Optimal Transport and Conformal Mappings for Registration and Surface Warping

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Medical Applications: IGT and IGS

- Biomedical Engineering principles to develop general-purpose software methods that can be integrated into complete therapy delivery systems.
- Four main components of image-guided therapy (IGT): localization, targeting, monitoring and control.
- Develop robust algorithms for:
  - Segmentation - automated methods that create patient-specific models of relevant anatomy from multi-modal data.
  - Registration – automated methods that align multiple data sets with each other and with the patient.

Surface Deformations and Flattening

- Conformal and Area-Preserving Maps
  - Optical Flow
- Gives Parametrization of Surface
  - Registration
- Shows Details Hidden in Surface Folds
- Path Planning
  - Fly-Throughs
- Medical Research
  - Brain, Colon, Bronchial Pathologies
  - Functional MR and Neural Activity
- Computer Graphics and Visualization
  - Texture Mapping

Mathematical Theory of Surface Mapping

- **Conformal Mapping:**
  - One-one
  - Angle Preserving
  - Fundamental Form $(E,F,G) \rightarrow \rho(E,F,G)$

- **Examples of Conformal Mappings:**
  - One-one Holomorphic Functions
  - Spherical Projection

- **Uniformization Theorem:**
  - Existence of Conformal Mappings
  - Uniqueness of Mapping

Deriving the Mapping Equation

Let p be a point on the surface Σ. Let

$$z : \Sigma \to S^2$$

be a conformal equivalence sending p to the North Pole.

Introduce **Conformal Coordinates** (u,v) near p,
with $u = v = 0$ at p.

In these coordinates, $ds^2 = \lambda(u,v)^2 \left(du^2 + dv^2\right)$

We can ensure that $\lambda(p) = 1$.

In these coordinates, the Laplace Beltrami operator takes the form

$$\Delta = \frac{1}{\lambda(u,v)^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

Deriving the Equation-Continued

Set $w = u + iv$. The mapping $z = z(w)$ has a simple pole at $w = 0$, i.e. at p.

Near p, we have a Laurent series $z(w) = \frac{A}{w} + B + C + Dw^2 + \ldots$

Apply Δ to get $\Delta z = A\Delta\left(\frac{1}{w}\right)$.

Taking $A = \frac{1}{2\pi}$,

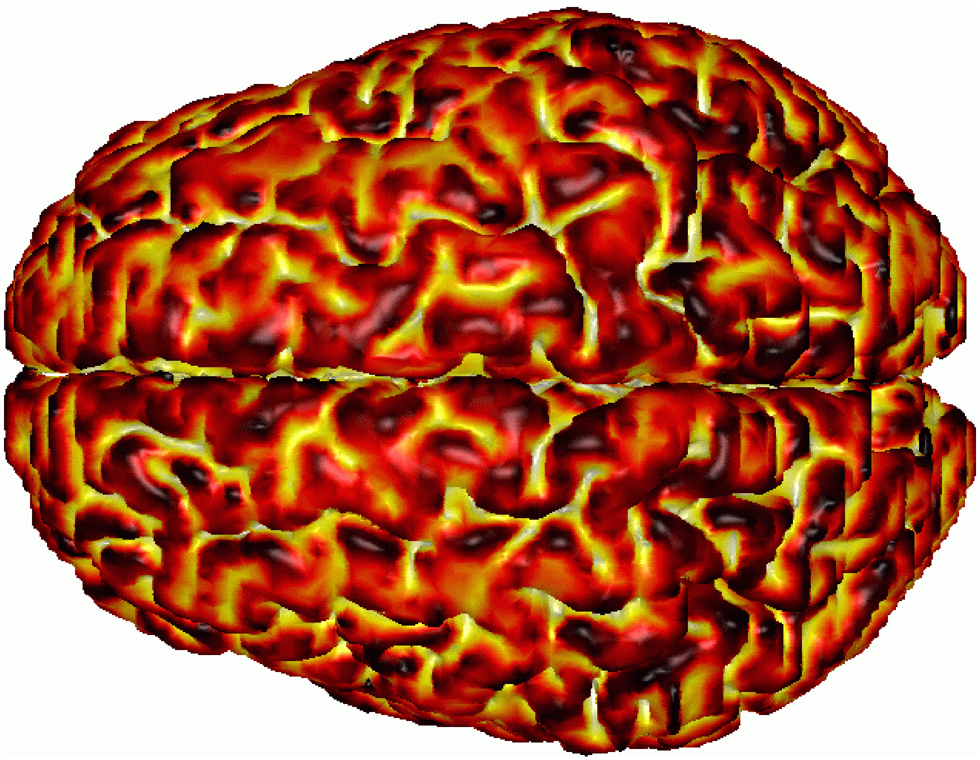
$$\begin{aligned}
\Delta z &= \frac{1}{2\pi}\Delta\left(\frac{1}{w}\right) \\
&= \frac{1}{2\pi}\Delta\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\log|w| \\
&= \frac{1}{2\pi}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\Delta\log|w| \\
&= \frac{1}{2\pi}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)(2\,\pi\,\delta_p)
\end{aligned}$$

