Gauge Choice in spacetime

Maximal slice asymptotic to linear space at spatial infinity.

(After gauge fixing at infinity, this is unique)

On this slice, one wants to figure out all apparent horizons.

(These are closed surfaces so that its mean curvature is equal to ± of trace of extrinsic curvature tensor (p_{ij}) on the surface)

Procedure to find apparent horizon

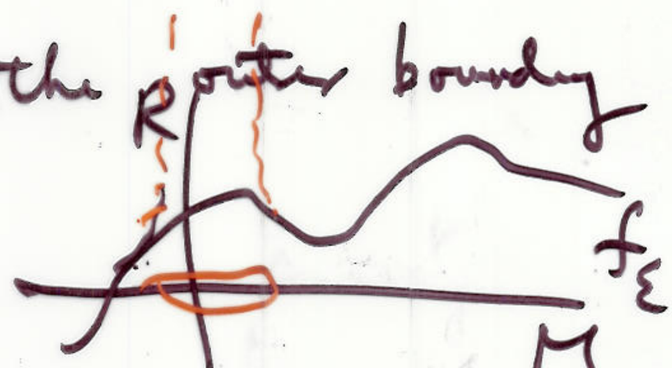
We want to find a function f

so that

$$\sum\left(g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2}\right)\left(\frac{f_{ij}}{\sqrt{1+|\nabla f|^2}} - p_{ij}\right) = \varepsilon f$$

when $f = 0$ at the outer boundary

(or at infinity)



We prove that f exists

in a neighborhood at infinity. But

along some closed surfaces, it

may go to $+\infty$ or $-\infty$. These

surfaces are apparent horizon,

(How do we choose gauge at spatial infinity?

Huisken-Yau proved that There is unique foliation of surfaces Σ_t in a neighborhood of infinity.

Such surfaces have constant mean curvature. Such surfaces converge exponentially to the coordinate sphere at infinity. (The Hawking mass is positive on such surfaces)

So these surfaces give a canonical choice of gauge at spatial infinity.

Given a closed surface with prescribed $g_{ij}|_\Sigma$ + $p_{ij}|\Sigma$

Suppose $\Sigma \hookrightarrow M^3$ with given metric g_{ij}. It is possible to find a procedure on M^3 to find p_{ij} on M^3 that extends $p_{ij}|\Sigma$ and satisfies the constrain equation. This is related to the discussion of quasi-local mass.

Quasi-local Conserved Quantities in General Relativity

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In order to help to find an efficient numerical calculation, I think it is important to understand conserved (or quasi-conserved) quantities that have physical relevance.

Conserved quantities: Mass, momentum, angular momentum.

Symmetries of spacetime give these quantities.

Asymptotic flat spacetime has asymptotic symmetries which give rise to global definitions of mass, momentum, and angular momentum.

However, for an object embedded into some other physical system, it is not trivial to define its mass.

Let us enclose the object by a surface Σ. Then we like to associate quasi-local mass to Σ. Such concept is important for understanding binding energy of the body, for example.

I am motivated by trying to understand another question.

Hoop Conjecture (K.Thorne)
(Formation of black hole)
A black hole forms when and only when a mass m gets compacted into a region whose circumference C in every direction is

$$C \leq 4\pi G m$$

Neither m nore C is clearly defined. Perhaps we are considerring a surface Σ bounding this region and we can define C to be

$$2\pi\sqrt{\frac{\text{Area}(\Sigma)}{4\pi}}.$$

Then we need quasi-local mass m?

In 1983, Schoen-Yau proved that in a region Ω, the matter density

$$\mu - |J| \geq \frac{3\pi^2}{2(\text{Rad}(\Omega))^2}$$

then an apparent horizon forms inside Ω.

In the theorem of Schoen-Yau, only matter density is used. In 2001, Yau, "Geometry of three manifolds and existence of black hole", ATMP, I observed that norm of mean curvature of Σ in spacetime can be used to create a mass to force black hole to form.

Hence I am interested in defining quasi-local mass using the norm of mean curvature. Later we found that Brown-York had similar definitions which are better motivated by Hamiltonian formulation.

We think that the quasi-local mass we define should be used to settle the hoop conjecture.

Quasi-local energy-momentum vector

A *spacetime* is a 4-manifold with a pseudo-metric of signature $(+,+,+,-)$. A hypersurface or a 2-surface in a spacetime is *spacelike* if the induced metric is positive definite. *Quasi-local energy-momentum vector* is a vector in $\mathbb{R}^{3,1}$ associated to a spacelike 2-surface which depends on its first and second fundamental forms, and the connection on its normal bundle in the spacetime. The time component of the four vector is called *quasi-local mass*.

We require the quasi-local energy-momentum vector to satisfy the following properties.

