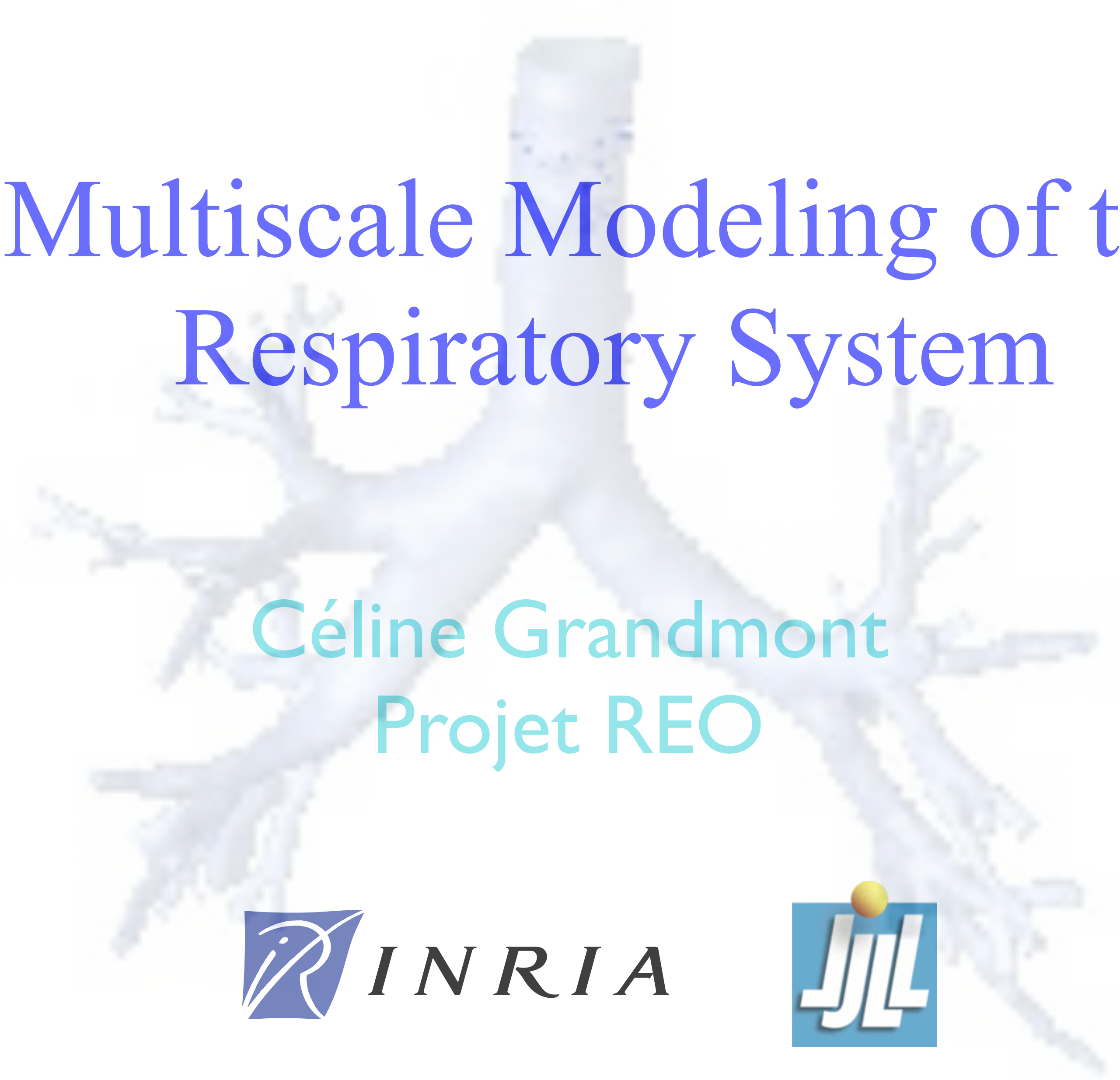
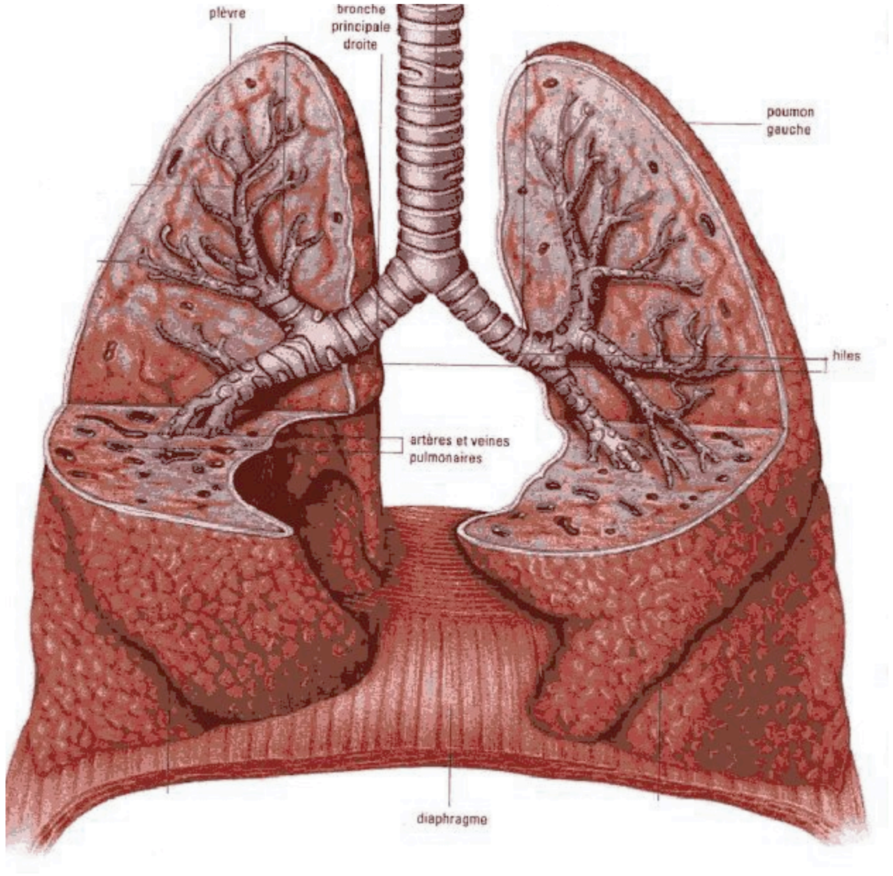
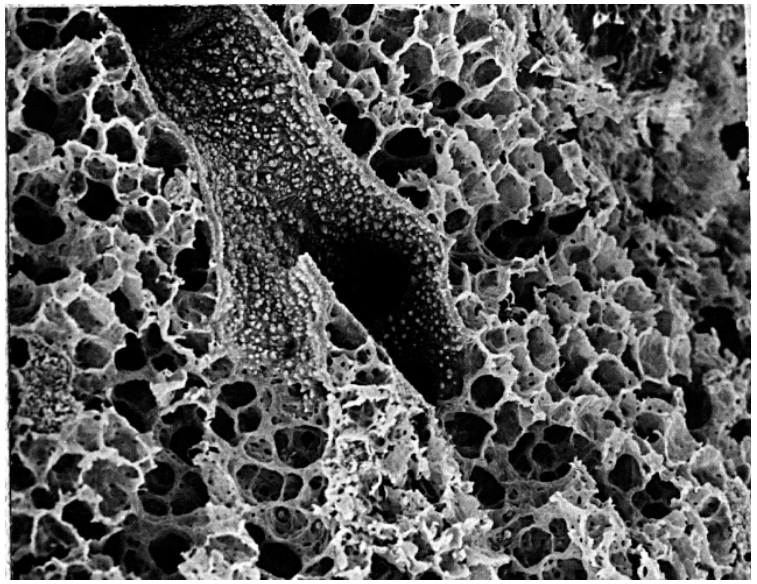


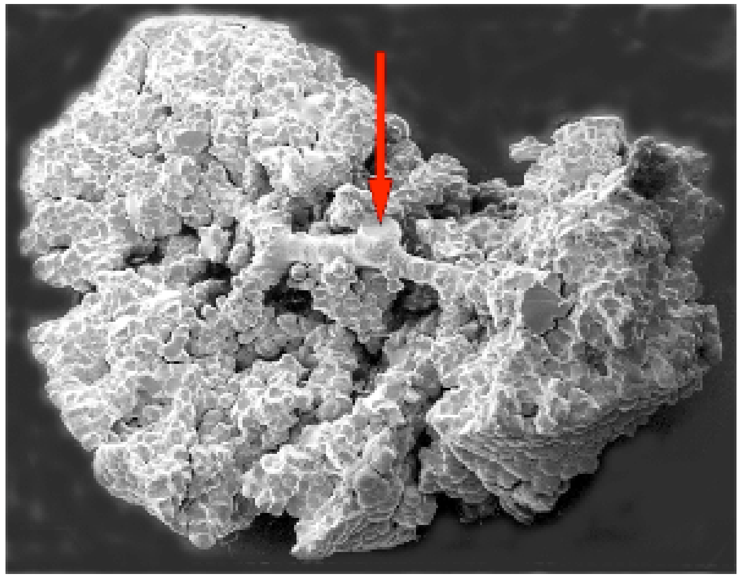
Multiscale Modeling of the Respiratory System

Céline Grandmont
Projet REO

Respiratory System Structure

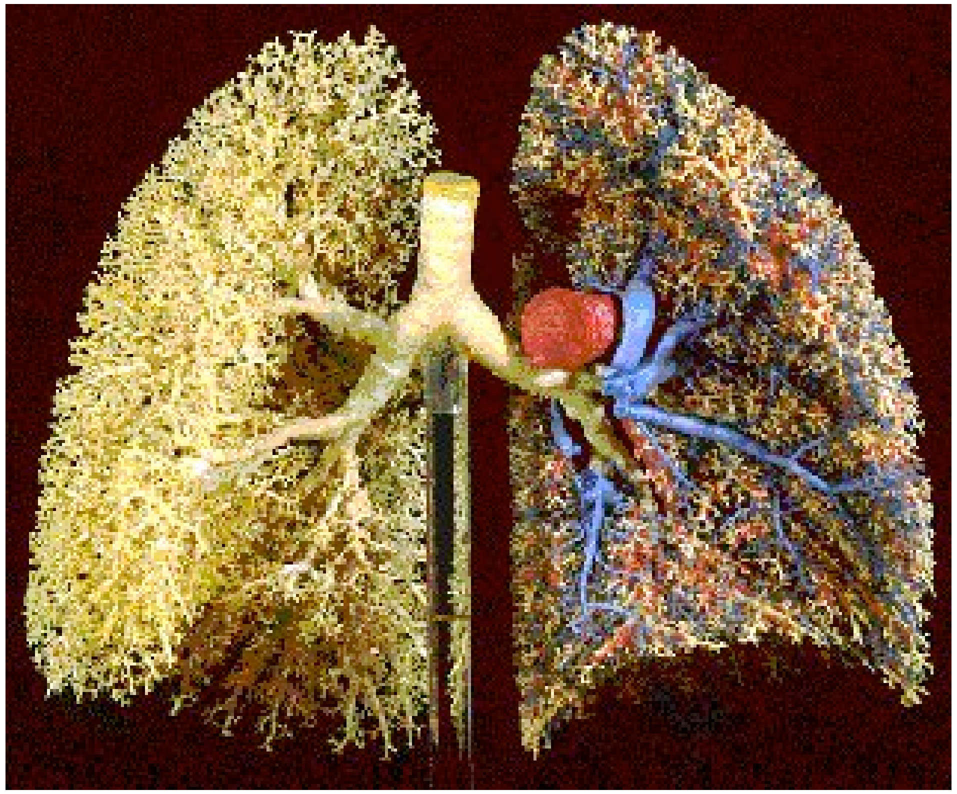






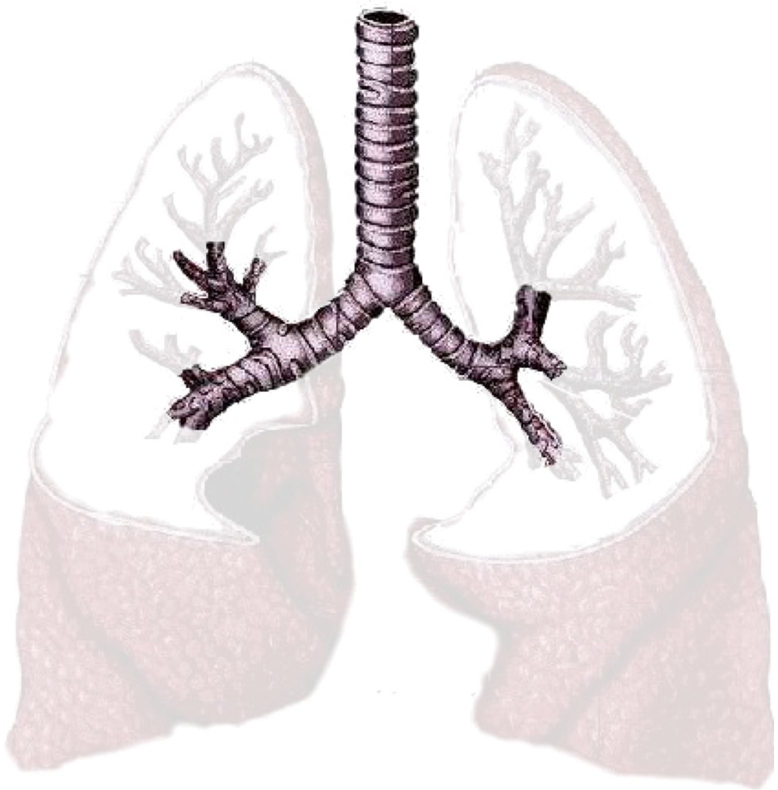
The Respiratory System

- Enable gas exchanges
- Pathologies:
  - Asthma
  - Fibrosis, emphysema
  - ...