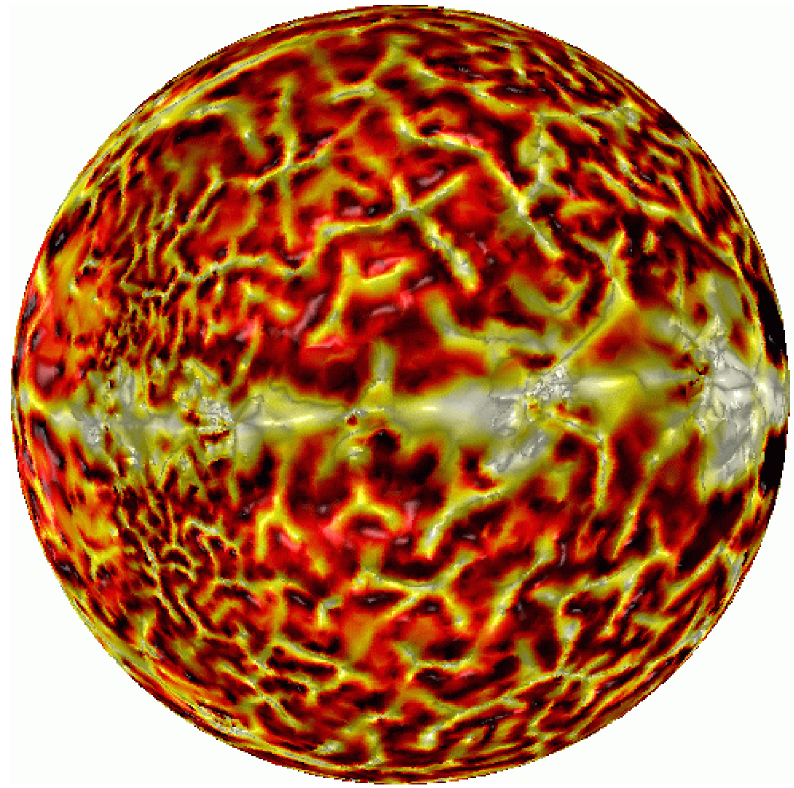
The Mapping Equation

$$\Delta z = \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p \; .$$

Simply a second order linear PDE. Solvable by standard methods.

Cortical Surface Flattening-Normal Brain





White Matter Segmentation and Flattening

Conformal Mapping of Neonate Cortex

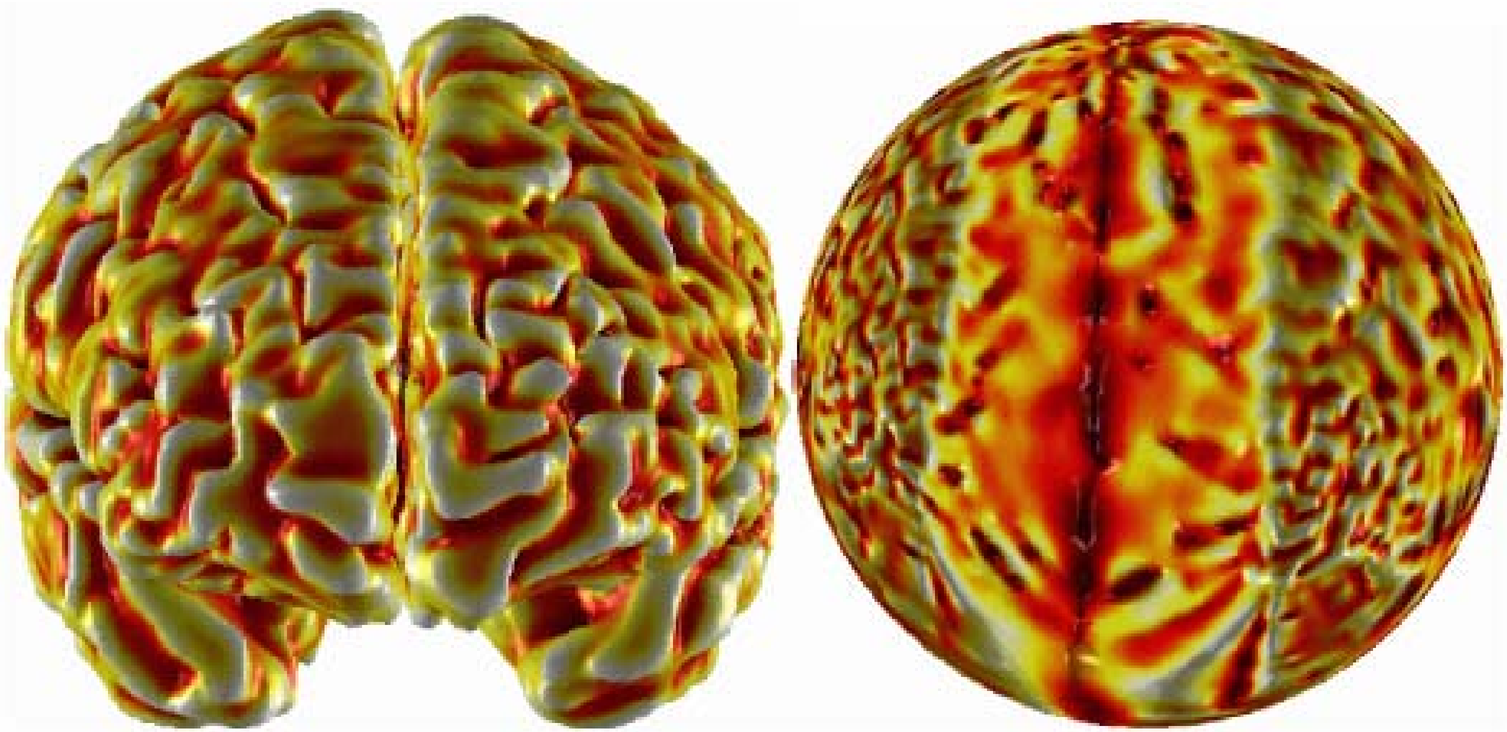
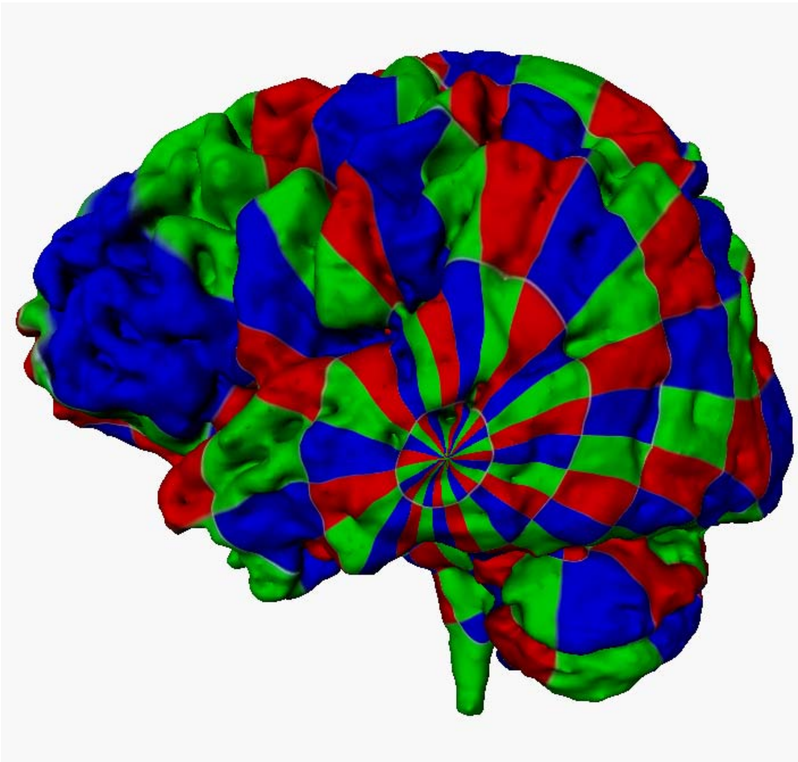
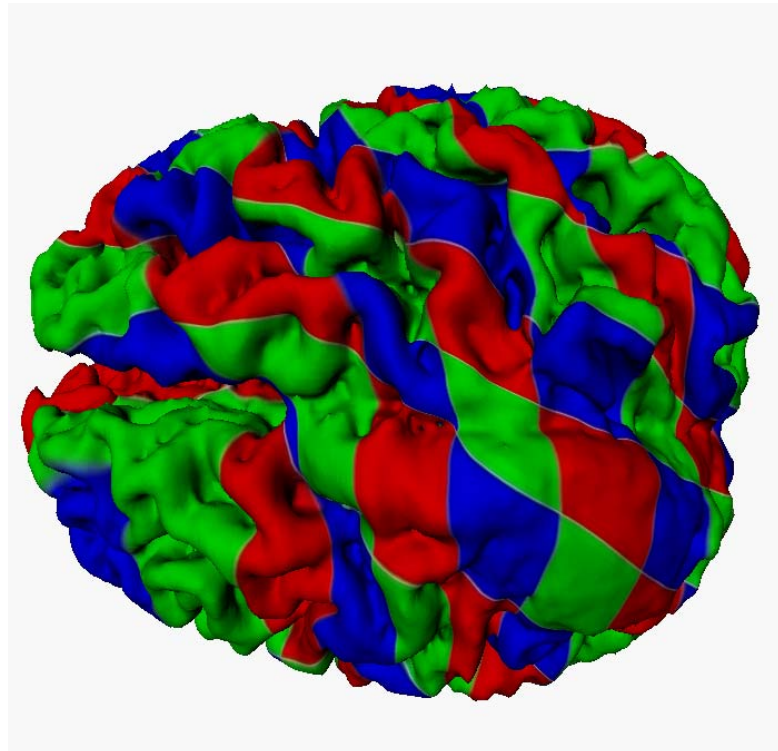