1. It should be zero for the flat spacetime, at least when the spacelike 2-surface lies in a spacelike hyperplane.

2. The quasi-local mass should be equivalent to the standard definition if the spacetime is spherically symmetric and quasi-local mass is evaluated on the round spheres. (We say two masses m_1 and m_2 are equivalent if there is a universal constant $c > 0$ such that $c^{-1}m_1 \leq m_2 \leq cm_1$.) In particular, for the centered spheres in the Schwarzschild spacetime, the quasi-local mass should be equivalent to the standard mass.

3. For an asymptotically flat slice, the quasi-local mass of the coordinate sphere should be asymptotic to the ADM energy-momentum vector.

4. For an asymptotically null slice, the quasi-local mass of the coordinate sphere should be asymptotic to the Bondi energy-momentum vector.

5. For an apparent horizon Σ, the quasi-local mass should be no less than a (universal) constant multiple of the irreducible mass which is $\sqrt{\text{Area}(\Sigma)/16\pi}$.

6. The quasi-local energy-momentum vector should be non-spacelike and the quasi-local mass should be non-negative.

Hawking mass

Let Σ be a spacelike 2-surface in a spacetime N. At each point of Σ, choose two null normals l, n such that $\langle l, n\rangle = -1$. Any other choice (l', n') is related to (l, n) by $l' = \lambda l, n' = \lambda^{-1} n$ or $l' = \lambda n, n' = \lambda^{-1} l$ for some function $\lambda : \Sigma \to \mathbb{R} \setminus \{0\}$. We denote the mean curvature with respect to l and n by

$$\begin{aligned} 2\rho &= -\langle \nabla_1 e_1 + \nabla_2 e_2, l\rangle, \\ -2\mu &= -\langle \nabla_1 e_1 + \nabla_2 e_2, n\rangle, \end{aligned}$$

respectively, where $\{e_1, e_2\}$ is a local orthonormal frame of Σ. The definitions of ρ and μ depend on choice of (l, n), but their product $\rho\mu$ is independent of choice of (l, n). More intrinsically,

$$8\rho\mu = \langle \mathbf{H}, \mathbf{H}\rangle$$

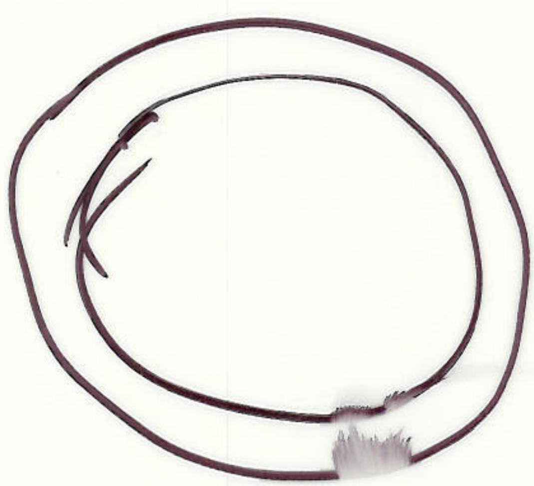
where $\mathbf{H}$ is the mean curvature vector of Σ in N.

Hawking mass is defined to be

$$\sqrt{\frac{\text{Area}(\Sigma)}{16\pi G^2}}\left(1+\frac{1}{2\pi}\int_\Sigma \rho\mu\right).$$

This mass is not positive in general. However, based on my works on estimation of eigenvalues of Laplacian on Riemann surfaces, Christodoulou and I proved that it is positive if Σ is a round sphere in a maximal slice.

"Round" means that it minimizes area among all the surfaces nearby which encloses the same volume in the maximal slice.



The Hawking mass is observed by Geroch to be monotone if we flow the surface Σ along the inverse mean curvature flow.

This enables Huisken-Ilmanen to prove the Penrose inequality (that ADM mass is not less than the irreducible mass of the apparent horizon) in the time symmetric case. Bray generalized this theorem, using different argument.

Definition of quasi-local mass

Our definition of quasi-local mass arises naturally from calculations in the works of Shoen-Yau and my recent work on blackholes, and is strongly motivated by our ability to prove its positivity. After I proposed our definition, we were informed of the existence of much earlier works by Brown-York and others (Lau, Kijowski, Epp).

We assume that $\rho\mu > 0$, or equivalently, the mean curvature vector $\mathbf{H}$ of Σ in N is spacelike. We also assume that Σ has positive Gaussian curvature so that Σ is topologically a 2-sphere. By Weyl's embedding theorem, Σ can be isometrically embedded into the Euclidean space $\mathbb{R}^3$ so that the second fundamental form $(H_0)_{ab}$ is positive definite.

The embedding $\Sigma \subset \mathbb{R}^3$ is unique up to an isometry of $\mathbb{R}^3$, so $(H_0)_{ab}$ is determined by the metric on Σ. Let ρ_0, μ_0 be the mean curvatures with respect to null normals l_0, n_0 of the embedding $\Sigma \subset \mathbb{R}^3 \subset \mathbb{R}^{3,1}$, with the normalization $\langle l_0, n_0 \rangle = -1$. Then

$$8\rho_0\mu_0 = H_0^2$$

where $H_0 > 0$ is the trace of $(H_0)_{ab}$. Define the *quasi-local mass* of Σ to be

$$\begin{aligned} E(\Sigma) &= \frac{1}{8\pi G}\int_\Sigma (\sqrt{8\rho_0\mu_0} - \sqrt{8\rho\mu}) \\ &= \frac{1}{8\pi G}\int_\Sigma (H_0 - \sqrt{8\rho\mu}) \end{aligned}$$

Properties of quasi-local mass

Recall (1)–(6).