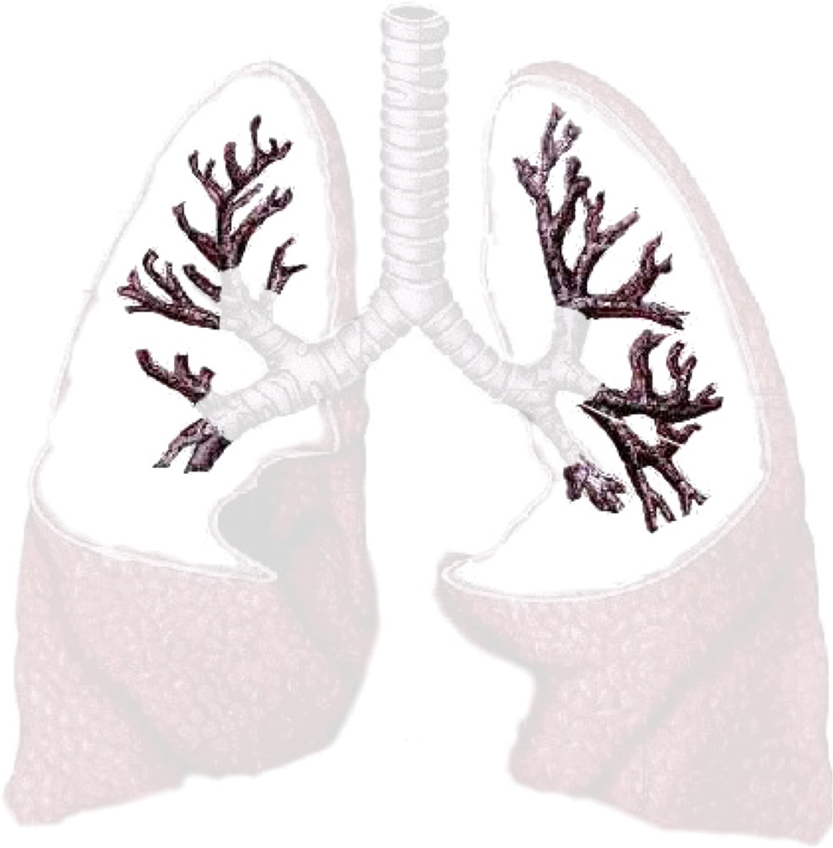


Weibel

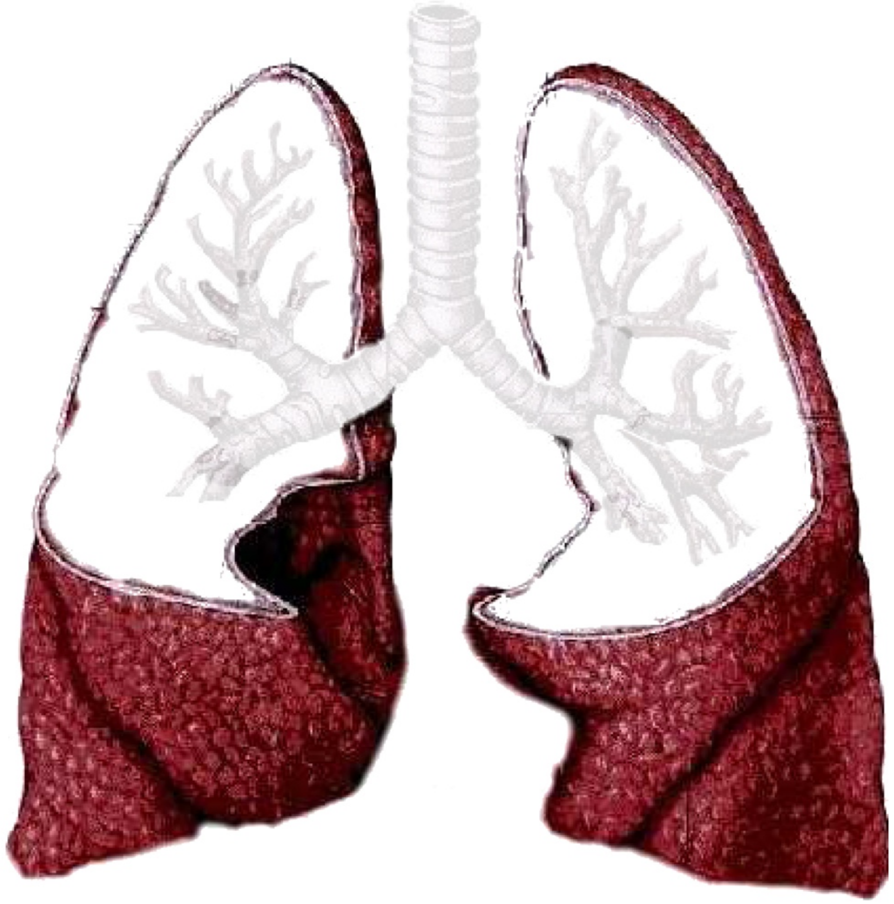
Multiscale Strategy



3D modeling
Viscous
incompressible flow



Laminar flow



Simple elastic
model

Goals

- Obtain simple but representative models
  - Modeling each zone ;
  - Coupling each model ;
  - or take into account the other parts thanks to well chosen boundary conditions ;

Outline

- Viscoleastic model of the lung parenchyma
  [CG, Maury, Meunier, Vannier]
- Air flow modeling in the proximal part
  [Baffico, CG, Maury, Soualah]
- Air - Spray modeling
  [Boudin, Desvillettes, Filoche, CG, Moussa]

Modeling of the lung parenchyma

C.G., B. Maury, N. Meunier

Condensation of the bronchial tree

Hypothesis

- Diadic tree of pipes
- In each tube the Poiseuille law is satisfied $\Longrightarrow$ the flow is caracterized by a resistance r_n at each generation

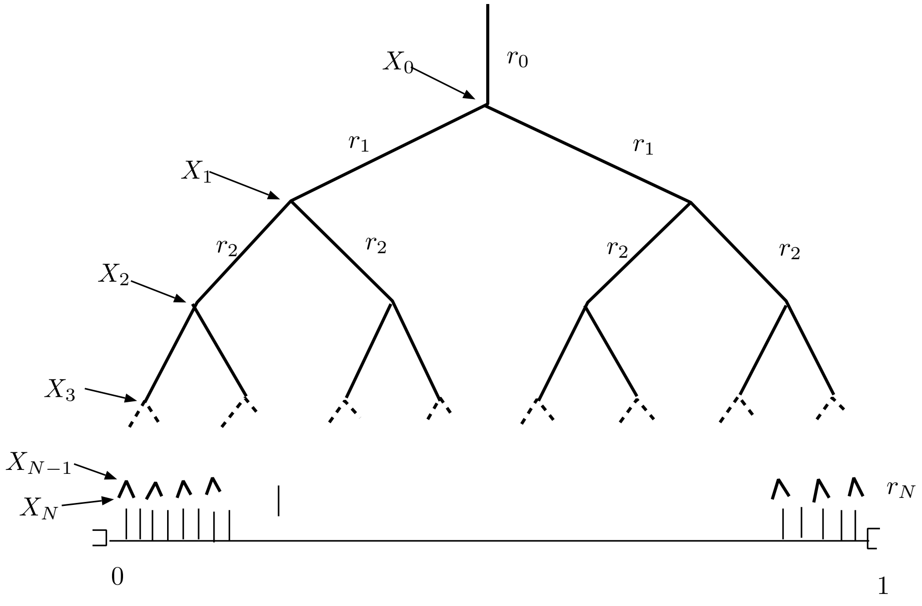


Figure 1. Dyadic tree

Condensation of the bronchial tree

$$p_i = \sum_{j=0}^{2^N-1} q_j R_{N-\nu_{ij}},$$

with

$$R_n = \sum_{j=0}^{n} r_j,$$

and

$$\nu_{ij} = \inf\{k \geq 0,\ \alpha_\ell = \beta_\ell \quad \forall \ell \geq k\},$$

$$i = \sum_{k=0}^{\infty} \alpha_k 2^k,\ j = \sum_{k=0}^{\infty} \beta_k 2^k \text{ with } \alpha_k, \beta_k \in \{0,1\} \quad \forall k.$$

Condensation of the bronchial tree

Let $u \in L^2(I)$, we define 2^N flux flows

$$\mathbf{q}^N = (q_i^N)_{i=0,\dots 2^N-1}, q_i^N = \int_{\frac{i}{2^N}}^{\frac{i+1}{2^N}} u(y)dy.$$

To the pressure vector $\mathbf{p}^N = (p_i^N)_{i=0,\dots 2^N-1}$, we associate the function p that takes the value p_i^N on $(\frac{i}{2^N}, \frac{i+1}{2^N})$.

Then

$$p(x) = \int_I K_N(x,y)u(y)dy, a.e.$$

where $K_N \in L^1(I \times I)$ is a function picewise constant and defined by

$$x \in (\frac{i}{2^N}, \frac{i+1}{2^N}), y \in (\frac{j}{2^N}, \frac{j+1}{2^N}) \mapsto R_{N-\nu_i j}.$$