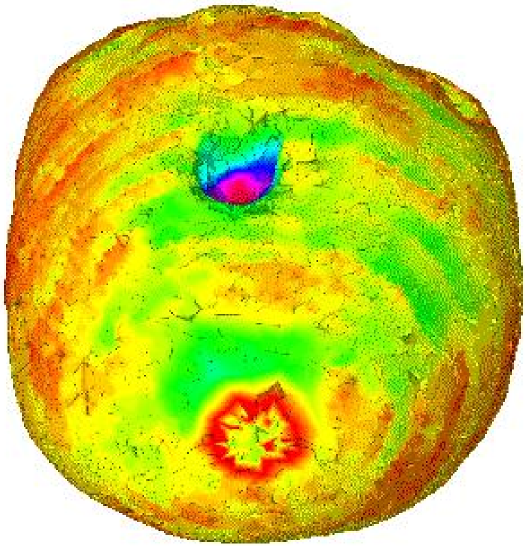
Figure 8.4.5-12
Conformal mapping of the neonate cortical surface to the sphere. The shading scheme represents mean curvature.

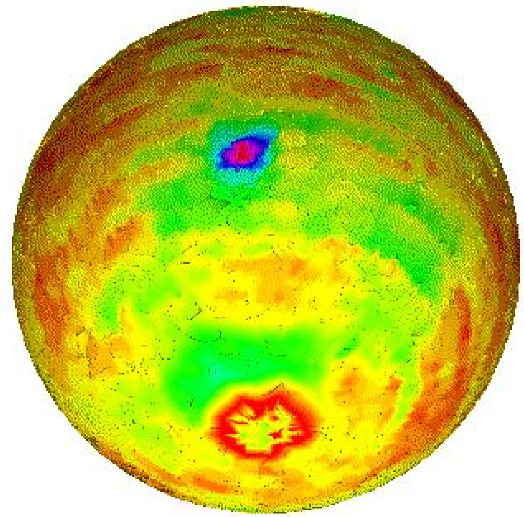
Coordinate System on Cortical Surface



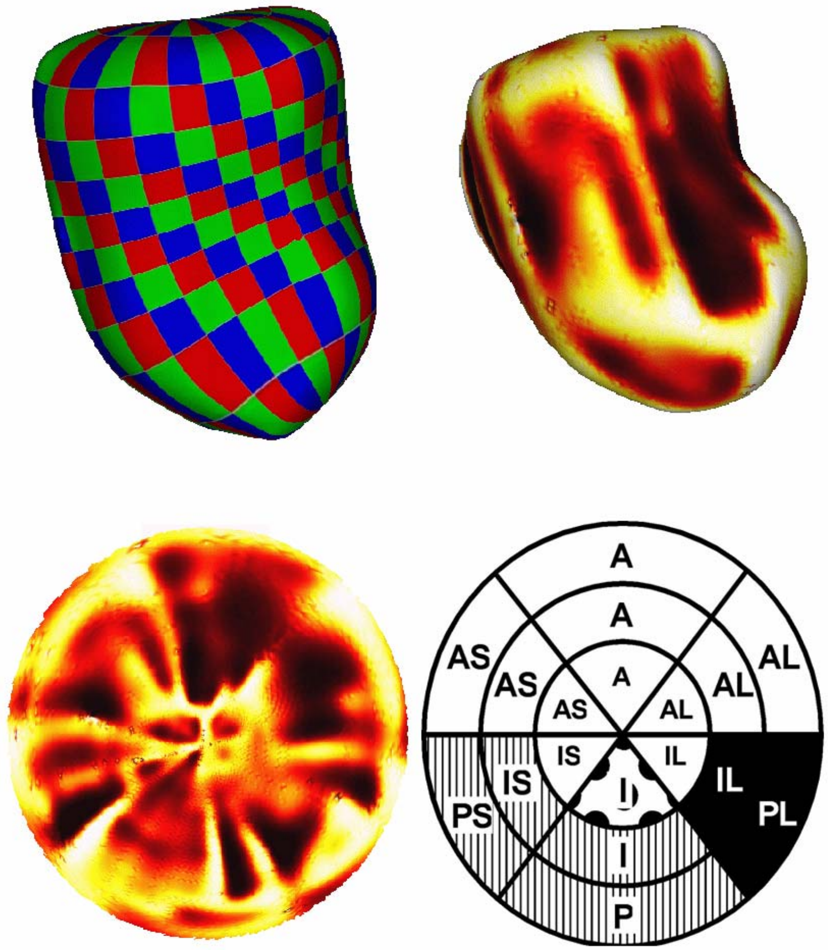


Bladder Flattening





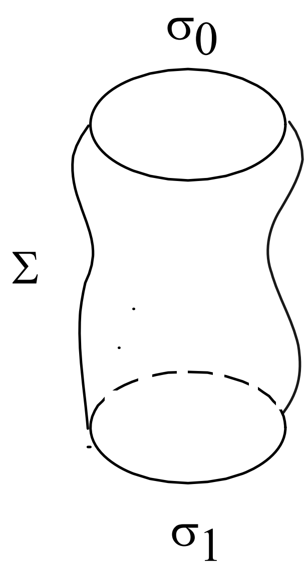
3D Ultrasound Cardiac Heart Map



Flattening a Tube

(1) Solve

$$\Delta u = 0 \quad on \quad \Sigma \setminus (\sigma_0 \cup \sigma_1)$$
$$u = 0 \quad on \quad \sigma_0$$
$$u = 1 \quad on \quad \sigma_1$$



(2) Make a cut from σ_0 to σ_1.

Make sure u is increasing along the cut.

Flattening a Tube-Continued

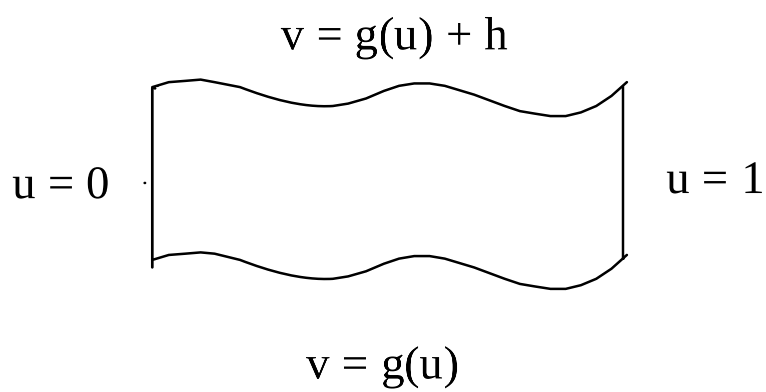
(3) Calculate v **on the boundary loop**

$$\sigma_0 \to cut \to \sigma_1 \to cut \to \sigma_0$$

by integration

$$v(\xi) = \int^{\xi} \frac{\partial v}{\partial s} ds = \int^{\xi} \frac{\partial u}{\partial n} ds$$

(4) Solve Dirichlet problem using boundary values of v **.**



If you want, scale so $h = 2\pi$**, take** e^{u+iv} **to get an annulus.**

Flattening Without Distortion-I

In practice, once the tubular surface has been flattened into a rectangular shape, it will need to be visually inspected for pathologies. We present a simple technique by which the entire colon surface can be presented to the viewer as a sequence of images or cine. In addition, this method allows the viewer to examine each surface point without distortion at some time in the cine. Here, we will say a mapping is without distortion at a point if it preserves the intrinsic distance there.

It is well known that a surface cannot in general be flattened onto the plane without some distortion somewhere. However, it may be possible to achieve a surface flattening which is free of distortion along some curve. A simple example of this is the familiar Mercator projection of the earth, in which the equator appears without distortion. In our case, the distortion free curve will be a level set of the harmonic function (essentially a loop around the tubular colon surface), and will correspond to the vertical line through the center of a frame in the cine. This line is orthogonal to the "path of flight" so that every point of the colon surface is exhibited at some time without distortion.

Flattening Without Distortion-II

Suppose we have conformally flattened the colon surface onto a rectangle

$R=[0,u_{\max}]\times[-\pi,\pi]$.

Let F be the inverse of this mapping, and let $\phi^2=\phi^2(u,v)$ be the amount by Which F scales a small area near (u,v), i.e. let $\phi>0$ be the "conformal factor" for F.

Fix $w>0$, and for each $u_0\in[0,u_{\max}]$ define a subset $R_0=([u_0-w,u_0+w]\times[-\pi,\pi])\cap R$ which will correspond to the contents of a cine frame. We define a mapping

$$(\hat{u},\hat{v})=G(u,v)=\left(\int_{u_0}^{u}\phi(\mu,v)d\mu,\int_{0}^{v}\phi(u_0,v)dv\right).$$

Flattening Without Distortion-III

We have

$$dG(u,v)=\begin{pmatrix}\hat{u}_u & \hat{u}_v\\ \hat{v}_u & \hat{v}_v\end{pmatrix}=\begin{pmatrix}\phi(u,v) & \int_{u_0}^{u}\phi_v(\mu,v)d\mu\\ 0 & \phi(u_0,v)\end{pmatrix},$$

$$dG(u_0,v)=\phi(u_0,v)\times\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

This implies that composition of the flattening map with G sends level set loop $\{u=u_0\}$ on the surface to the vertical line $\{\hat{u}=0\}$ in the $\hat{u}-\hat{v}$ plane without distortion. In addition, it follows from the formula for dG that lengths measured in the $\hat{u}$ direction accurately reflect the lengths of corresponding curves on the surface.

Introduction: Colon Cancer

- US: 3rd most common diagnosed cancer
- US: 3rd most frequent cause of death
- US: 56.000 deaths every year

- Most of the colorectal cancers arise from preexistent adenomatous polyps

- Landis S, Murray T, Bolden S, Wingo Ph.Cancer Statitics 1999. Ca Cancer J Clin. 1999; 49:8-31.

