Quantifying Stochasticity in Geoscience

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Outline

Is the evolution of the surface of the Earth stochastic or deterministic?

- Most geoscientists believe that several aspects of the Earth's surface are stochastic
- There is however no universal agreement on the relative importance of stochastic versus deterministic forces
- In recent years it has been possible to simulate on computational clusters physically based model for erosion where the water and sediment flow are coupled

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Ill-posedness makes a deterministic system stochastic!

- It was discovered that whereas these dynamical models are strongly ill-posed (i.e. smallest scales grow the fastest), it was still possible to simulate the solutions
- The reason is that the nonlinearities saturate the exponential growth of the instabilities
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A Statistical Theory of Landsurfaces

- The solutions of the equations describing the evolution of the landsurface are therefore not deterministic, but stochastic processes
- These stochastic processes are characterized by their scaling properties
- In particular by β and χ which are called respectively the temporal and spatial roughness exponents and z the dynamic exponent $\chi = \beta z$

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The Scaling of the Variogram for Landsurfaces

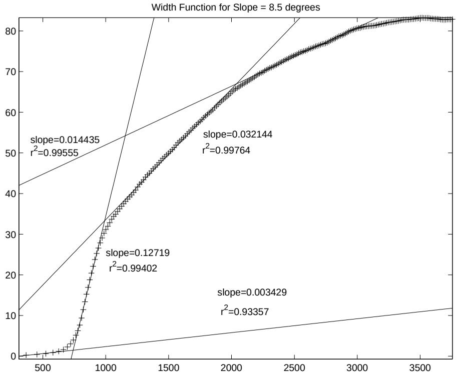

Figure: The scaling exponents of the variogram are shown as a function of time, for an initial landsurface with a slope of 8.5 degrees, on a log-log plot. Four different exponents (slopes) are shown, along with their regression coefficients, and a statistically stationary state (with slope zero) is emerging, furthest to the right.

The scalings of the statistical quantities

- There is an equivalence between time and distance in space $t \sim |x|^z$ given by the dynamic exponent z
- The system (width function) roughens initially as t to the power β
- Eventually the system gets into a statistically stationary state where it does not roughen any more, but spatial fluctuations scale with the lag variable (correlation length), to the power χ

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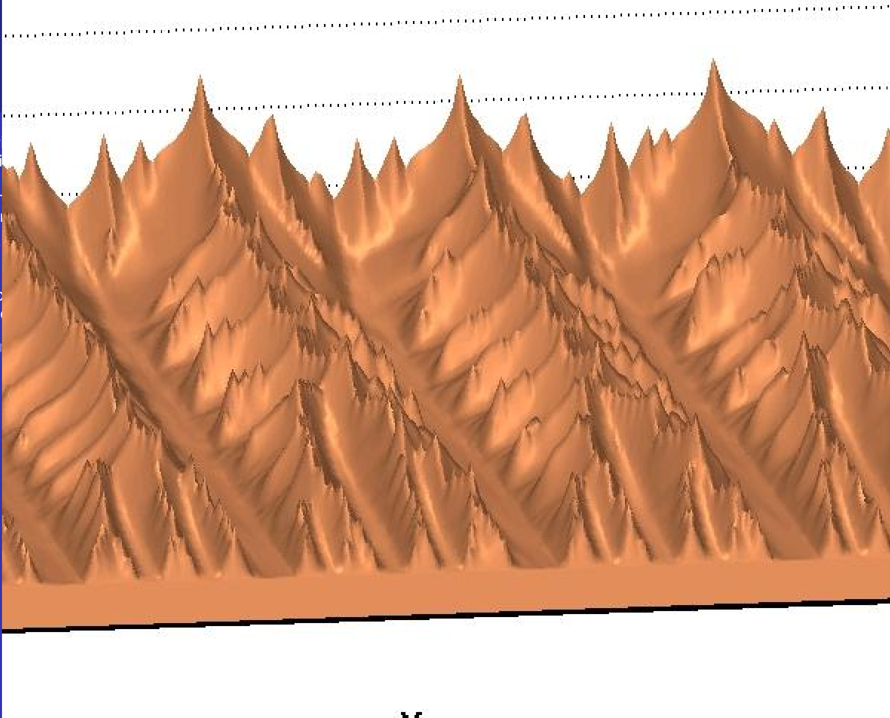