Condensation of the bronchial tree

Limit when $N \to +\infty$

$$K_N \longrightarrow K, a.e.$$

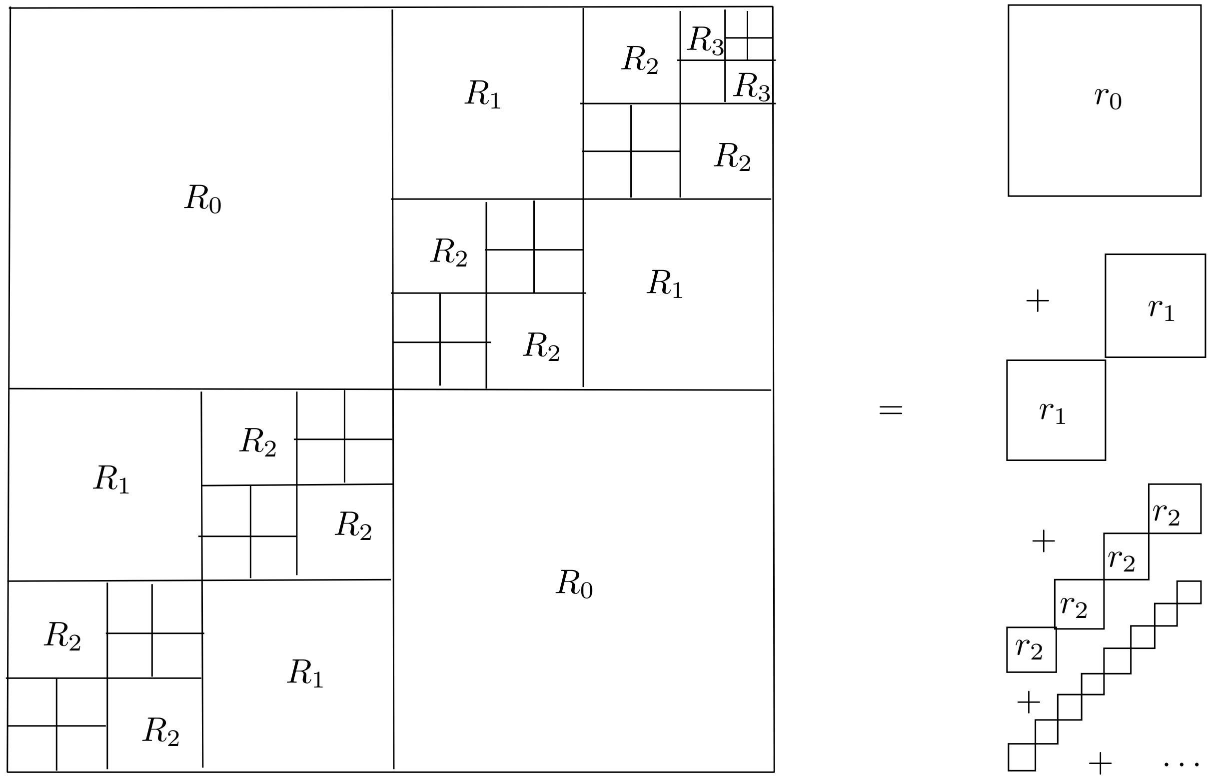


FIGURE 1. Kernel $K(x, y)$

Condensation of the bronchial tree

Properties of the limit operator

- $\mathcal{R}_N \longrightarrow \mathcal{R}$ in $\mathcal{L}(L^2(I))$ iff $\sum_{n\geq 0} \frac{r_n}{2^n} < +\infty$.

- $\|\mathcal{R}_N u\|_{L^p(I)} \leq \|K_N\|_{L^1(I\times I)} \|u\|_{L^p(I)}$.

- $\mathcal{R}$ self–adjoint, compact, monotone.

- The Haar basis is a familly of eigenfunctions of $\mathcal{R}$ associated to the eigenvalues$\lambda_N = \sum_{n=N}^{+\infty} \frac{r_n}{2^n}$.

- if $\sum_{n\geq 0} \frac{r_n}{2^n} = +\infty$ then $\displaystyle\int_I \frac{K_N(\cdot, y)}{\|K_N\|_{L^1(I\times I)}} q(y)dy \longrightarrow q(\cdot)$.

Condensation of the bronchial tree

Geometric Tree

We consider the case $r_n = r_0 \alpha^n$. [Human lung: $\alpha = 1.63$]

- $\mathcal{R}_N \longrightarrow \mathcal{R}$ in $\mathcal{L}(L^2(I))$ if $\alpha < 2$.

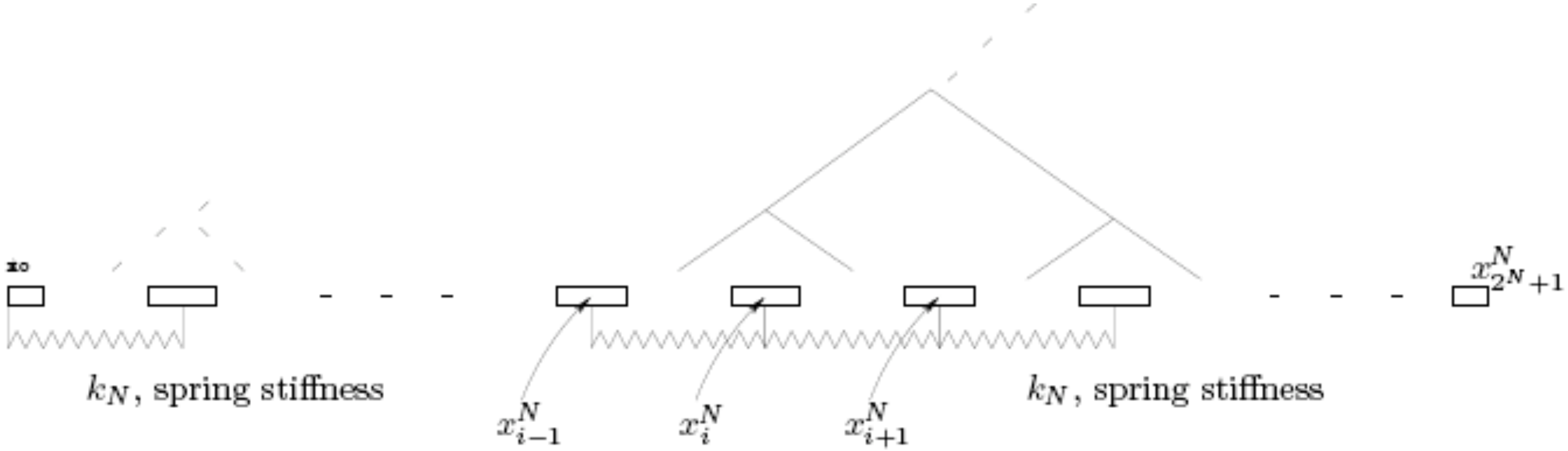
- if $\alpha = 2$ then $\displaystyle\int_I \frac{K_N(\cdot, y)}{\|K_N\|_{L^1(I\times I)}} q(y)dy \longrightarrow q(\cdot)$.

Remark: $\alpha = 2$ corresponds to an optimal coefficient that is the minimizer of the global resistance under a volume constrain.
[B. Mauroy, M. Filoche, E. R. Weibel, B. Sapoval]

Viscoelastic model

Spring–mass sytem

We consider a sucession of masses (lung parenchyma) and of air pockets (alveoli), each mass beeing linked by springs. Thus each air pocket is connected to each other through the tree.



Viscoelastic model

Spring–mass sytem

The motion of the $i-th$ mass is described by:

$$m_N \ddot{u}_i^N(t) - k_N(u_{i+1}^N(t) - 2u_i^N(t) + u_{i-1}^N(t)) + p_i^N(t) - p_{i-1}^N(t) = m_N f_i^N(t),$$

with

$$p_i^N = \sum_{j=0}^{2^N-1} R_{N-\nu_{ij}} q_j^N,$$

and

$$q_j^N = \dot{u}_j^N(t) - \dot{u}_{j+1}^N(t).$$

Viscoelastic model

A priori Estimates

(u^N) is bounded in $W^{1,\infty}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1_0(0,1))$.

Limit equation in the case $\sum \frac{r_n}{2^n} < +\infty$

$$\rho\,\alpha_s\,\partial_{tt}u(x,t) - k\,\partial_{xx}u(x,t) - \partial_x \int_0^1 K(x,y)\,\partial_y\partial_t u(y,t)dy = \rho\,\alpha_s f(x,t),$$

where k satisfies:

$$k = \frac{k_N}{2^N}.$$

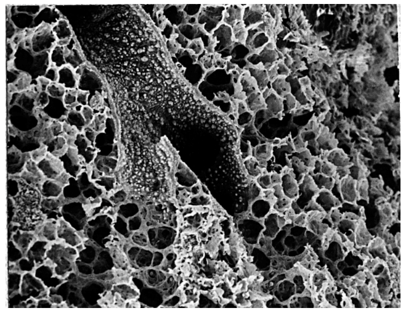
Limit in the case $\sum \frac{r_n}{2^n} = +\infty$

$$u = u_0, \text{ where } u_0 \text{ is the initial displacement.}$$

Ongoing and Related Works

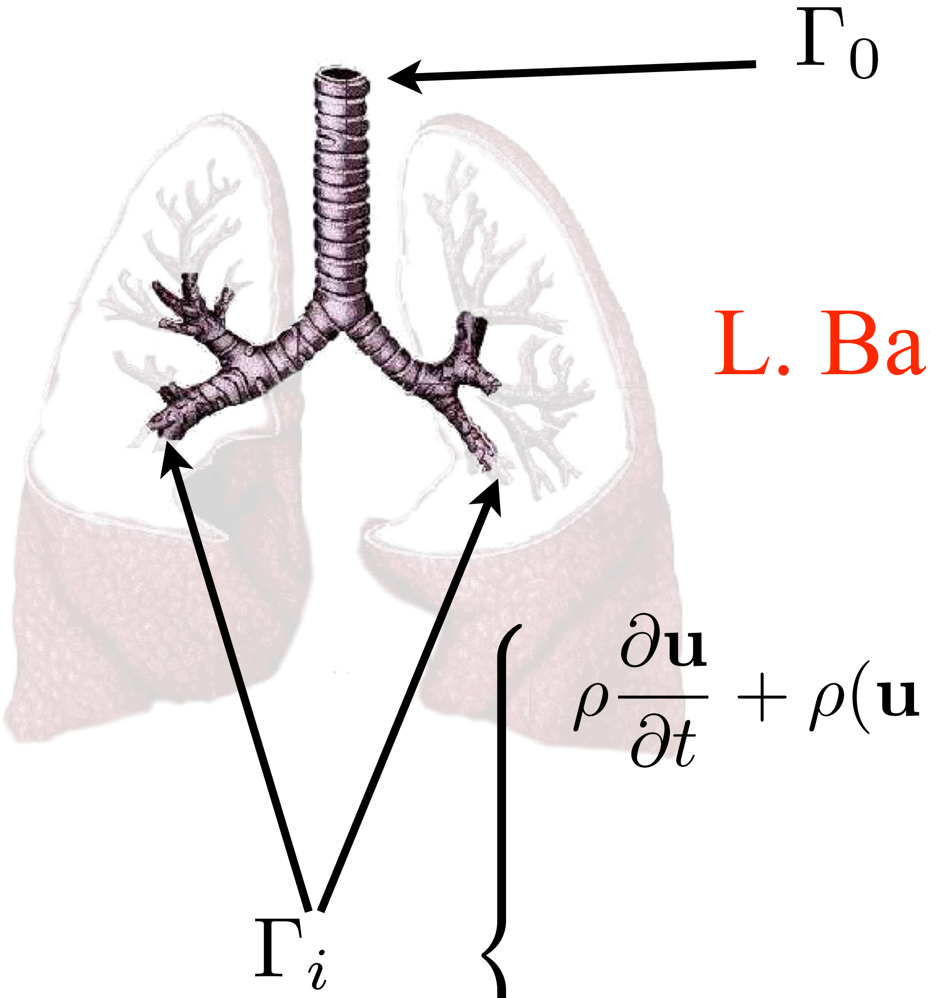
- L. Baffico, C. G., Y. Maday, A. Osses,

  Homogenized foam model;



- C.G., C. Vannier, long time behavior;
- B. Maury, D. Salort, C. Vannier, diadic tree;
- Y. Achdou, C. Sabot, N. Tchou, boundary conditions for self similar geometries.