Problems of CT Colonography

- ❑ Proper preparation of bowel
- ❑ How to ensure complete inspection

→ Nondistorting colon flattening program

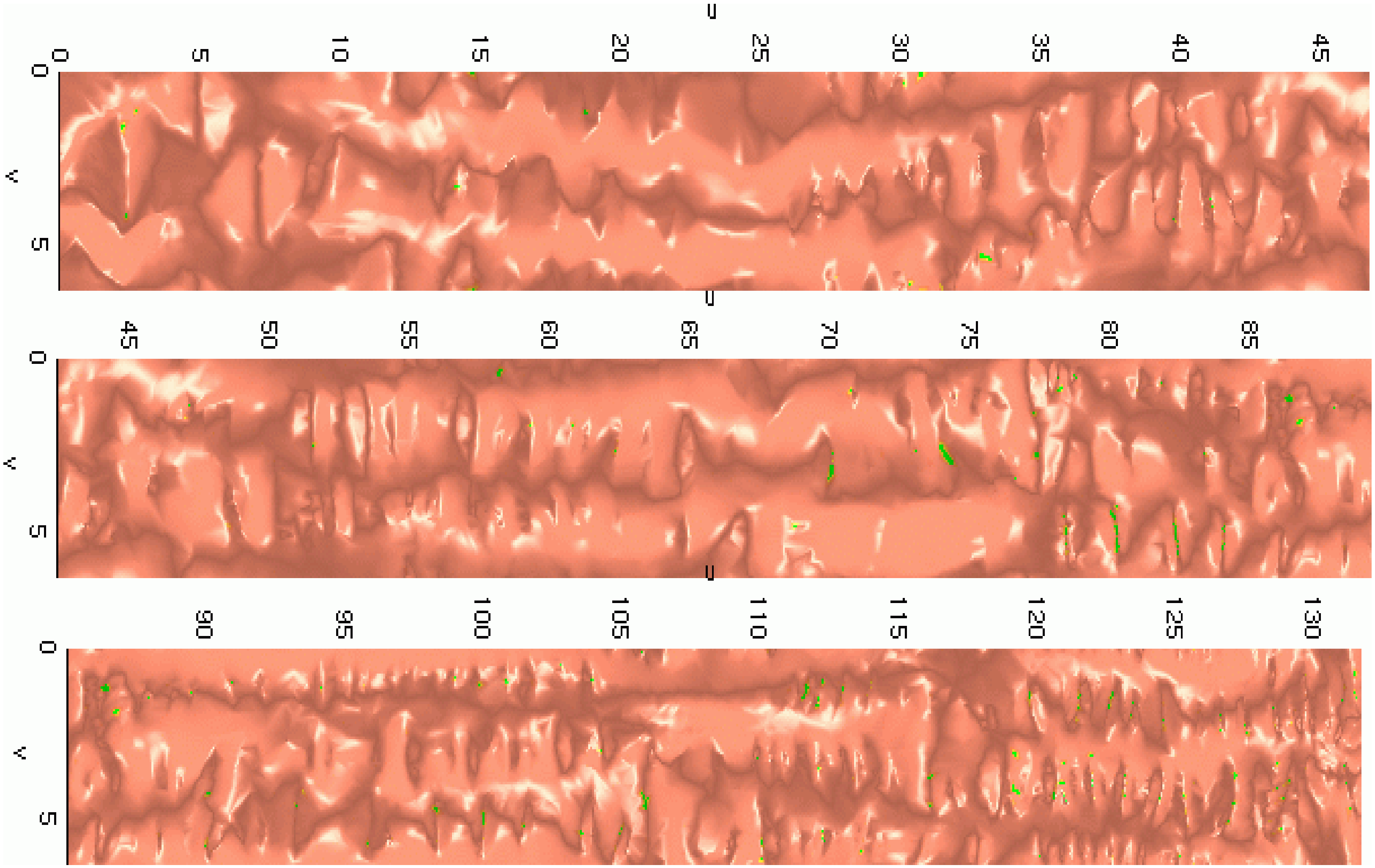
Nondistorting colon flattening

- Simulating pathologist' approach
- No Navigation is needed
- Entire surface is visualized