For (1), Murchadha, Szabados, and Tod gave examples of $\Sigma \subset \mathbb{R}^{3,1}$ but $E(\Sigma) > 0$. In these examples, Σ does not lie in a spacelike hyperplane.

For (2), recall that the Schwarzschild spacetime metric on $\mathbb{R}^4$ is given by

$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

where $r > 2M$, r, θ, ϕ are the spherical coordinates on $\mathbb{R}^3$. Let $S_a \subset (\mathbb{R}^4, g)$ be the round sphere defined by $t = 0, r = a$, and let $m(r) = E(S_r)$. Then

$$m(r) = r\left(1-\sqrt{1-\frac{2M}{r}}\right).$$

Note $m(r)$ is decreasing (for $r \geq 2M$), $m(2M) = 2M$, and $m(\infty) = M$, which is consistent with (2).

For (5), on an apparent horizon Σ we have $\rho\mu = 0$, so

$$E(\Sigma) = \frac{1}{8\pi}\int_{\Sigma} H_0 \geq \sqrt{\frac{\text{Area}(\Sigma)}{4\pi}}$$

by Minkowski inequality of convex bodies. Therefore, $E(\Sigma)$ satisfies (5). By the main result of this paper, $E(\Sigma)$ is nonnegative as required in (6) and is strictly positive when the spacetime is not flat along Σ.

Positivity of quasi-local mass

Let Ω be a compact spacelike hypersurface in a time orientable four dimensional spacetime N. Let g_{ij} denote the induced metric on Ω, and let p_{ij} denote the second fundamental form of Ω in N.

The local mass density μ and the local current density J^i on Ω are related to g_{ij} and p_{ij} by the constraint equations

$$\begin{aligned}
\mu &= \frac{1}{2}\left(R - \sum_{i,j} p^{ij} p_{ij} + (\sum_i p_i^i)^2\right) \\
J^i &= \sum_j D_j \left(p^{ij} - (\sum_k p_k^k) g^{ij}\right),
\end{aligned}$$

where R is the scalar curvature of the metric g_{ij}.

Theorem (Positivity of quasi-local mass) Let Ω, μ, J be as above. We assume that μ and J^i satisfies the local energy condition

$$\mu \geq \sqrt{J^i J_i}$$

and the boundary $\partial\Omega$ has finitely many connected components $\Sigma^1, \ldots, \Sigma^\ell$, each of which has positive Gaussian curvature and has space-like mean curvature vector in N.

Let $E(\Sigma^\alpha)$ be the quasi-local energy defined as above. Then $E(\Sigma^\alpha) \geq 0$ for $\alpha = 1, \ldots, \ell$. Moreover, if $E(\Sigma^\alpha) = 0$ for some α, then M is flat spacetime along Ω, $\partial\Omega$ is connected and will be embedded into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ by the well-known Weyl embedding theorem.

Riemannian case

When the second fundamental form of Ω in N vanishes, the local energy condition reduces to $R \geq 0$ and the condition $\rho\mu > 0$ reduces to $H > 0$, where H is the mean curvature of the spacelike 2-surface in Ω with respect to the outward unit normal. Shi and Tam proved positivity of quasi-local mass in this case.

Theorem (Shi-Tam) Let (Ω^3, g) be a compact manifold of dimension three with smooth boundary and with nonnegative scalar curvature. Suppose $\partial\Omega$ has finitely many connected components Σ^α so that each connected component has positive Gaussian curvature and positive mean curvature H with respect to the unit outward normal. Then for each boundary component Σ^α,

$$\int_{\Sigma^\alpha} H d\sigma \leq \int_{\Sigma^\alpha} H_0^\alpha d\sigma$$

where H_0^α is the mean curvature of Σ^α with respect to the outward normal when it is isometrically embedded in $\mathbb{R}^3$, $d\sigma$ is the volume form on Σ^α induced from g. Moreover, if equality holds in for some Σ^α, then $\partial\Omega$ has only one connected component and Ω is a domain in $\mathbb{R}^3$.

