Colorado State University
Knowledge to Go Places

Inclined Plane Flows and Meandering

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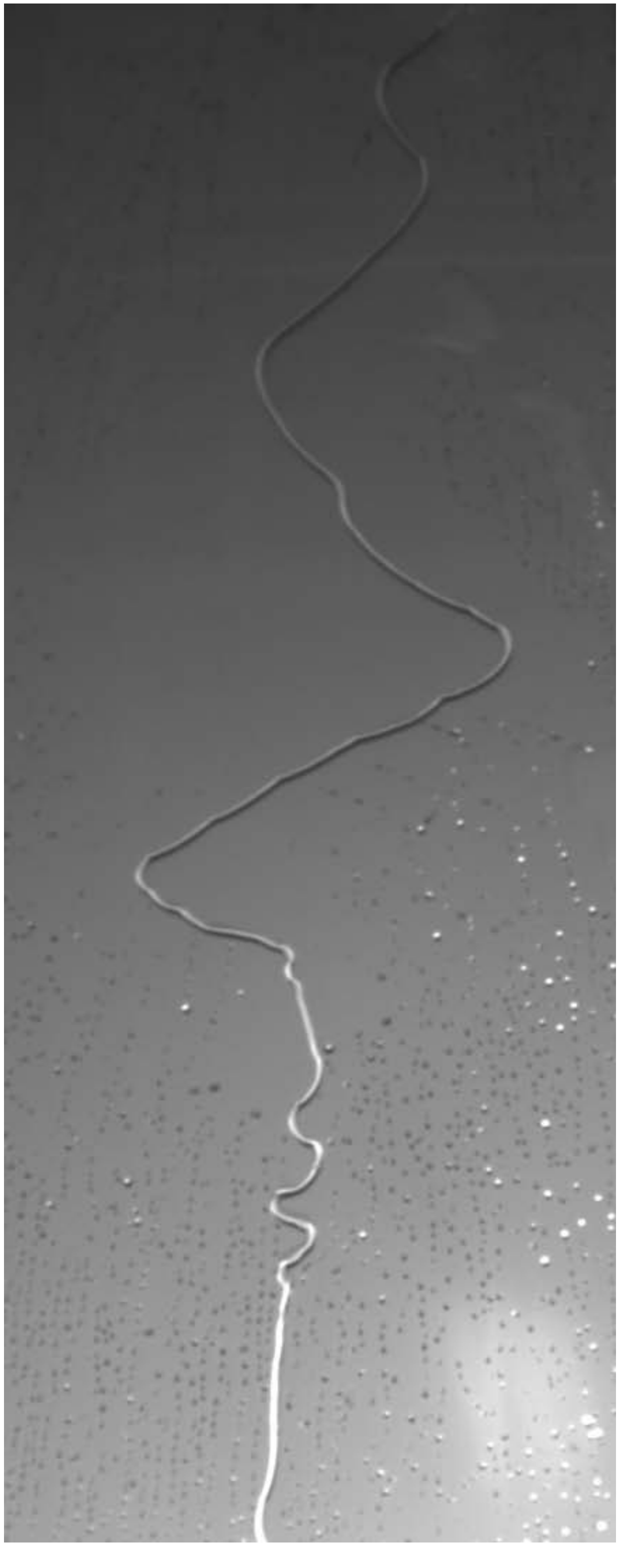
K. Mertens, V. Putkaradze and P. Vorobieff, *Nature* ***430***, 165 (2004), *J. Fluid Mech.* ***531***, 49 (2005)

B. Birnir, K. Mertens, V. Putkaradze and P. Vorobieff, *J. Fluid Mech* (to appear) (2008), *Phys. Rev. Lett., under consideration (2008).*



Brief Problem History:

Previous work: J. Thompson *(1876)*, A. Einstein *(1926)*, Gorycki *(1973)*, Parker *(1976)*, Ikeda, *JFM (1981)*

Long wavelength theory: (= complex equations) S.H. Davis, *and co-workers JFM,* ***98*** *, (1980), JFM* ***107****, (1981), JFM* ***176*** *, (1987).*

More Recently: Mizumura, Schmuki, Nakagawa *(1990's),*
Kim, Perazzo, Le-Grand-Piteira, *(2000-)*.

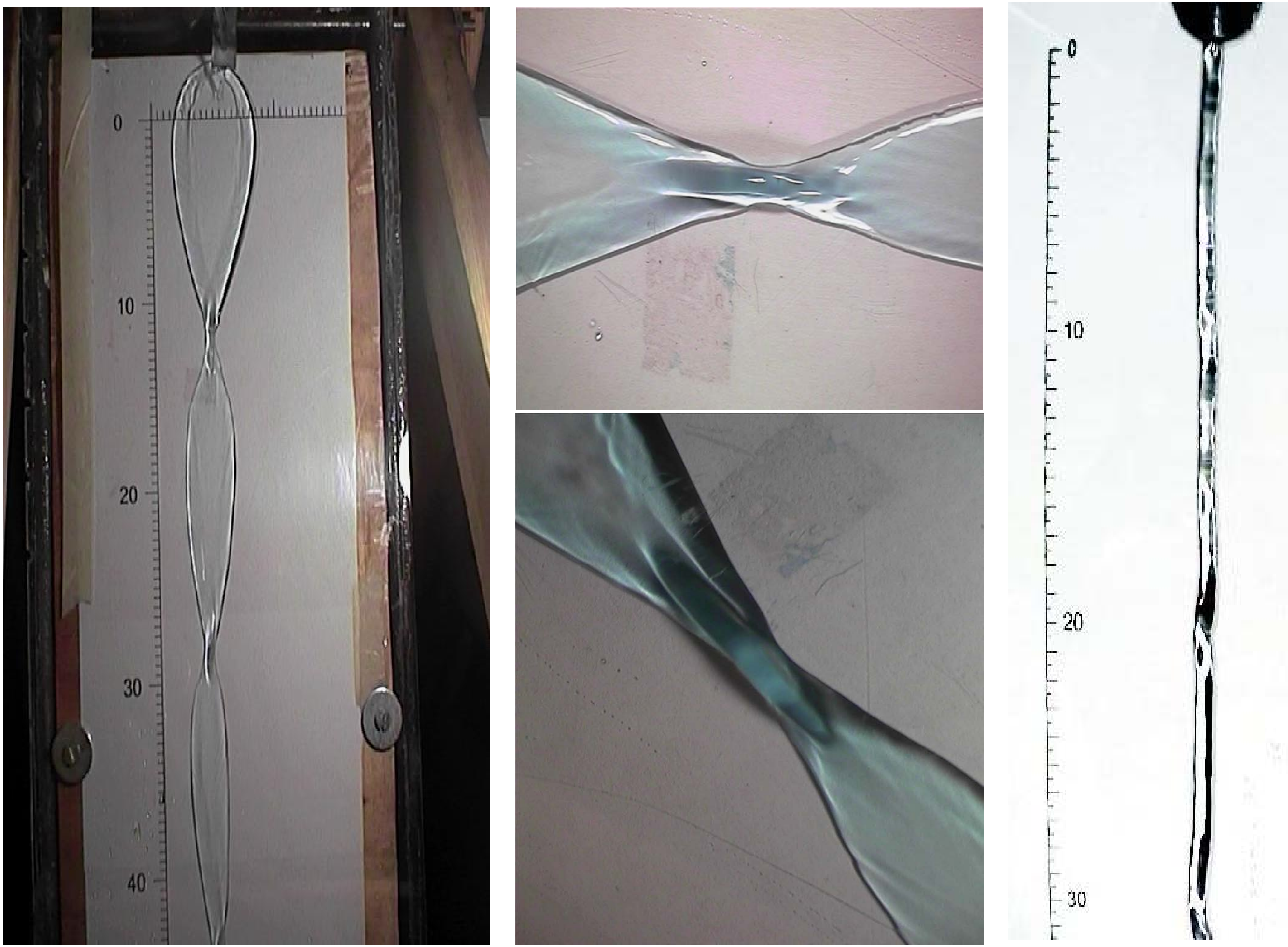
Meandering is a fundamental instability of the stream

Inertial effects in thin-film flows:
Bohr *et. al. PRL 79****,*** *(1997),* Lopez *et. al. Phys Fluids* ***9***, (1997).

Contact Angle: We assume contact angle to be constant; for more accurate assumptions see De Gennes, *Rev Mod Phys* ***57***, (1985).