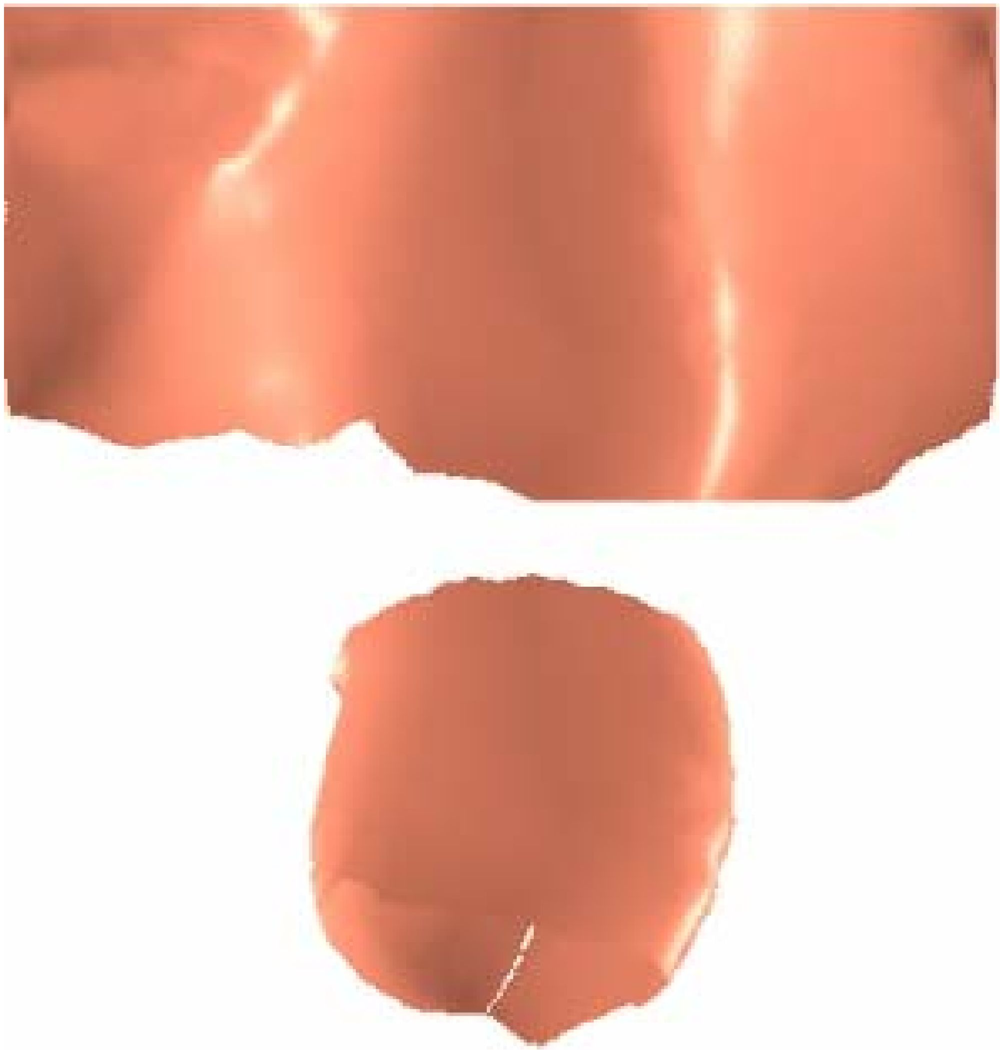
Nondistorting Colon Flattening

- Using CT colonography data
- Standard protocol for CT colonography
- Fifty-three patients (28 m, 25 f)
- Mean age 70.2 years (from 50 to 82)