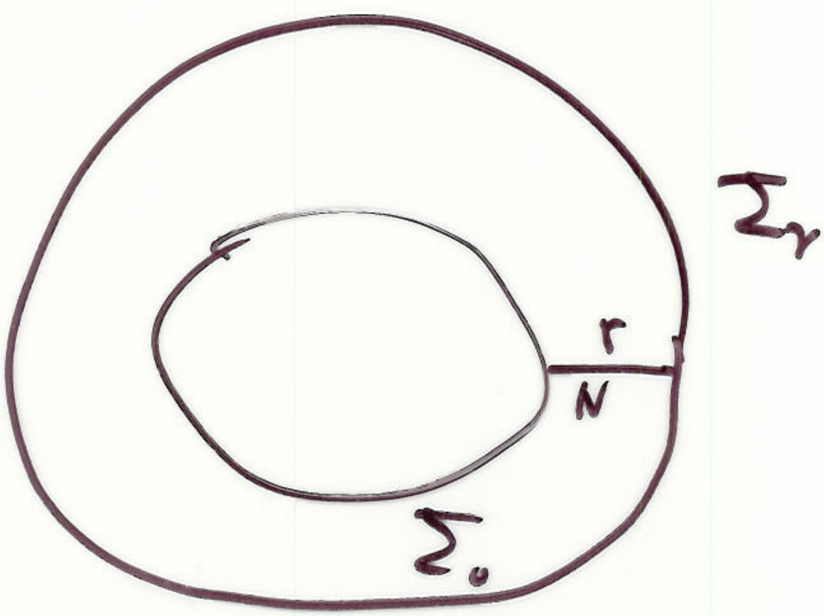
Proof of the Riemannian case

Let Σ^α be a connected component of $\partial\Omega$. By hypothesis, it has positive Gaussian curvature, so it can be isometrically embedded to $\mathbb{R}^3$ by the well-known Weyl embedding theorem. Moreover, the embedding is unique up to an isometry of $\mathbb{R}^3$. Let $\Sigma_0^\alpha \subset \mathbb{R}^3$ be the image of such an embedding. Then Σ_0^α is a strictly convex hypersurface diffeomorphic to S^2.

Let $\mathbf{X}$ be the position vector of a point on Σ_0^α, and let $\mathbf{N}$ be the unit outward normal of Σ_0^α at $\mathbf{X}$. Let Σ_r^α be the surface described by $\mathbf{Y} = \mathbf{X} + r\mathbf{N}$, with $r \geq 0$. Let D^α be the region of $\mathbb{R}^3$ outside Σ_0^α, and let $E^\alpha = D^\alpha \cup \Sigma_0^\alpha$ be the closure of D^α in $\mathbb{R}^3$. Then E^α can be represented by

$$(\Sigma^\alpha \times [0, \infty), g^0 = dr^2 + g_r),$$

where g_r is the induced metric on Σ_r^α, $g^0 = dr^2 + g_r$ is the standard Euclidean metric on $E^\alpha \subset \mathbb{R}^3$. Note that $g_r = (a + r)^2(d\theta^2 + \sin^2\theta d\phi^2)$ if Σ_0^α is a round sphere of radius $a > 0$.



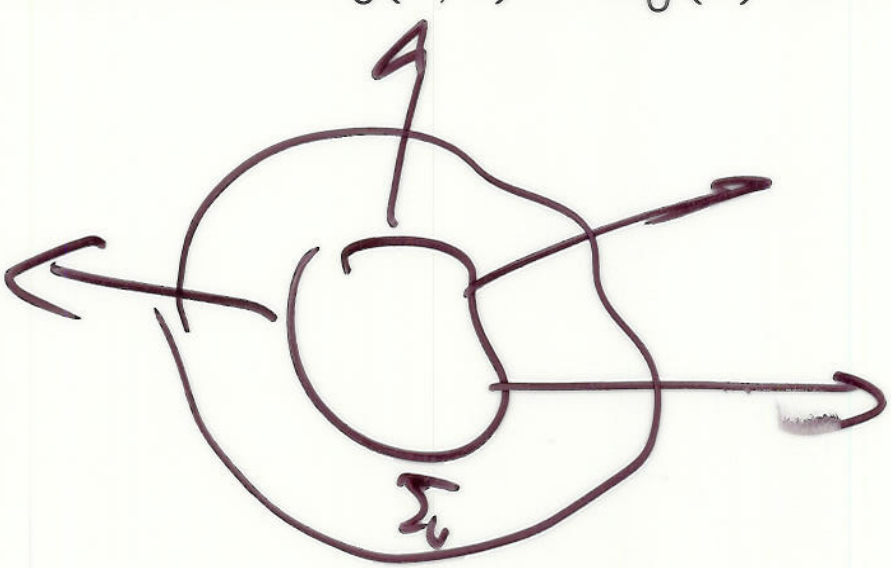
Consider a Riemannian metric on E^α of the form

$$g = h^2 dr^2 + g_r.$$

where h is a smooth positive function. This is a special case of Bartnik's construction. Note that g and g^0 induce the same metric on each Σ_r^α. The mean curvature H and H_0 of Σ_r^α with respect to g and g^0 are related by

$$H = h^{-1} H^0.$$

Note that $H_0(x, 0) = H_0^\alpha(x)$ for $x \in \Sigma_0^\alpha \cong \Sigma^\alpha$.