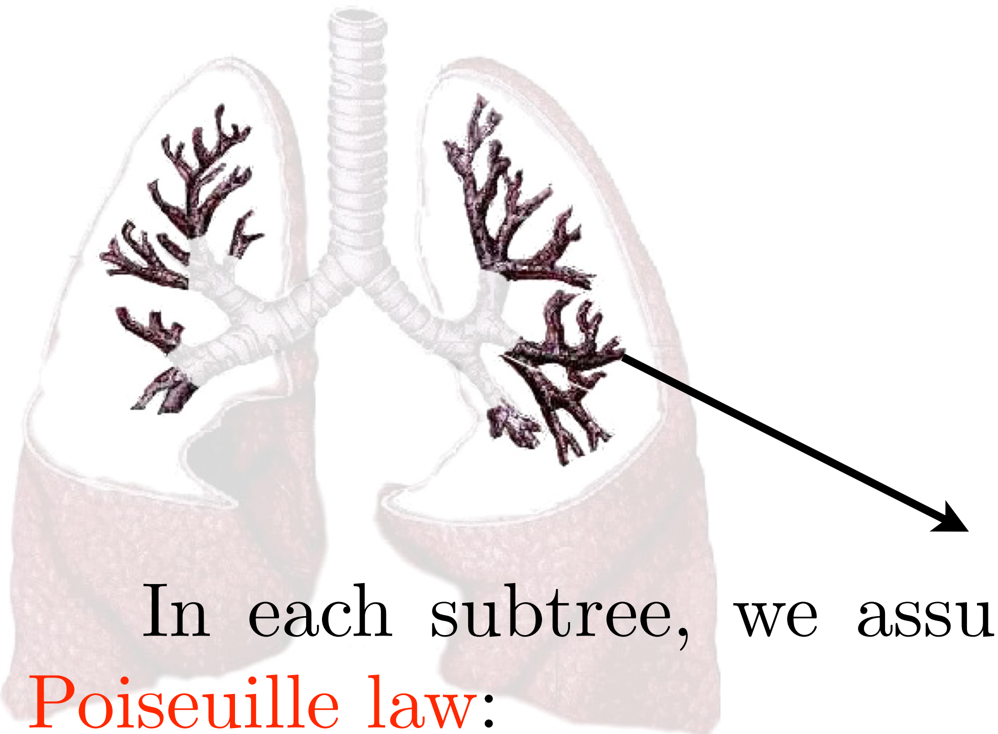
Air flow in the proximal part



L. Baffico, C. G., Y. Maday B. Maury, A. Soualah

$$\left\{\begin{array}{rcll}
\rho\dfrac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\triangle\mathbf{u}+\nabla p & = & 0, & \text{in } \Omega\,, \\
\nabla\cdot\mathbf{u} & = & 0, & \text{in } \Omega\,, \\
\mathbf{u} & = & 0, & \text{on } \Gamma_l\,, \\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_0\mathbf{n} & \text{on } \Gamma_0\,, \\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -\Pi_i\mathbf{n} & \text{on } \Gamma_i \quad i=1,\ldots,N,
\end{array}\right.$$

Air flow in the distal part



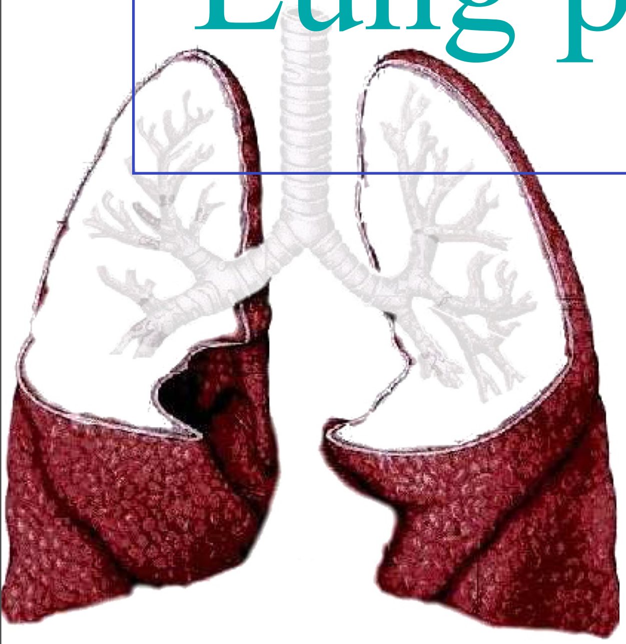
In each subtree, we assume that the flow is laminar and thus satisfies a Poiseuille law:

$$\Pi_i - P_i = R_i \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n},\ \ R_i \geq 0,$$

where R_i denotes the equivalent resistance of the distal tree and P_i is an alveola pressure. The boundary conditions at the outlets Γ_i writes

$$\mu \nabla \mathbf{u} \cdot \mathbf{n} - p\mathbf{n} = -P_i\mathbf{n} - R_i \left(\int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \right) \mathbf{n} \text{ on } \Gamma_i \quad i = 1, \ldots, N.$$

Lung parenchyma - Diaphragm muscle



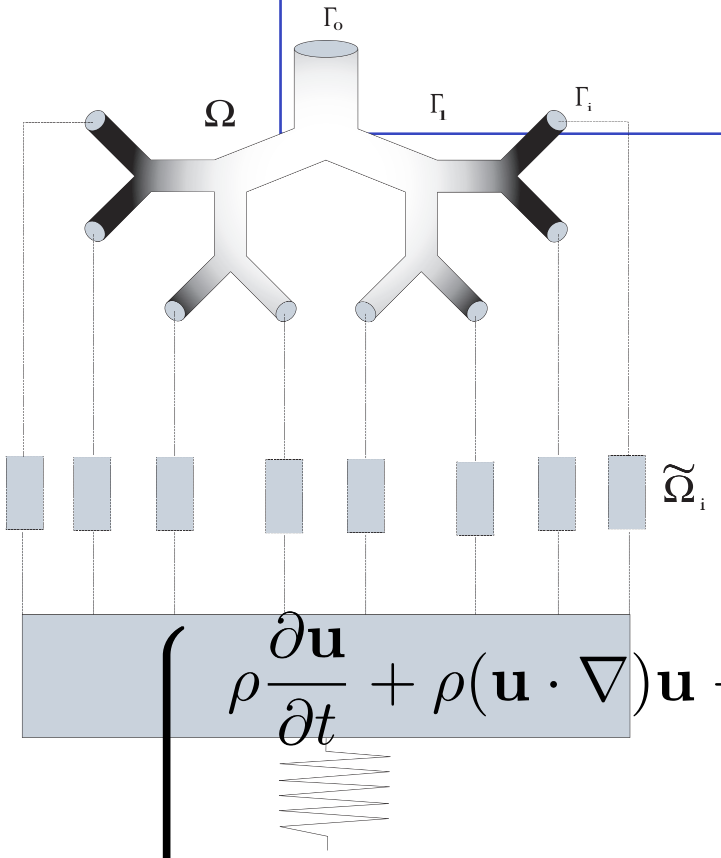
All the alveola pressures are equal: $P_i = P_a$. The diaphragm and parenchyma motion are govern by:

$$m\ddot{x} = -kx + f_{ext} + SP_a,$$

Moreover, since the flow and the parenchyma are incompressible

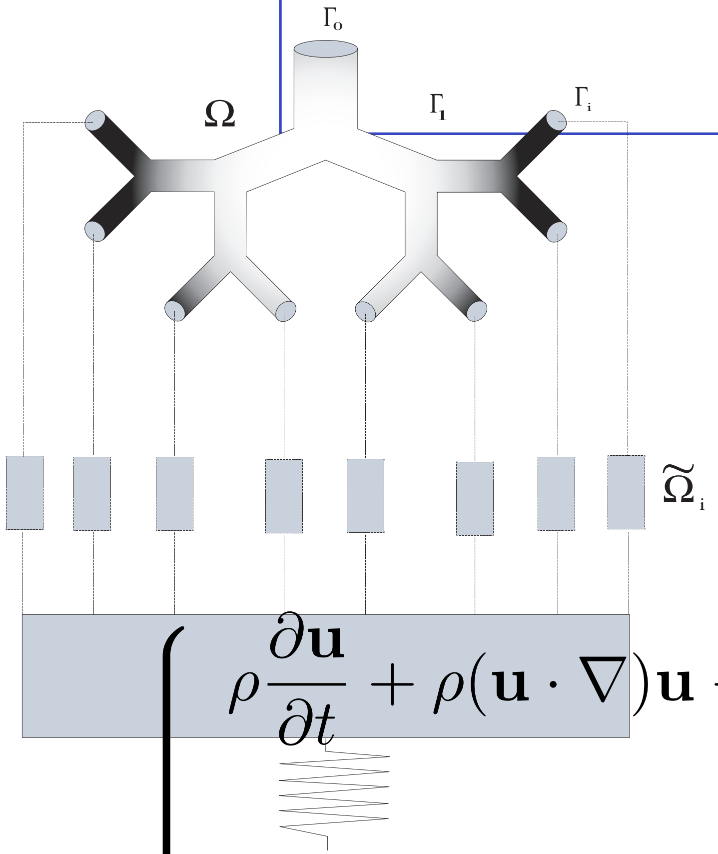
$$S\dot{x} = \sum_{i=1}^{N} \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} = -\int_{\Gamma_0} \mathbf{u} \cdot \mathbf{n},$$

Coupled System



$$\left\{\begin{array}{rcll}
\rho\dfrac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\triangle\mathbf{u}+\nabla p & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\nabla\cdot\mathbf{u} & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\mathbf{u} & = & 0\,, & \text{on } (0,T)\times\Gamma_l\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_0\mathbf{n} & \text{on } (0,T)\times\Gamma_0\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_a\mathbf{n}-R_i\left(\displaystyle\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}\,, & \text{on } (0,T)\times\Gamma_i\,,\\
 & & & i=1\,,\ \ldots\,,\ N\,,\\
m\ddot{x}+kx & = & f_{ext}+SP_a\,, & \\
S\dot{x} & = & \displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}=-\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n}. &
\end{array}\right.$$

Coupled System



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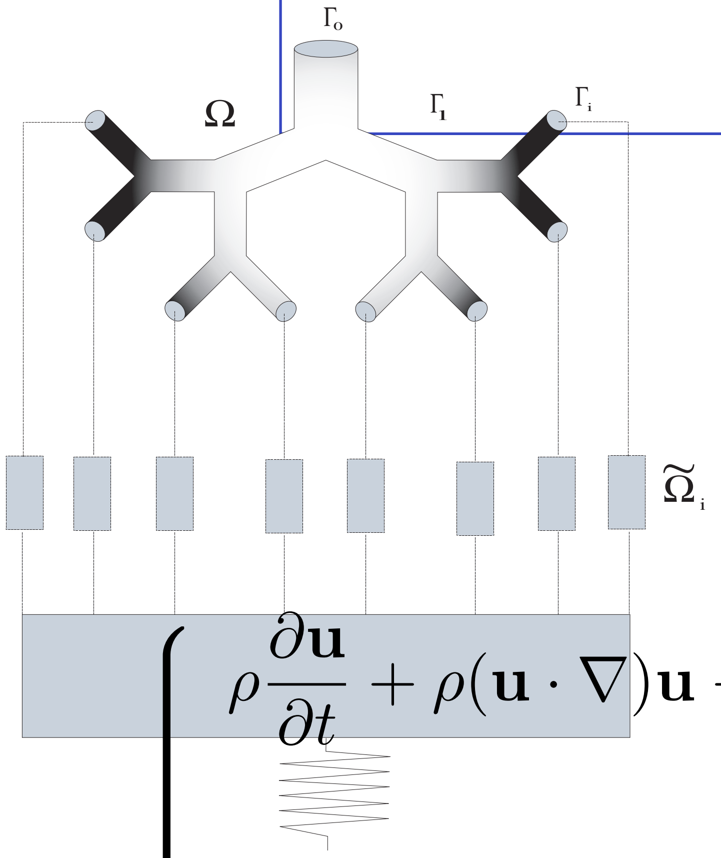
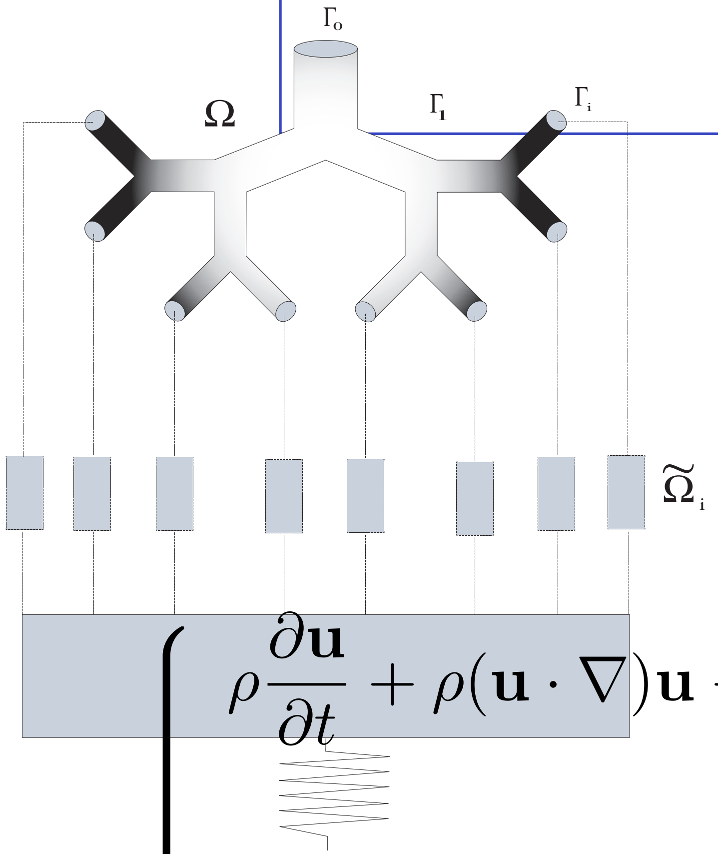
\rho\dfrac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\triangle\mathbf{u}+\nabla p & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\nabla\cdot\mathbf{u} & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\mathbf{u} & = & 0\,, & \text{on } (0,T)\times\Gamma_l\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_0\mathbf{n} & \text{on } (0,T)\times\Gamma_0\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_a\mathbf{n}-R_i\left(\displaystyle\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}\,, & \text{on } (0,T)\times\Gamma_i\,,\\
 & & & i=1\,,\ \ldots\,,\ N\,,\\
m\ddot{x}+kx & = & \boxed{f_{ext}}+SP_a\,, & \\
S\dot{x} & = & \displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}=-\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n}. &
\end{array}\right.$$