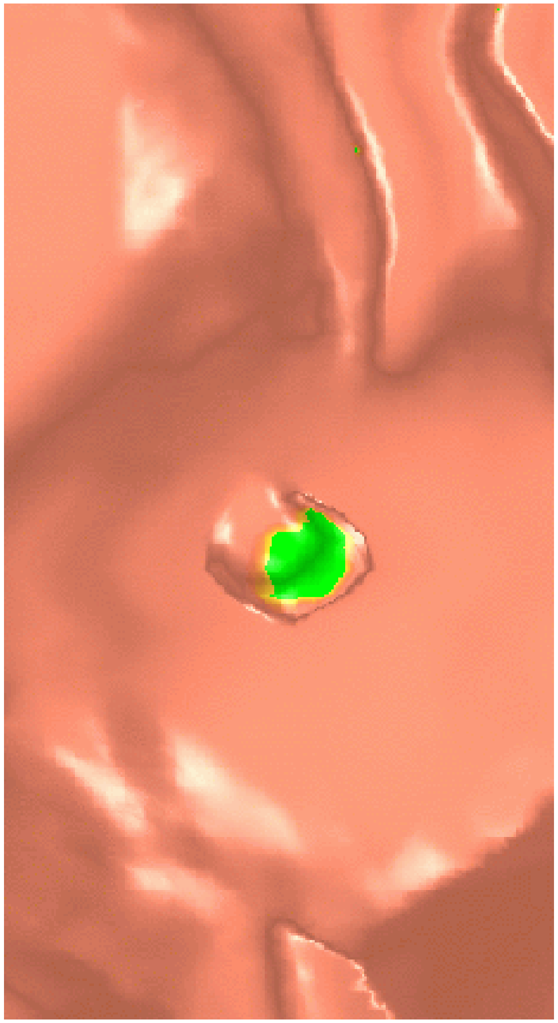
Flattened Colon

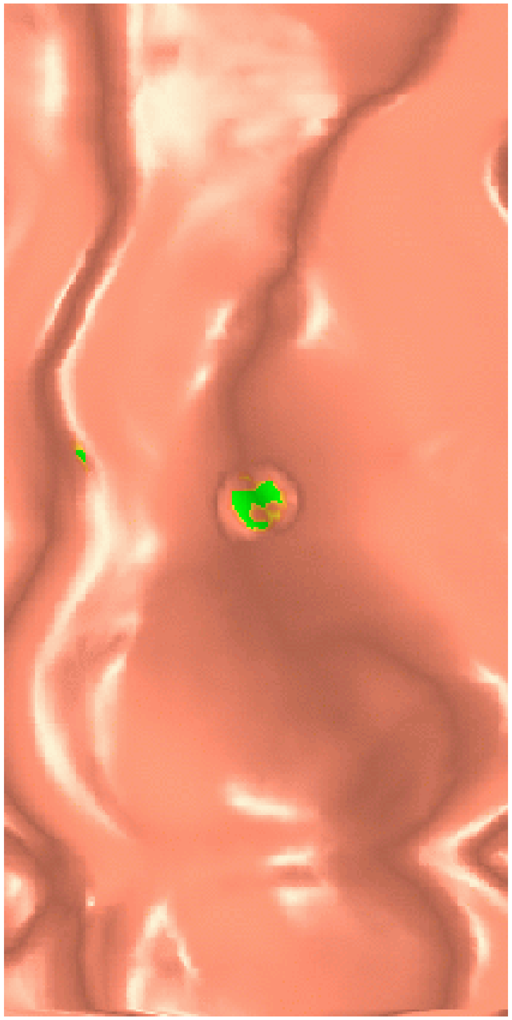


Colon Fly-Through Without Distortion

