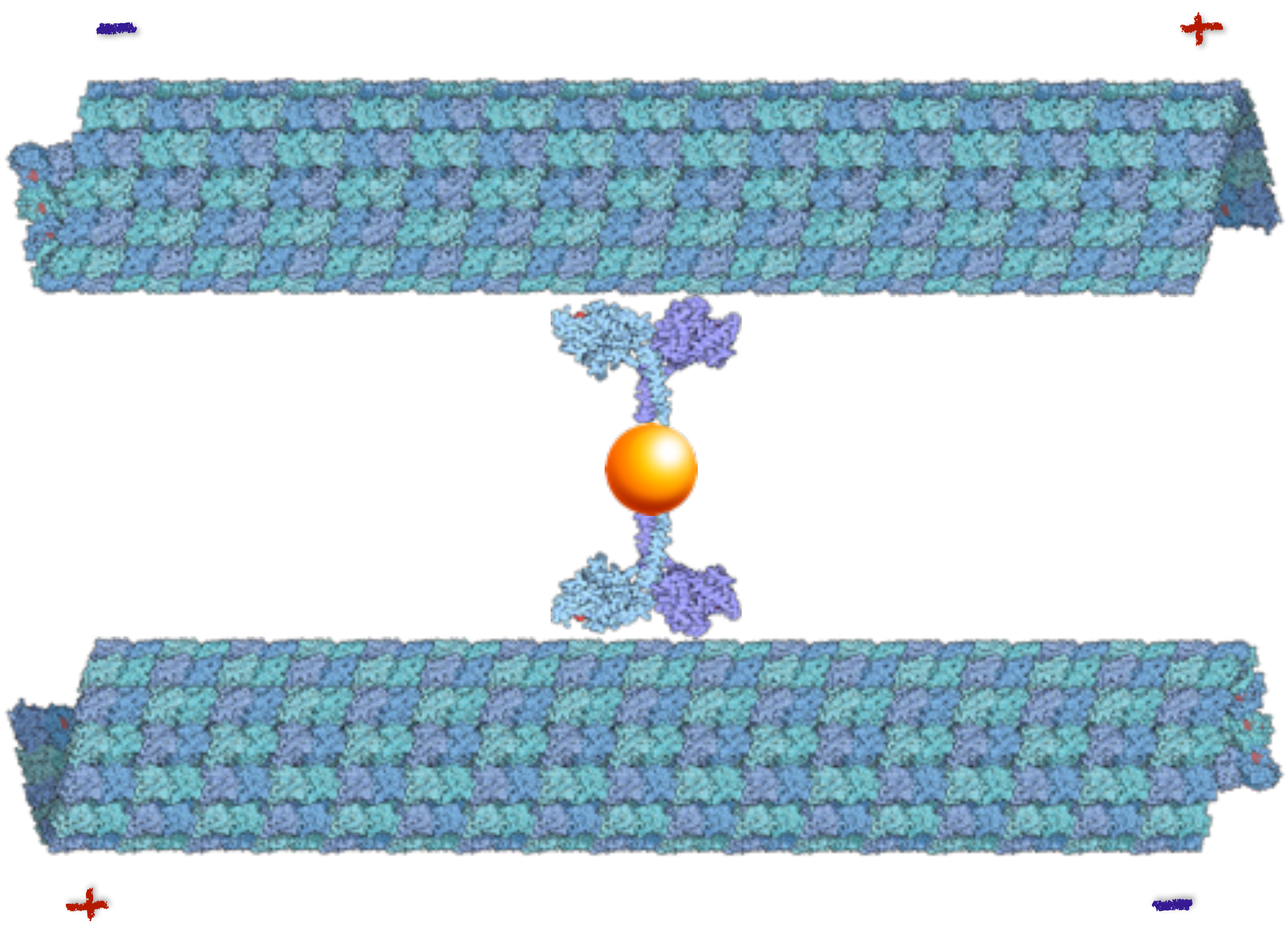

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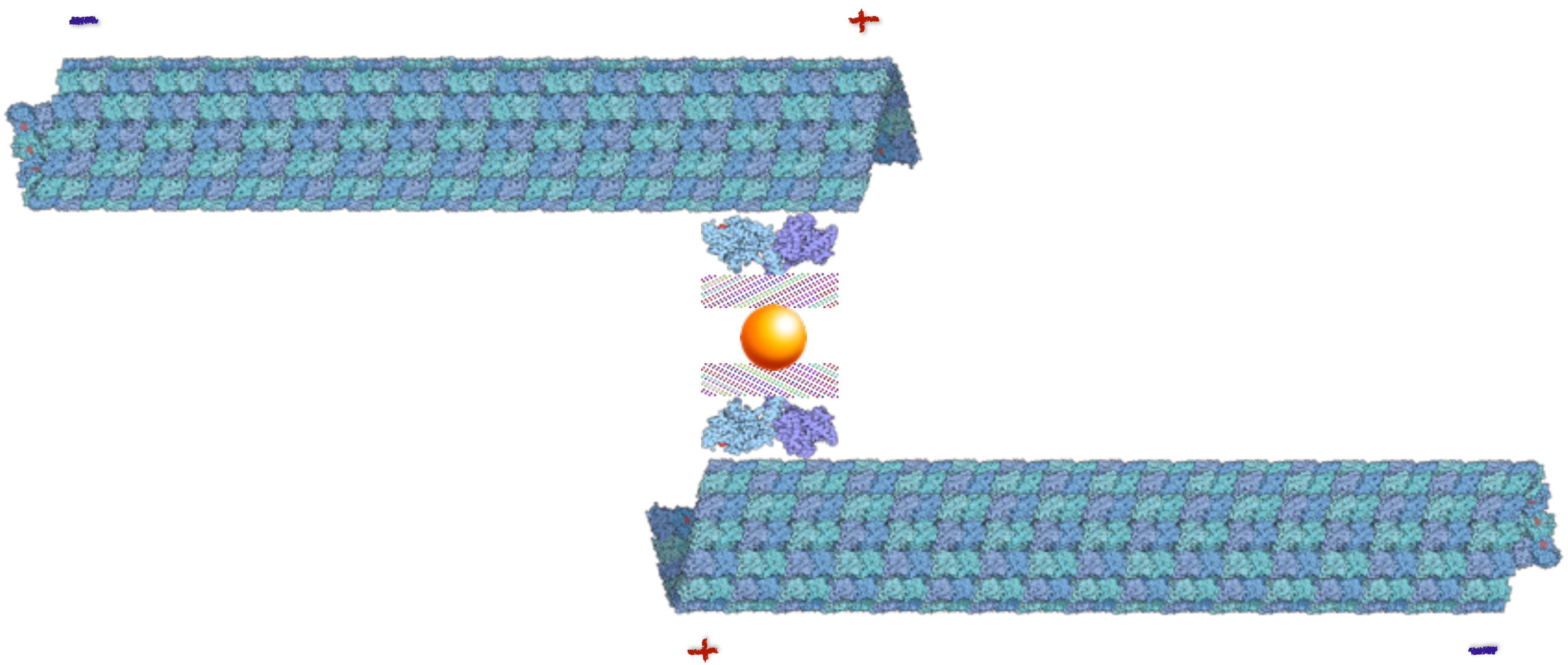

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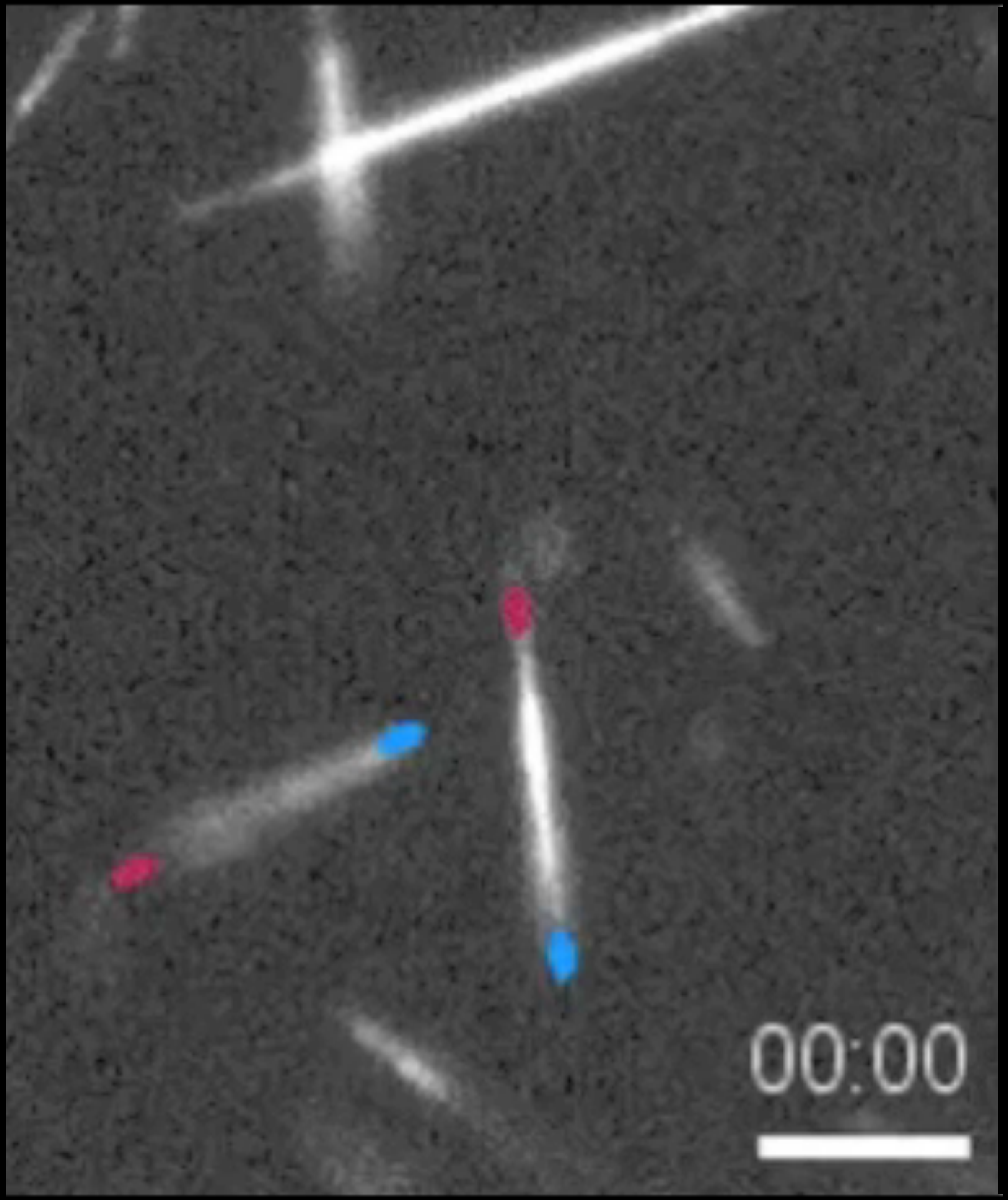


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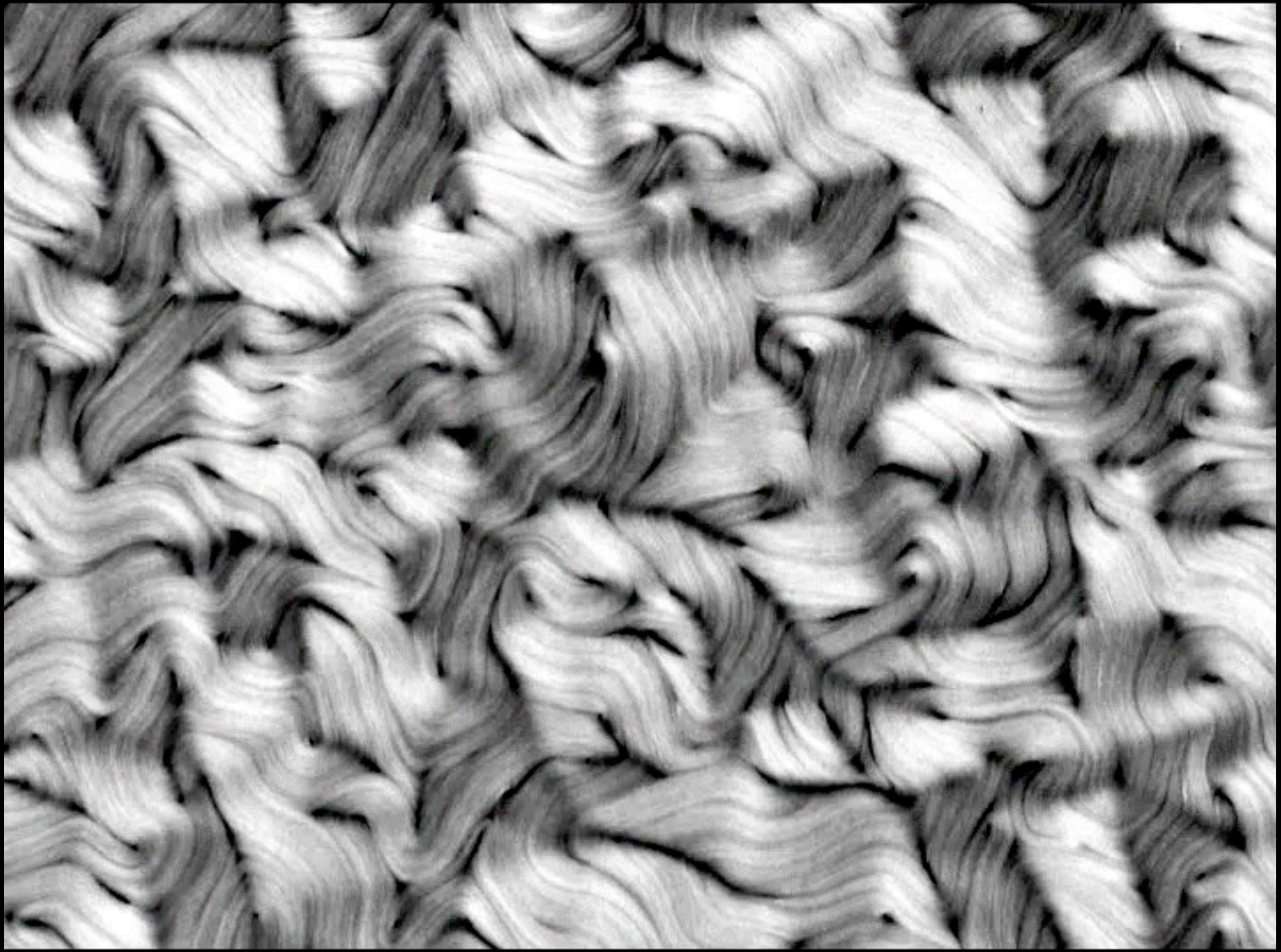