River Meandering and Landscape Evolution: (Controversial)
Physics is different, but some scalings are similar
Birnir, *Comm. Pure App. Math* (2006), *Nonlinear Sci.* **17** (2007),
Dodds & Rothman, *Phys Rev E, Ann Rev EPS* (1999, 2000),
Hack, *Geological Survey Paper* (1957),

Braids and Rivulets: No Meandering



The Experiment

•Ranges Investigated:

- ➔ Flow Rates: 100ml/min-2L/min
- ➔ Viscosities: 0.01-0.05 cm^2/sec
- ➔ Inclination Angles: 15-60 degrees
- ➔ Jet Diameters: 0.05-0.3cm

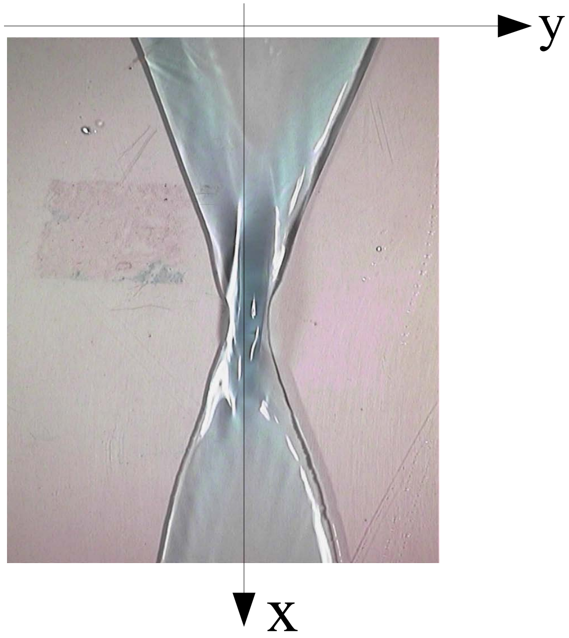
Parameters in this system:

1. Inclination angle: α
2. Fluid density: ρ
3. Kinematic viscosity: ν
4. Coefficient of surface tension: γ
5. Gravity: g
6. Contact angle: θ
7. Volume flow rate: Q

•Additionally Note:

- ✓ Fluid nozzle kept approximately few mm (or direct contact) from the plate to reduce pearling instabilities
- ✓ Upper reservoir is 1.5 meters tall to reduce flow disturbances in the flow
- ✓ Transient times can be quite large, of order hours. All our results are valid only after transitional times have passed

Assumptions for a Stationary State Model:



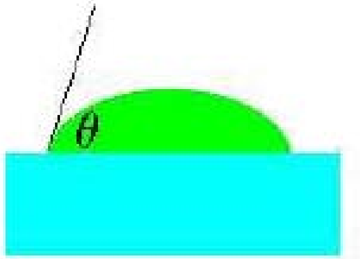
1. Stationary State: $\Rightarrow \frac{\partial}{\partial t} = 0$

2. Symmetry Condition: h(x,y)=h(-x,y)

3. Lubrication approximation in *y* and *z* directions

4. Flux in downstream dimension is conserved $\langle U \rangle(x)\, A(x) = Q = constant$

5. Contact Angle Condition: $\frac{dh}{dy}(\pm w) = \mp \tan(\theta)$



Details on the Model Construction: Kinematics I

$$\frac{d(\boldsymbol{U})}{dt}+\boldsymbol{U}\nabla\boldsymbol{U}=\frac{1}{\rho}\nabla P+g sin(\alpha)\hat{e}_x+\nu\nabla^2\boldsymbol{U}$$

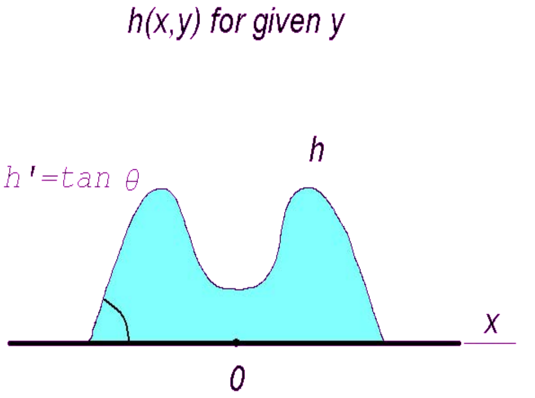
$$\nabla\cdot\boldsymbol{U}=0$$

We start with the boundary layer approximation parallel to the plane along with the incompressibility condition

We then postulate a fourth order free sufrace profile based on observation:

$$h(x,y)=(w^2-y^2)(a-by^2)$$

With the use of assumptions 3-5 we can explicitly determine the free parameters a and b in term of real system variables.



First, using contact angle condition $h'(w)=-\tan\theta$ we can eliminate a:

$$a=\frac{\tan\theta}{2w}+2bw^2$$

Details on the Model Construction: Kinematics II

Given the width *w(x)*, cross-section area at a given x is

$$A(x)=\int_{-w}^{w} h(y)\,dy=\frac{w^2\tan(\theta)}{3}+\frac{8\mathrm{b}\mathrm{w}^5}{15}$$

The conservation of flux

$$Q=AU_x=constant$$

allows to solve for the second parameter b:

$$b=\frac{15}{8\mathrm{w}^5}\left(\frac{Q}{U_x}-\frac{\tan(\theta)w^2}{3}\right)$$

Next, knowledge of *h(x,y)* allows calculation of the average surface tension force acting on each cross section:

$$F_{st}(U_x,w)=\gamma\,\partial_x\kappa=\int_{dS}\gamma\,h(x,y)'''h(x,y)\,dS$$
$$=3\mathrm{b}\,\gamma\tan(\theta)w^3+4\mathrm{b}^2\gamma\,w^6$$

Equations of motion (component form):

Downstream equation (*x*)

$$U_x \frac{\partial U_x}{\partial x} = g sin(\alpha) - \nu \frac{\partial^2 U_x}{\partial z^2}$$

Transverse direction (*x*)

$$(\rho A) U_x \frac{\partial}{\partial x} U_y = F(w, U_x) - \nu \frac{\partial^2 U_y}{\partial z^2} (\rho A)$$

Kinematic condition for each cross-section gives estimate for the transverse velocity

$$U_y = U_x \frac{dw}{dx}$$

Approximations on 2nd Derivatives

Because the free surface height h(x,y) is much smaller than the other characteristic length scales of the problem we can use the lubrication approximation to estimate the second derivative terms:

Velocity profile chosen such that:

1. no slip is satisfied on fluid/substrate boundary
2. shear stress $U'(z)$ is zero on free surface $z=h$.

$$U_x(z)=U_s z(h-z/2)$$

In this section, we further approximate $U_x(z)$ using the average value

$$\frac{1}{h}\int_0^h U_x(z)\,dz=\langle U_x(z)\rangle:=U_x$$

This *lubricaiton approximation* gives second derivatives as

$$\frac{\partial^2 U_x}{\partial z^2} \approx -3\,\frac{U_x}{h_{avg}^2}$$

Now using the kinematic relation between U_x and U_y velocities we also get

$$\frac{\partial^2 U_y}{\partial z^2} \approx -3\,\frac{U_y}{h_{avg}^2} = -3\,\frac{U_x}{h_{avg}^2}\frac{dw}{dx}$$

Where

$$h(0) \approx h_{avg} \approx \frac{A}{w} \approx \frac{Q}{U_x w}$$

Equations of motion, complete form

$$U_x \frac{dU_x}{dx} = g\sin(\alpha) - 3\nu \frac{U_x}{h_{avg}^2}$$

$$(\rho A) U_x \frac{d}{dx}\left(U_x \frac{dw}{dx}\right) = F(w, U_x) - \nu \frac{U_x}{h_{avg}^2} \frac{dw}{dx} (\rho A)$$

Additionally, use geometric relationship

$$h_{avg}^2 \tan\theta \approx A$$

Rescaling

We rescale velocity and length to make equations dimensionless
We choose the following scales

$$V = \frac{g \sin\alpha \, \rho \, Q}{\gamma} \qquad L = \frac{g \sin\alpha \, \rho^2 Q^3}{\gamma^2}$$

There is also an implicit length scale present in this problem to characterize the height of the flow. This can be thought of as a modified capillary length

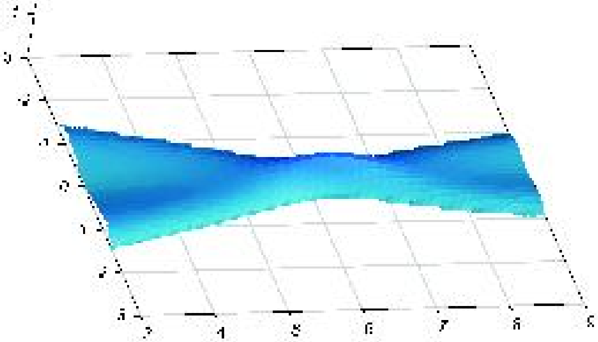
$$Z = \sqrt{\frac{\gamma}{\rho \, g \sin\alpha}}$$

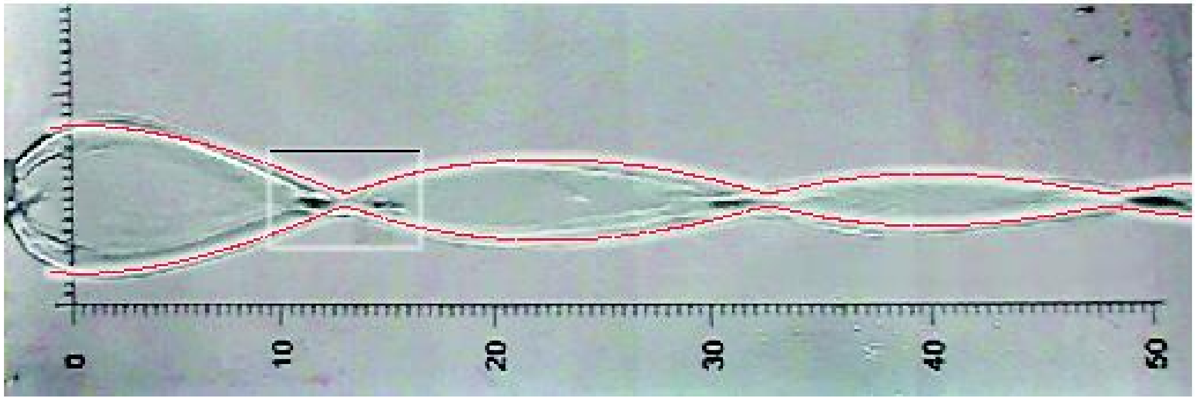
Our scales define two dimensionless parameters