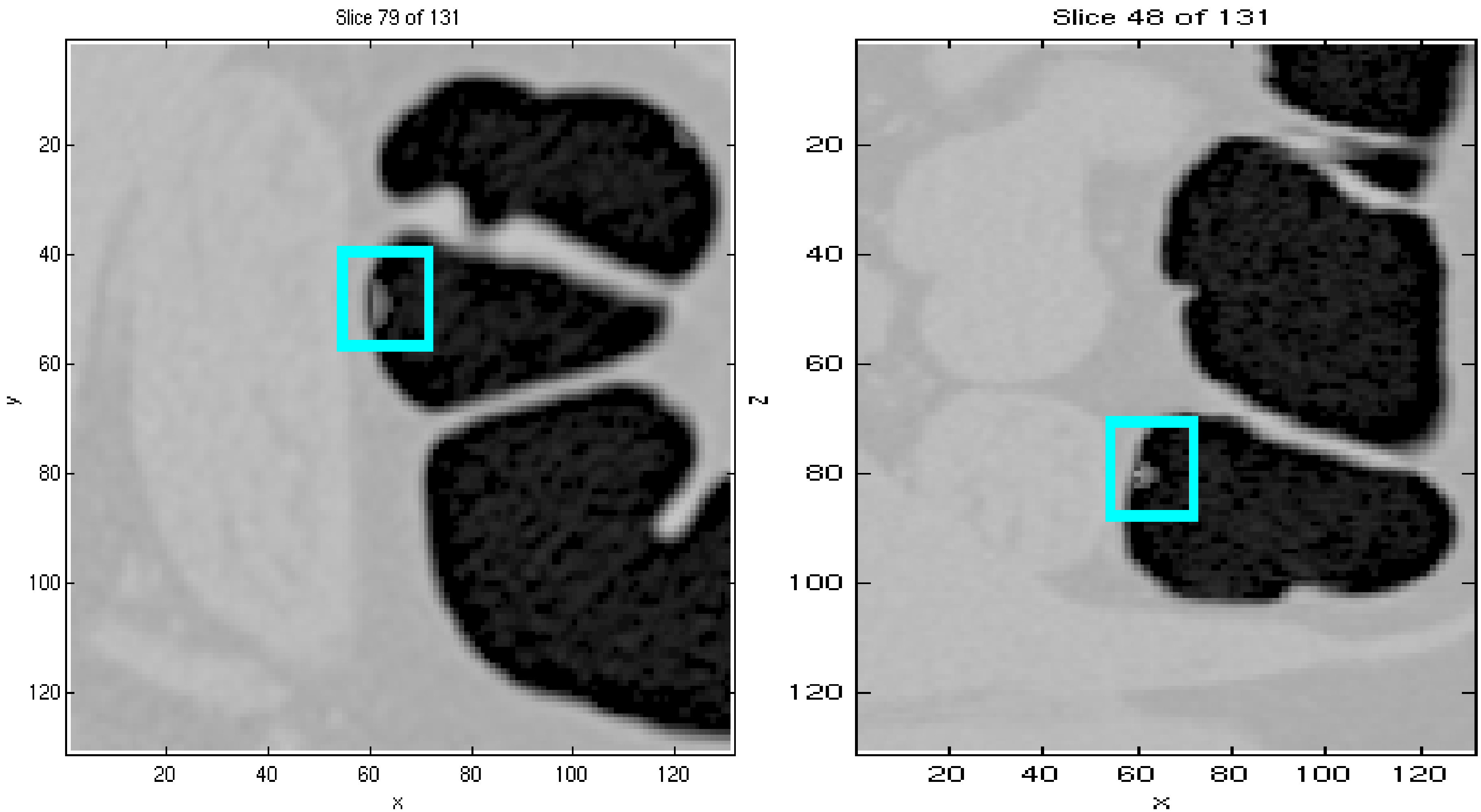


Polyps Rendering

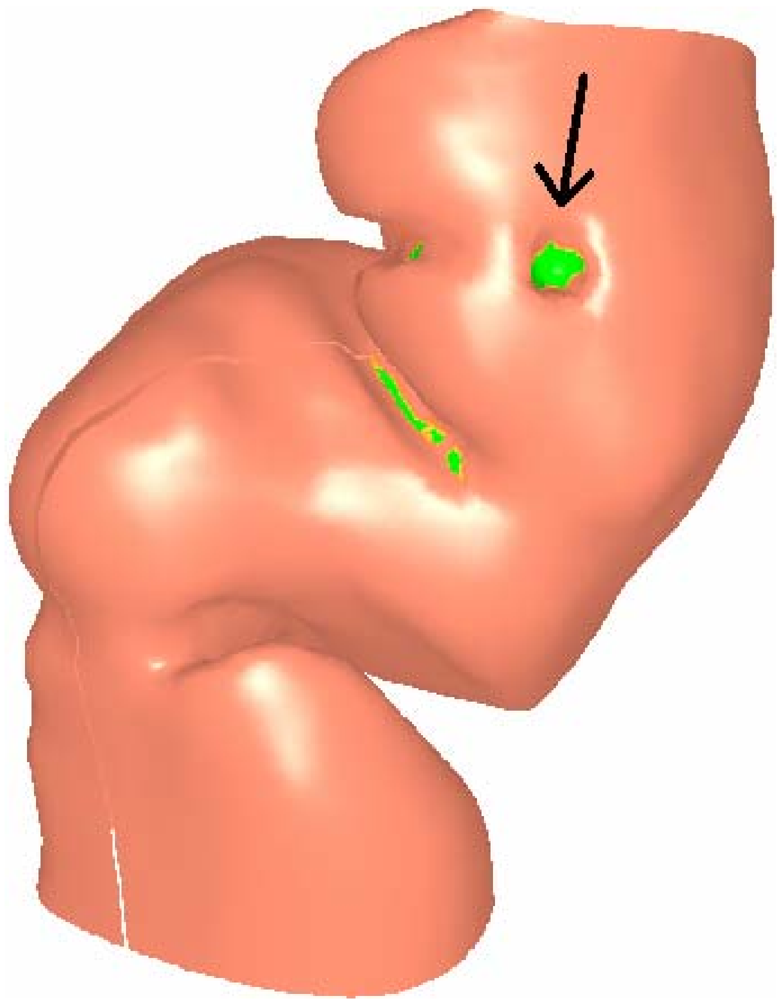