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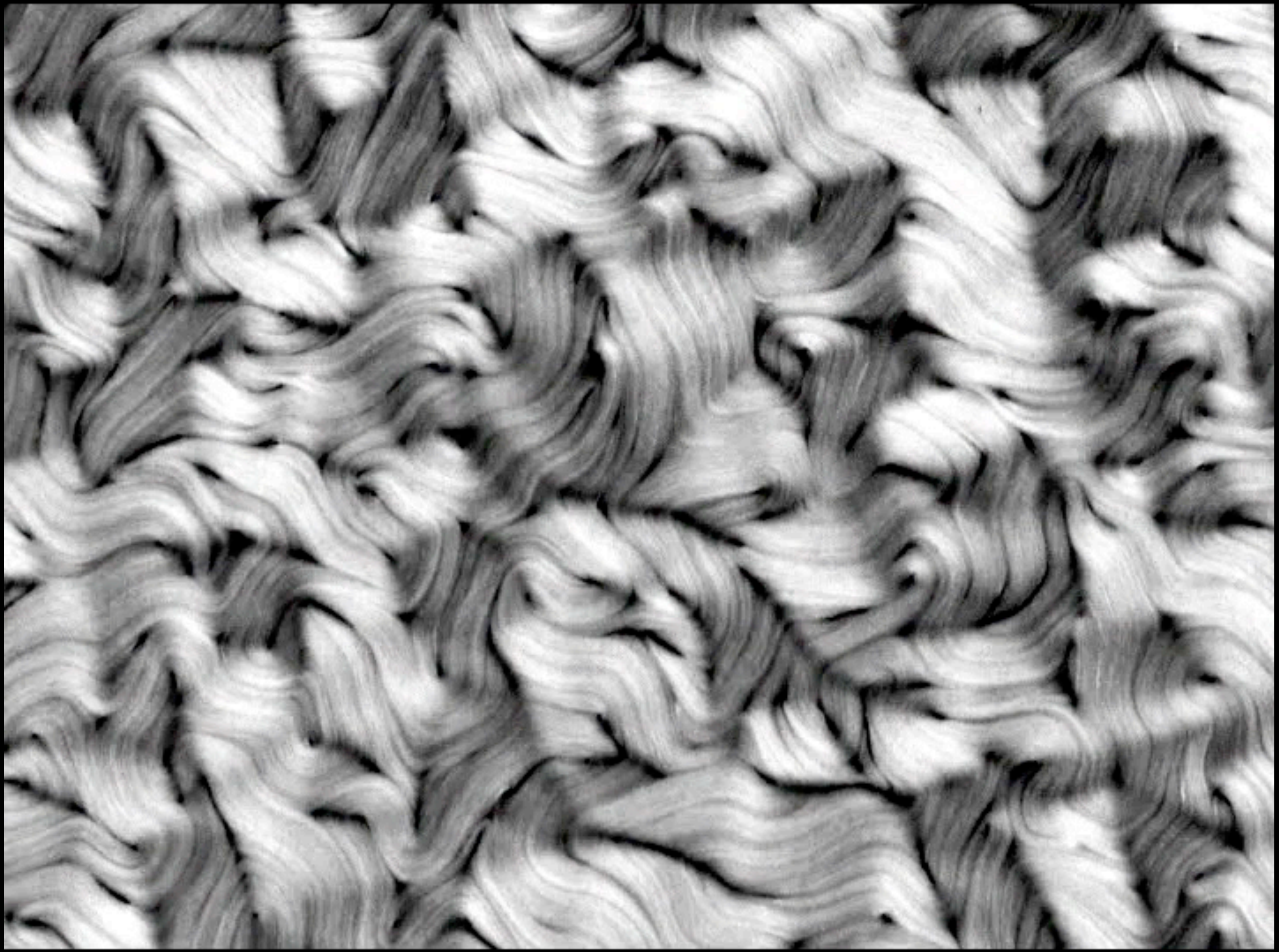
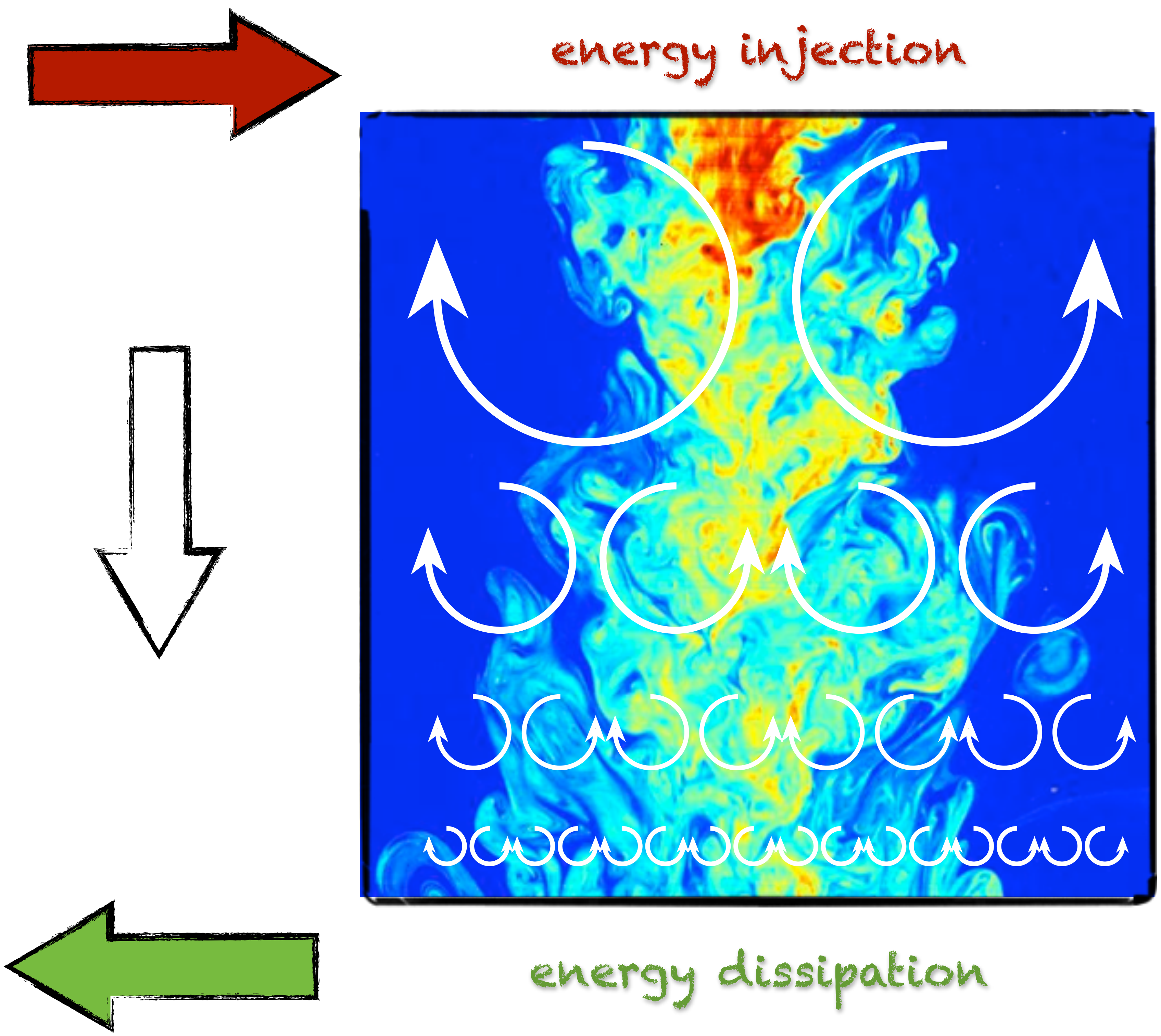
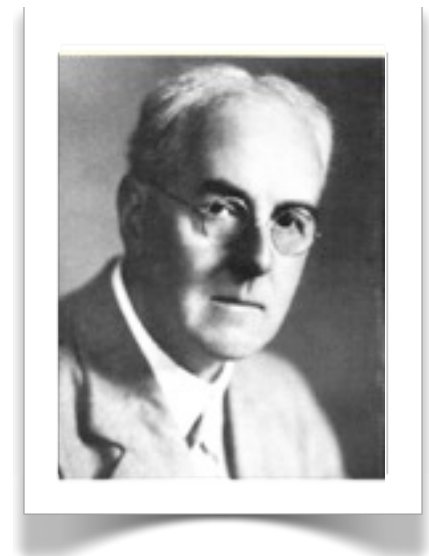


Reynolds number = $10^{-5}$ – $10^{-4}$

# Richardson cascade

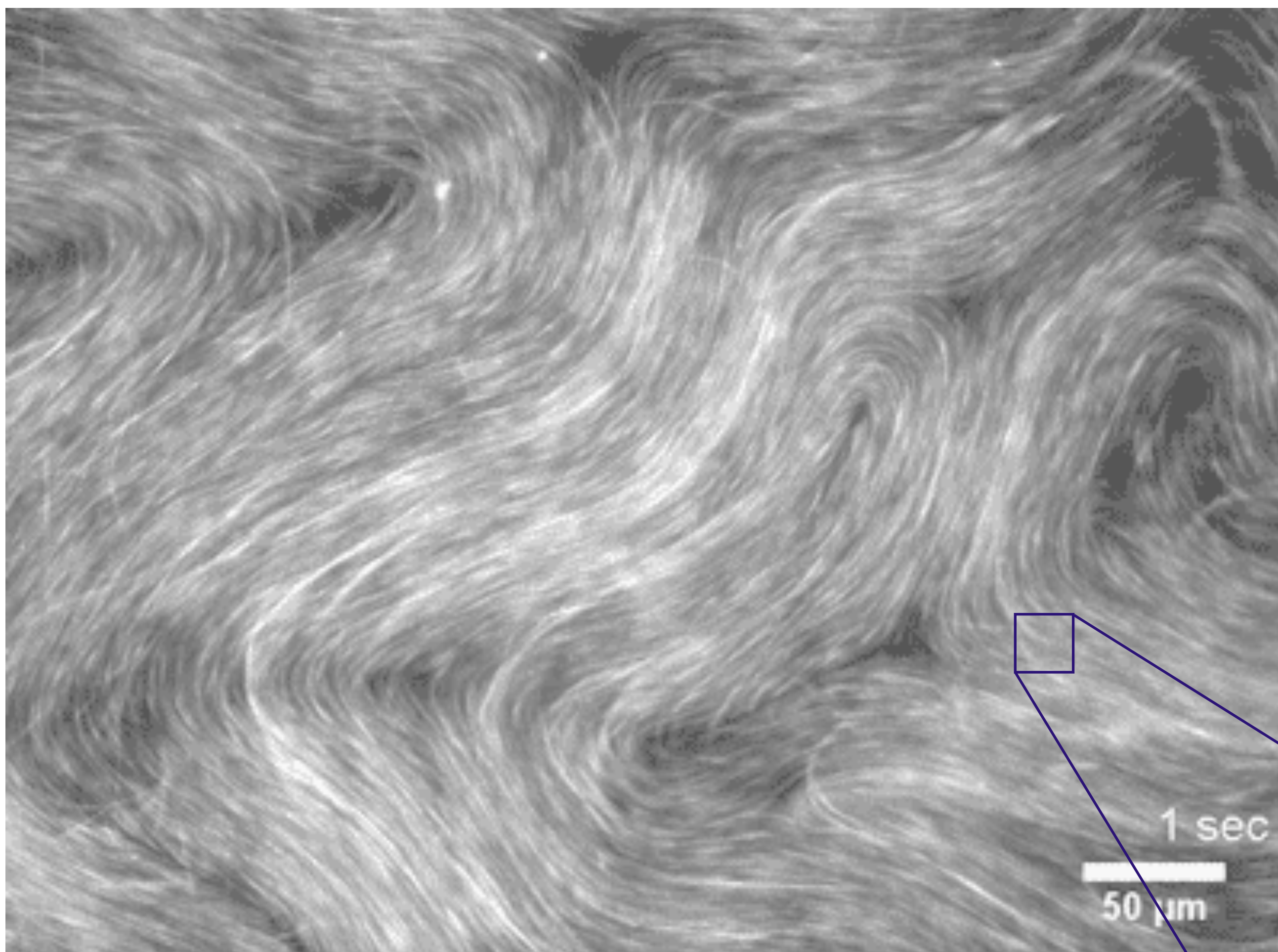


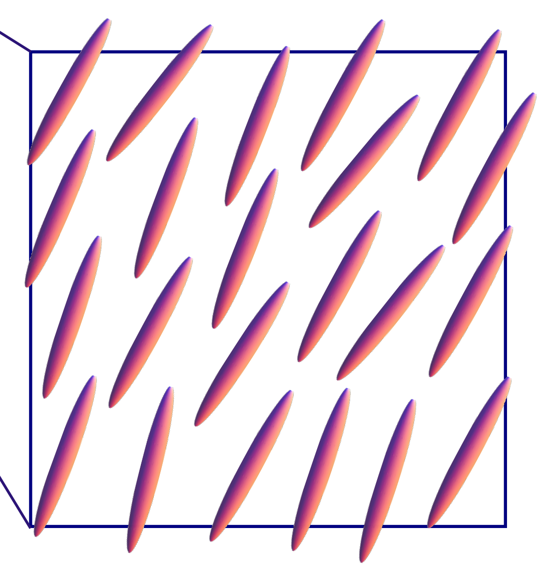
*Big whorls have little whorls*
*That feed on their velocity;*
*And little whorls have lesser whorls*
*And so on to viscosity.*

# Modeling active LCs

In nematics, a material element is characterized by four physical quantities:




$$\boldsymbol{Q} = S\left(\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} - \tfrac{1}{2}\boldsymbol{I}\right)$$



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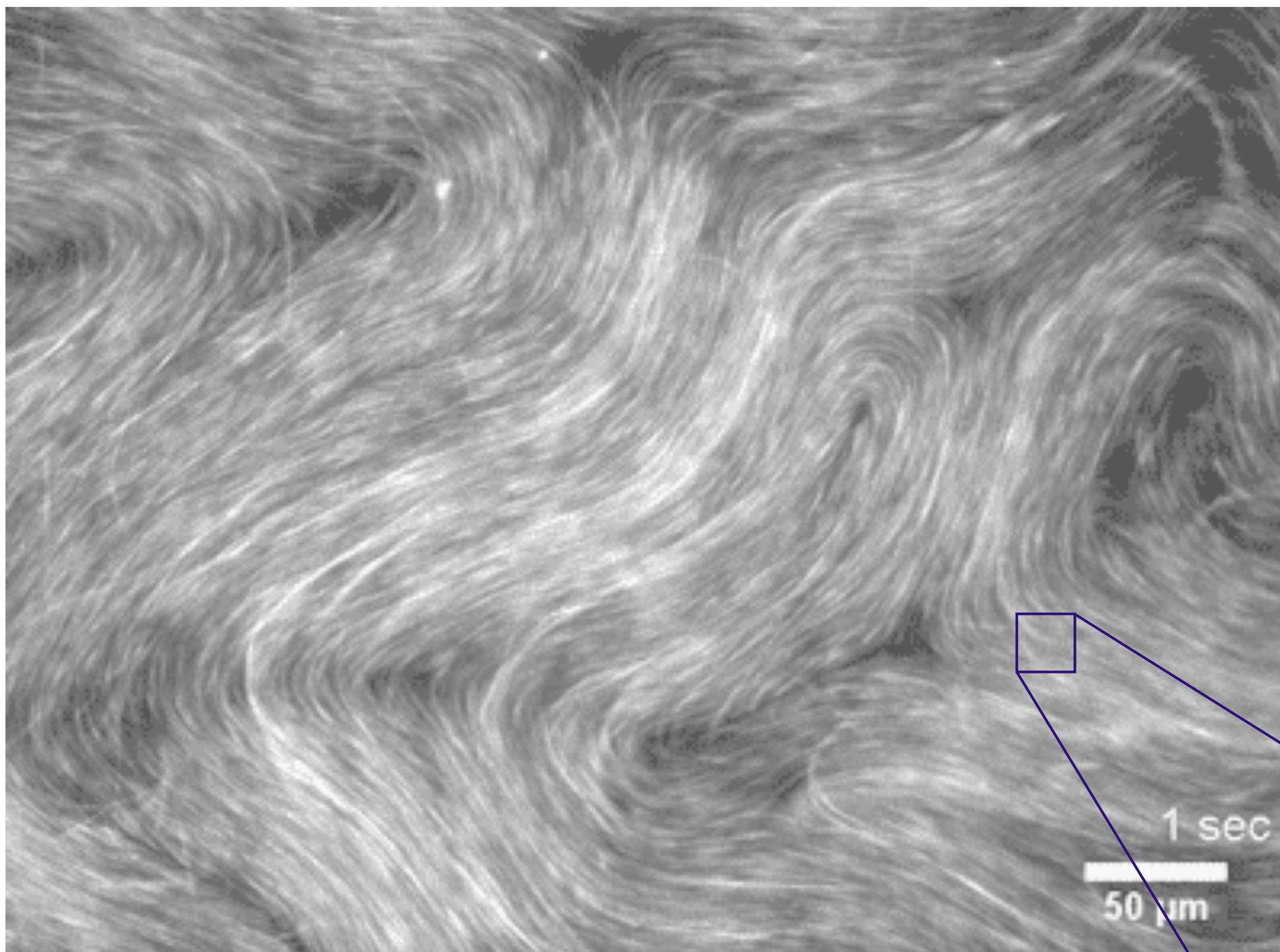


Density $\rho$

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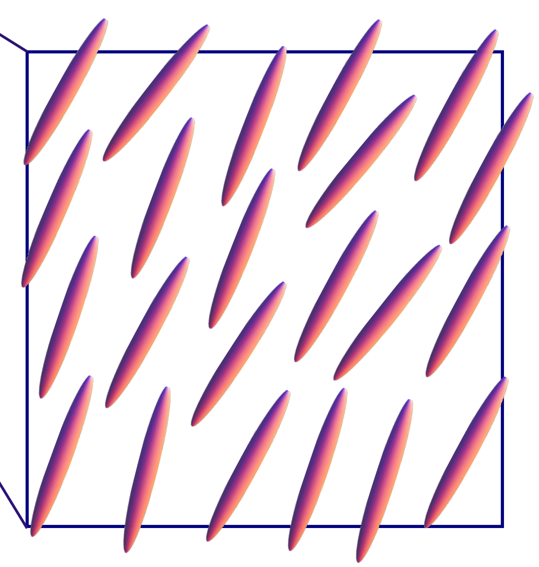