Coupled System



$$
\left\{
\begin{array}{rcll}
\rho \dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\triangle\mathbf{u} + \nabla p & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\nabla\cdot\mathbf{u} & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\mathbf{u} & = & 0\,, & \text{on } (0,T)\times\Gamma_l\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n} - p\mathbf{n} & = & -P_0\mathbf{n} & \text{on } (0,T)\times\Gamma_0\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n} - p\mathbf{n} & = & -P_a\mathbf{n} - R_i\left(\displaystyle\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}\,, & \text{on } (0,T)\times\Gamma_i\,,\\
 & & & i = 1\,,\ \ldots\,,\ N\,,\\
m\ddot{x} + kx & = & f_{ext} + SP_a\,, & \\
S\dot{x} & = & \displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n} = -\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n}. &
\end{array}
\right.
$$

Coupled System



$$\left\{\begin{array}{rcll}
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\nabla\cdot\mathbf{u} & = & 0\,, & \text{in } (0,T)\times\Omega\,,\\
\mathbf{u} & = & 0\,, & \text{on } (0,T)\times\Gamma_l\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_0\mathbf{n} & \text{on } (0,T)\times\Gamma_0\,,\\
\mu\nabla\mathbf{u}\cdot\mathbf{n}-p\mathbf{n} & = & -P_a\mathbf{n}-\boxed{R_i}\left(\displaystyle\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}\,, & \text{on } (0,T)\times\Gamma_i\,,\\
 & & & i=1\,,\ \ldots\,,\ N\,,\\
m\ddot{x}+\boxed{kx} & = & \boxed{f_{ext}}+SP_a\,, & \\
S\dot{x} & = & \displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}=-\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n}. &
\end{array}\right.$$

Coupled System

By setting $\overline{p} = p - P_a$, by using the expression of the alveola pressure $P_a = \frac{m}{S}\ddot{x} + \frac{k}{S}x - \frac{f_{ext}}{S}$ and the volume preserving equation $S\dot{x} = -\int_{\Gamma_0} \mathbf{u}\cdot\mathbf{n}$, the coupled system can be written as follows:

$$\left\{\begin{array}{rcll}
\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\triangle\mathbf{u} + \nabla\overline{p} & = & 0\,, & \text{in } (0,T)\times\Omega\,, \\
\nabla\cdot\mathbf{u} & = & 0\,, & \text{in } (0,T)\times\Omega\,, \\
\mathbf{u} & = & 0\,, & \text{on } (0,T)\times\Gamma_l \\
\mu\nabla\mathbf{u}\cdot\mathbf{n} - \overline{p}\mathbf{n} & = & -P_0\mathbf{n} - \frac{f_{ext}}{S}\mathbf{n} & \\
 & & -\frac{m}{S^2}\frac{d}{dt}\left(\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n} - \frac{k}{S^2}\left(\int_0^t\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n} - Sx_0\right)\mathbf{n}\,, & \text{on } (0,T)\times\Gamma_0 \\
\mu\nabla\mathbf{u}\cdot\mathbf{n} - \overline{p}\mathbf{n} & = & -R_i\left(\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}\,, & \text{on } (0,T)\times\Gamma_i \\
 & & & i = 1\,,\ \ldots\,,\ N
\end{array}\right. \tag{1}$$

Wellposedness

Energy Balance

$$\frac{d}{dt}\underbrace{\left(\frac{\rho}{2}\int_\Omega |\mathbf{u}|^2 + \frac{m}{2}|\dot{x}|^2 + \frac{k}{2}|x|^2\right)}_{\text{Total energy}} + \underbrace{\mu\int_\Omega |\nabla\mathbf{u}|^2}_{\text{Dissipation within }\Omega} + \underbrace{\sum_{i=1}^{N} R_i\left(\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)^2}_{\text{Dissipation in the subtrees}}$$

$$= -\underbrace{\frac{\rho}{2}\sum_{i=0}^{N}\int_{\Gamma_i}|\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{n})}_{\text{In/out-come of kinetic energy}} + \underbrace{P_0 S\dot{x}}_{\text{Power of inlet pressure}} + \underbrace{f_{ext}\dot{x}}_{\text{Power of ext. forces}} .$$

Wellposedness

- C. G., B. Maury, Y. Maday, p is replaced by the total pressure $p + \rho|\mathbf{u}|^2/2$
- C. G., B. Maury, A. Soualah, with $\mathbf{u} = \lambda_i \mathbf{U}_i$ on Γ_i, existence of weak solutions
  - locally in time for any data
  - for all time for small enough data
- L. Baffico, C. G., B. Maury, existence of "strong" solutions
  - locally in time for any data
  - for all time for small enough data
- Related works: [Heywood, Rannacher, Turek] [Quarteroni, Veneziani], [Vignon et al]...

$$V = \{\mathbf{v} \in H^1(\Omega)^d\,,\ \nabla \cdot \mathbf{v} = 0, \mathbf{v} = 0 \text{ on } \Gamma_\ell\}.$$

$$H = \overline{V}^{L^2},$$

We denote by $(\cdot,\cdot)_0$ the scalar product on $H \times H$ defined by

$$(\mathbf{v},\mathbf{w})_0 = \rho \int_\Omega \mathbf{v} \cdot \mathbf{w} + \frac{m}{S^2} \left(\int_{\Gamma_0} \mathbf{v} \cdot \mathbf{n} \right) \left(\int_{\Gamma_0} \mathbf{w} \cdot \mathbf{n} \right),$$

and by $\|\cdot\|_0$ the associated norm. Next we set

$$a_1(\mathbf{v},\mathbf{w}) = \mu \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} + \sum_{i=1}^{N} R_i \left(\int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n} \right) \left(\int_{\Gamma_i} \mathbf{w} \cdot \mathbf{n} \right),$$

Finally we introduce an operator A whose eigenvectors will constitute the Galerkin basis used to build our sequence of approximated solutions.

The operator A is defined on H as follows:

$$D(A) = \{\mathbf{v} \in V\,,\ |a_1(\mathbf{v},\mathbf{w})| \leq C\|\mathbf{v}\|_0, \forall \mathbf{w} \in V\},$$

$$(A\mathbf{v},\mathbf{w})_0 = a_1(\mathbf{v},\mathbf{w}),\ \forall (\mathbf{v},\mathbf{w}) \in D(A) \times V.$$