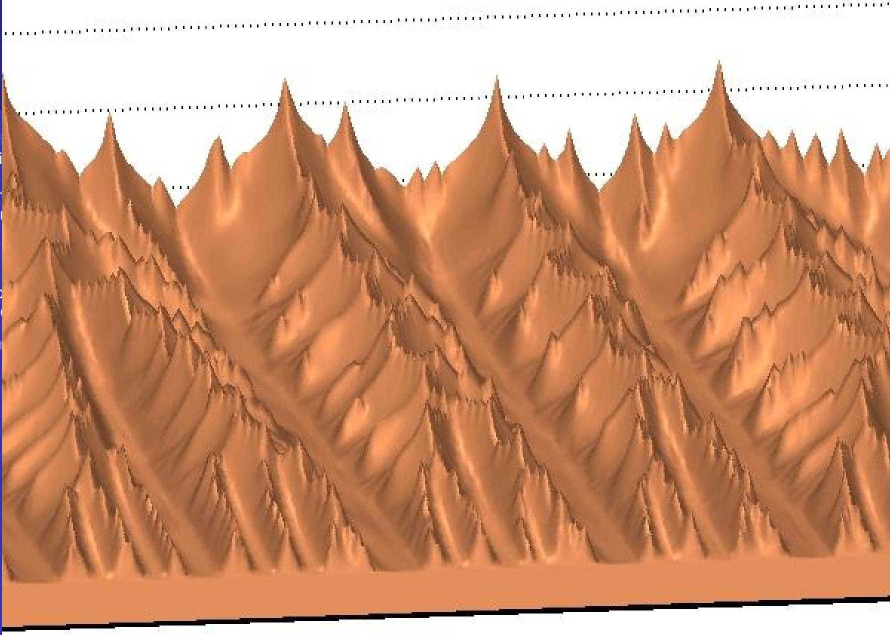

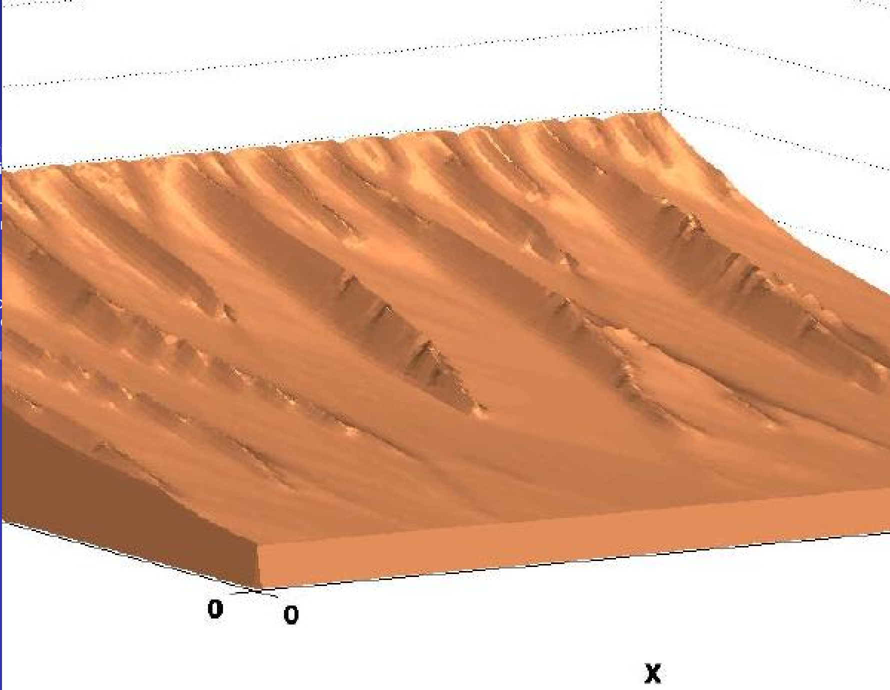
0
0
x

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What did we learn from the simulations?