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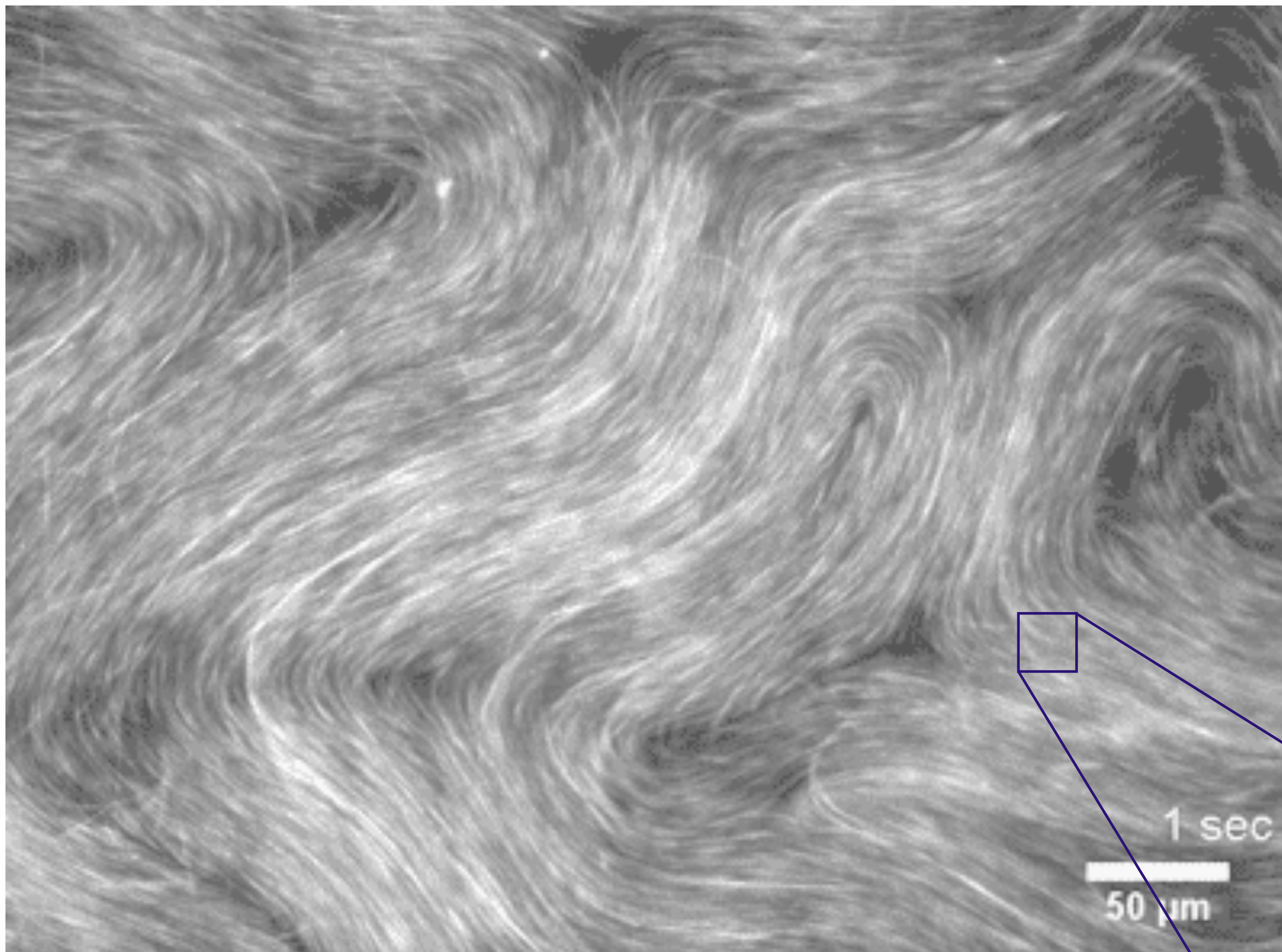


Density $\rho$

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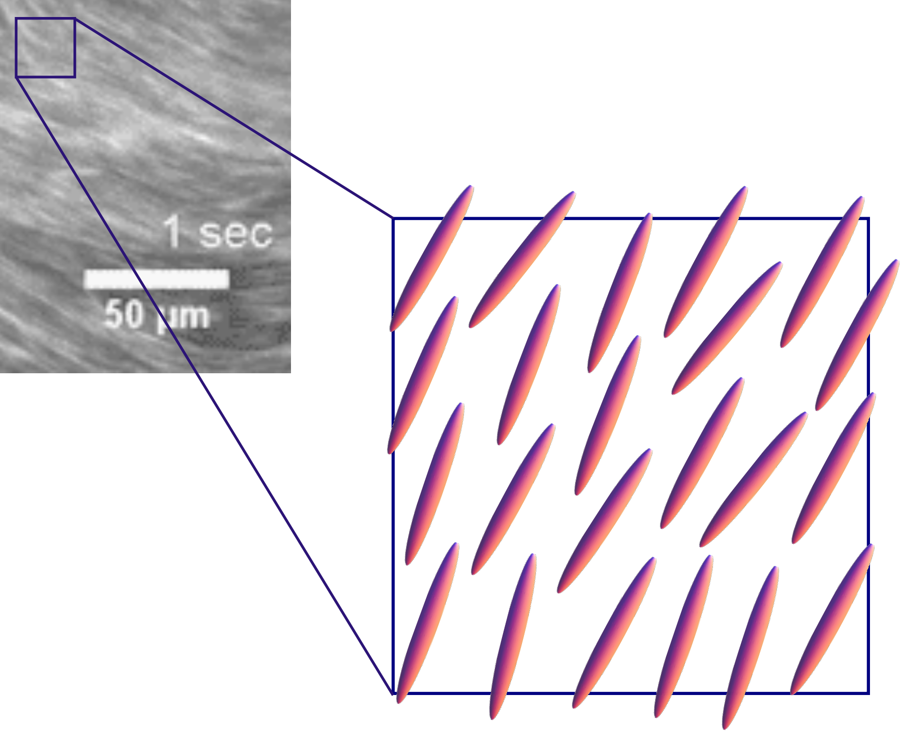

$\boldsymbol{n}$

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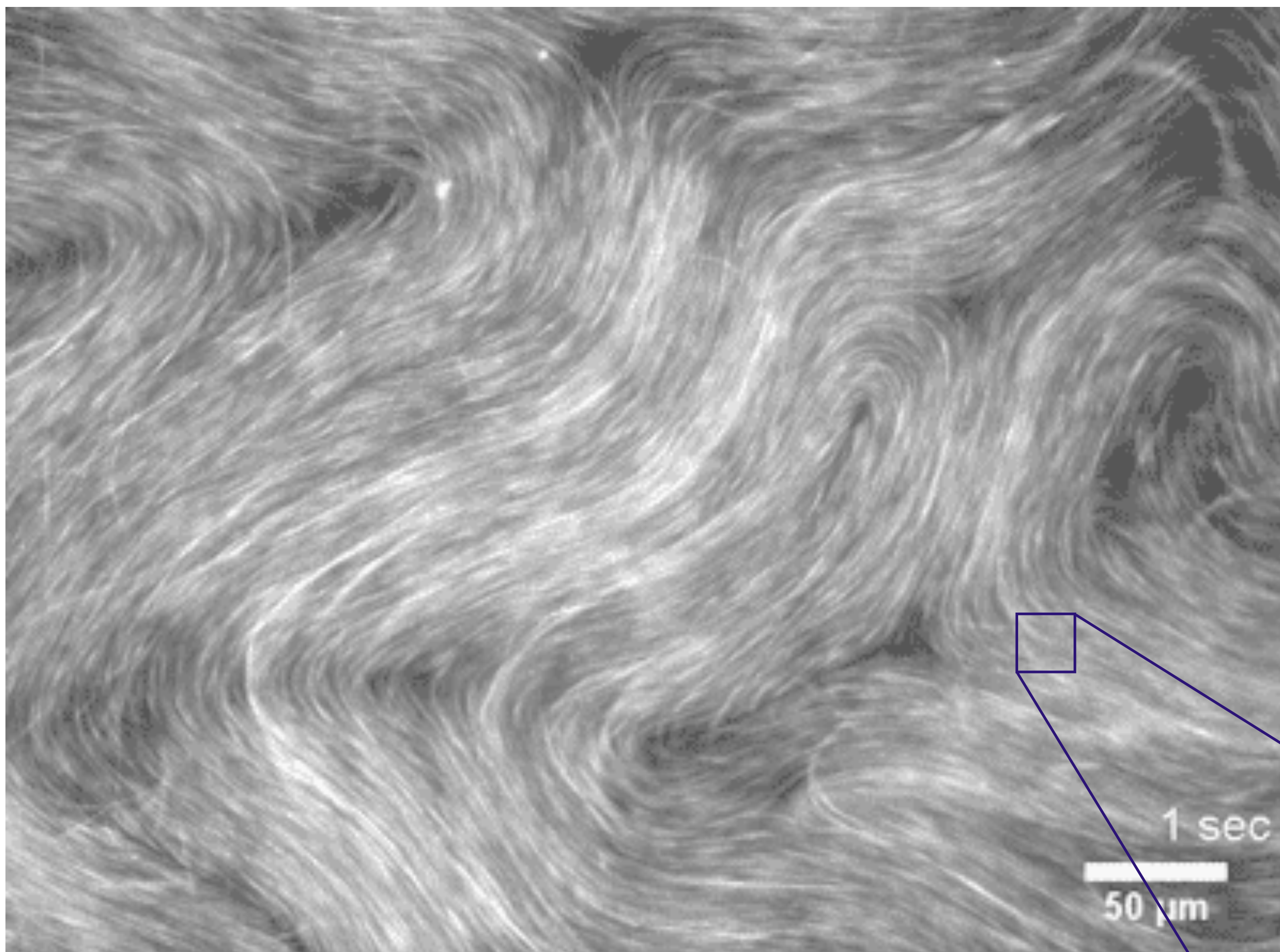


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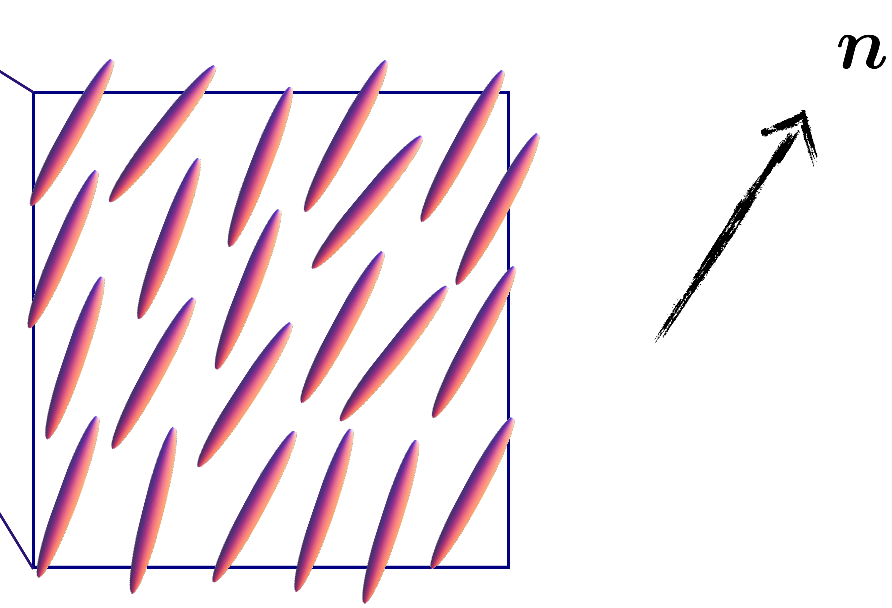

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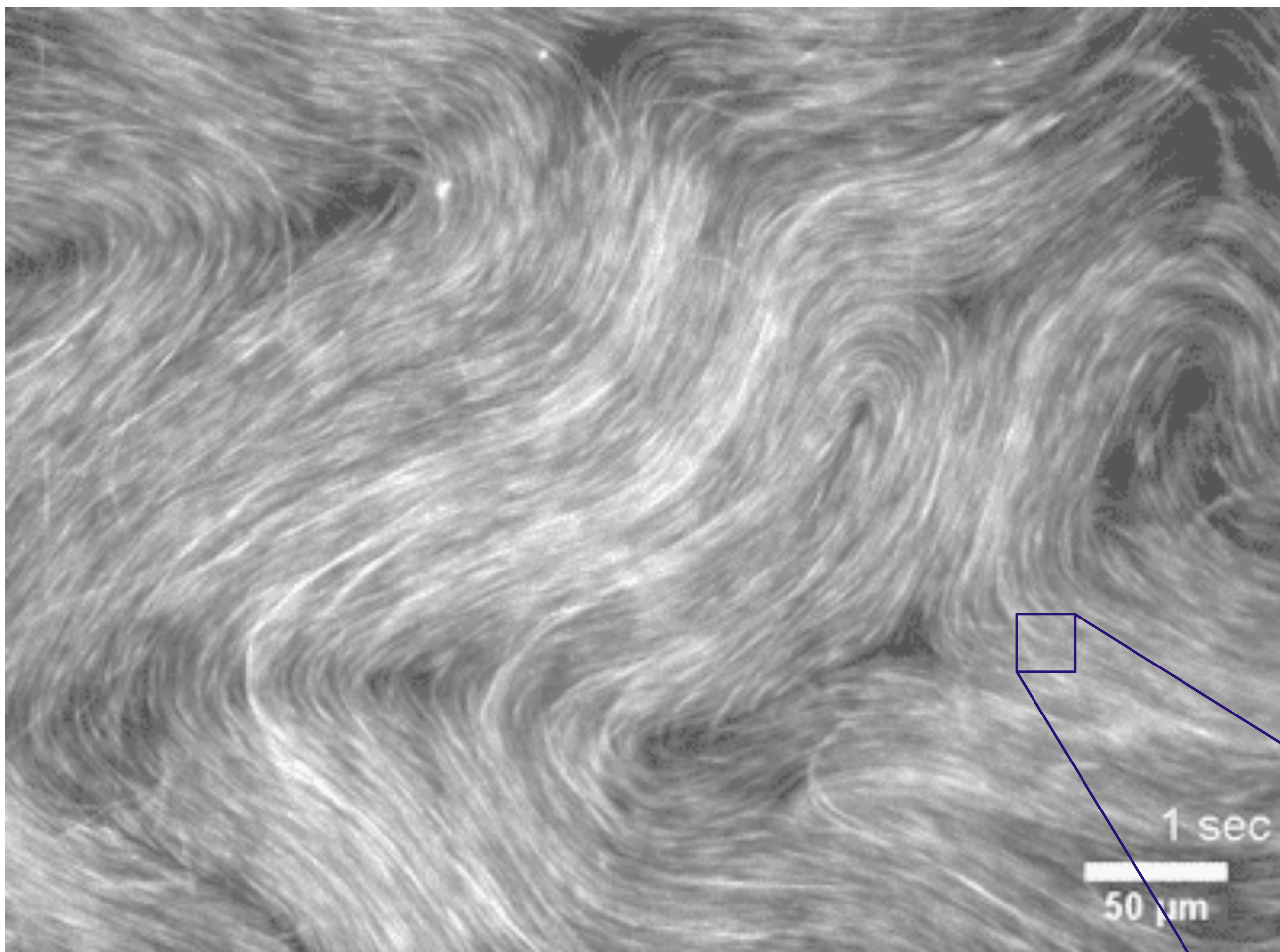


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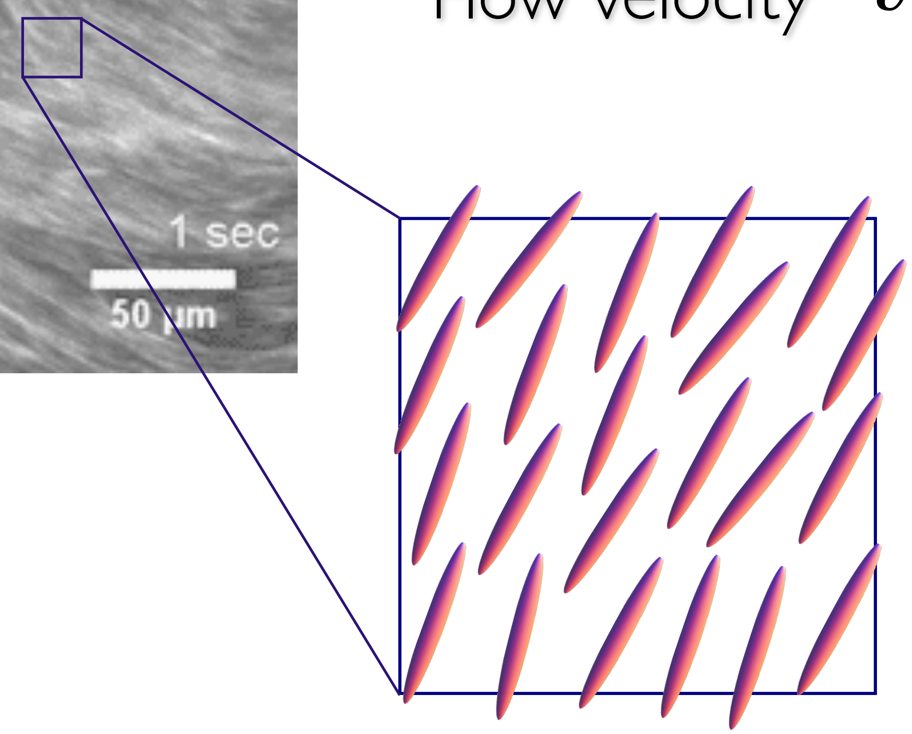
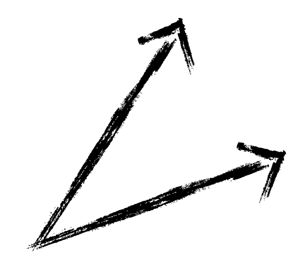
Flow velocity $\boldsymbol{v}$





# Active and passive stresses

The flow velocity obeys to the Navier-Stokes equation:

$$\rho \frac{D\boldsymbol{v}}{Dt} = \nabla \cdot \left( \boldsymbol{\sigma}^{\mathrm{p}} + \boldsymbol{\sigma}^{\mathrm{a}} \right)$$