The scalar curvature R of g is given by

$$2H_0\frac{\partial h}{\partial r} = 2h^2\Delta_r h + (h - h^3)R^r + h^3 R,$$

where R^r is the scalar curvature of Σ_r^α, and Δ_r is the Laplacian operator on Σ_r^α. So a solution to the parabolic partial differential equation

$$2H_0\frac{\partial h}{\partial r} = 2h^2\Delta_r h + (h - h^3)R^r \qquad (1)$$

on $E^\alpha \cong \Sigma^\alpha \times [0, \infty)$ with the initial condition

$$h(x, 0) = \frac{H_0^\alpha}{H}. \qquad (2)$$

defines a metric on E^α such that the scalar curvature $R = 0$ and the mean curvature of Σ_0^α coincides with the restriction of H to $\Sigma^\alpha \cong \Sigma_0^\alpha$.

Equation (1) with the initial condition (2) has a unique solution such that

(a) $h = 1 + m_0\rho^{-1} + \kappa$, where m_0 is a constant and the function κ satisfies

$$|\kappa| = O(\rho^{-2}), \quad |\nabla_0 \kappa| = O(\rho^{-3}),$$

where ∇_0 is Levi-Civita connection of the Euclidean metric on $\mathbb{R}^3$.

(b) The metric $g^\alpha = h^2 dr^2 + g_r$ on E^α is asymptotically flat with zero scalar curvature.

(c) The ADM mass of (E^α, g^α) is given by

$$m^\alpha_\infty = \lim_{r\to\infty} m^\alpha(r),$$

where $m^\alpha(r) = E(\Sigma^\alpha_r)$ is the quasi-local mass of Σ^α_r.

The function $m^\alpha(r)$ satisfies the following monotonicity formula:

$$\frac{dm^\alpha}{dr}(r) = \frac{-1}{16\pi G}\int_{\Sigma_r^\alpha} R^r u^{-1}(1-u)^2 \leq 0.$$

So

$$E(\Sigma^\alpha) = m^\alpha(0) \geq m_\infty^\alpha.$$

Gluing (E^α, g^α) to (Ω, g) along Σ^α, one obtains a complete noncompact 3-manifold M with a continuous Riemannian metric $\tilde{g}$ such that

1. $\tilde{g}$ is smooth on $M \setminus \Omega$ and $\bar{\Omega}$, and is Lipschitz near $\partial\Omega$.

2. The mean curvatures of Σ^α w.r.t. $g = \tilde{g}|_\Omega$ and $g^\alpha = \tilde{g}|_{E^\alpha}$ are the same for each α.

3. Each end E^α of M is asymptotically Euclidean.

4. The scalar curvature R of $M \setminus \partial\Omega$ is nonnegative and is in $L^1(M)$.

Using Witten's argument, Shi and Tam proved that the positive mass theorem holds for such a metric, so the ADM mass

$$m_\infty^\alpha = \lim_{r \to \infty} m^\alpha(r)$$

is nonnegative for each end E^α, and m_∞^α vanishes for some α if and only if M has only one end and M is flat.

Proof of the general case

Let (Ω, g_{ij}, p_{ij}) and $\Sigma^1, \ldots, \Sigma^\ell$ be as in the main Theorem (positivity of quasi-local mass).

We first deform the metric g_{ij} on Ω by a procedure used in the proof of the positive energy theorem of Schoen and myself, and also in my work on blackholes. This procedure consists of two steps. The first step is to deform g_{ij} to a new metric

$$\bar{g}_{ij} = g_{ij} + f_i f_j$$

where f is a solution to Jang's equation

$$\sum_{i,j=1}^{3} \left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2} \right) \left(\frac{f_{ij}}{\sqrt{1 + |\nabla f|^2}} - p_{ij} \right) = 0$$

on Ω with the Dirichlet boundary condition

$$f|_{\partial\Omega} \equiv 0.$$

Most of the estimates were made in my paper with Schoen. To solve the boundary value problem, I constructed a barrier and concluded that there exists a solution f to the Jang's equation with $f|_{\partial\Omega} = 0$ when (Ω, g_{ij}, p_{ij}) has no *apparent horizon*.

Definition 1 *Let (Ω, g_{ij}, p_{ij}) be an initial data set. Given a smooth compact surface S embedded in Ω, let H_s be the mean curvature of S with respect to the outward unit normal vector, and let P_s be the trace of the restriction of p_{ij} to S. A smooth 2-sphere S embedded in Ω is an* apparent horizon *of the initial data (Ω, g_{ij}, p_{ij}) if $H_s + P_s = 0$ or $H_s - P_s = 0$.*

We first assume that (Ω, g_{ij}, p_{ij}) has no apparent horizon so that there exists a solution f to the Jang's equation on Ω such that $f|_\Omega = 0$. The induced metric of the graph $\Omega_f \cong \Omega$ of f in $(\Omega \times \mathbb{R}, g_{ij}dx^i dx^j + dt^2)$ is

$$\bar{g}_{ij} = g_{ij} + f_i f_j$$

which can be viewed as a deformation of the metric g_{ij} on Ω. Note that the new metric $\bar{g}$ coincides with the old metric g when restricted to $\partial\Omega$.