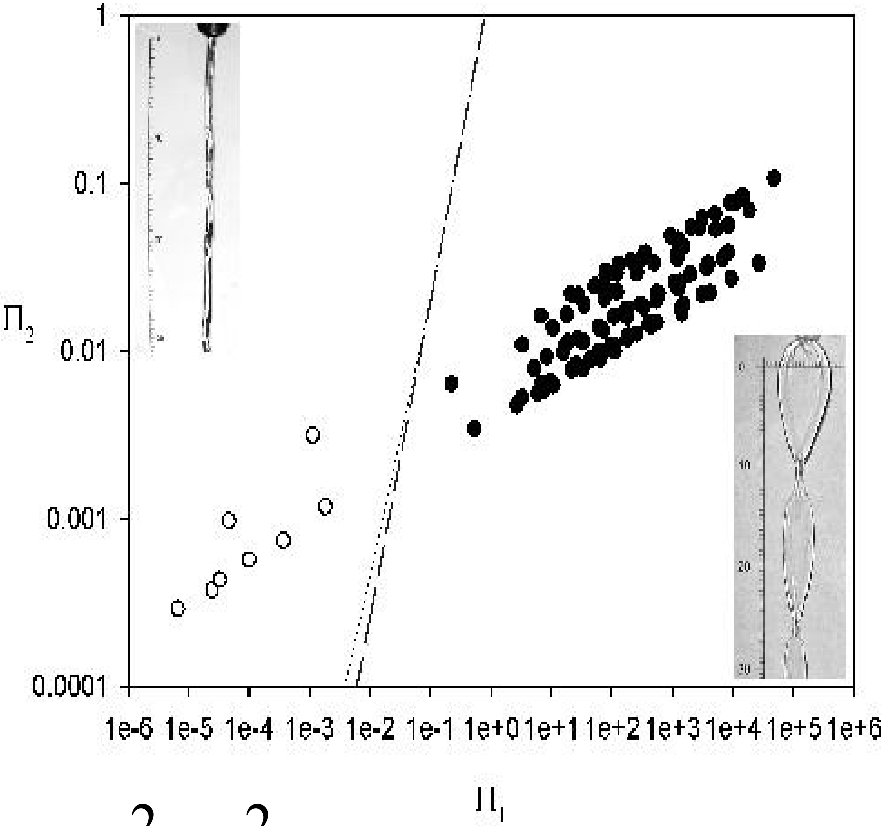
$$\pi_1 = \frac{3 \nu \rho^7 Q^5 (g \sin\alpha)^4}{\gamma^7} \qquad \pi_2 = \frac{3 \nu \rho^2 Q \, g \sin\alpha}{\gamma^2}$$

We can also define the material constant $\pi_1 \pi_2^{-5}$

Results for stationary states (braiding)







$$(u \cdot w')' = F(u, w) - \pi_1 u^2 w^2 w'$$

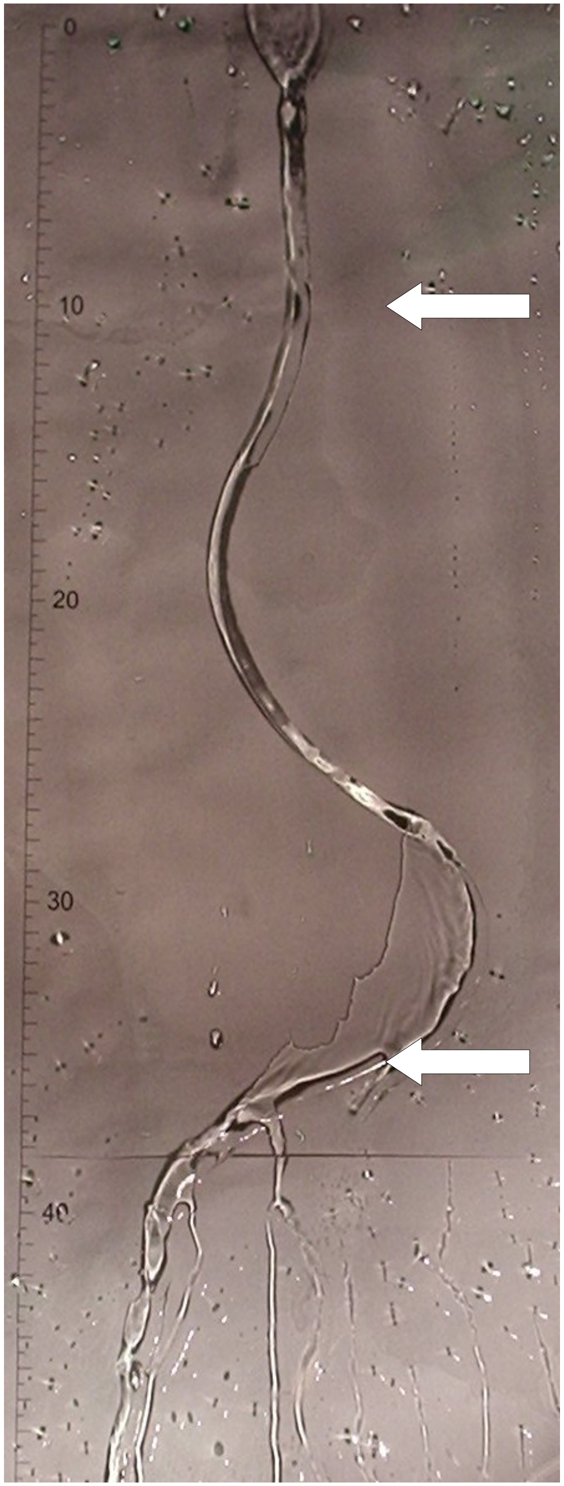
$$u \cdot u' = 1 - \pi_2 u^3 w^2$$

Stationary states (*u′=w′=…=0)* are always stable (rivulets).
Stability changes from damped oscillations (braids)
to over-damped oscillations (rivulets)

Meandering

Goals

1) Understand basic driving mechanism.
2) Experimentally quantify meandering exponent.
3) Generalize the stationary equations of motion
   to create a model
4) Establish any universal properties of these flows
   and/or similarities to flows in nature.



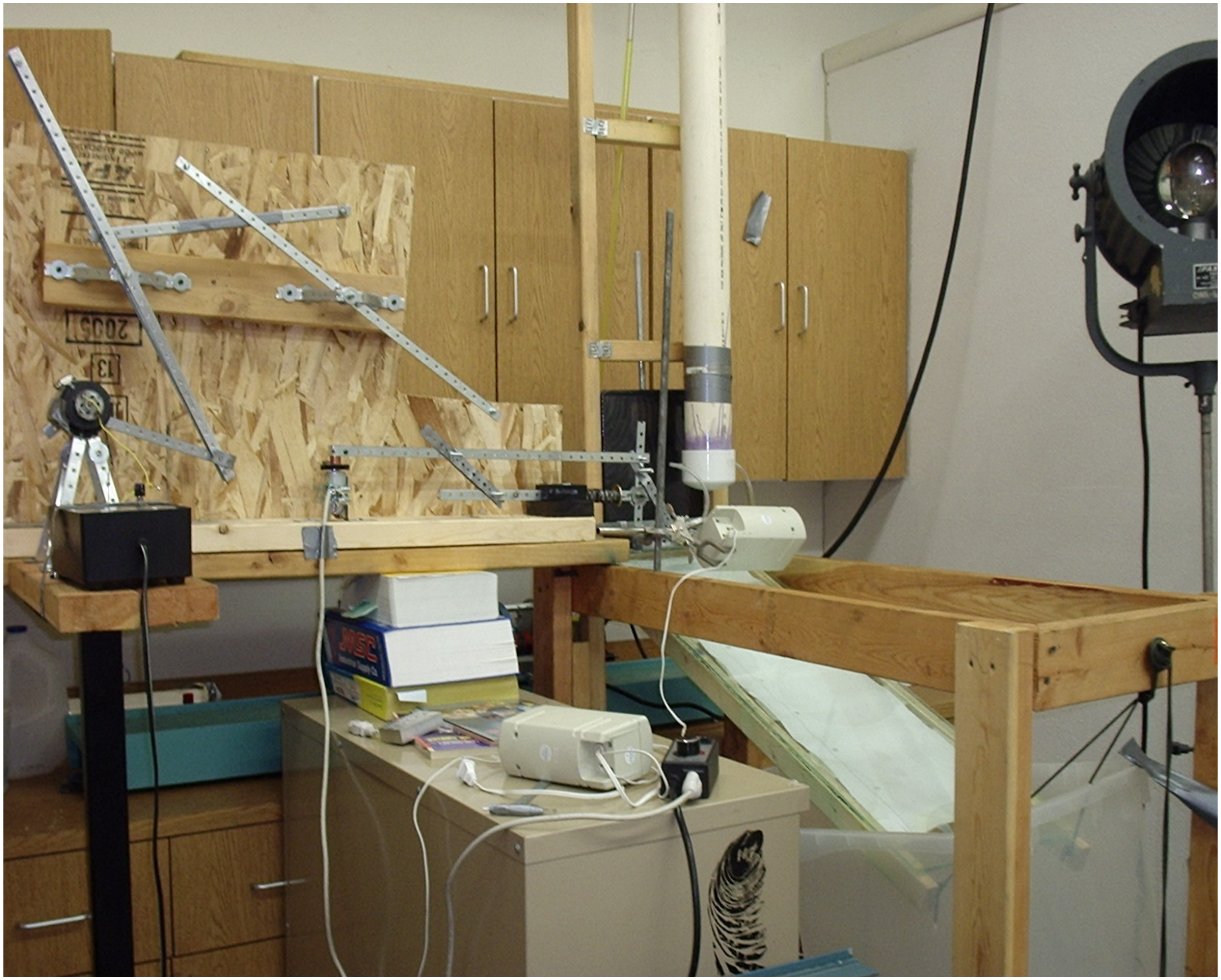
B. Birnir, K. Mertens, V. Putkaradze, P. Vorobieff, *Meandering Streams*, *Phys Rev Lett*, under consideration