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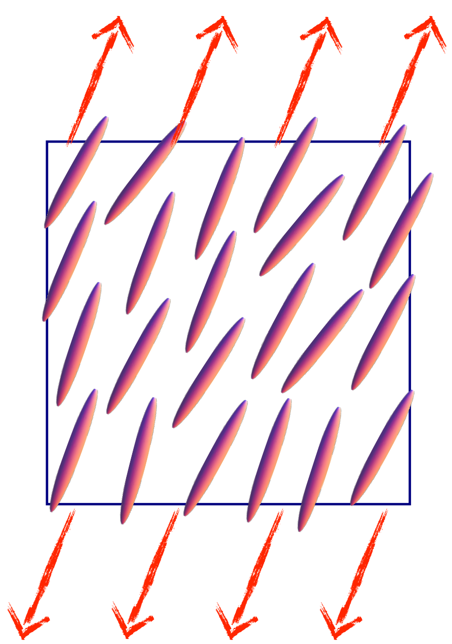
Active stress:



$$\mathbf{F}^{\mathrm{a}} \sim \mathbf{n}$$

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Pedley & Kessler (1992),
Simha & Ramaswamy (2002)

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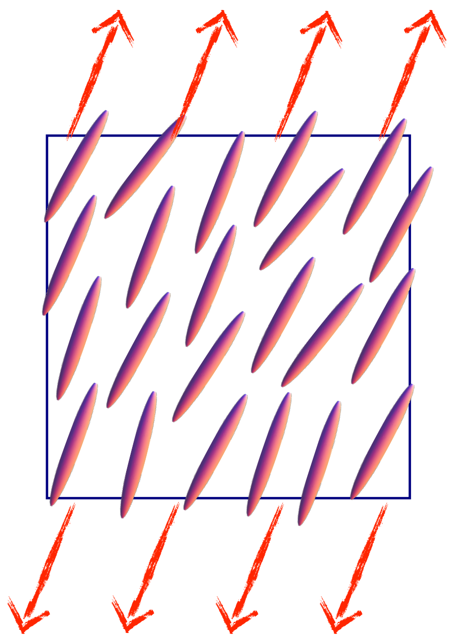

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Activity constant
(= biochemestry)

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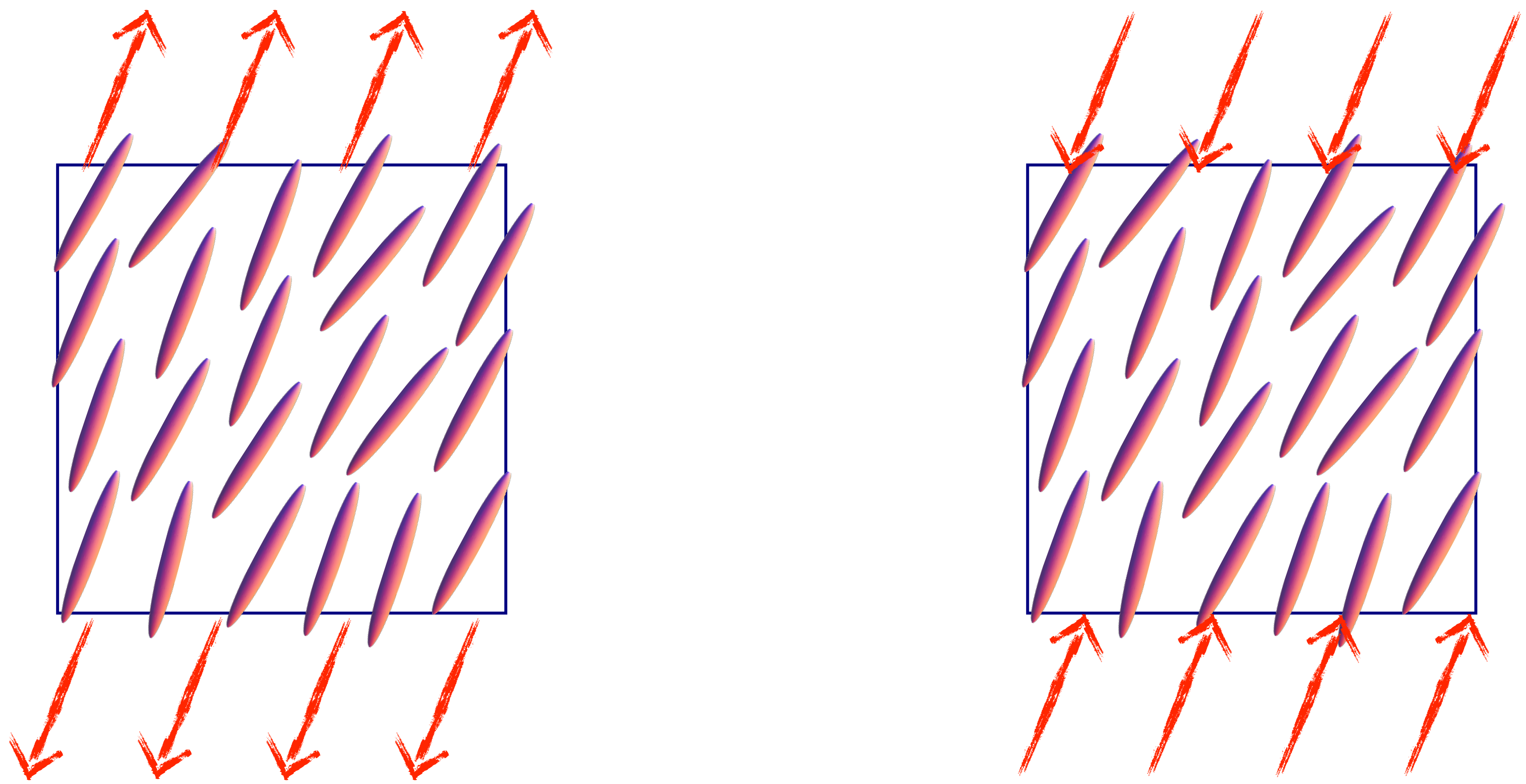
# Contractile vs extensile active stress



Contractile
$\alpha > 0$

Extensile
$\alpha < 0$

# Nematic tensor equation

An hydrodynamic equation for the nematic tensor $\boldsymbol{Q}$ is obtained phenomenologically (Olmsted & Goldbart '92):

$$\frac{D\boldsymbol{Q}}{Dt} = \lambda S \boldsymbol{u} + [\boldsymbol{Q}, \boldsymbol{\omega}] + \gamma^{-1} \frac{\delta F_{\mathrm{LdG}}}{\delta \boldsymbol{Q}}$$

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relaxational dynamics

$\boldsymbol{u} = \frac{1}{2}\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\mathrm{T}}\right)$ strain-rate

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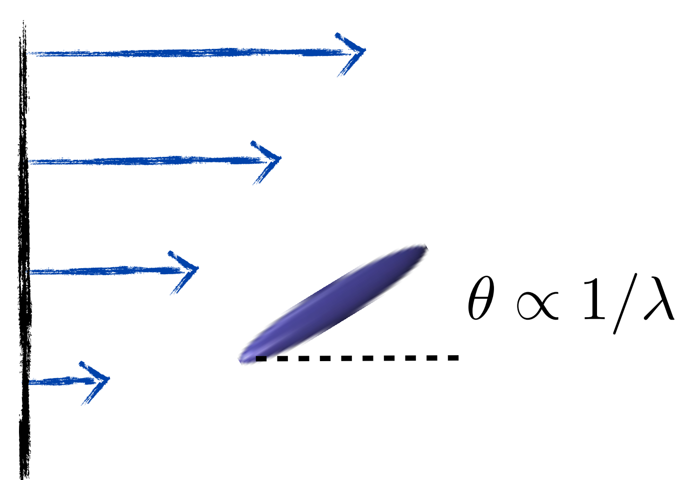
flow-alignment parameter



# Active nematic hydrodynamics

Pedley & Kessler (1992), Hatwalne *et al.* (2004), Voituriez *et al.* (2005), Liverpool & Marchetti (2006), Marenduzzo *et. al.* (2007), Edwards & Yeomans (2009), Giomi *et al.* (2011).

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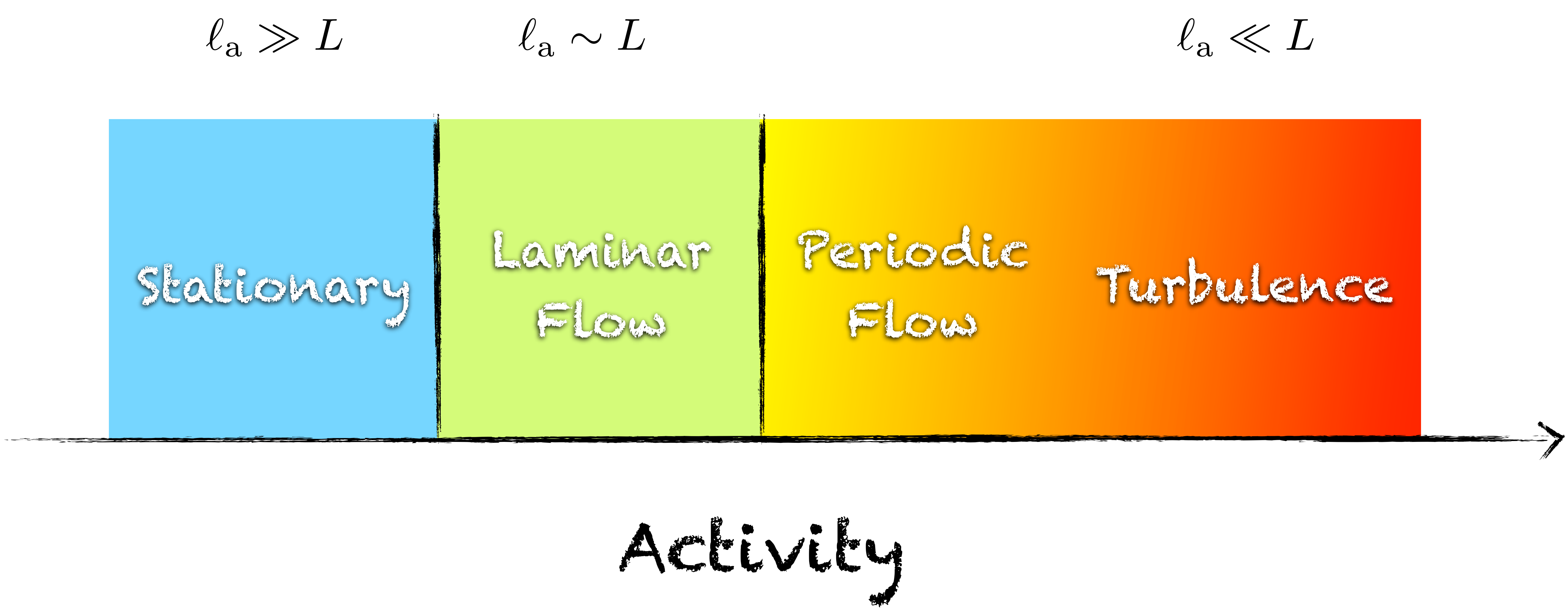
$$\ell_{\mathrm{a}} = \sqrt{\frac{K}{|\alpha|}}$$