Numerical methods

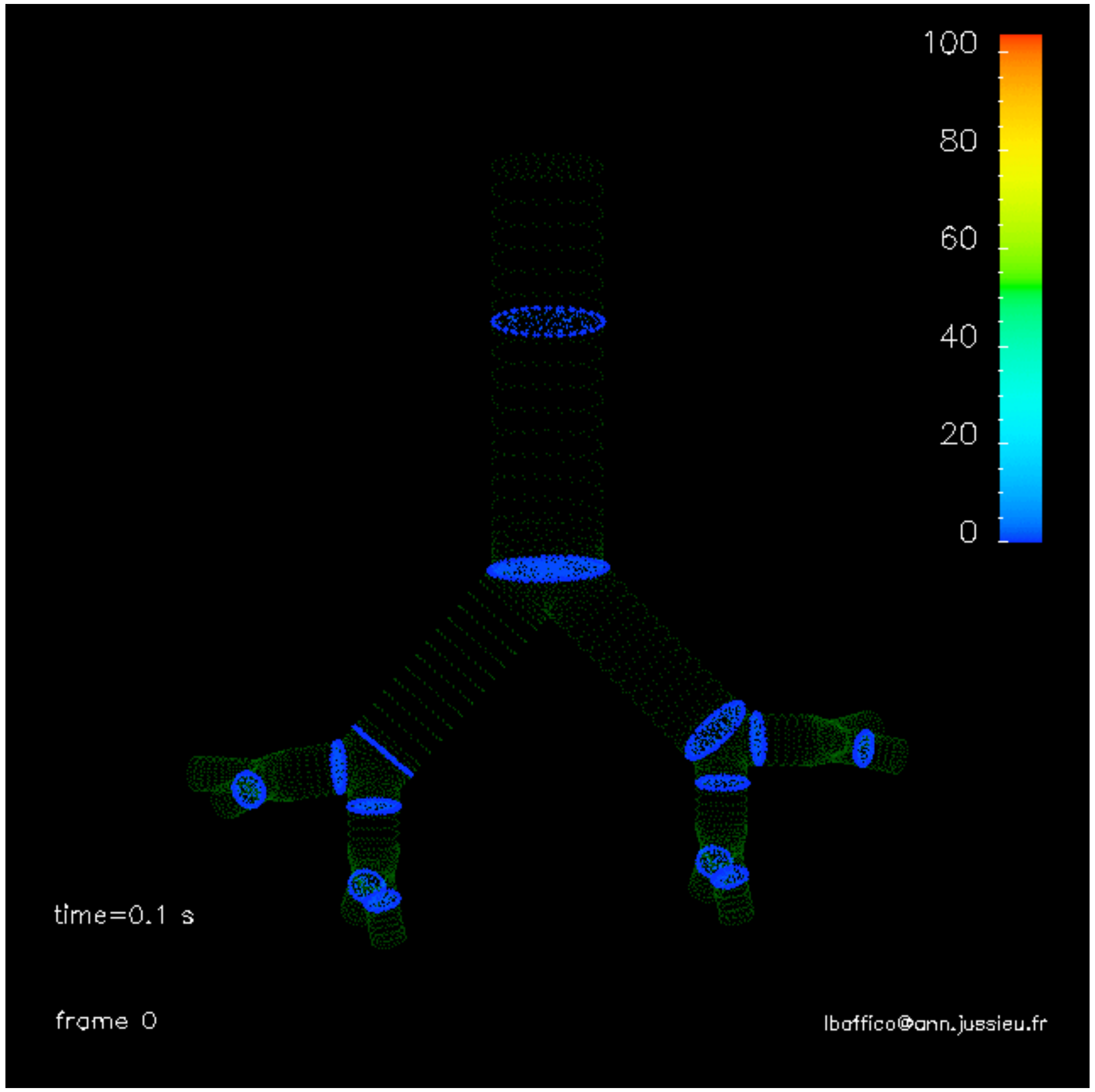
Discretization

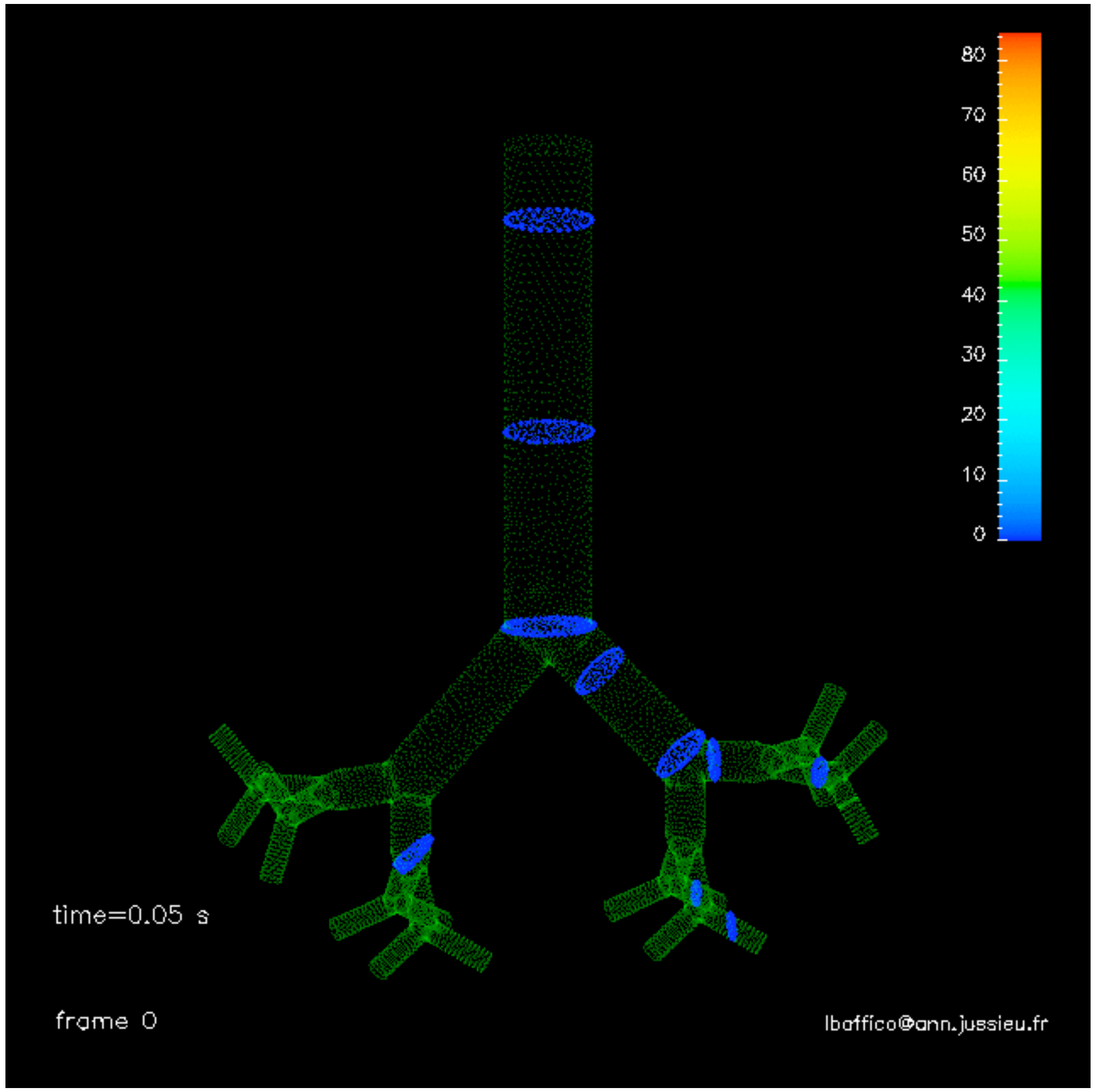
$$\begin{aligned}
a(\mathrm{u}^{n+1}, \mathrm{v}) + b(\mathrm{v}, p^{n+1}) &= l^{n+1}(\mathrm{v}) \\
b(\mathrm{u}^{n+1}, q) &= 0 \ \forall q \in L^2(\Omega), \\
x^{n+1} &= x^n - \frac{\delta t}{S} \int_{\Gamma_0} \mathrm{u}^{n+1} \cdot \mathrm{n}
\end{aligned}$$

where

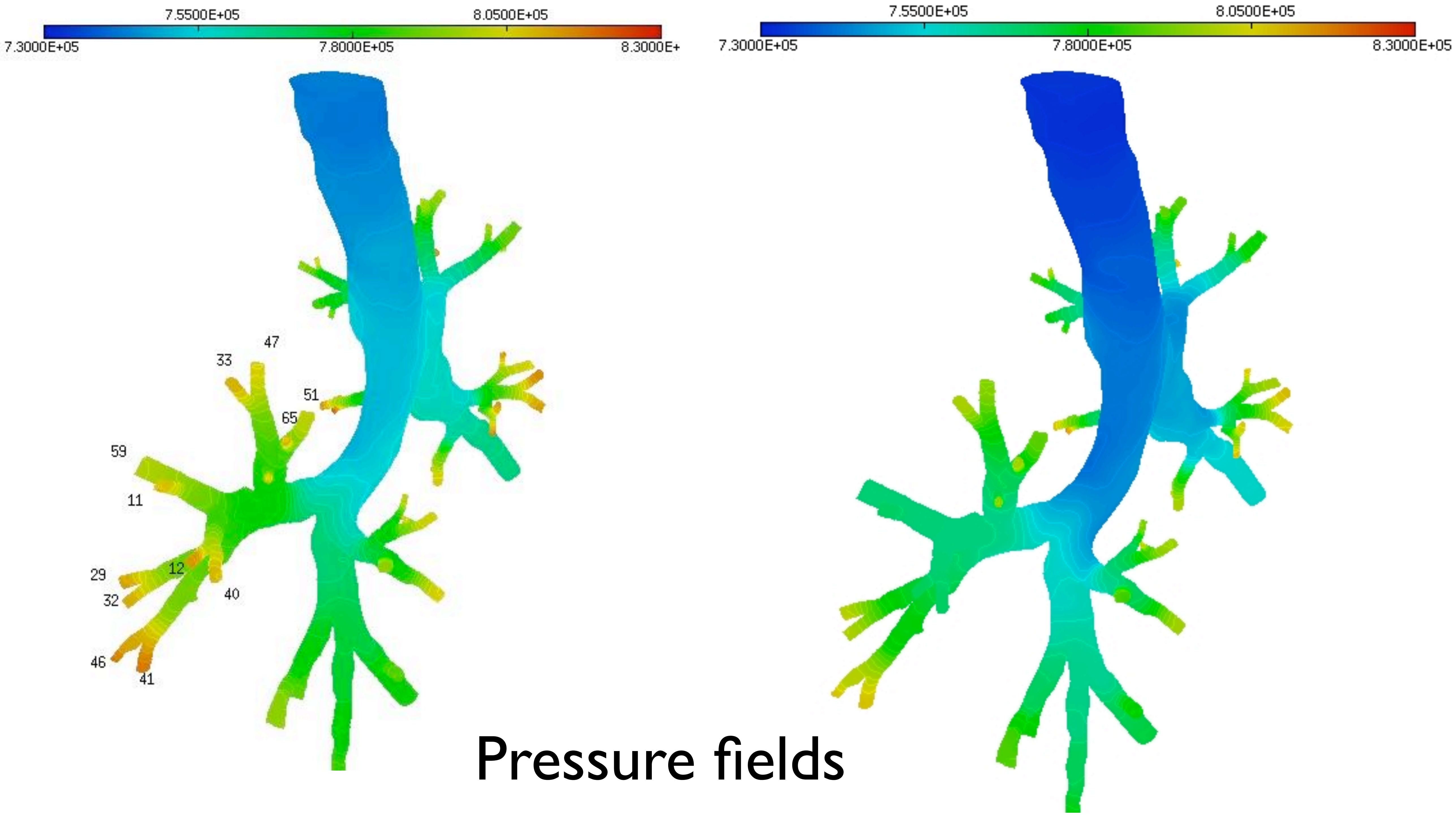
$$\begin{aligned}
a(\mathrm{u}, \mathrm{v}) = {} & \frac{\rho}{\delta t} \int_\Omega \mathrm{u} \cdot \mathrm{v} + 2\mu \int_\Omega \nabla \mathrm{u} : \nabla \mathrm{v} \\
& + \sum_{i=1}^{N} R_i \left(\int_{\Gamma_i} \mathrm{u} \cdot \mathrm{n} \right) \left(\int_{\Gamma_i} \mathrm{v} \cdot \mathrm{n} \right) \\
& + \frac{1}{S^2} \left(\frac{m}{\delta t} + k\delta t \right) \left(\sum_{i=1}^{N} \int_{\Gamma_i} \mathrm{u} \cdot \mathrm{n} \right) \left(\sum_{i=1}^{N} \int_{\Gamma_i} \mathrm{v} \cdot \mathrm{n} \right),
\end{aligned}$$