B. Birnir, K. Mertens, V. Putkaradze, P. Vorobieff, *Morphology of a Stream Flowing Down an Inclined Plane. Part2: Stream Meandering, Journal of Fluid Mechanics,* to appear


FIRE EXTINGUISHER
ACID
FLAMMABLE
KEEP FIRE AWAY

Initial Results on Stability

Perturbations:



Sound
Frequency: <4000kHz
Power: ~10W

Mechanical
Amplitude: several values between 1mm and 5cm

Frequency: all spectra 0-100Hz

The flow is surprisingly stable...

The Return of Meandering

We re-introduce disturbances in the flow by squeezing the nozzle tube with a relay switch

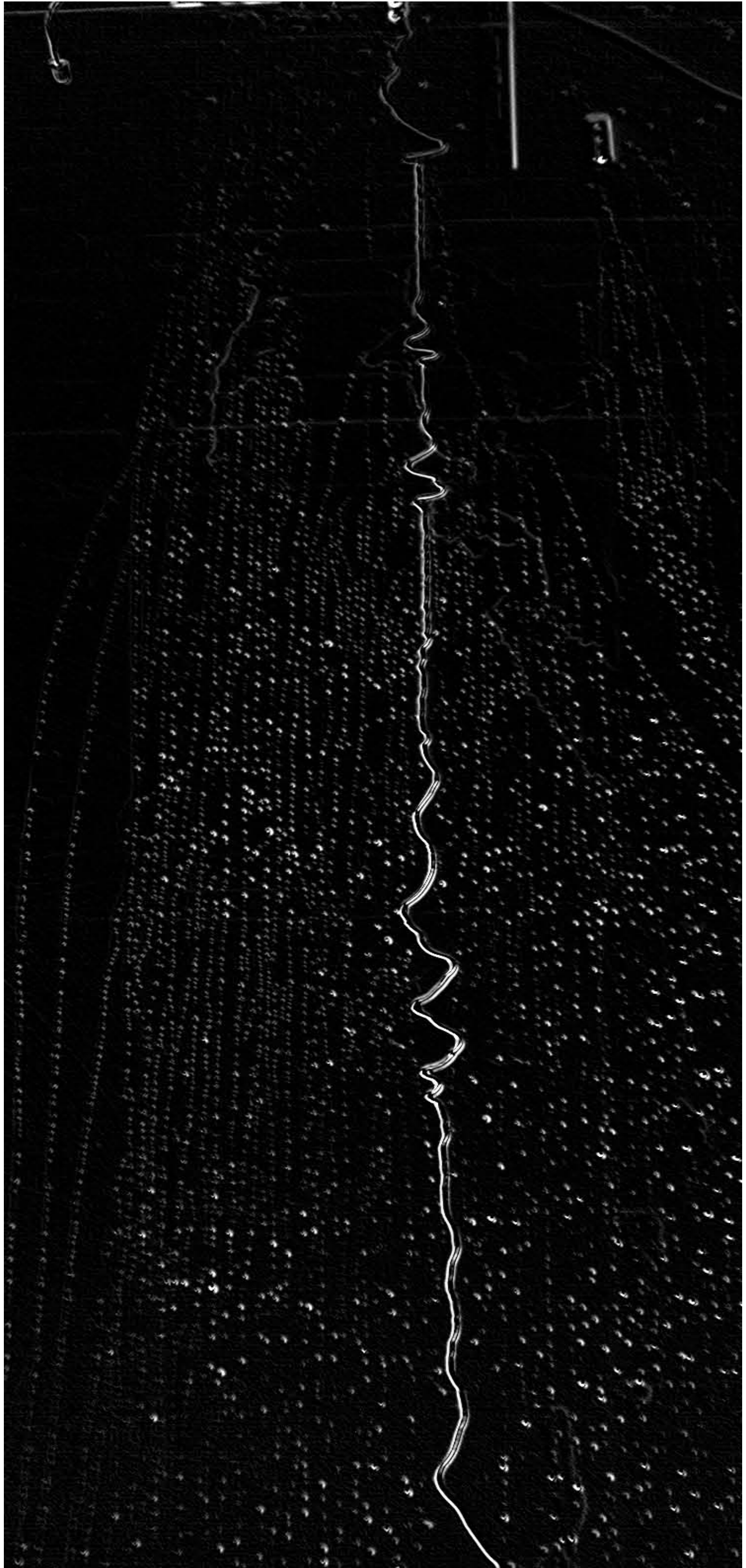
Frequency of squeezing is 6-240 times per minute

Each squeeze blocks about 10% of the flow

Upon the introduction of pulses into the flow rate destabilization occurs almost immediately and indefinitely.

This gives strong indication that pulsation in the flow is responsible for the onset of meandering.

Observe the now meandering stream has the additional complication of droplets left from where it has been.



Foundations for a Model

$$\frac{d\mathbf{U}}{dt}+\mathbf{U}\nabla\mathbf{U}=\frac{1}{\rho}\nabla P+g\sin(\alpha)\hat{e}_x+\nu\nabla^2\mathbf{U}$$

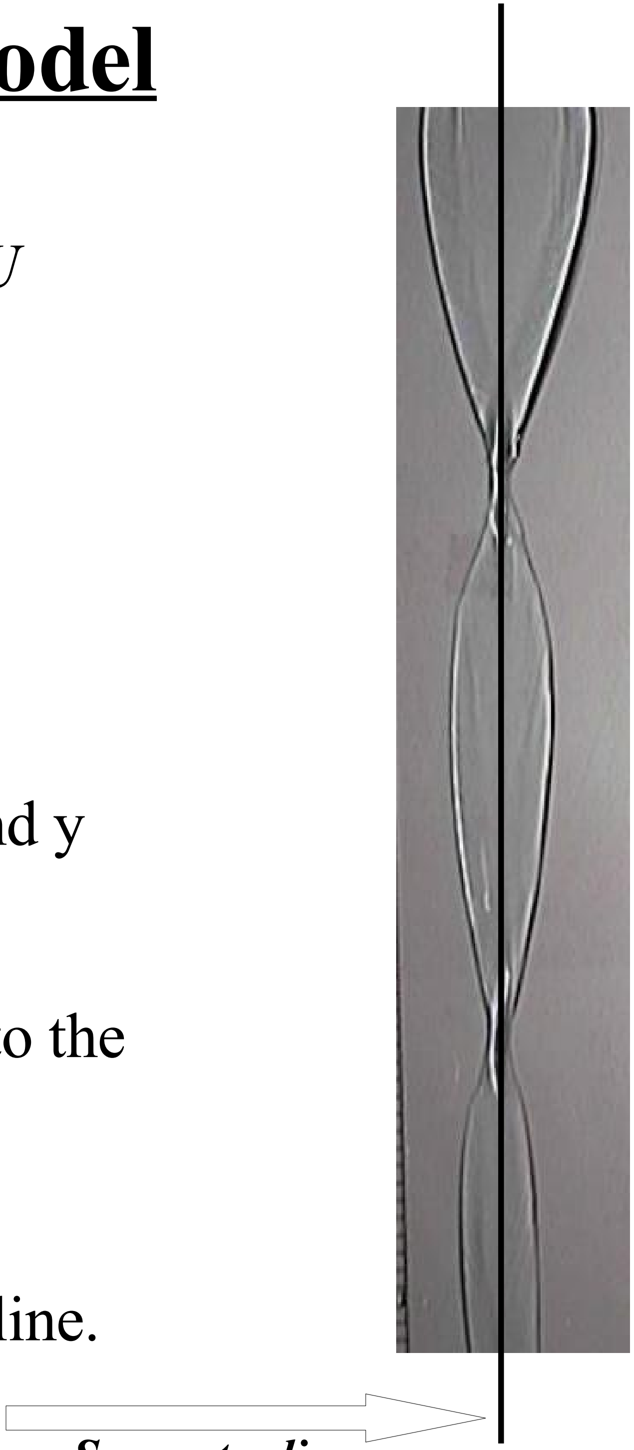
Dominant contributions to force balance will again come from surface tension and inertia under the influence of gravity

Initial Considerations:

1. We will use lubrication approximation in z and y to reduce to 1+1 dimensional model (PDEs).

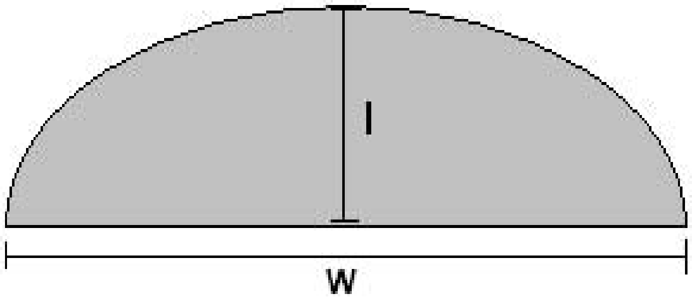
2. Droplets are modeled as a stochastic forcing to the PDE by colored noise.