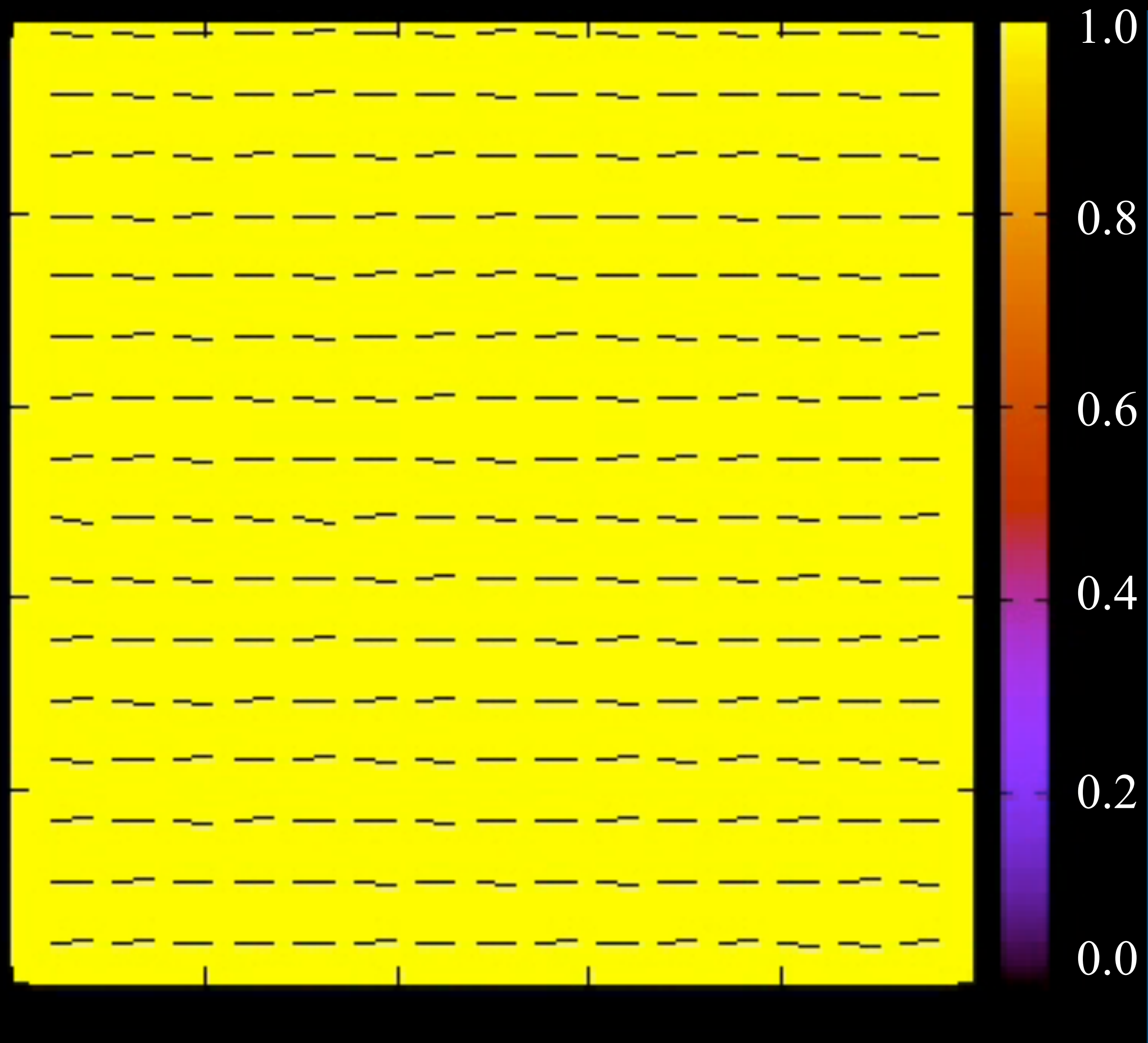
active length scale

where active and elastic stresses balance

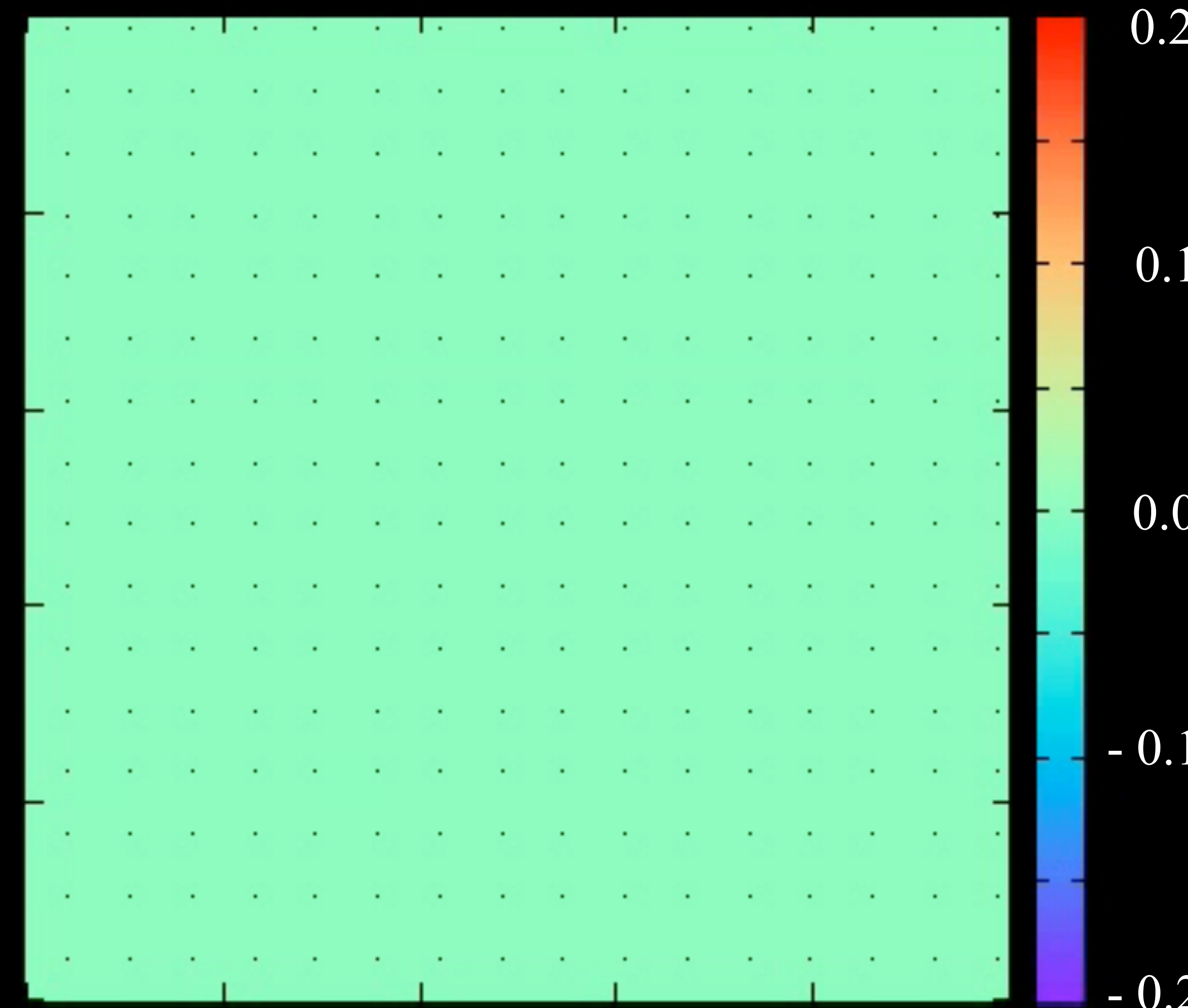
# Dynamics of active nematics



Director / Order parameter

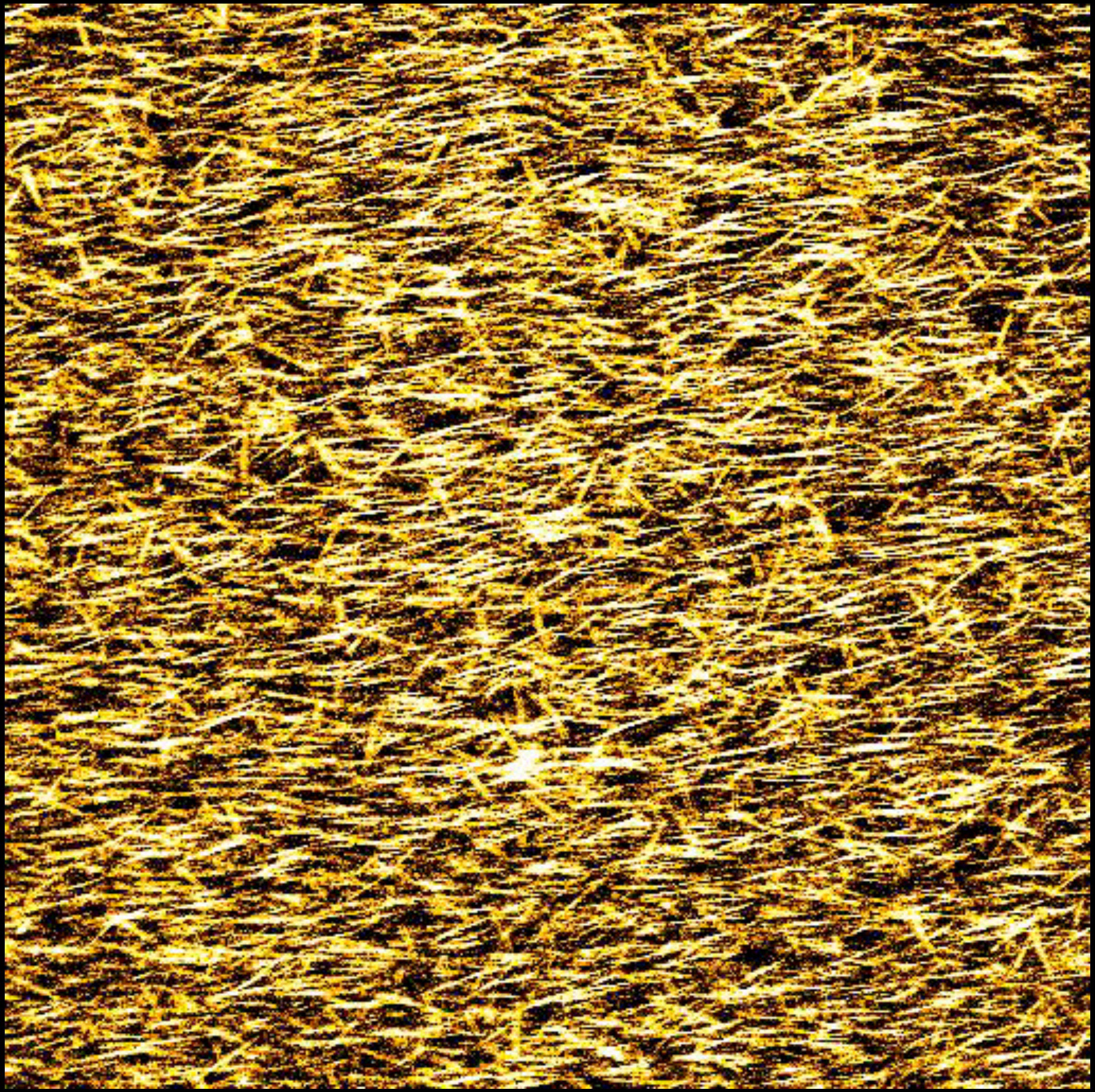



Velocity / Vorticity




Extensile: $\gamma = 10, \alpha = -3$

Curtesy of Zvonimir Dogic



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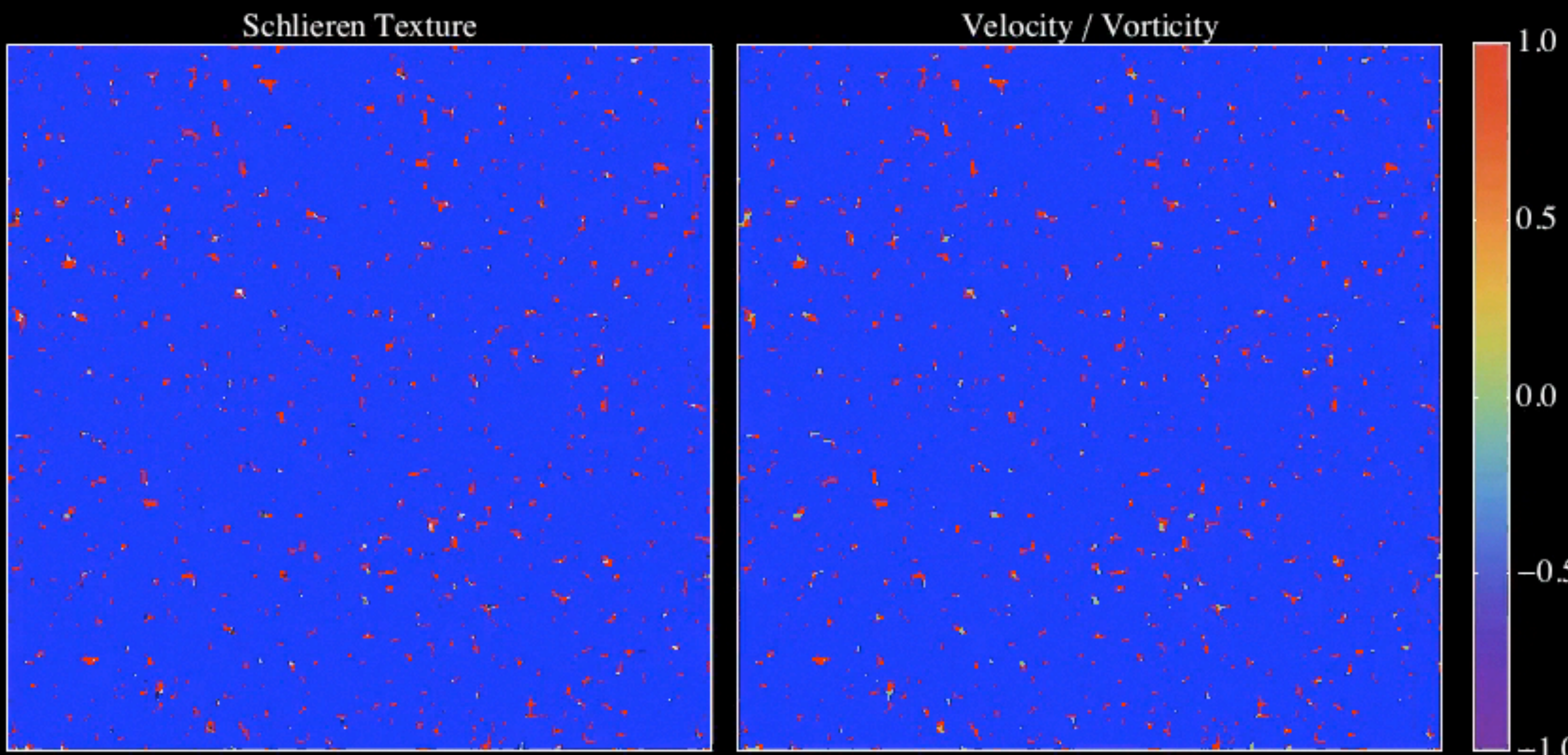


Topological defects play a pivotal role ! (Thampi, Golestanian & Yeomans 2013)

# Vortex statistics

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Vortex areal density: $n(a)$

$$da\, n(a) = \text{\# of vortices whose area is in [a, a+da]}$$

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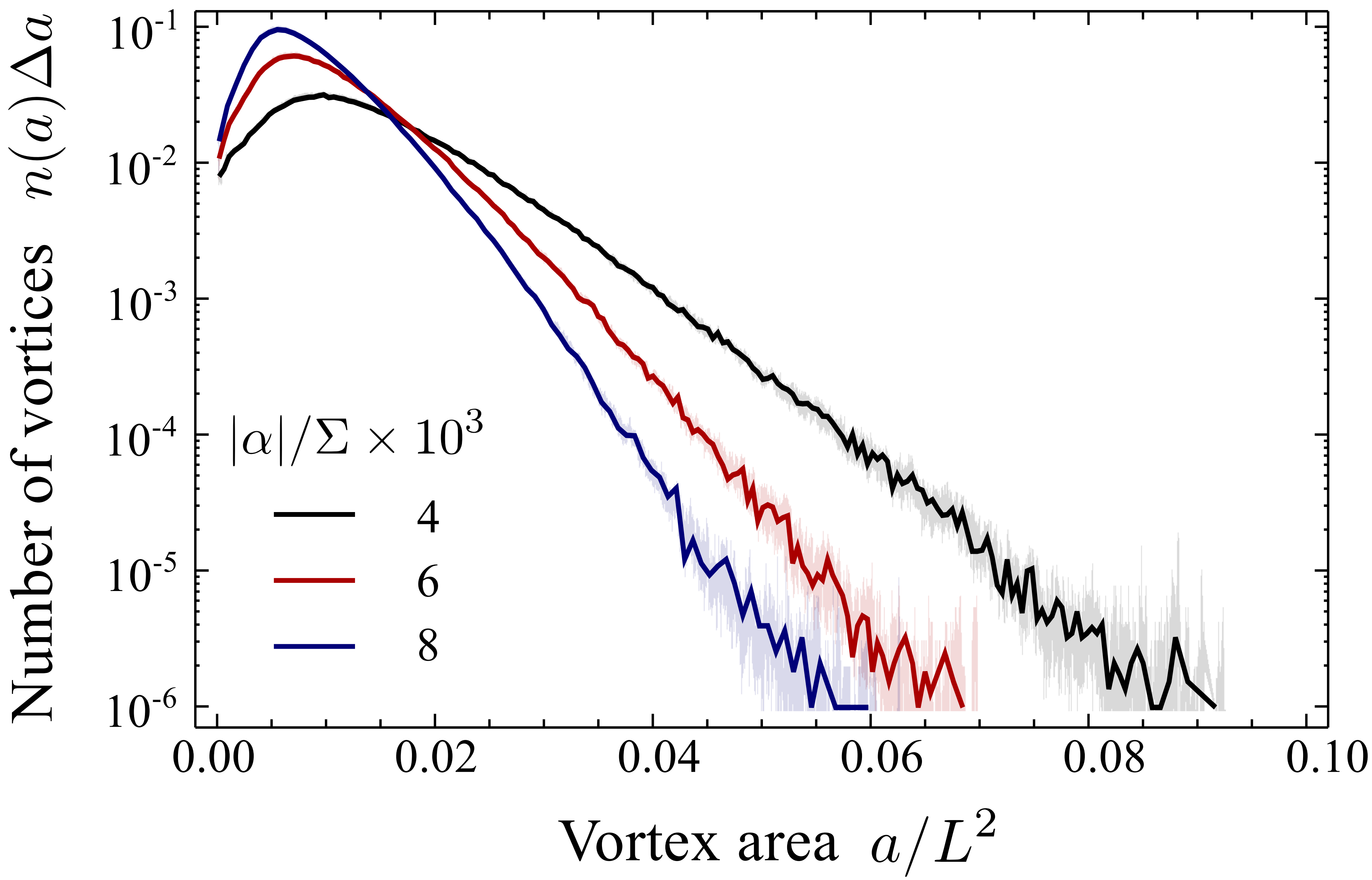

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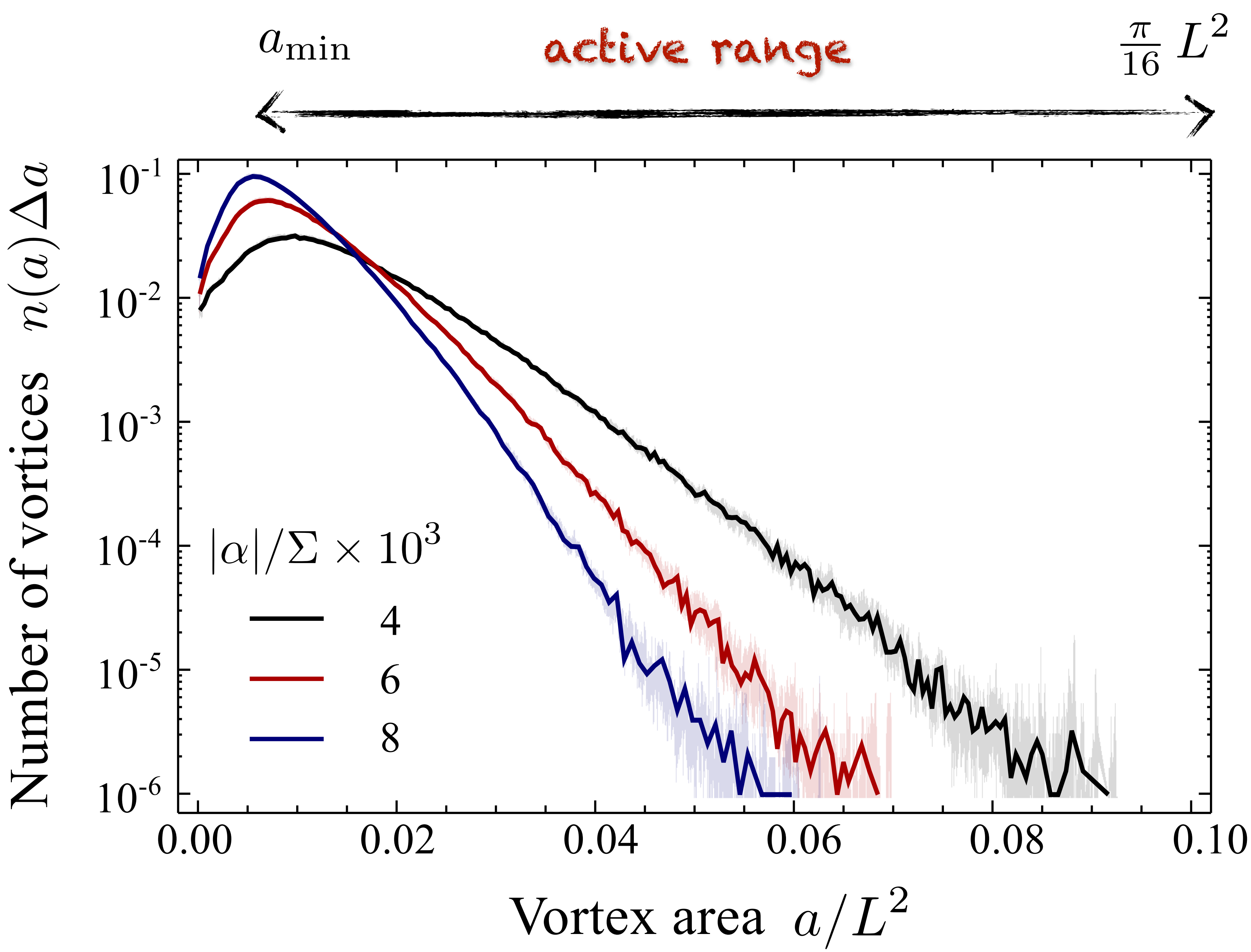