Let $\bar{e}_4$ be the downward unit normal to Ω_f in $\Omega \times \mathbb{R}$, and let $\bar{e}_1, \bar{e}_2, \bar{e}_3$ be a local orthonormal frame of Ω. We define h_{i4} by

$$\bar{\nabla}_4 \bar{e}_4 = h_{i4} \bar{e}_i.$$

where $\bar{\nabla}$ denotes the Levi-Civita connection of the metric $g_{ij} dx^i dx^j + dt^2$ on $\Omega \times \mathbb{R}$. Let $h_{ij} = \langle \bar{e}_i, \bar{\nabla}_j \bar{e}_4 \rangle$ be the second fundamental form of Ω_f in $\Omega \times \mathbb{R}$. Let $\bar{R}$ be the scalar curvature of $\bar{g}$, and extend p_{ij}, μ, J^i parallely along the $\mathbb{R}$ factor. The following inequality was derived in Scheon-Yau:

$$\begin{aligned} 2(\mu - |J|) \ \le \ & \bar{R} - \sum_{i,j} (h_{ij} - p_{ij})^2 - 2(h_{i4} - p_{i4})^2 \\ & + 2 \sum_i D_i (h_{i4} - p_{i4}), \end{aligned}$$

where D_i denotes the covariant derivative of $\bar{g}$.

$\bar{g} = g_{ij} + f_i f_j$.

The second step is to deform $\bar{g}_{ij}$ conformally to a metric with zero scalar curvature.

Proposition 2 *Let $(\Omega, \bar{g})$ be a compact Riemannian manifold of dimension three with smooth boundary. Suppose that the scalar curvature $\bar{R}$ of $\bar{g}$ satisfies*

$$\bar{R} \geq c|X|^2 - 2\mathrm{div}X,$$

for some constant $c > \frac{1}{2}$ and some smooth vector field X on Ω. Then there is a unique metric $\hat{g}_{ij}$ on Ω such that

1. *The metric $\hat{g}_{ij}$ is conformal to $\bar{g}_{ij}$.*

2. *The scalar curvature of $\hat{g}_{ij}$ is zero.*

3. *The metric $\hat{g}_{ij}$ coincides with $\bar{g}_{ij}$ on $\partial\Omega$.*

In particular,

$$\bar{R} \geq 2|X|^2 - 2\mathrm{div}X, \tag{3}$$

where $X = \sum(h_{i4} - p_{i4})e_i$, and the divergence is defined by $\bar{g}$. The inequality (3) is an equality only if $p_{ij} = h_{ij}$.

In general, the solution f and the metric $\bar{g}$ are defined on Ω', the complement of union of apparent horizon, but one can extend $\bar{g}$ to a metric on Ω'' which is obtained by adding a point on each end of Ω'.

4. Let $\bar{H}$ and $\hat{H}$ denote the mean curvatures with respect the metric $\bar{g}$ and $\hat{g}$, respectively, and let $\bar{\nu}$ denote the outward unit normal of $\partial\Omega$ in $(\Omega, \bar{g})$. Then

$$\int_{\partial\Omega} \hat{H} \geq \int_{\partial\Omega} (\bar{H} - \langle X, \bar{\nu} \rangle),$$

where the equality holds if and only of $\bar{R} = 0$, $X = 0$, and $\hat{g}_{ij} = \bar{g}_{ij}$.

Lemma 3

$$\bar{H} - \langle X, \bar{\nu} \rangle \geq \sqrt{H^2 - P^2} \quad = \sqrt{\langle H, H \rangle}$$

Proof: Let $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ be a local orthonormal frame of $\Omega \times \mathbb{R}$ along the graph Ω_f so that $\bar{e}_1, \bar{e}_2$ is tangent to $\partial\Omega$ and $\bar{e}_3 = \bar{\nu}$. Let w be the outward unit normal of $\partial\Omega_0$ in Ω_0, the graph of the zero function. In my work on blackholes, I computed that

$$\bar{H} - \langle X, \bar{\nu} \rangle = -\frac{\langle \bar{e}_4, w \rangle}{\langle \bar{e}_3, w \rangle} P + \frac{1}{\langle e_3, w \rangle} H.$$

Recall that $H > 0$, so the above equation is equivalent to

$$(-\langle \bar{e}_4, w \rangle P + H)^2 \geq \langle \bar{e}_3, w \rangle^2 \left(H^2 - P^2 \right)$$

which is equivalent to

$$\begin{aligned} &(\langle e_4, w \rangle^2 + \langle e_3, w \rangle^2) P^2 - 2 \langle e_4, w \rangle P H \\ &+ (1 - \langle e_3, w \rangle^2) H^2 \geq 0 \end{aligned}$$

But $|w|^2 = 1 = \langle e_4, w \rangle^2 + \langle e_3, w \rangle^2$, this inequality holds trivially.