- Some features of the landsurface are deterministic, for example, the width of the valleys
- Other features are stochastic, for example, the roughness of the surface
- The stochastic features are characterized by the scalings of the statistical laws

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Hack's Law

- The length ℓ of the main river in a river basin scales with the area A of the river basin as

$$\ell \sim A^{0.58}$$

The Amazon River Basin

Strahler's Order of Streams

- Streams without tributaries have order 1.
- When two streams of order n join, they form a stream of order n+1
- When two streams of different orders join, the resulting stream inherits the higher order of the two.

Horton's Bifurcation Ratios

- Horton discovered that if $N(n)$denotes the number of streams of order n and $L(n)$ is their mean length then the length and bifurcation ratios
  - $R_l = N(n)/N(n+1)$
  - $R_b = L(n+1)/L(n)$

  are constant over the entire river basin.

Probability of Exceedance

- The probability that the area of a subbasin exceeds a certain value *a*, scales with *a*

$$P(\text{area of subbasin} > a) \sim a^{-0.42}$$

- All the known scaling laws for landsurfaces are determined by just two exponents (assuming uniform drainage density)
- Hack's exponent and the meandering exponent of rivers (Dodds and Rothman)

How can these statistical quantities and their scalings be used?

- We can compute the transport of water, chemicals, and sediment by a river network
- The residue of inorganic CO_2 in glacier rivers scales with discharge with the same exponent as the probability of exceedance (Karidiĉ Univ. of Iceland, 2008)
- If the turbulent flow is releasing CO_2, we might be able to estimate the total release of CO_2 from the Amazon basin

Outline

Turbulent Flow in Rivers

- The flow in the river satisfies the Navier-Stokes Equation
- $$v_t + v \cdot \nabla v = \Delta v + \nabla\{\Delta^{-1}[trace(\nabla v)^2]\} \qquad (1)$$
- Let $v = U + u$

$$u_t + Uu = \nu u_{xx} - uu_x + \partial_x^{-1}(u_x)^2 + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k$$

- Each $e_k = e^{2\pi ikx}$ comes with its own independent Brownian motion β_t^k, h_ks are coefficients giving the color

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The Statistical Theory of Rivers

Theorem

In one dimension the Navier-Stokes equation (with pressure) driven by colored noise, has a unique solution if U is sufficiently large. Moreover, there exists a unique measure left invariant by the flow. The flow is ergodic and strongly mixing and the second structure function (variogram) scales with roughness exponent $\chi = 3/4$ in the statistically stationary state

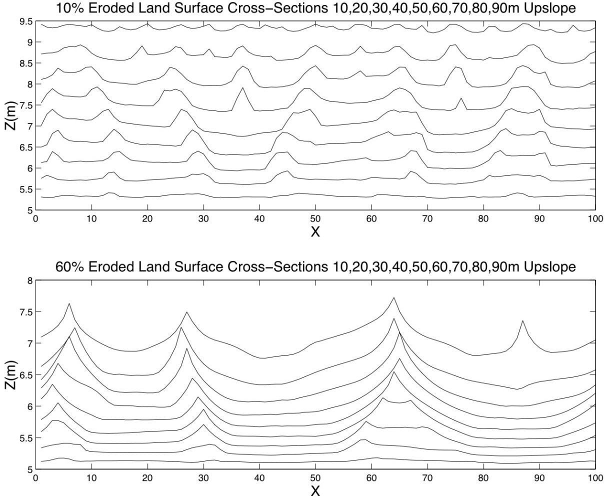
$$S_2(x) \sim |x|^{3/2}$$

All the statistical properties of the solution are determined by the invariant measure.

How does one get the invariant measure for the sediment flow?