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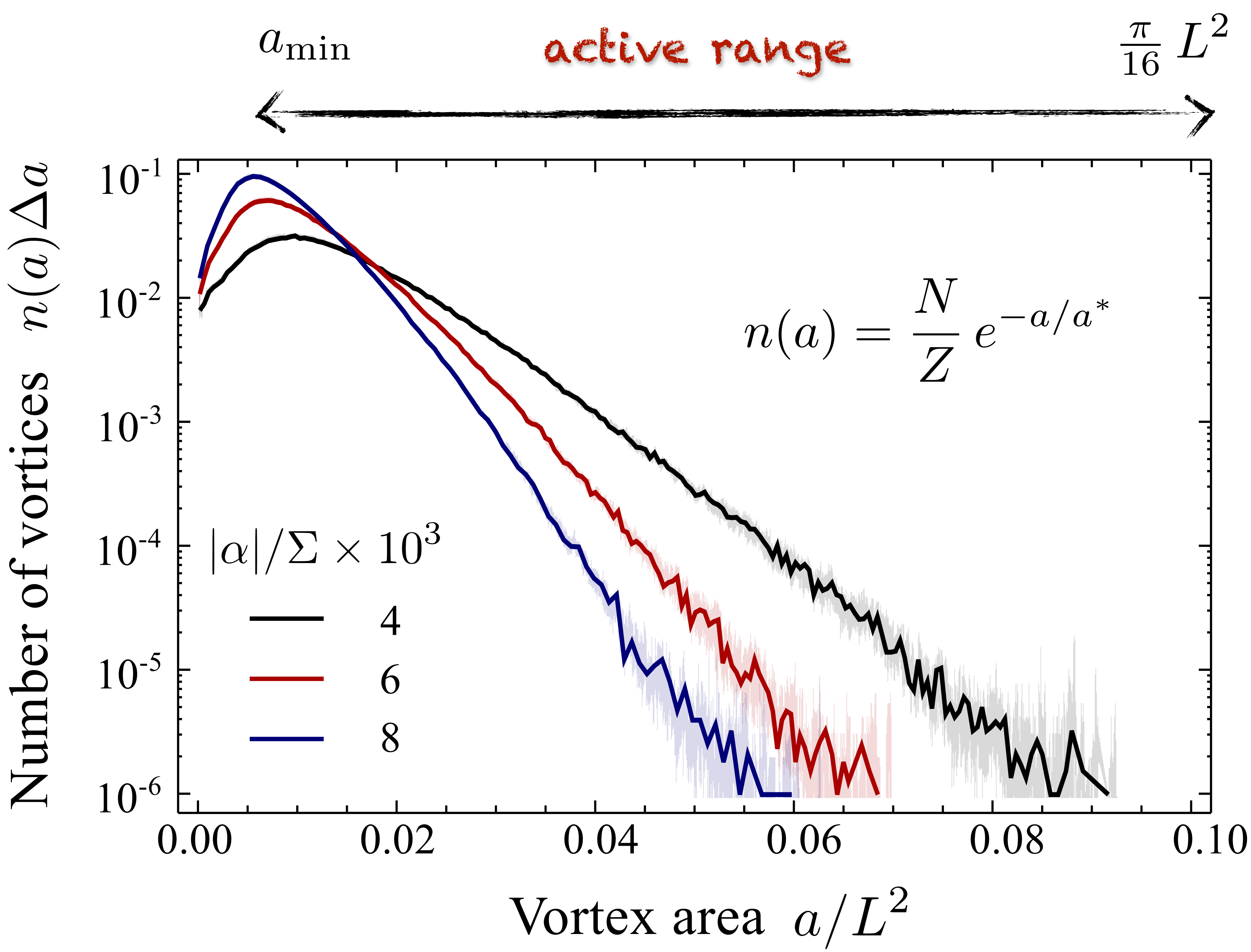
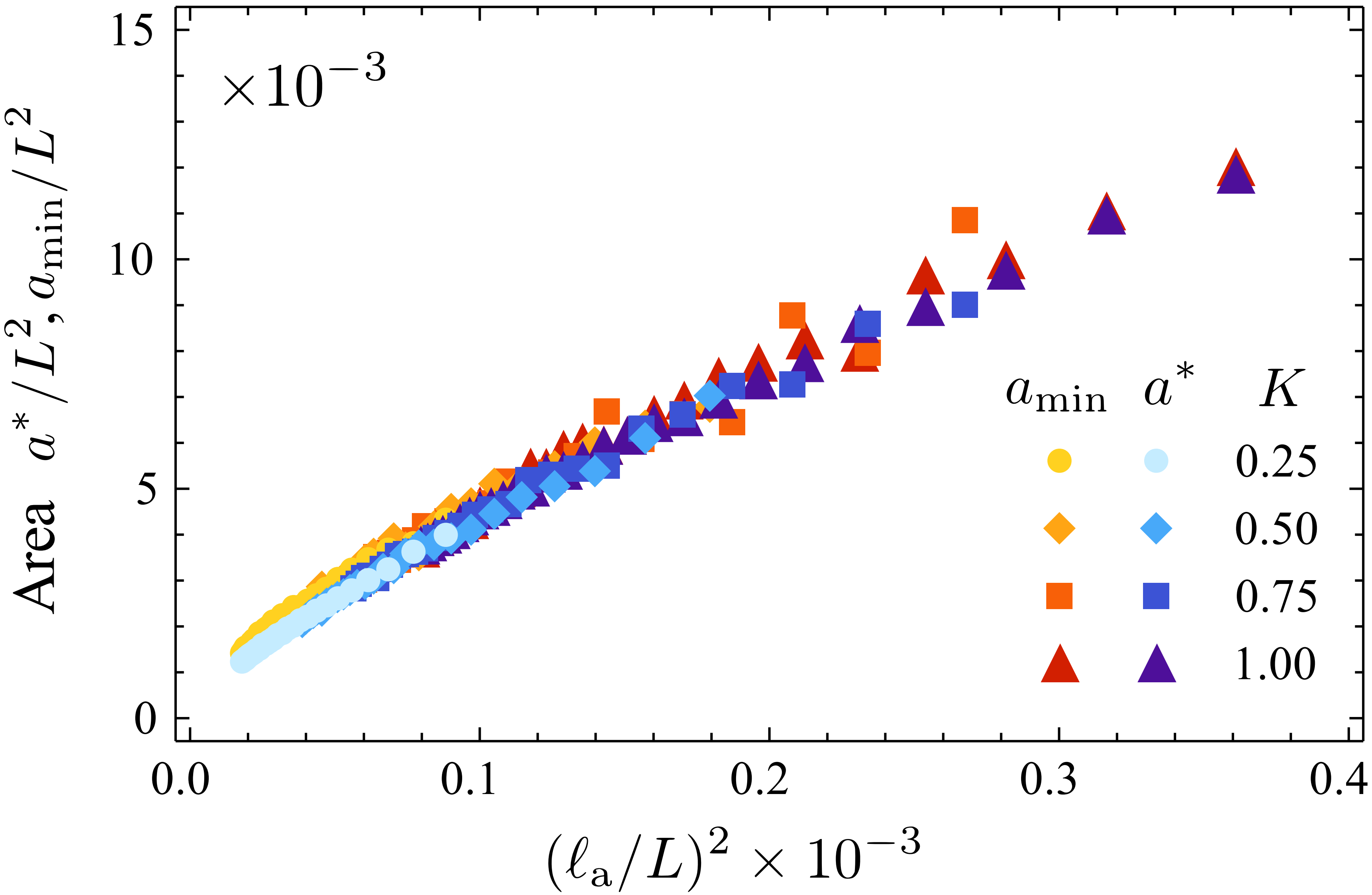