Let $\Sigma^\alpha, \Sigma^\alpha_r, E^\alpha, g^\alpha = h^2 dr^2 + g_r$ be defined as in Shi-Tam's proof. There is a unique solution to

$$2H_0 \frac{\partial h}{\partial r} = 2h^2 \Delta_r h + (h - h^3) R^r$$

on $E^\alpha \cong \Sigma^\alpha \times [0, \infty)$ with the initial condition

$$h(x, 0) = \frac{H_0^\alpha}{\bar{H} - \langle X, \bar{\nu} \rangle}$$

such that

$$|h(x, r) - 1| \leq \frac{C}{r}$$

for $r \geq 1$. Equip E^α with the metric $g^\alpha = h^2 dr^2 + g_r$. Then g^α has zero scalar curvature, and the mean curvature of Σ^α in (E^α, g^α) is $\bar{H} - \langle X, \bar{\nu} \rangle$.

Let

$$m^\alpha(r) = \frac{1}{8\pi G}\int_{\Sigma_r^\alpha}(H_0 - H)d\sigma_r$$

be the quasi-local mass of $E(\Sigma_r^\alpha)$, and let

$$m_\infty^\alpha = \lim_{r\to\infty} m^\alpha(r)$$

be the ADM mass of the asymptotically flat end E^α. By monotonicity formula of $m^\alpha(r)$, $m_\infty^\alpha \le m^\alpha(0)$. By Lemma 3,

$$\begin{aligned} m^\alpha(0) &= \frac{1}{8\pi G}\int_\Sigma (H_0^\alpha - (\bar{H} - \langle X, \bar{\nu}\rangle))d\sigma \\ &\le \frac{1}{8\pi G}\int_\Sigma (H_0^\alpha - \sqrt{H^2 - P^2})d\sigma = E(\Sigma^\alpha). \end{aligned}$$

Gluing (E^α, g^α) to $(\Omega, \hat{g})$ along Σ^α, one obtains a complete noncompact 3-manifold M with a continuous Riemannian metric $\tilde{g}$ such that

1. $\tilde{g}$ is smooth on $M \setminus \Omega$ and $\bar{\Omega}$, and is Lipschitz near $\partial\Omega$.

2. Each end E^α of M is asymptotically Euclidean.

3. The scalar curvature R of $M \setminus \partial\Omega$ is nonnegative and is in $L^1(M)$.

The mean curvatures of Σ with respect to $\hat{g} = \tilde{g}|_\Omega$ is $\hat{H} = \bar{H} + 4\bar{\nu}(u)$, and the mean curvature of Σ with respect to $g^\alpha = \tilde{g}|_{E^\alpha}$ is $\bar{H} - \langle X, \bar{\nu}\rangle$. They are not necessarily the same.

This causes the following problem which is absent in the case considered by Shi and Tam: the zeroth order term of the Dirac operator can be discontinuous along $\partial\Omega$, so there is an extra term when we integrate the Weitzenböck-Lichnerowicz formula.

Note that $(M, \tilde{g})$ is uniquely determined by $(\Omega, \hat{g})$, which is uniquely determined by $(\Omega, \bar{g})$. Our main result follows from the following positive mass theorem of $(M, \tilde{g})$.

Theorem 4 *The ADM mass* m_∞^α *of the end* (E^α, g^α) *is nonnegative for* $\alpha = 1, \ldots, \ell$, *and* $m_\infty^\alpha = 0$ *for some* α *if and only if* $\ell = 1$ *and* M *is the Euclidean space.*

To prove Theorem 4 by Witten's argument, we need the following existence and uniqueness of Dirac spinor with prescribed asymptotics.

Theorem 5 *Let $\psi^1, \ldots, \psi^\ell$ be constant spinors defined on the ends $E^1, \ldots, E^\ell$. Then there exists a unique spinor $\psi \in W^{1,2}_{\mathrm{loc}}(M, S)$ such that*

1. $\mathbf{D}\psi = 0$.

2. $\psi \in C^\infty(M \setminus \partial\Omega, S)$.

3. $\psi \in W^{1,p}_{\mathrm{loc}}(M, S)$ *for any* $2 \leq p < \infty$.

4. *On each end* E^α,

$$\lim_{\rho \to \infty} \rho^{1-\epsilon} |\psi - \psi^\alpha| = 0$$

for any $\epsilon > 0$.

Proof: We modify Parker and Taubes's version of Witten's argument. In Parker-Taubes, the hypersurface Dirac operator is studied, so the second fundamental form of the spacelike hypersurface in the spacetime is involved in estimates. Here we consider the Riemannian case, so the estimates are simpler in certain steps. The main difficulty in our case comes from the discontinuity of the zeroth order term of the Dirac operator along $\partial\Omega$.