3. Equations are sought for the velocities and deviation of stream center from the symmetry line.



Symmetry line

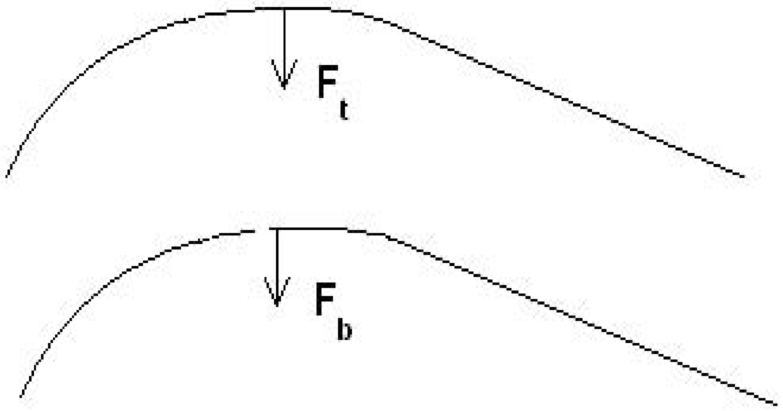
Surface Tension Forces



Let us assume the pressure is a result of the difference of surface tensions on either side of the stream.

Further assume the average cross-sectional area can be approximated by:

$$A = l\, w$$



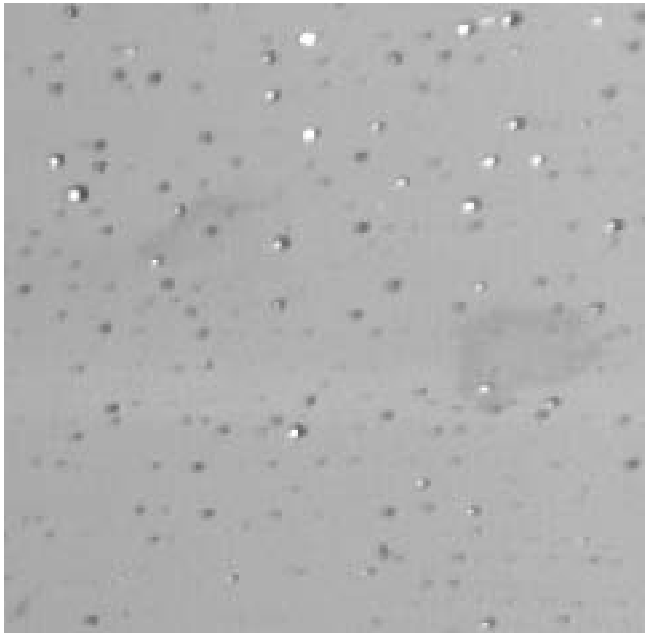
The energy per unit length is given by

$$E = \gamma\, w \int_{x_1}^{x_2} \sqrt{1+h_x^2}\, dx$$

The force/volume can then be found as:

$$\frac{1}{A}(F_t - F_b)\, dx = \frac{1}{A}\frac{\delta E}{\delta h} = -\frac{\gamma}{l}\frac{\partial}{\partial x}\left(\frac{h_x^2}{\sqrt{1+h_x^2}}\right)$$

Modeling the droplets



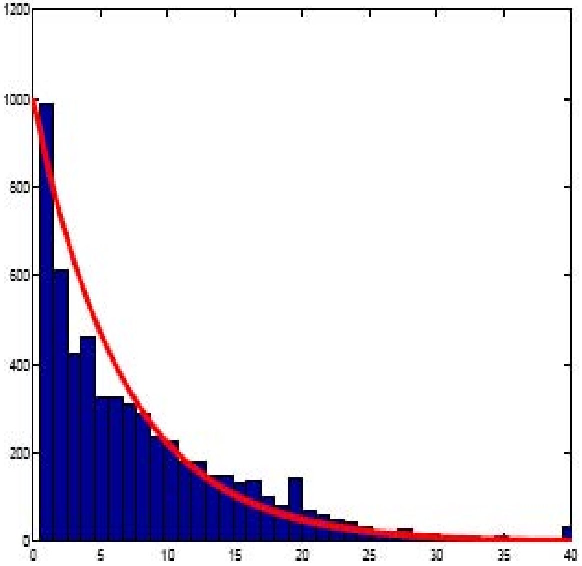
What is the proper way to model the influence of droplets all over the surface?

There are two types of distributions to consider:

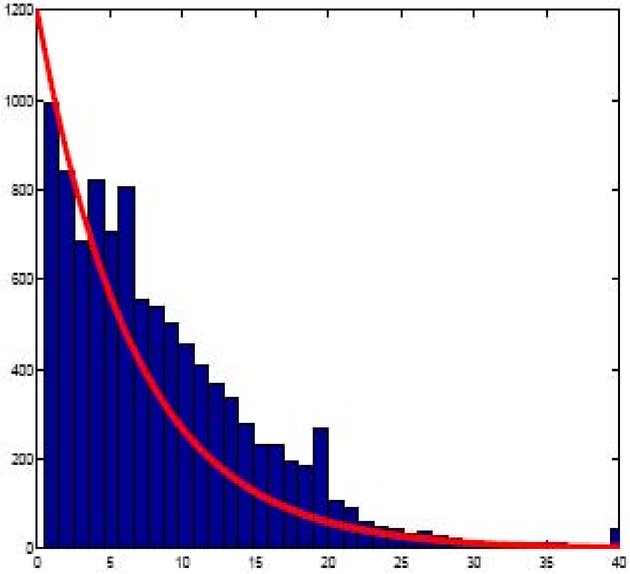
1. Droplet distribution in size
2. Droplet distribution in space

For simplicity our *spatial droplet distribution* was assumed roughly *uniform*.
In simulation's the *average drop size* has been used rather than introducing an extra droplet size distribution parameter

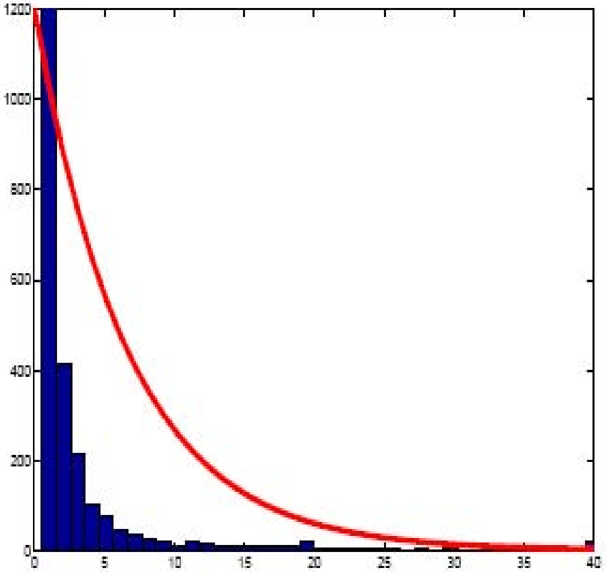
Acrylic (57deg)



Rain-X acrylic (74 deg)



Polypropylene (99 deg)



Equations of motion

The equations of motion for the two velocity components (*u,v*) coupled with an equation for the deviation from the symmetry line:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=gsin(\alpha)-\lambda u^2+\nu\frac{\partial^2 u}{\partial x^2}+\frac{1}{A\rho}\frac{\delta E}{\delta h}\cos\theta+\eta_x$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}=-\lambda u v+\nu\frac{\partial^2 v}{\partial x^2}+\frac{1}{A\rho}\frac{\delta E}{\delta h}\sin\theta+\eta_y$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}=v$$

where $\lambda=\frac{18\nu}{Q\tan\phi}\left(1+\cot^2\theta\right)$ is the dissipation coefficient

(Use lubrication in both *y* and *z* directions)