- One way is to find it for the coupled nonlinear PDEs describing the water and sediment flow
- If we know surface structures, young landsurface concave hills or mature landsurface convex mountains, we can linearize about them
- Then it is enough to know the color for the water flow that constitutes the noise driving the sediment flow

Sections of Landsurface
Top concave hills, bottom convex mountains



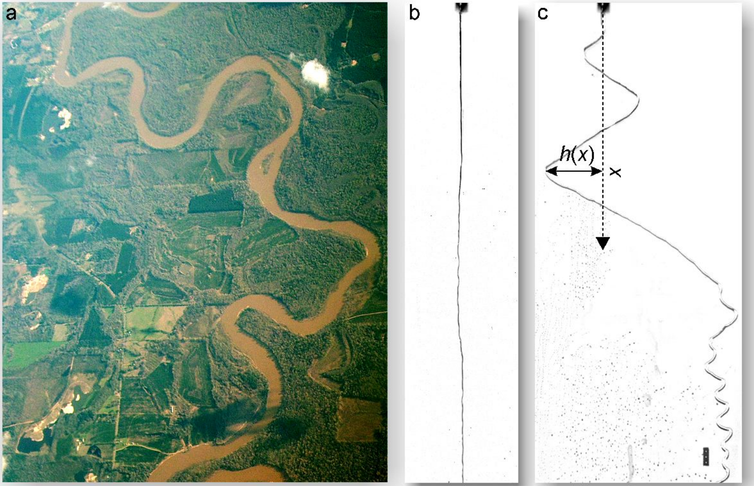
Hack's Law

$$A \sim \ell^D$$

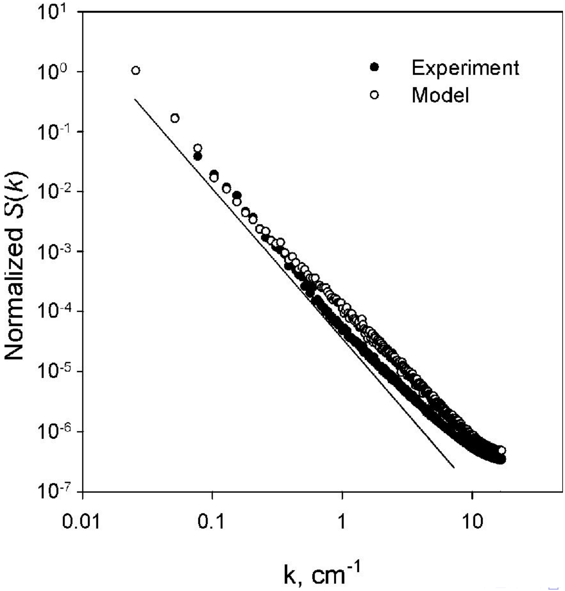
- The avalanche dimension is $D = 1 + \chi, \ell$ being the length of the main river. Then the width of the basin in the direction perpendicular to the main river, is ℓ^χ, $\chi = 3/4 = 0.75$, whereas along the main river it is ℓ, hence

$$\ell \sim A^{\frac{1}{1+\chi}}$$
$$\approx A^{0.58}$$

Meanderings of the Mississippi



The Meandering Exponent (in the lab) is Determined by Turbulent Flow, $S(k) \sim k^{-5/2}$



The Scalings of a River

- There are actually three ranges in Hack's law that have been identified:
- Scaling exponent 1/2, Channelization
- Scaling exponent 2/3, Evolution of concave to convex surfaces
- Scaling of Shocks, Bores, Hydraulic Jumps
- River turbulence causes the scaling exponent 3/4

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Conclusions

- All the scaling laws of landsurface theory and river meanderings are determined by the roughness coefficient of turbulent flow in rivers
- The theory gives an invariant measure that determines all the statistics of river flow and landsurface evolution
- The turbulent flow in rivers possesses a scaling corresponding to Hölder continues functions of order 3/4. This scaling corresponds to Hack's law and river meanderings that cover the whole river basin

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