Numerical simulations





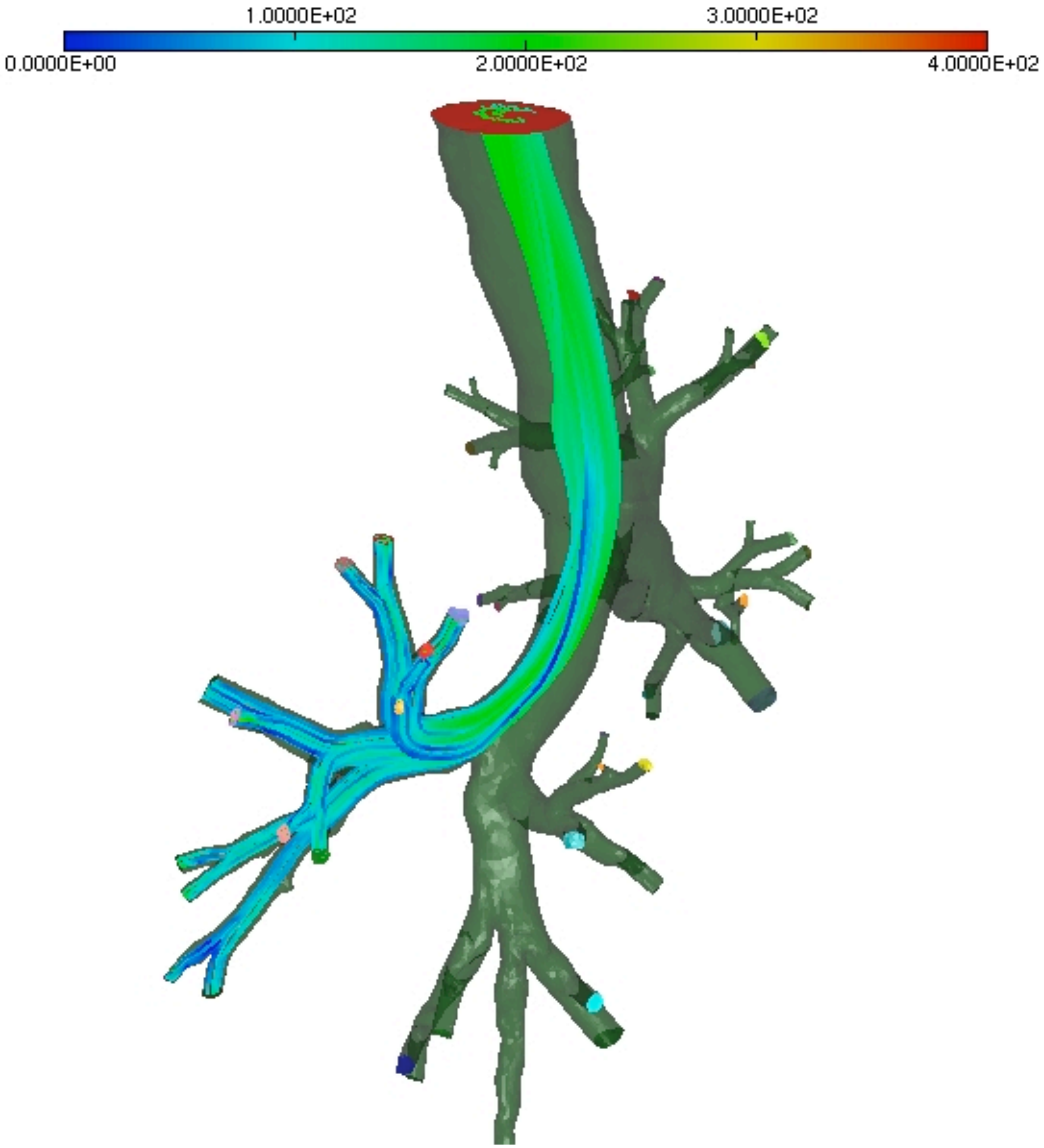
Numerical simulations

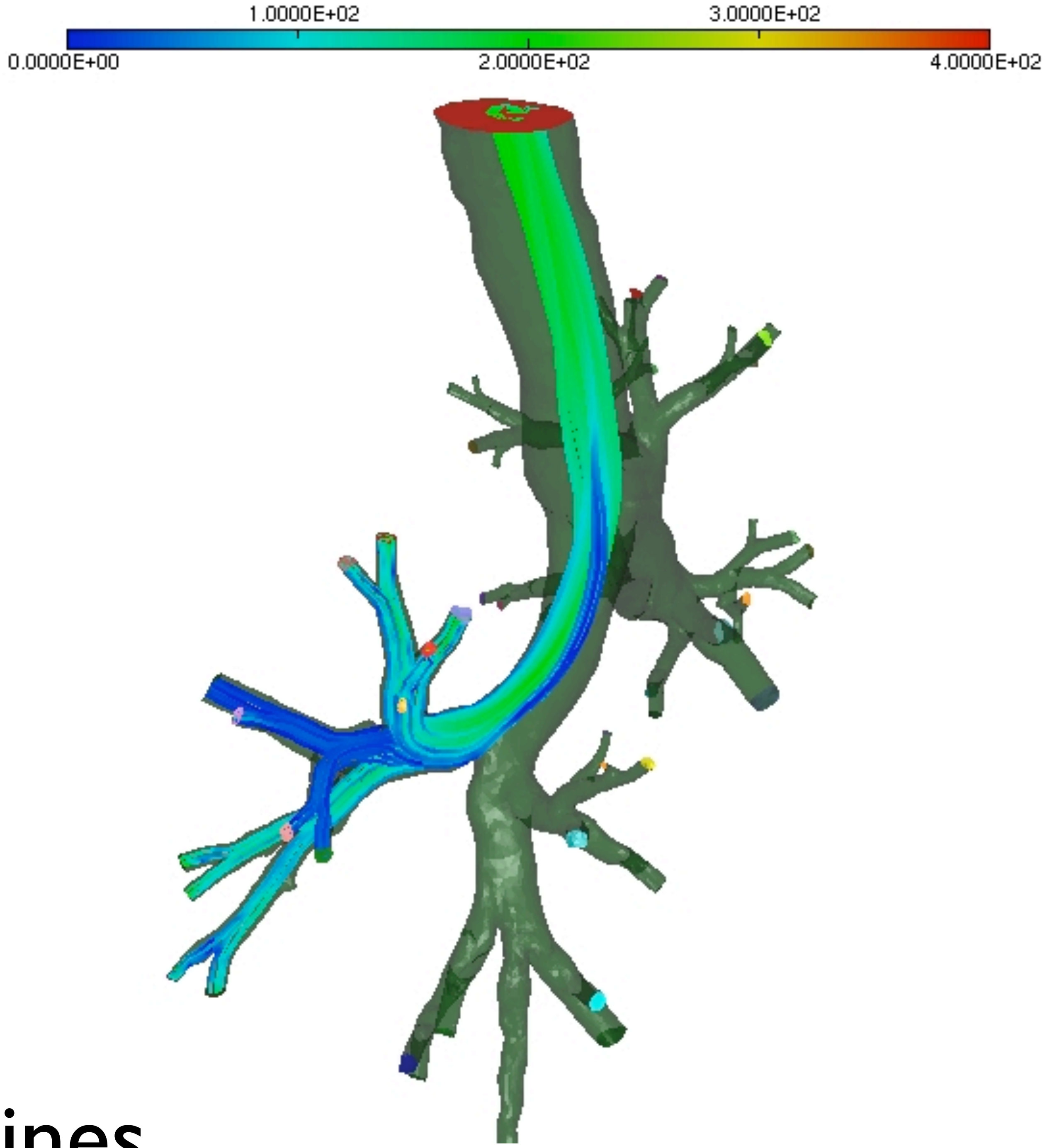


Pressure fields

Same geometry as in [C. Fetita, S. Mancini, D. Perchet, F. Preteux, M. Thiriet, L. Vial]