Simplifications through re-scaling

Assuming $u >> v$, we can rescale:

$$v \rightarrow \epsilon v$$

Keeping only lowest order terms gives simplified system

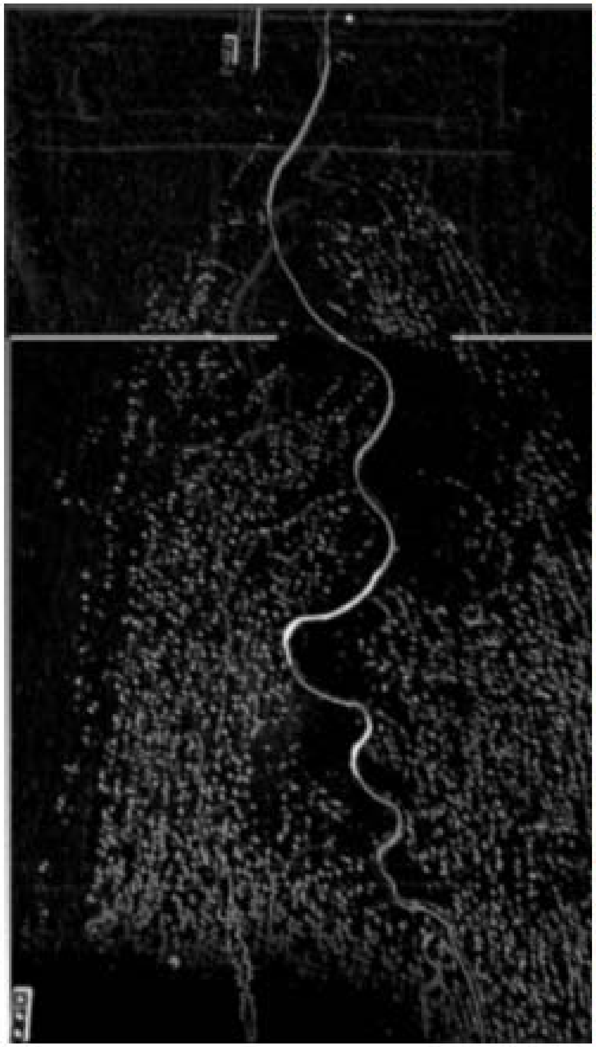
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g\sin(\alpha) - \frac{\lambda u^2}{Q} + \nu\frac{\partial^2 u}{\partial x^2} + \eta_x$$

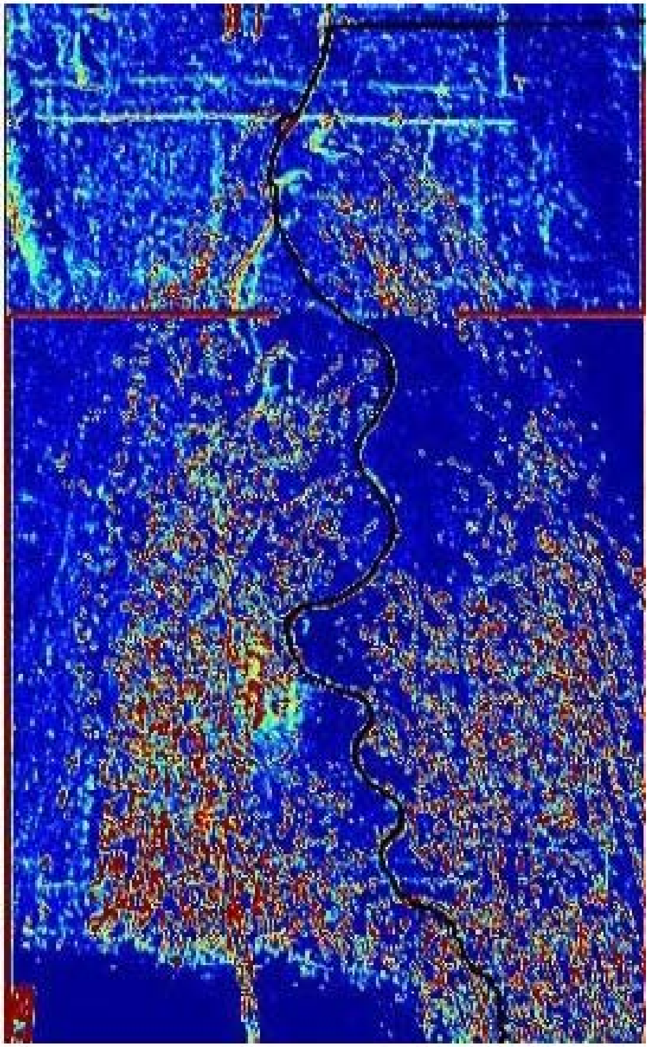
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = -\frac{\lambda u v}{Q} + \nu\frac{\partial^2 v}{\partial x^2} + \frac{\gamma}{\rho l}\frac{\partial^2 h}{\partial x^2} + \eta_y$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v$$

Simulate this system with the noise being produced by few fixed size droplets appearing at random time points and random positions along the stream.