Let $L > 0$ be such that E^α contains $\mathbb{R}^3 \setminus B_L$. For $r > L$, let

$$E_r^\alpha = E^\alpha \setminus B_r, \quad S_r^\alpha = \partial E_r^\alpha, \quad M_r = M \setminus \bigcup_{\alpha=1}^{\ell} E_r^\alpha.$$

Choose a smooth function β_L^α on each end E^α such that (i) $0 \leq \beta_L^\alpha \leq 1$, (ii) $\beta_L^\alpha \equiv 1$ on E_{3L}^α, (iii) $\beta_L^\alpha \equiv 0$ on $E^\alpha \cap M_{2L}$, and (iv) $|\nabla \beta_L^\alpha| \leq 2/L$. Then $\beta_L^\alpha \psi^\alpha$ extends to a smooth section of S over M. Define

$$\psi_0 = \sum_{\alpha=1}^{\ell} \beta_L^\alpha \psi^\alpha \in C^\infty(M, S). \tag{4}$$

We wish to find $\psi_1 \in W^{1,2}(M, S)$ such that

$$\mathbf{D}\psi_1 = -\mathbf{D}\psi_0 \tag{5}$$

and

$$\lim_{r \to \infty} r^{1-\epsilon} |\psi_1| = 0.$$

Then $\psi = \psi_0 + \psi_1$ is the desired solution.

We derive the following subtitute of the integral form of Weitzenböck-Lichnerowicz formula:

Proposition 6 *For $r > L$ and $\psi \in W^{1,2}_{\mathrm{loc}}(M, S) \cap C^\infty(M \setminus M_L, S)$, we have*

$$2\int_{M_r} |\mathbf{D}\psi|^2 \geq \frac{1}{10}\int_{M_r} |\nabla\psi|^2 + \frac{1}{16}\int_{\Omega} u^{-2}|du|^2|\psi|^2 + \sum_{\alpha=1}^{\ell}\int_{S_r^\alpha}\langle\frac{H}{2}\psi - c(\nu)\check{\mathbf{D}}\psi, \psi\rangle$$

where $\check{\mathbf{D}}$ is the Dirac operator on S_r^α.

This is a crucial ingredient of the proof of Theorem 5.

Let ψ be the unique spinor given by Theorem 5. By Proposition 6,

$$0 \geq \frac{1}{10}\int_{M_r} |\nabla\psi|^2 + \frac{1}{16}\int_{\Omega} u^{-2}|du|^2|\psi|^2 + \sum_{\alpha=1}^{\ell} \int_{S_r^\alpha} \langle \frac{H}{2}\psi - c(\nu)\check{\mathbf{D}}\psi, \psi\rangle.$$

By calculations similar to those in Parker-Taubes, we have

$$\lim_{r\to\infty} \int_{S_r^\alpha} \langle \frac{H}{2}\psi - c(\nu)\check{\mathbf{D}}\psi, \psi\rangle = -4\pi G m_\infty^\alpha |\psi^\alpha|^2$$

where m_∞^α is the ADM mass of the end E^α. So

$$4\pi G \sum_{\alpha=1}^{\ell} m_\infty^\alpha |\psi^\alpha|^2 \geq \frac{1}{10}\int_M |\nabla\psi|^2 + \frac{1}{16}\int_\Omega u^{-2}|du|^2|\psi|^2.$$

In particular, take $|\psi^\beta| = 1$ and $\psi^\alpha = 0$ for $\alpha \neq \beta$, we have

$$4\pi G m_\infty^\beta \geq \frac{1}{10}\int_M |\nabla\psi|^2 + \frac{1}{16}\int_\Omega u^{-2}|du|^2|\psi|^2 \geq 0.$$

If $m_\infty^\beta = 0$ for some β, we have $du = 0$, so $u \equiv 1$, which implies $\hat{g} = \bar{g}$. We also have $\nabla\psi = 0$ on M. We conclude that $\partial\Omega$ is connected, and M is the Euclidean space.

More general case

Theorem 7 (Wang-Yau) *Let* (Ω, g_{ij}, p_{ij}) *be a compact 3-manifold that satisfies the dominant energy condition. Suppose the boundary of* Ω *is a smooth surface* Σ *with Gaussian curvature* K *and mean curvature (with respect to outward normal)* H*. Suppose* $\kappa > 0$ *satisfies* $K > -\kappa^2$ *and* $H > P - 2\kappa$*, where* $P = tr_\Sigma p$*. Let* F_0 *be the (essentially unique) isometric embedding of* Σ *in* $\mathbb{H}^2_{-\kappa^2}$*. Then there exists a positive function* μ *which depends on* $\sqrt{H^2 - (P - 2\kappa)^2}$ *and the isometric embedding* F_0 *such that*

$$\int_\Sigma H_0 \mu \geq \int_\Sigma \sqrt{H^2 - (P - 2\kappa)^2} \mu$$

where H_0 *is the mean curvature* $F_0(\Sigma)$ *inside* $\mathbb{H}^3_{-\kappa^2}$*. Moreover, the function* μ *approaches the constant function as* $\kappa \to 0$*.*