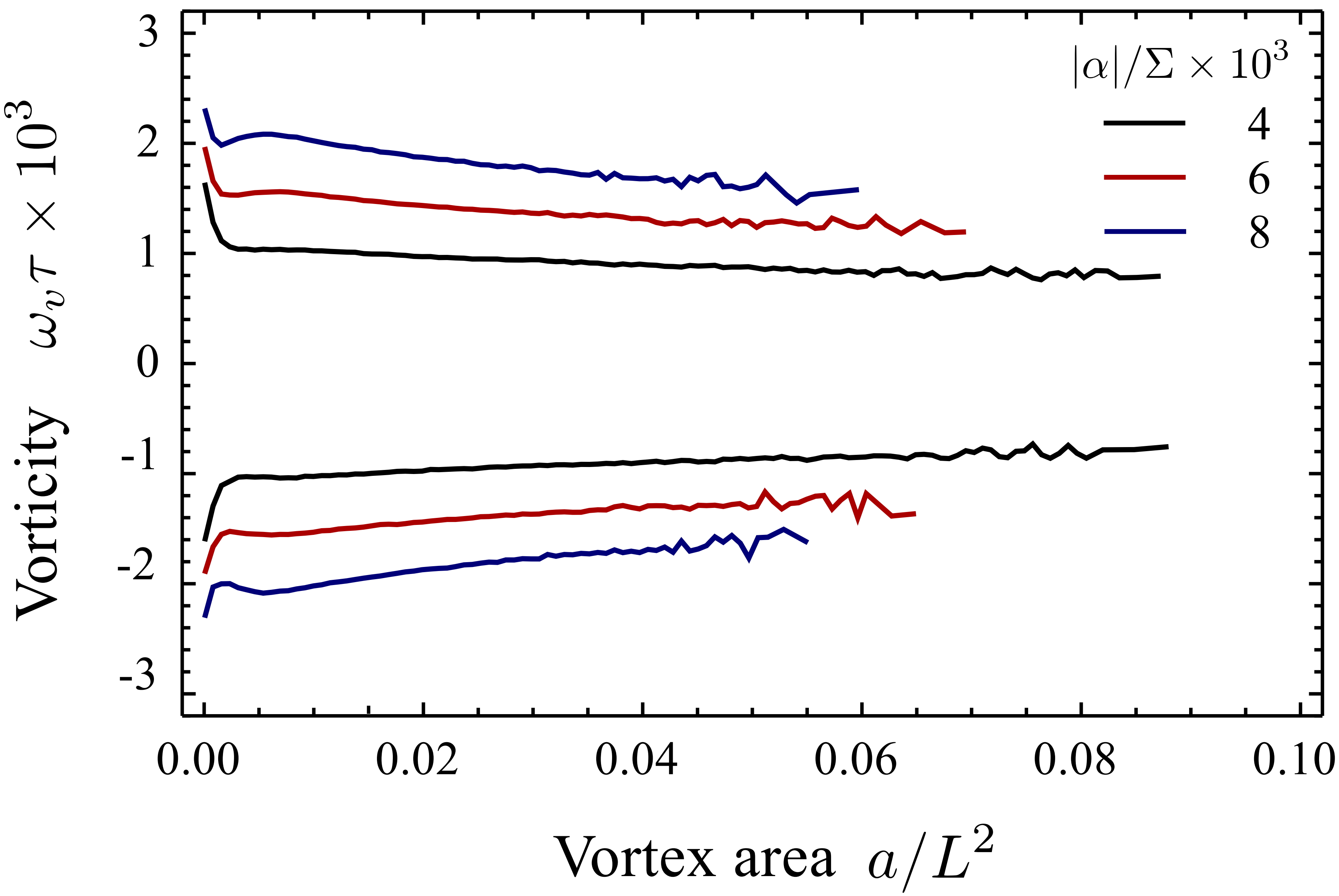
# Vortex statistics

$$a_{\mathrm{min}} = a^* \sim \ell_{\mathrm{a}}^2$$

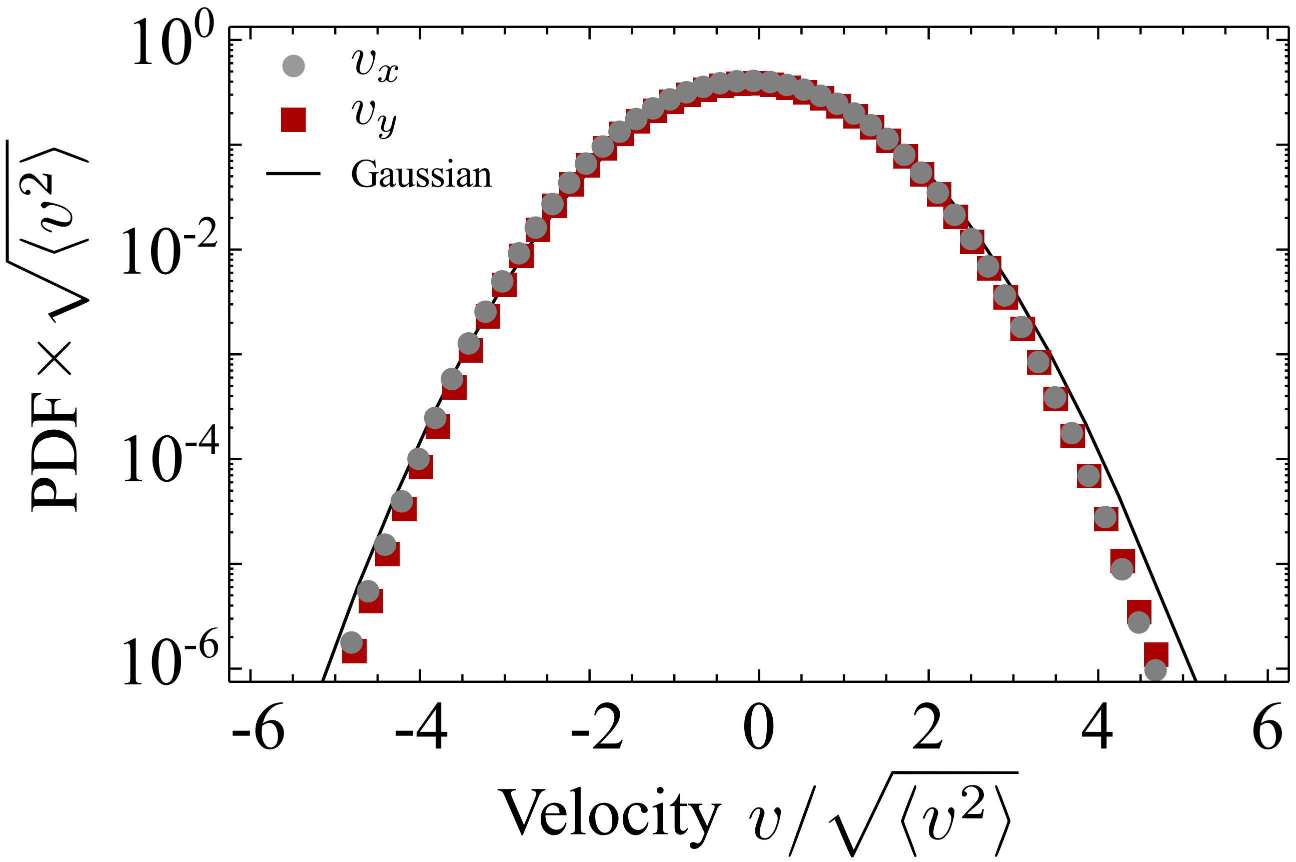


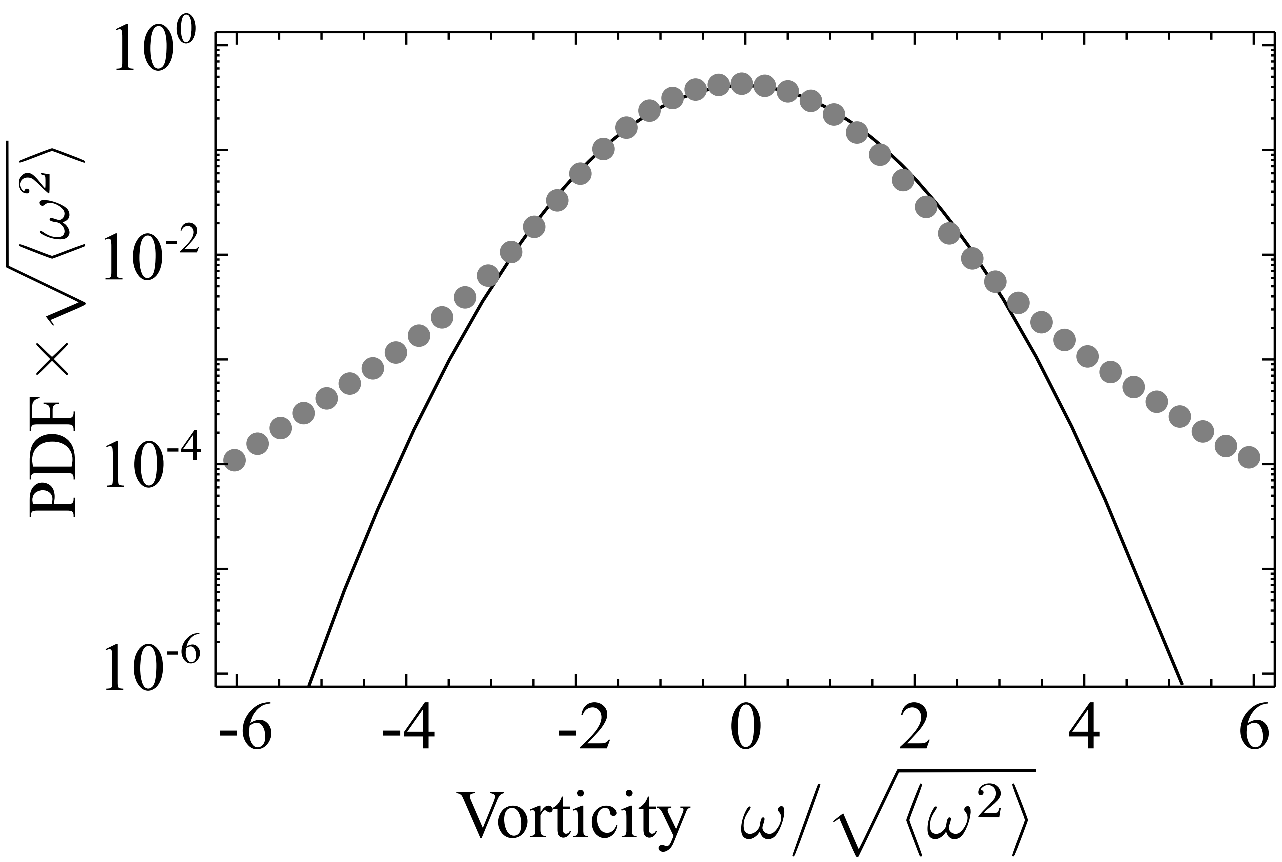
# Average vorticity

Average vorticity of a individual vortices: $\omega_v \approx \alpha/\eta$

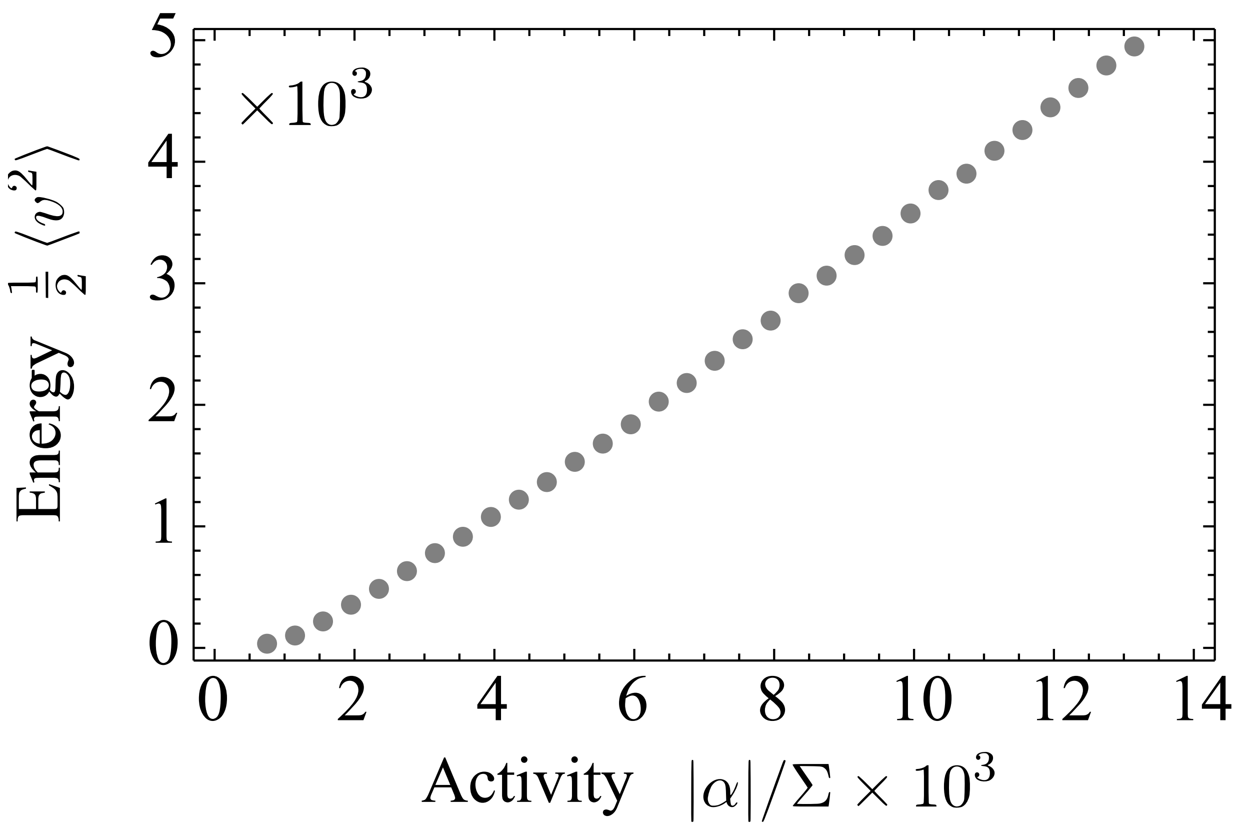


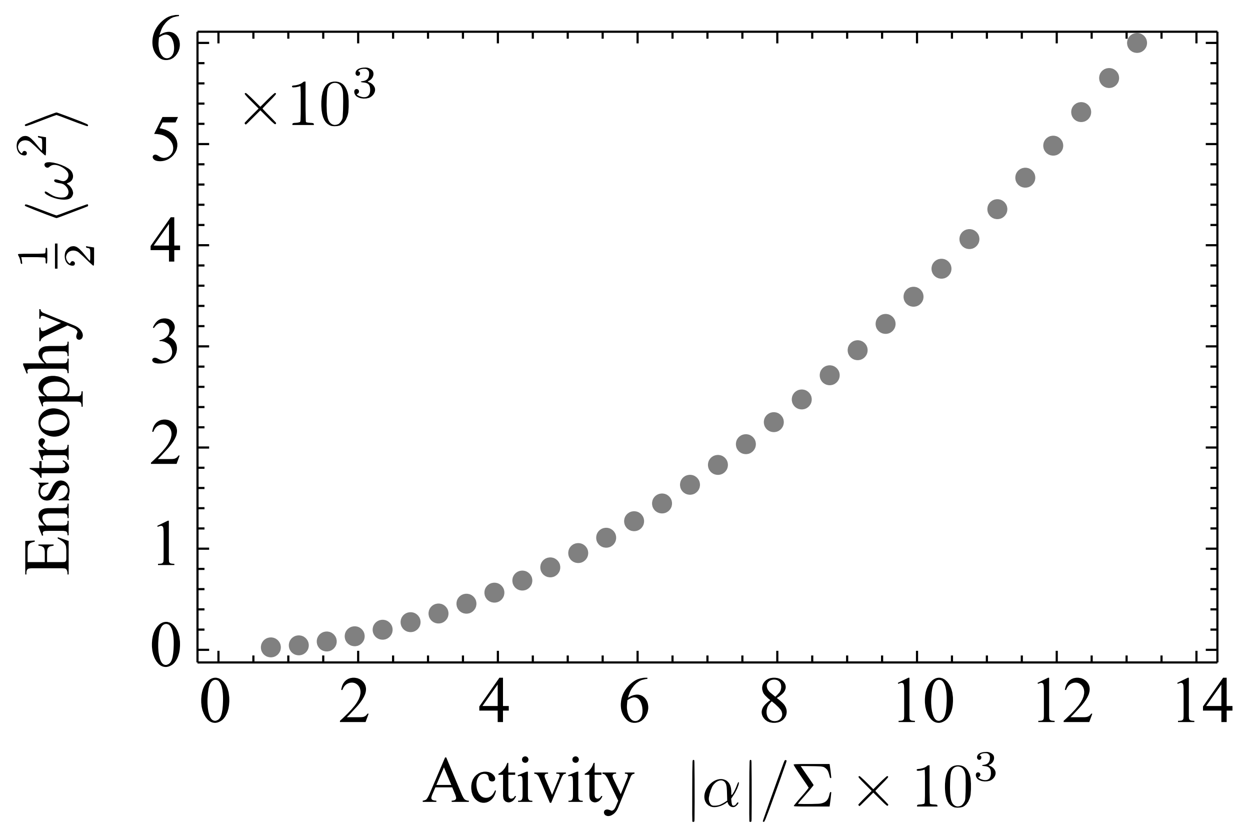
# Velocity and vorticity PDF



# Energy and enstrophy




Notice that for active laminar flows: $v \sim \alpha$

# Energy and enstrophy

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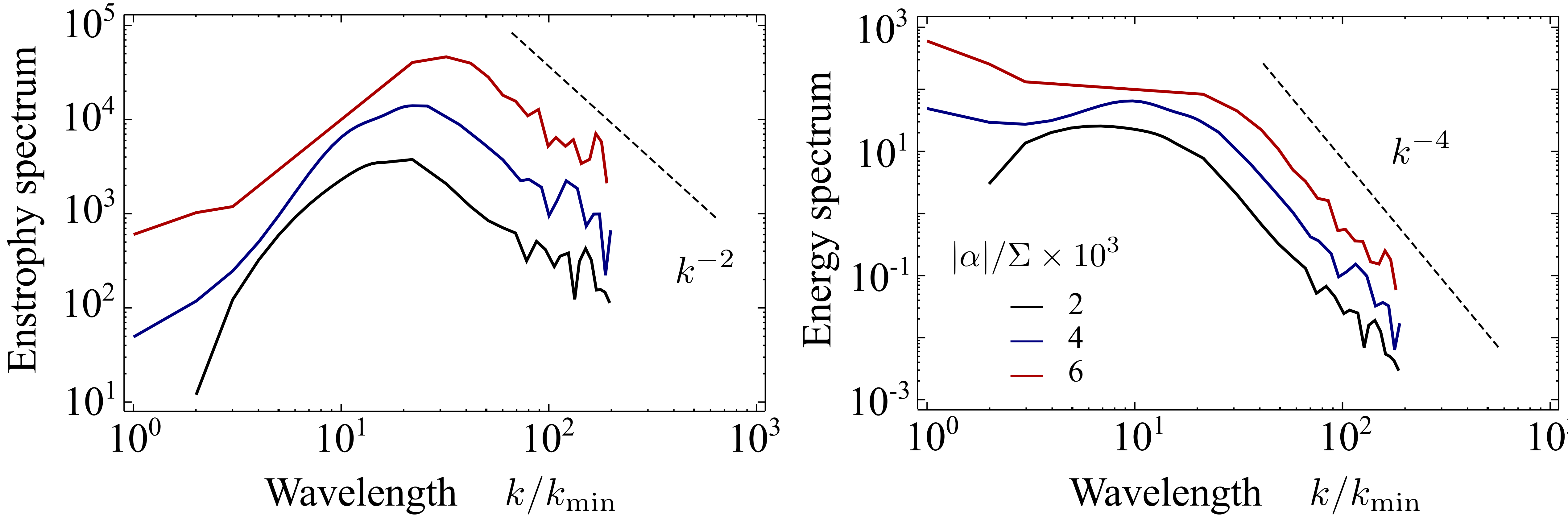
Upon increasing the activity, the vortices becomes faster but smaller.

# Energy and enstrophy spectra




For 2D inertial turbulence $\Omega(k) \sim k^{-1},\ E(k) \sim k^{-3}$

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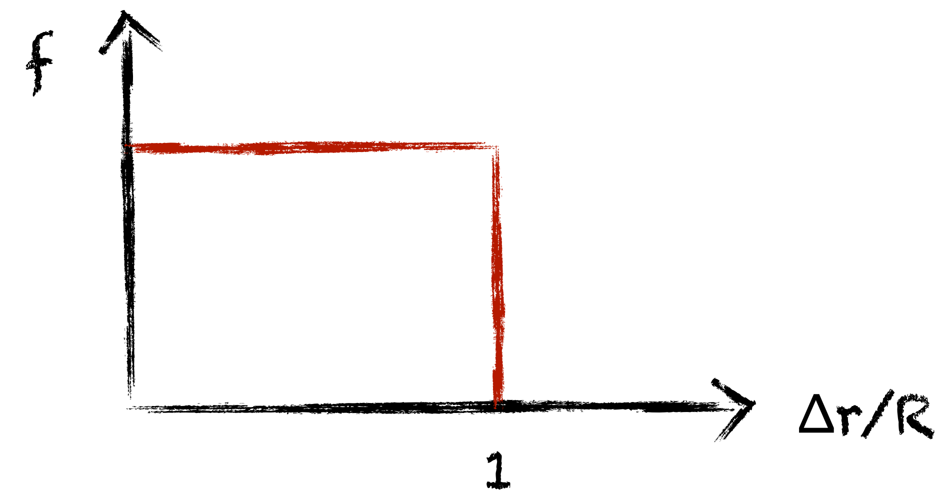

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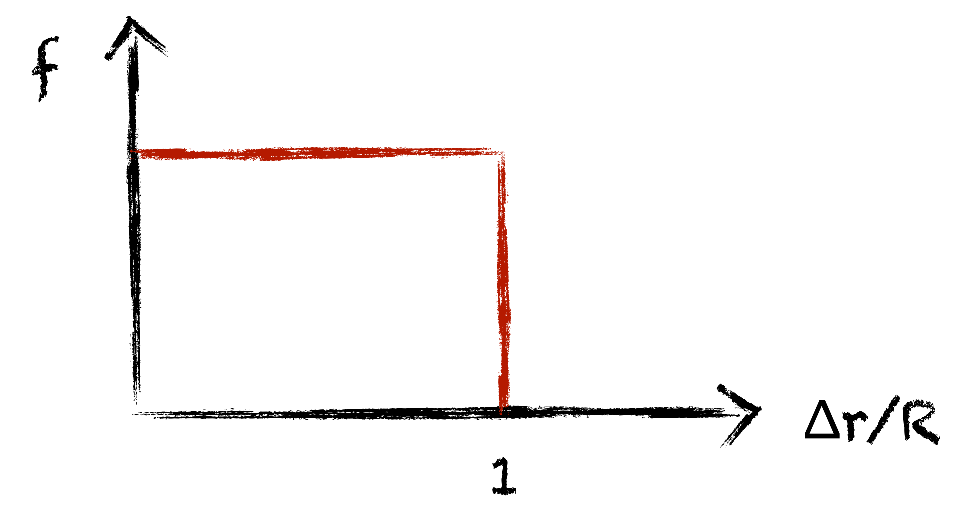


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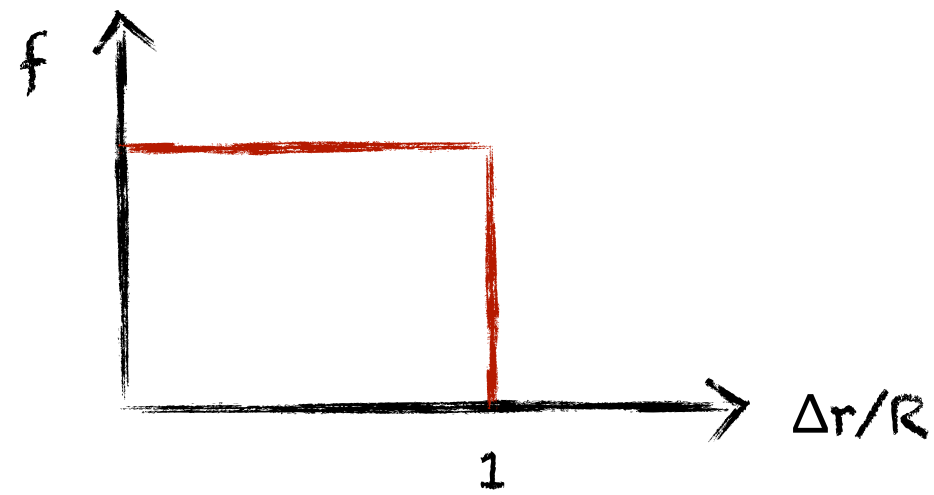


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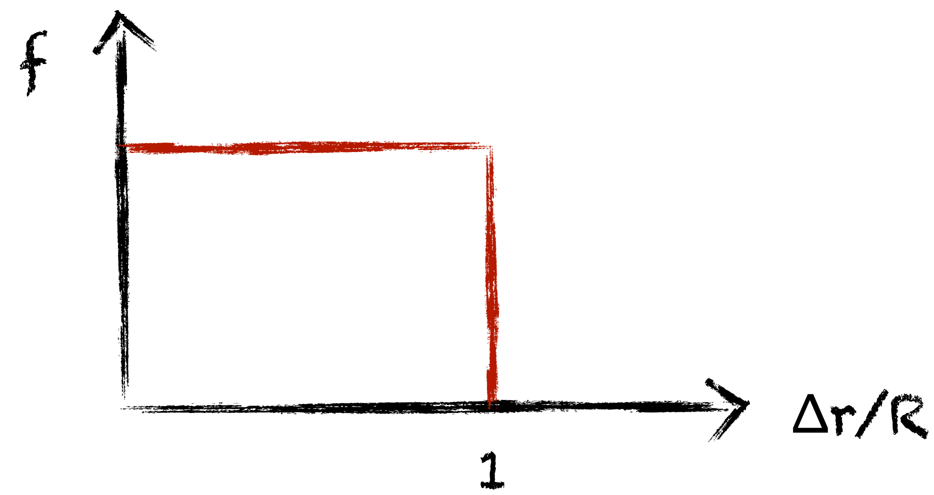