Data Analysis

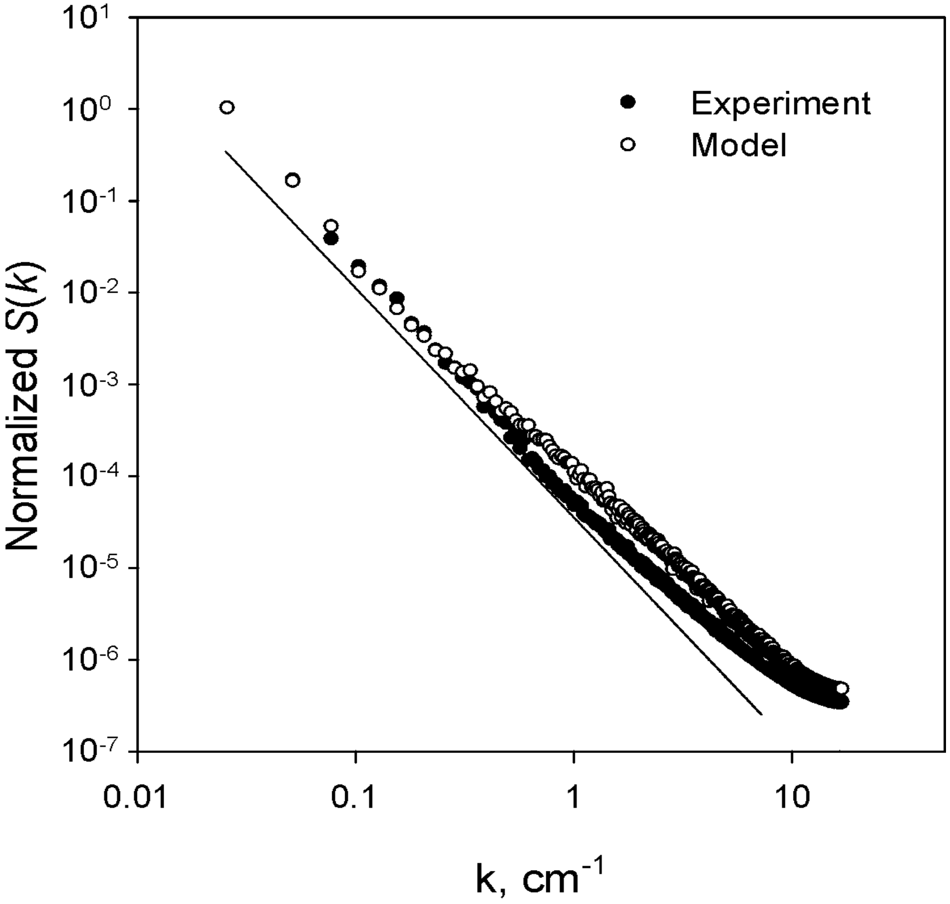


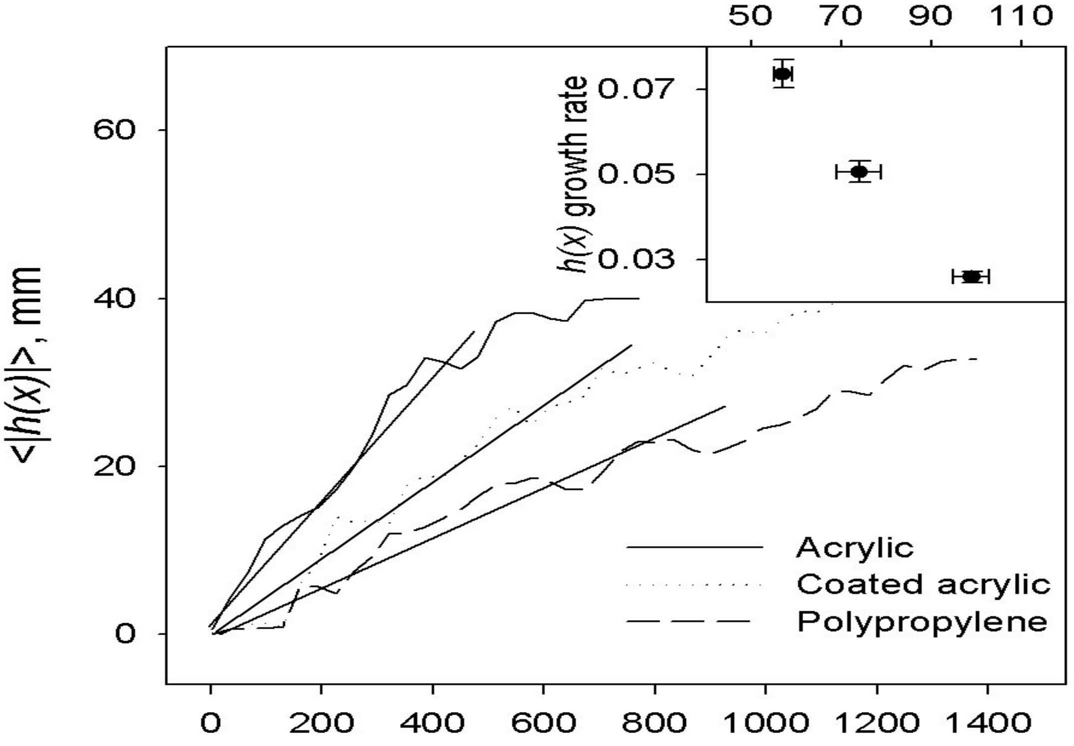


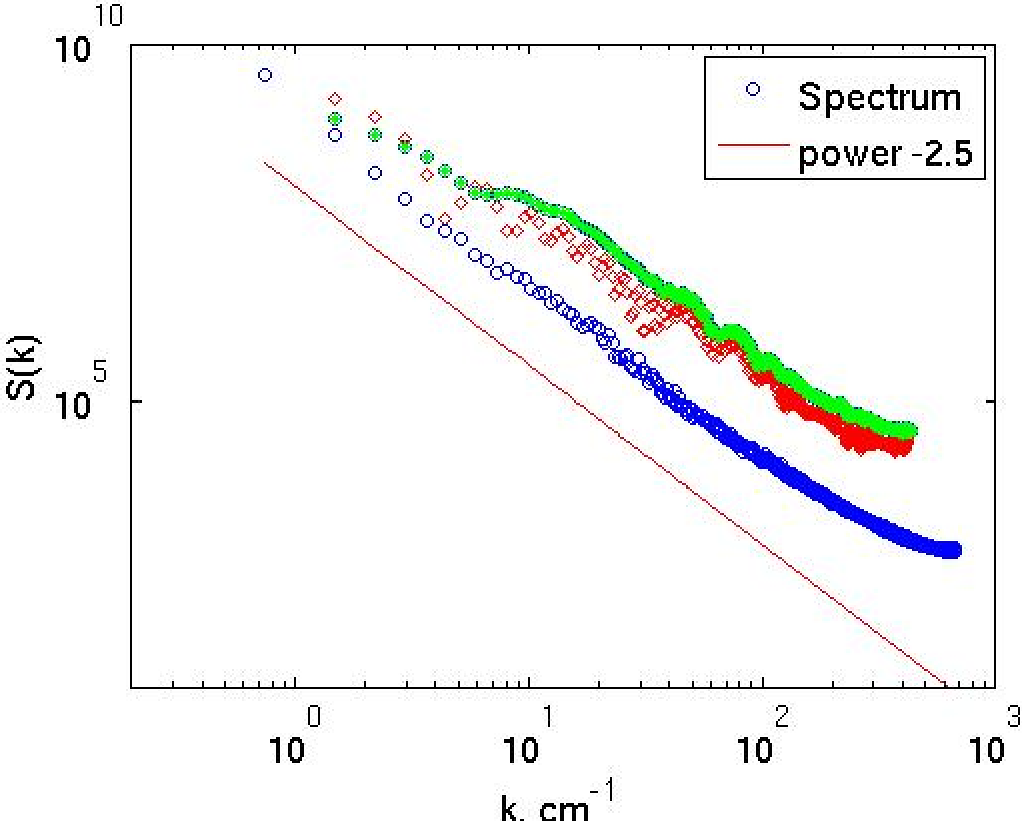
BMP Image → Gray-scale map + Matlab → Centerline Curve

Then we take FFT of each digitized centerline curve and do an ensemble average

Meandering Spectrum







Above data shows a -5/2 power law. For this experiment, this definitively answers the question about the existence of a preferred wavelength.

Contact angles: Green= 57deg
Red= 74deg
Blue= 99deg

Contact Angle and Surface Properties

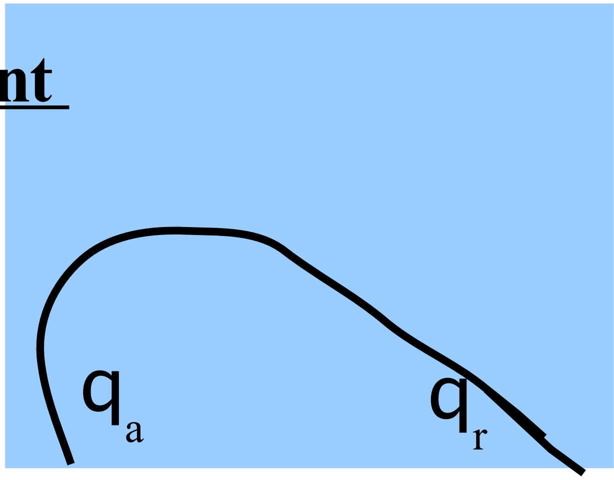
Does this mean the contact angle doesn't play a significant roll in meandering?

While the surface properties do affect both the amplitude and time scale of the meandering rivulets, the spectrum remains unchanged

This meandering spectrum appears to be a ***universal characteristic*** of flows on partially wetting surfaces.

Contact angles: left to right 57deg, 74deg, 99deg

Taking the hysteresis of contact angle into account



Use more detailed energy methods to minimize surface energy for each y-z cross-section.

This again implies finding a curve which minimizes the functional

$$E=\gamma\int_{-w}^{w} y\sqrt{1+y_z^2}\,dy$$

With the use of Lagrange multipliers this means solving the Euler Equation

$$\frac{\partial\phi}{\partial y}+\frac{d}{dy}\frac{\partial\phi}{\partial y_z}=0$$

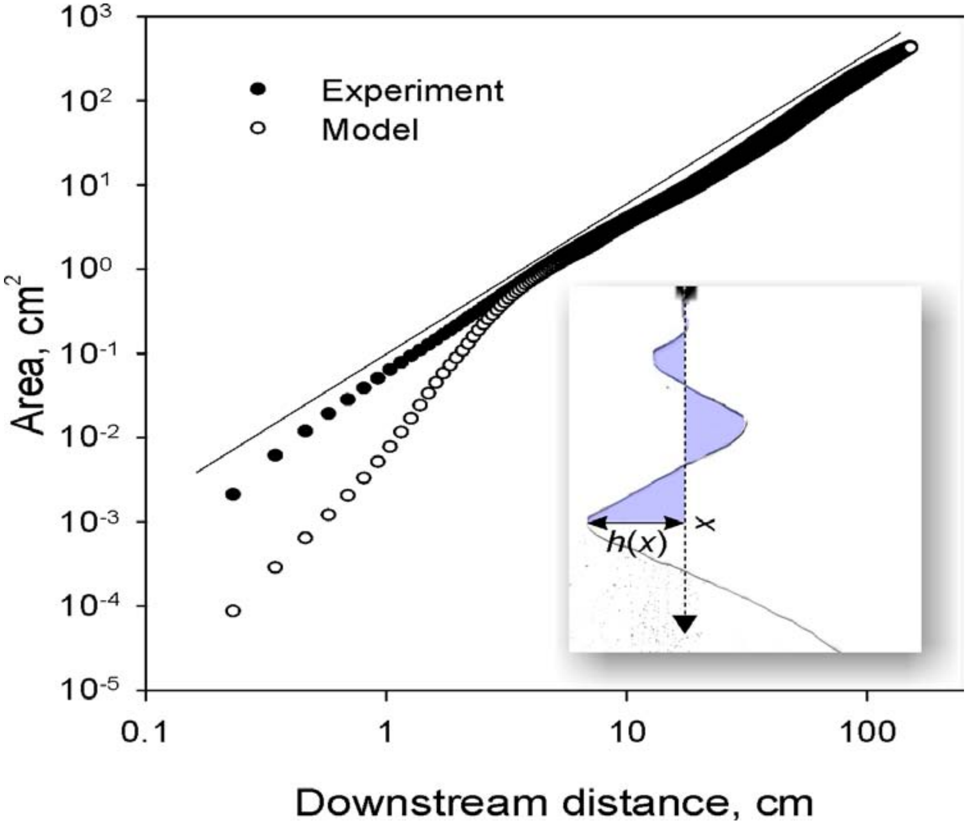
$$\phi(\lambda)=E-\gamma\int_{-w}^{w}\lambda\sqrt{1+y_z^2}\,dy$$

Additionally we need boundary conditions

This is not easy!