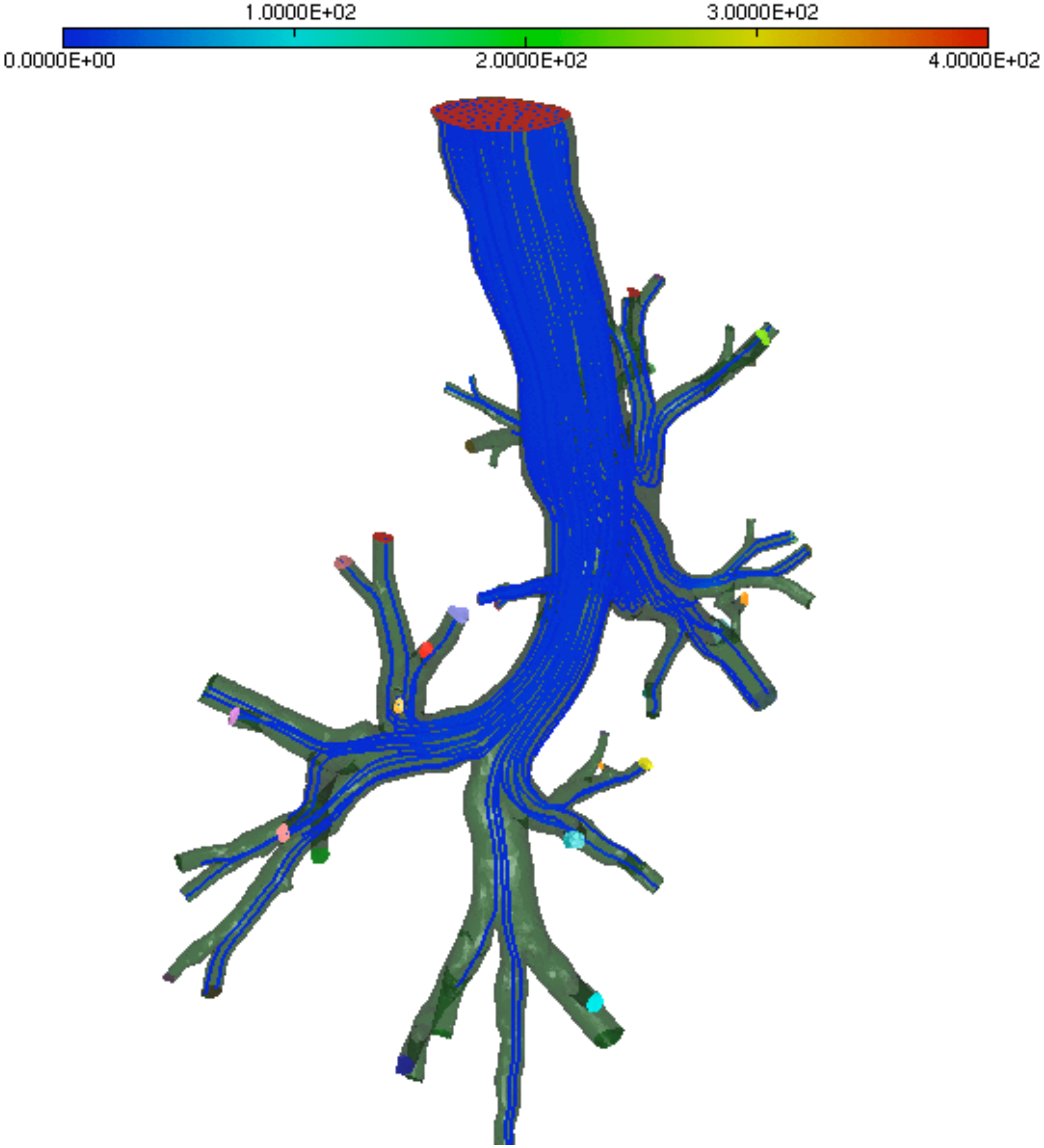
Numerical Simulations



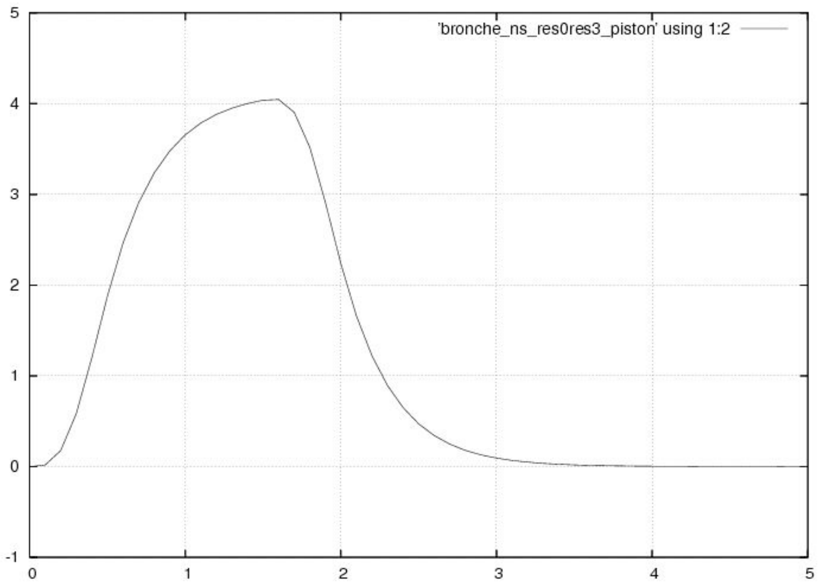


Streamlines

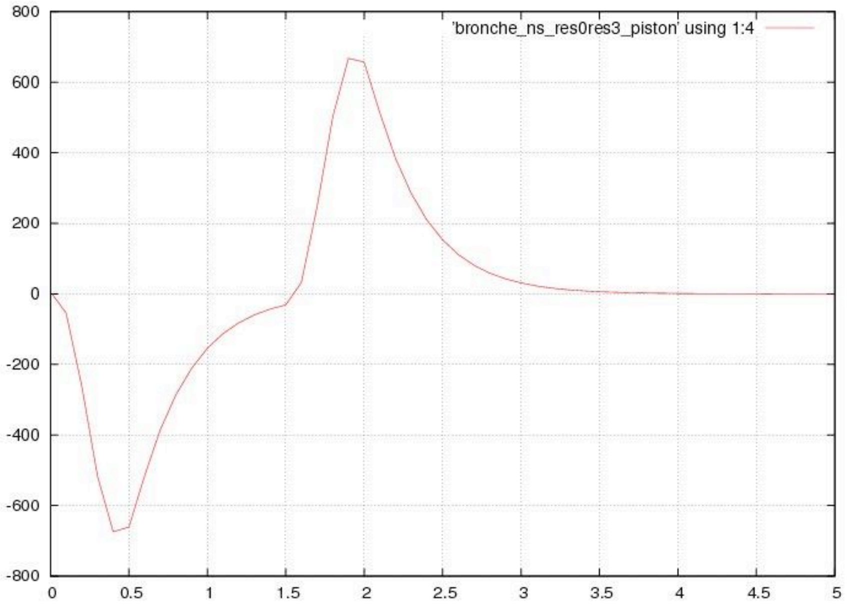
Numerical simulations



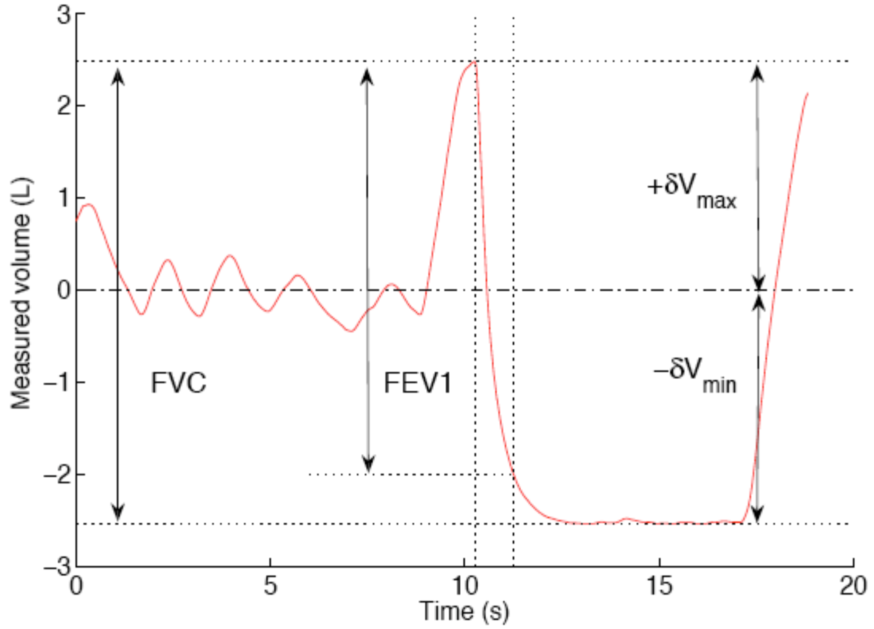
Displacement

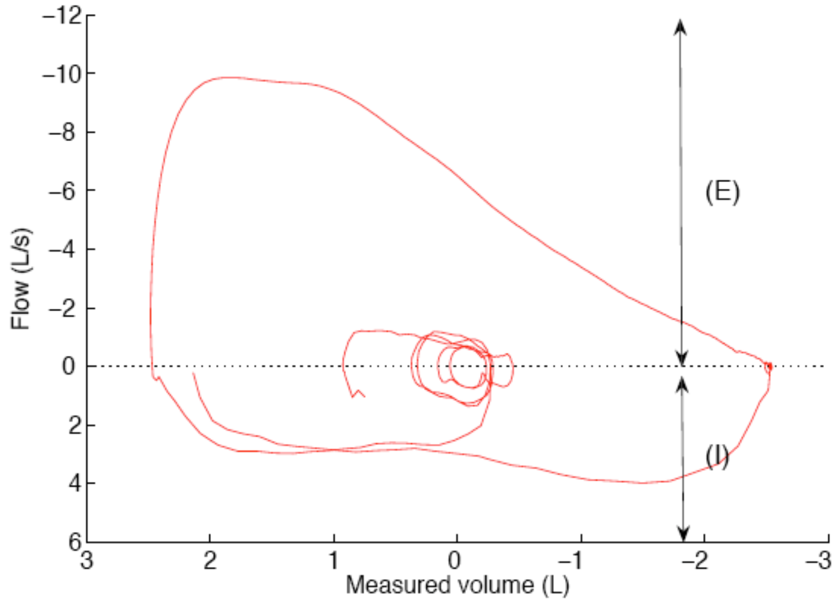


Flux



Improvements





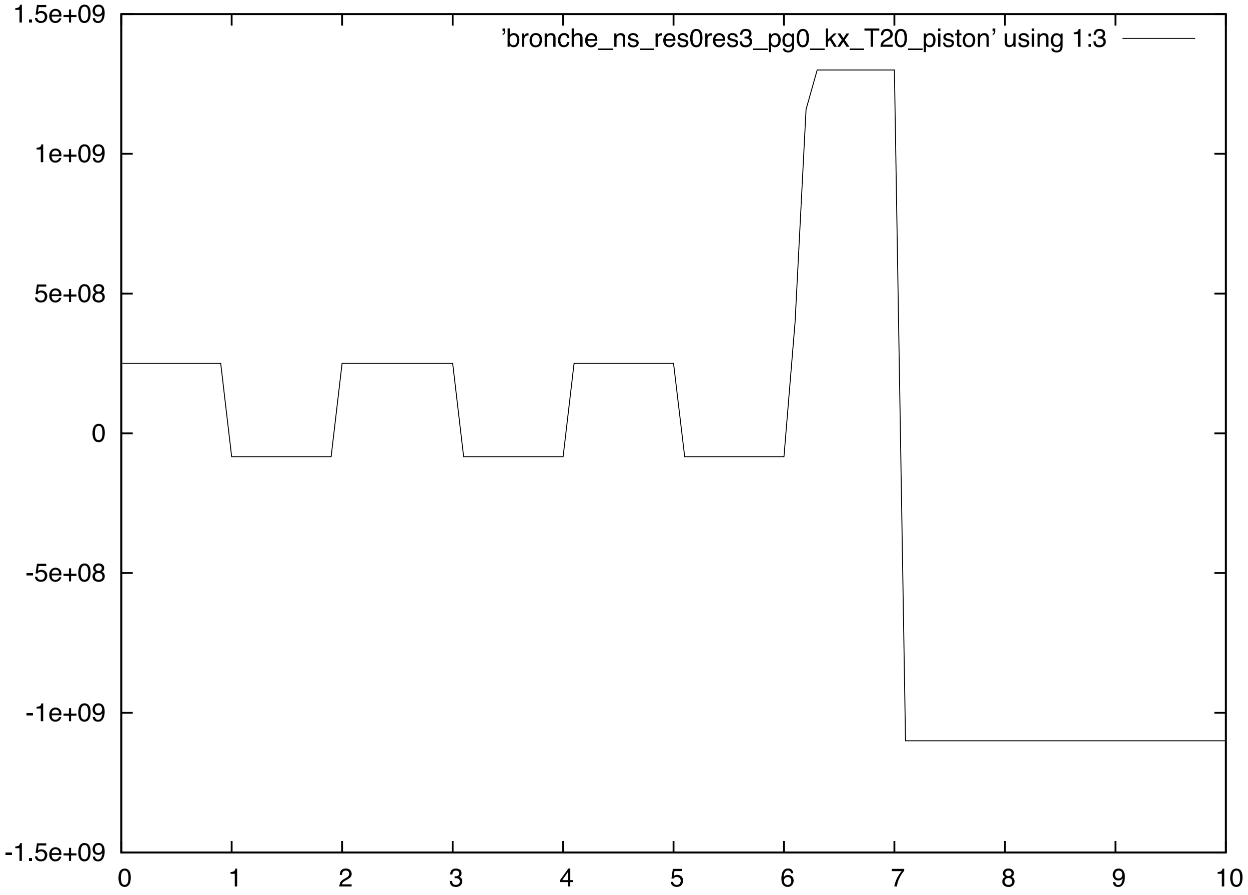
Following the work of S. Martin, M. Maury, T. Similovski, C. Strauss and in order to take into account the limitation of the lung volume we can consider a nonlinear spring constant

$$k(x) = k_0 + k^{(e)}(x) + k^{(i)}(x),$$

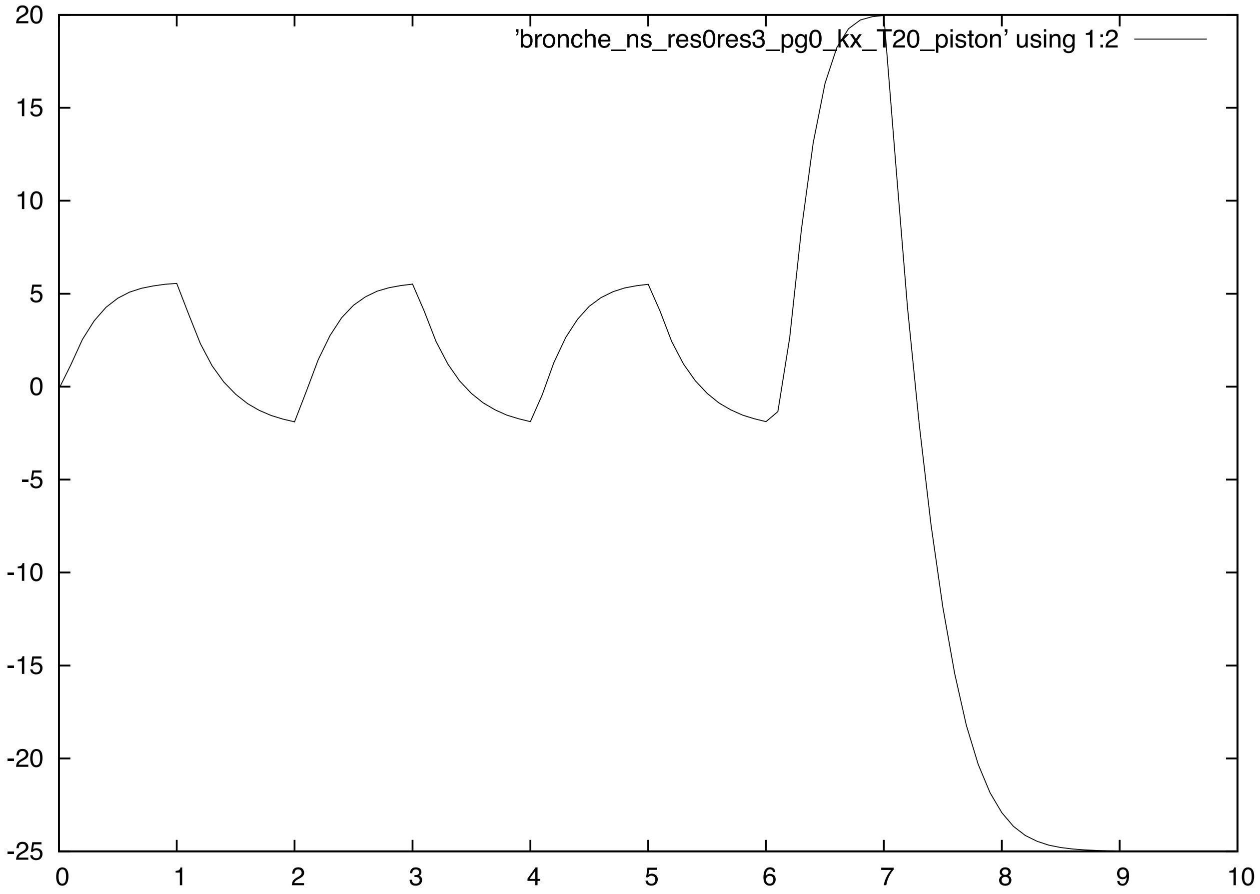
with

$$k^{(e)}(x) = \begin{cases} (f_{min}/x_{min} - k_0)x/x_{min}, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0, \end{cases}$$

$$k^{(i)}(x) = \begin{cases} 0, & \text{if } x < 0 \\ (f_{max}/x_{max} - k_0)x/x_{max}, & \text{if } x \geq 0 \end{cases}$$



External force



Air volume

Air - spray coupling

Ph.D. thesis of A. Moussa (supervised by L. Desvillettes and M. Filoche)

- Fluid Flow: Navier-Stokes equations

- Aerosol: Kinetic approach - Deposition on the bronchial wall, no secondary dropplets

Air - Spray Coupling

- Spray: probability density function $f(t, x, v, r, \dots)$ solving a kinetic equation (Vlasov, Boltzmann...)
- Fluid: density $\varrho(t, x)$, velocity $u(t, x)$, pressure $p(t, x)$, $\dots$ solving Navier-Stokes system

$$\begin{cases} \partial_t(\alpha\varrho) + \nabla_x \cdot (\alpha\varrho u) = 0 \\ \partial_t(\alpha\varrho u) + \nabla_x \cdot (\alpha\varrho u \otimes u) - \Delta_x u + \nabla_x p = \mathcal{F}_{\text{spray}} \\ \partial_t f + \nabla_x \cdot (fv) + \nabla_v \cdot (fa) + \partial_r(f\Phi) = Q(f, f) \end{cases}$$

Classification

- $Q(f,f)(t,x,v,r)$ droplets interaction operator
- $\alpha(t,x) = 1 - \displaystyle\int_{v,r} \frac{m(r)}{\varrho_{\text{drop}}}\, f(t,x,v,r)\; \mathrm{d}v\; \mathrm{d}r$
  fluid volume fraction (must remain near 1)
- $\mathcal{F}_{\text{spray}}(t,x) = - \displaystyle\int_{v,r} m(r)\; a(t,x,v,r)\; f(t,x,v,r)\; \mathrm{d}v\; \mathrm{d}r$
  force of the spray on the fluid

spray	$Q(f,f)$	α	$\mathcal{F}_{\text{spray}}$
very thin	0	1	0
thin	0	1	$\mathcal{F}_{\text{spray}}$
mod. thick	0	α	$\mathcal{F}_{\text{spray}}$
thick	$Q(f,f)$	α	$\mathcal{F}_{\text{spray}}$
two fluids with a mixture behaviour			

Are respiratory aerosols very thin or only thin sprays?

- Experimental studies give various answers!
- they are very thin sprays [Grotberg...].
- they are thin sprays [Chigier, Corcoran...].

Numerical Method

Numerical particle method

$$f(t,x,v) = \sum_{p=1}^{N} \omega_p(t)\ \delta_{x_p(t)}(x)\ \delta_{v_p(t)}(v)$$

- N total number of numerical particles,
- $t \mapsto (x_p(t), v_p(t))$ trajectory of the numerical p in the phase space,
- $\omega_p(t)$ representativity of particle p.

Coupling

Stagered scheme

Futur Work

- Use this model to obtain physiological results in the proximal part (by modifying the resistance and the spring stiffness)
- Couple it with aerosol transport

→ Comparison with experimental results
(INSERM Tours)