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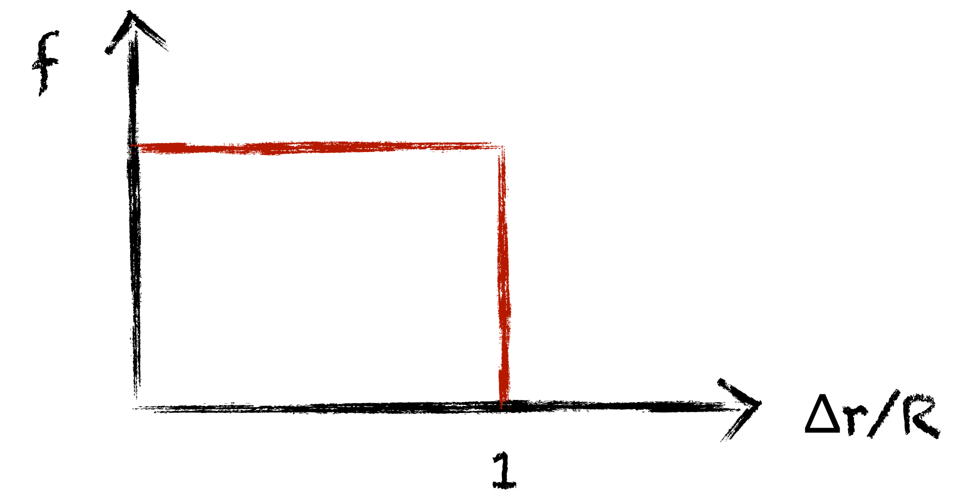


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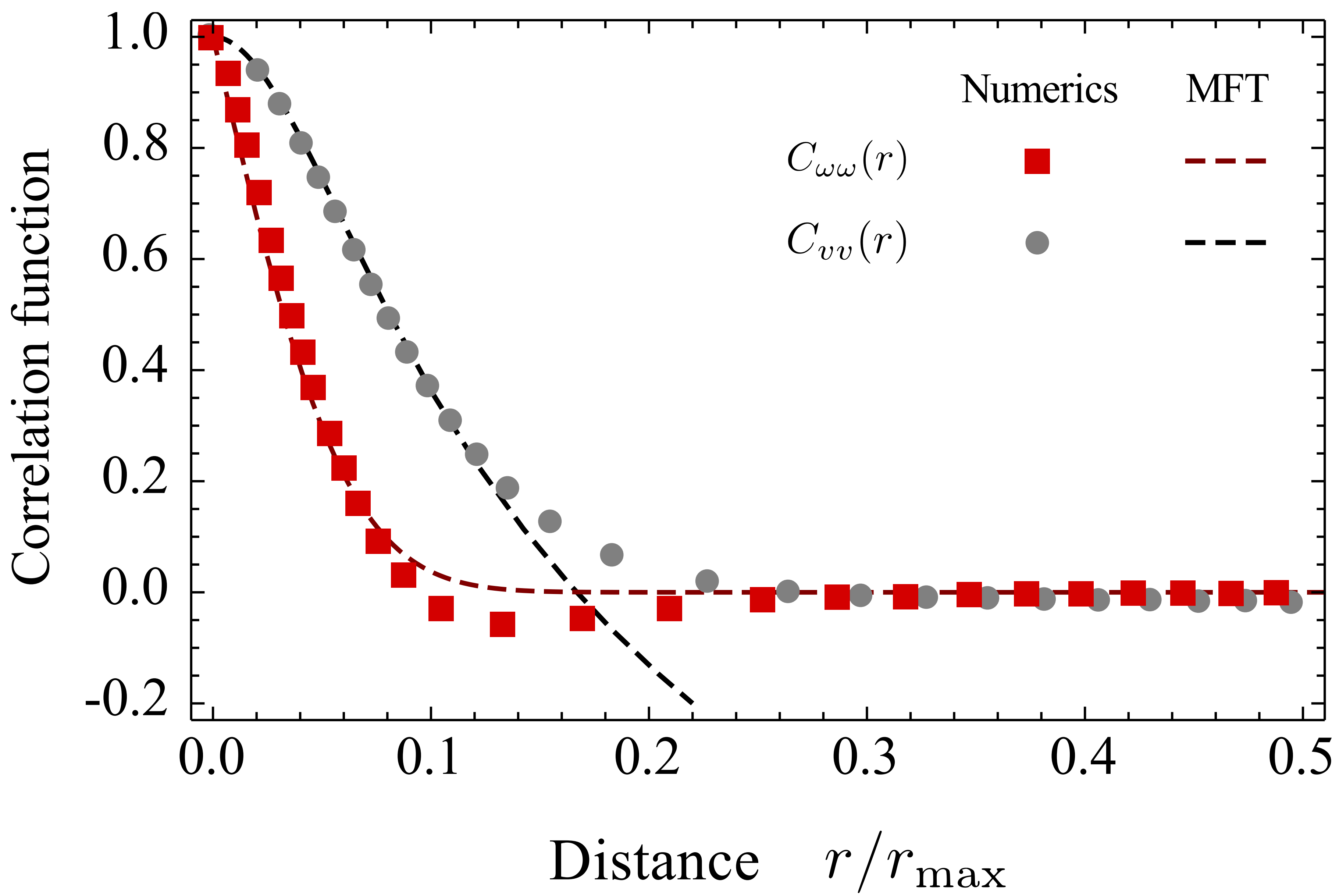
$$E(\kappa) = \kappa^{-2}\,\Omega(\kappa) \sim \kappa^{-4}$$

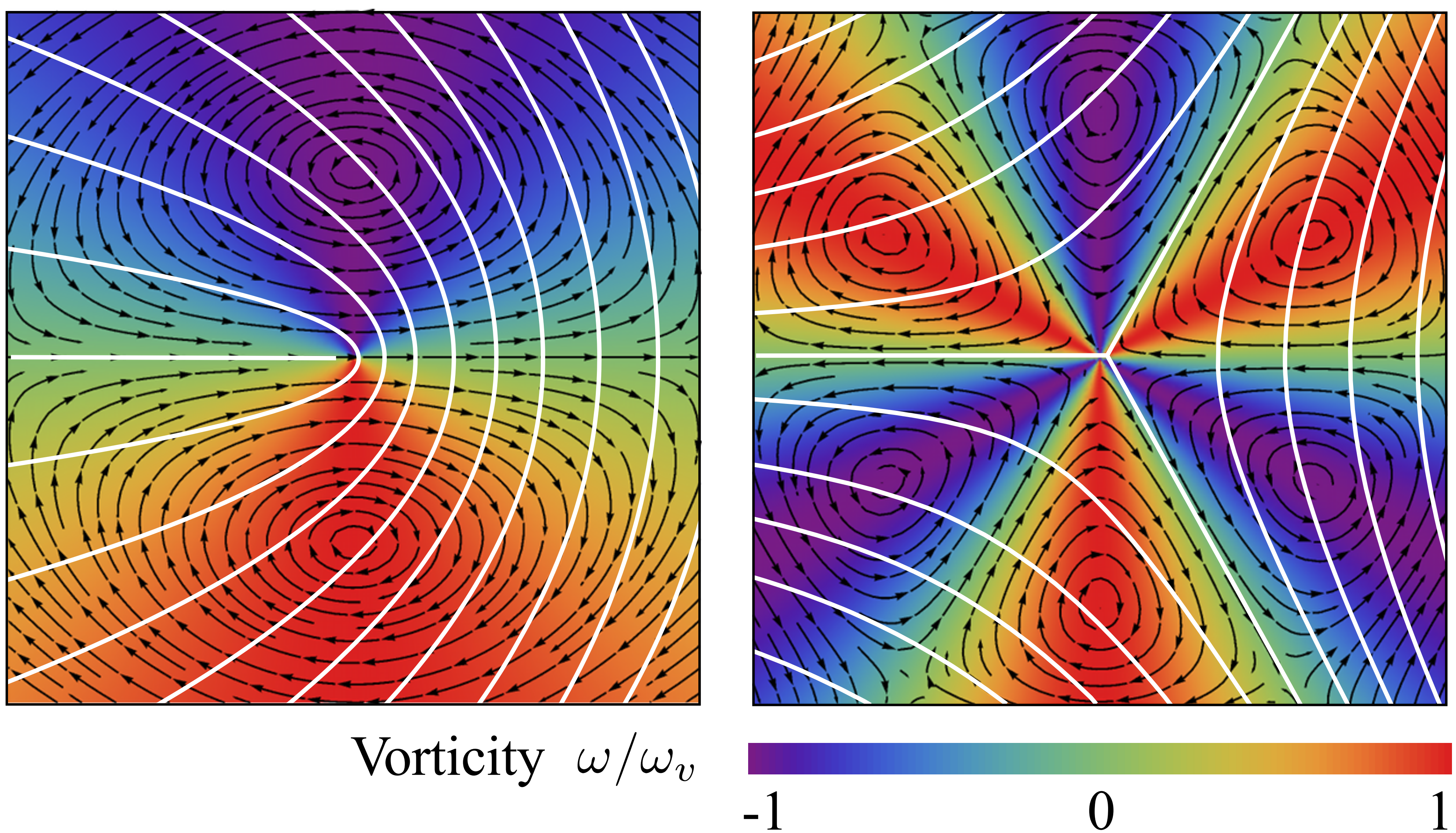
# Correlation function



# Topological defects

The local vortex geometry is intrinsically coupled with the topological structure of the director. Thus, topological defects have the same statistics of active vortices.

$$\eta \nabla^2 \boldsymbol{v} - \nabla p + \alpha \nabla \cdot \boldsymbol{Q} = 0$$



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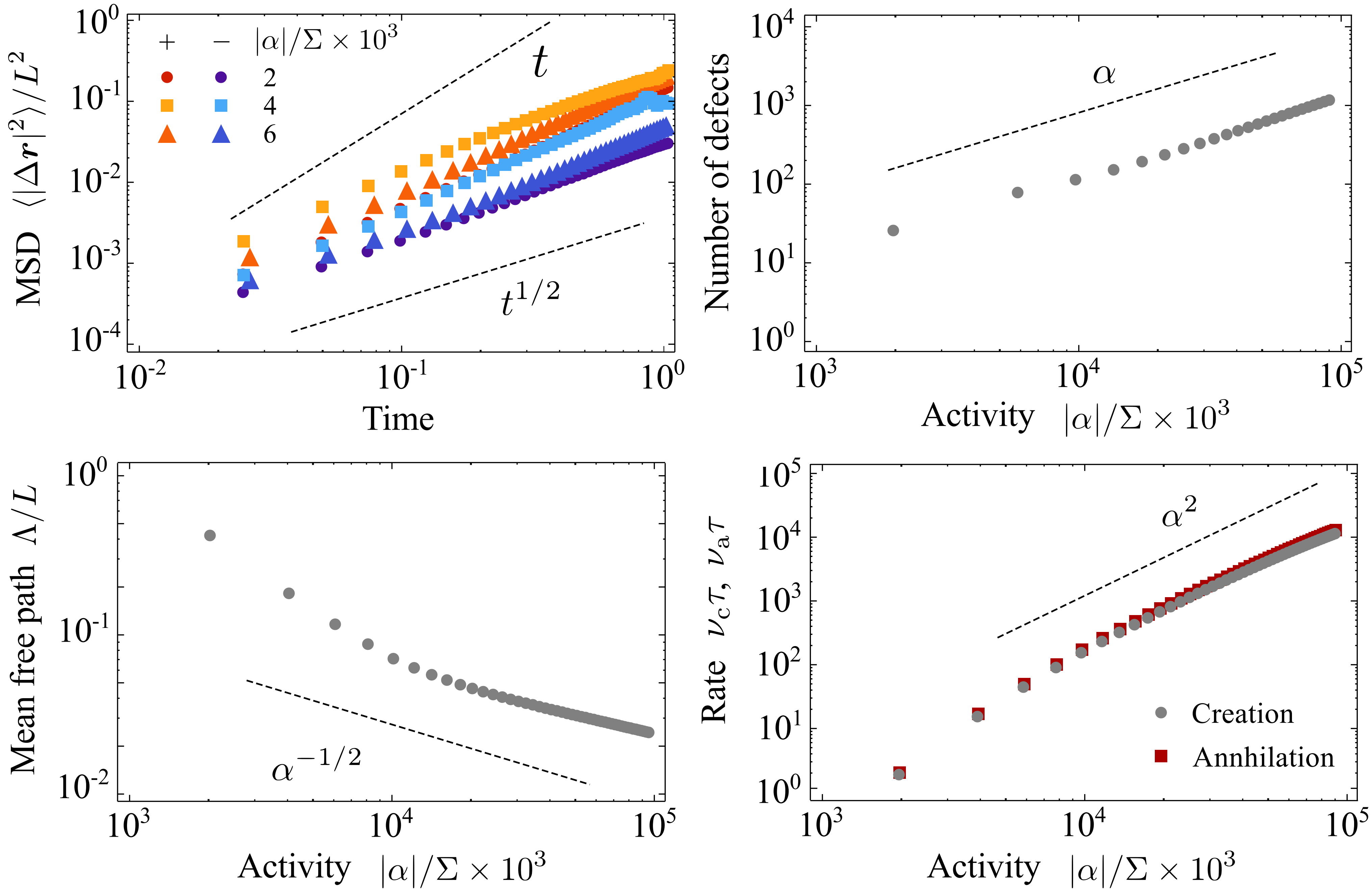
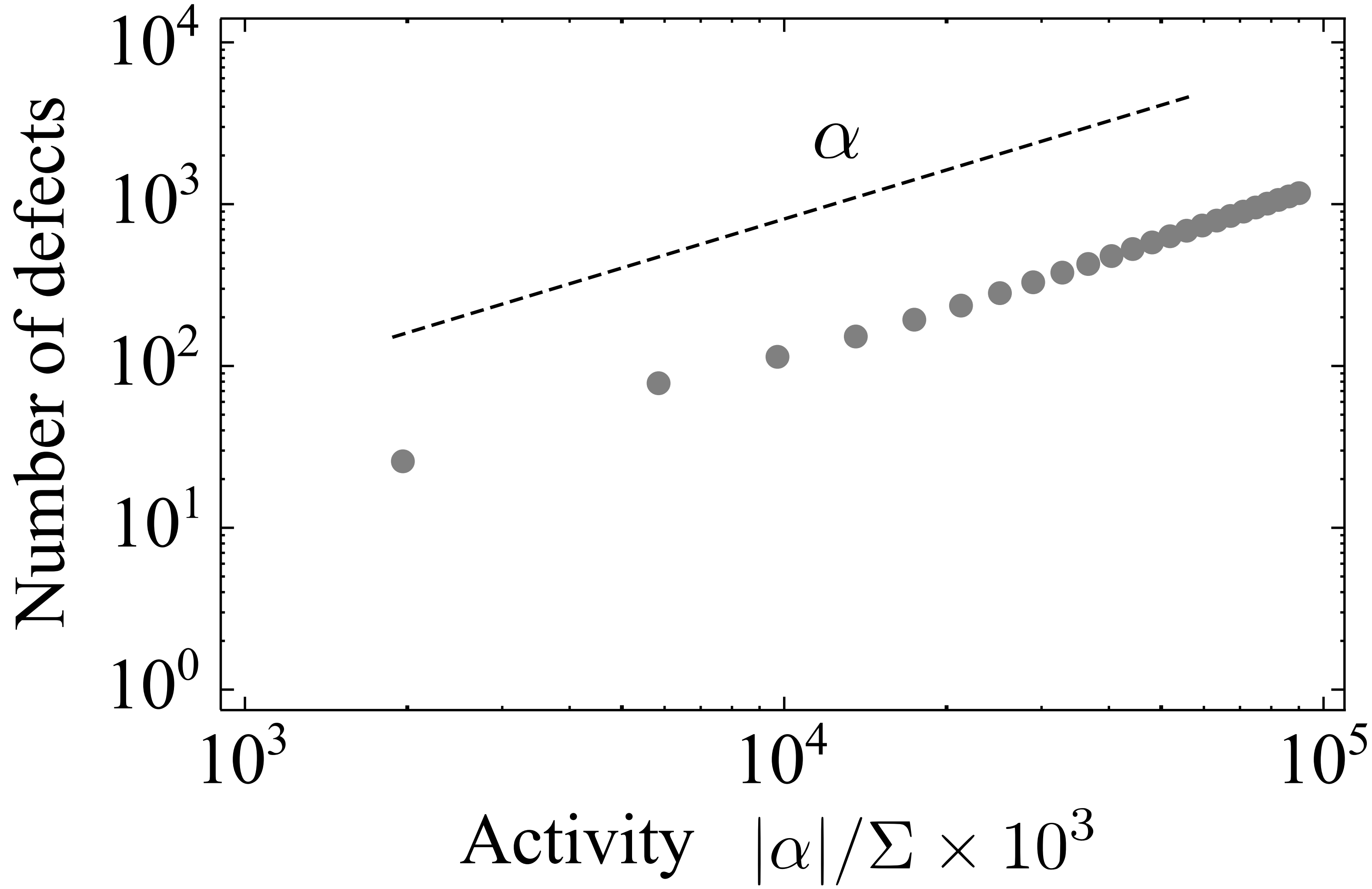


# Defects density

$$N_{\text{defects}} \sim N_{\text{vortices}} \sim \ell_{\text{a}}^{-2} \sim \alpha$$



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Thanks!

http://wwwhome.lorentz.leidenuniv.nl/~giomi