$$\frac{dy}{dz}(-w)=\theta_a(t)\quad\frac{dy}{dz}(w)=\theta_r(t)$$

Hack's Law



$$A_{basin} \sim l^{(1+\xi)}$$

3 Regimes of Hack's Law:

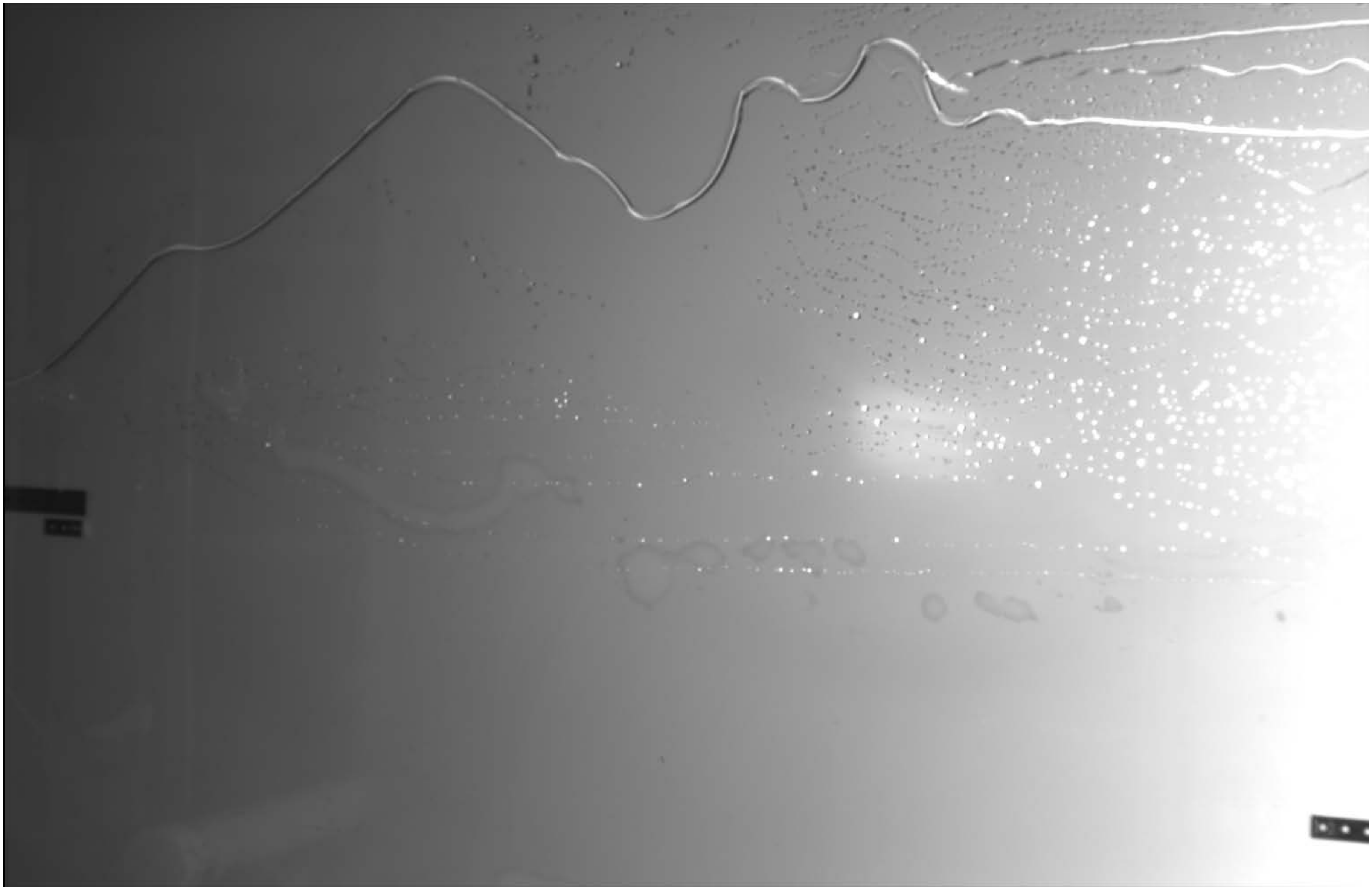
1. Channelization with exponent 1/2
2. Channel evolution with exponent 2/3
3. Mature Landscape with exponent 3/4

For both our model and experiment we obtain the same ¾ exponent

Appalachian Mountains/Susquehanna River

Dodds and Rothman, *Phys. Rev. E & Ann Rev. Earth Plan. Sci* (1999, 2000)
Birnir, *Quat. J. of Applied Math*, (2008)

Stream Splitting and Finite-Time Singularities



Whenever a stream develops a 'kink' (curvature singularity) it splits in two
Formation of kinks and there dynamics (experiments and model)?
(Schmuki and Laso, *JFM* 1990)

Further Simplifications and Analytical Results

Suppose we further assume that the time scale of meandering is longer than that of associated with the typical velocity L/V, so

$$\frac{dh}{dx} \approx \frac{v}{u}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g\sin(\alpha) + \nu_f \frac{\partial^2 u}{\partial x^2} + \eta_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = \nu_f \frac{\partial^2 v}{\partial x^2} + \frac{\gamma}{\rho l u^2}\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + \eta_y$$

In the case of rivers this further simplification and neglecting lubrication terms is reasonable.

Notice the downstream equation is now just a decoupled randomly forced Burgers equation. Assuming noise is Brownian we can solve this.

Analytic Solutions

Through the use of the Cole-Hopf transformation this can be found equivalent to the Feynman-Kac equation with the analytic solution:

$$w(x,t)=E\left(f(x_t)\,e^{-\phi\int_0^t x_s\,ds}\right)$$

where $u=-\lambda\frac{\partial}{\partial x}\ln(w)$ and $u(x,t=0)=f(x)$

The transverse equation is now of the form of a Cameron-Martin equation
The analytic solution is of the form (*g* denotes the initial condition for *v*):

$$v(x,t)=E\left(g(x_t)\,e^{\int_0^t \alpha\,d\eta-\int_0^t(\alpha^2/2+\beta)\,ds}\right)$$

Calculating long time ensemble average the meandering exponent for the stream deviation (h) yields $\kappa=1/6=.166667$
More precisely:

$$\sqrt{\langle h(x)\,h(x+l)\rangle}\sim l^{1+\kappa}$$

Global river data experimentally supports $.1 \le \kappa \le .2$

Conclusion

- This problem demonstrates the delicate balance between surface tension, gravity and boundary layer friction
- We could not reproduce previous results on the stability and preferred wavelength; discrepancy is presumably due to substrate.
- A universal power law is shown to exists across the entire parameter regime investigated

- Through the right approximations a simple models with only 1 fitting parameter can predict complex flow behavior
- These types of theories can provide excellent building blocks to understand more complex problems.

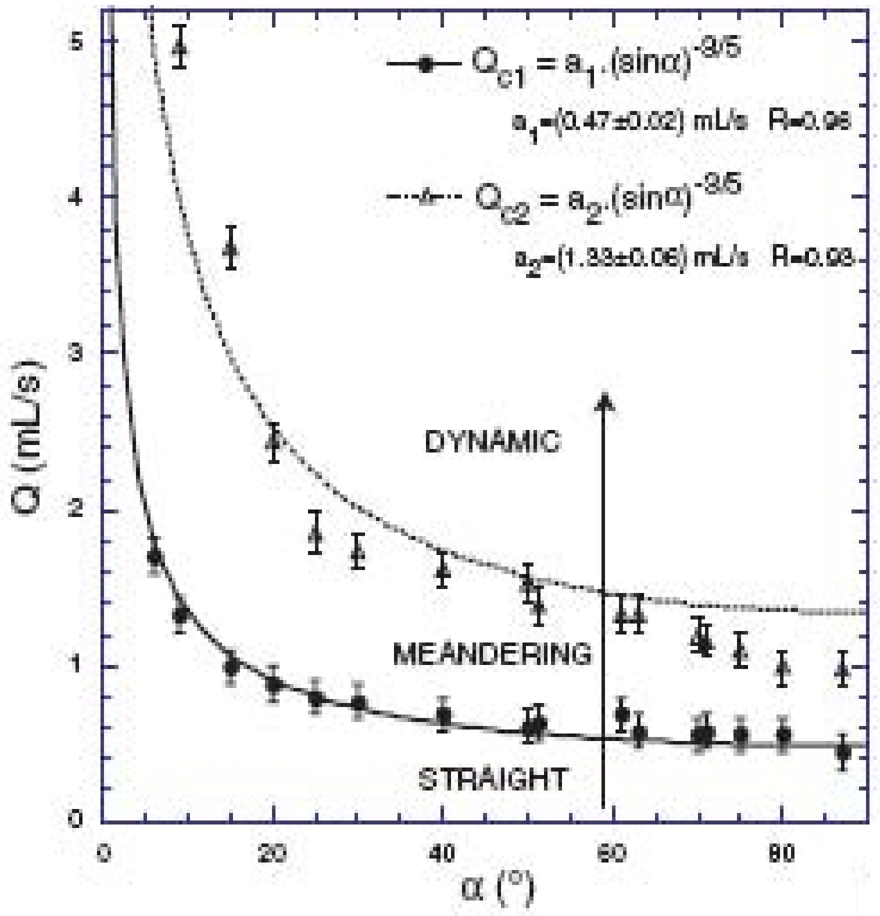
Colorado River, Arizona

Slow Flows

Experiments in literature use plastics which have the ability to hold large amounts of static charge (kV/cm^2).

Water is highly influenced by charge.

Our model assumes boundary effects to be inversely proportional to flow rate



N. Le Grand-Piteira, *JFM* **96**, 2006

In Reality:
There should be a coupled interaction between flow and surface charge but...
for large flows it will not be observable.

Q: **Is pinning a result of electrostatic interaction at small flow rates?**

Pressure estimates: hydrostatic and surface tension

Consider: $$\frac{\partial P}{\partial z} = g\sin(\alpha)$$

Integrating we get: $$P = g\sin(\alpha)(z-h) + F(x,y)$$

Now at z=h this implies:

$$P = F(x,y)$$

Using normal boundary conditions:

$$P = P_o + \gamma\kappa$$

A conservative estimate gives 1:5 ratio between hydrostatic pressure and surface tension forces

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}(\gamma\kappa)=\gamma h(x,y)'''$$

By now averaging across the surface we recover the average capillary force.

$$F(u,w)=\int_{dS}\gamma h(x,y)'''h(x,y)\,dS$$
$$=3\mathrm{b}\gamma\tan(\theta)w^{3}+4\mathrm{b}^{2}\gamma w^{6}$$

With these results and use of a few averaging techniques we will now be able to reduce the Boundary Layer Equations to a pair of coupled differential equations for the stream width and averaged downstream velocity component as a function of downstream distance.