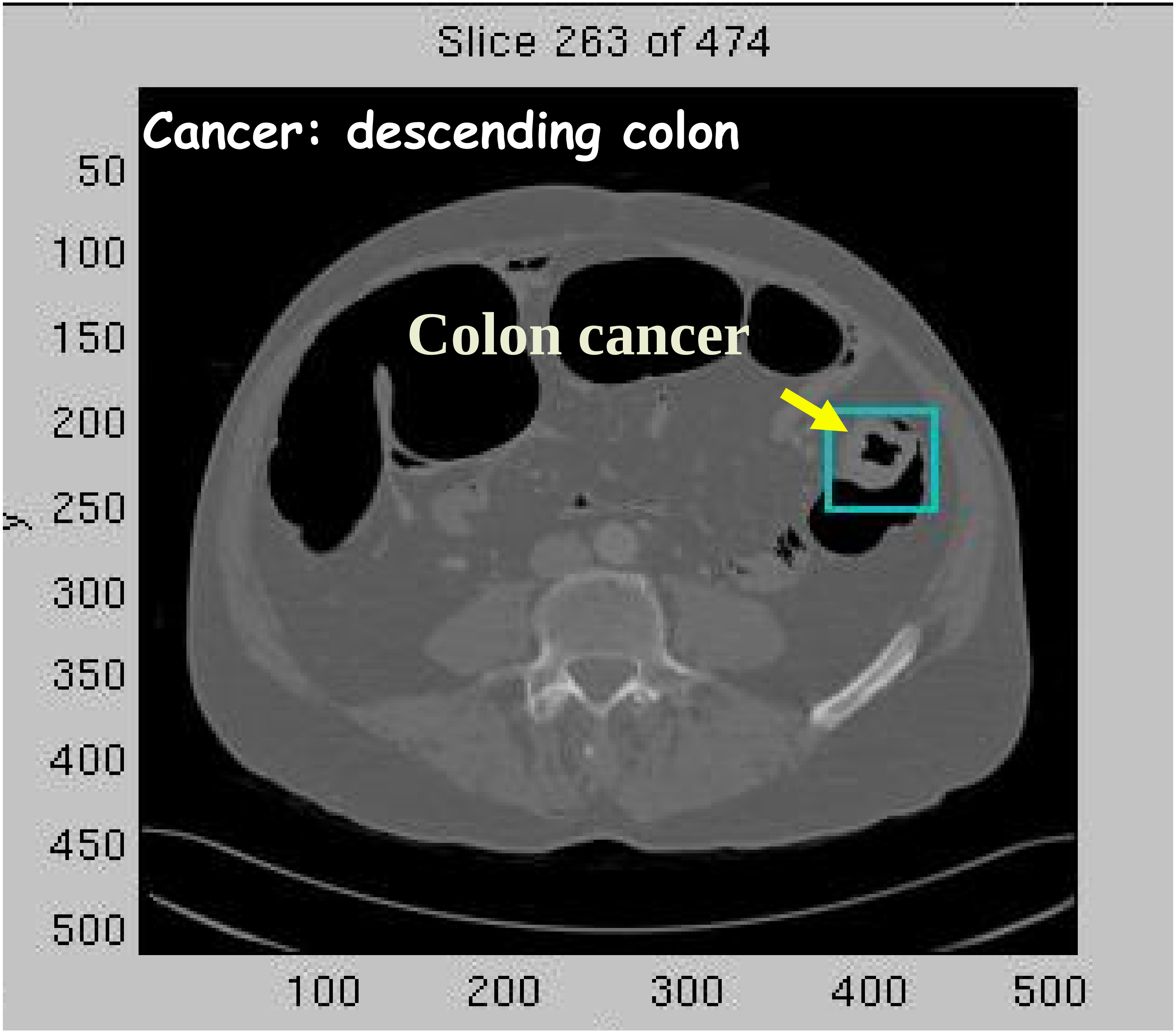


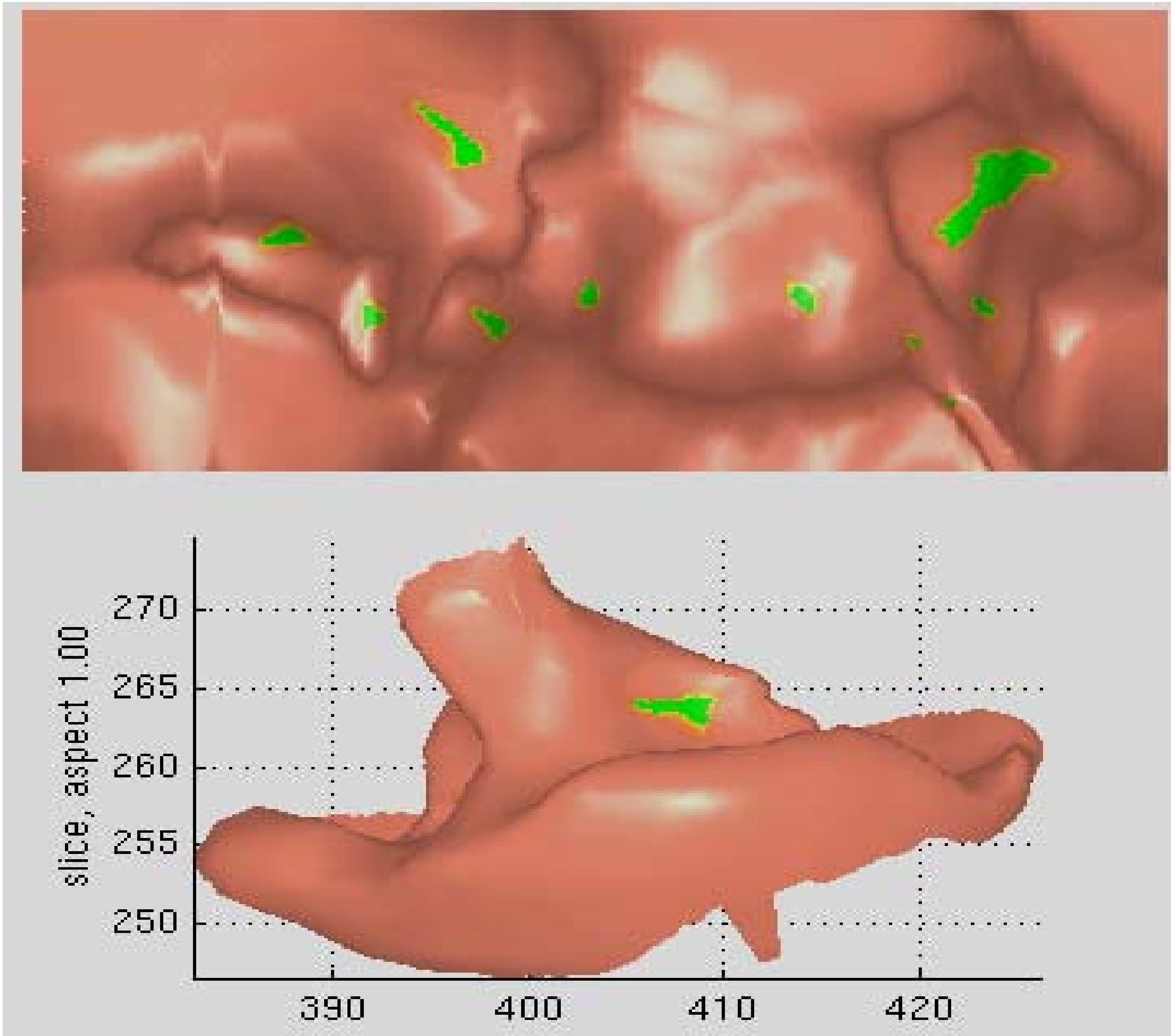
Finding Polyps on Original Images



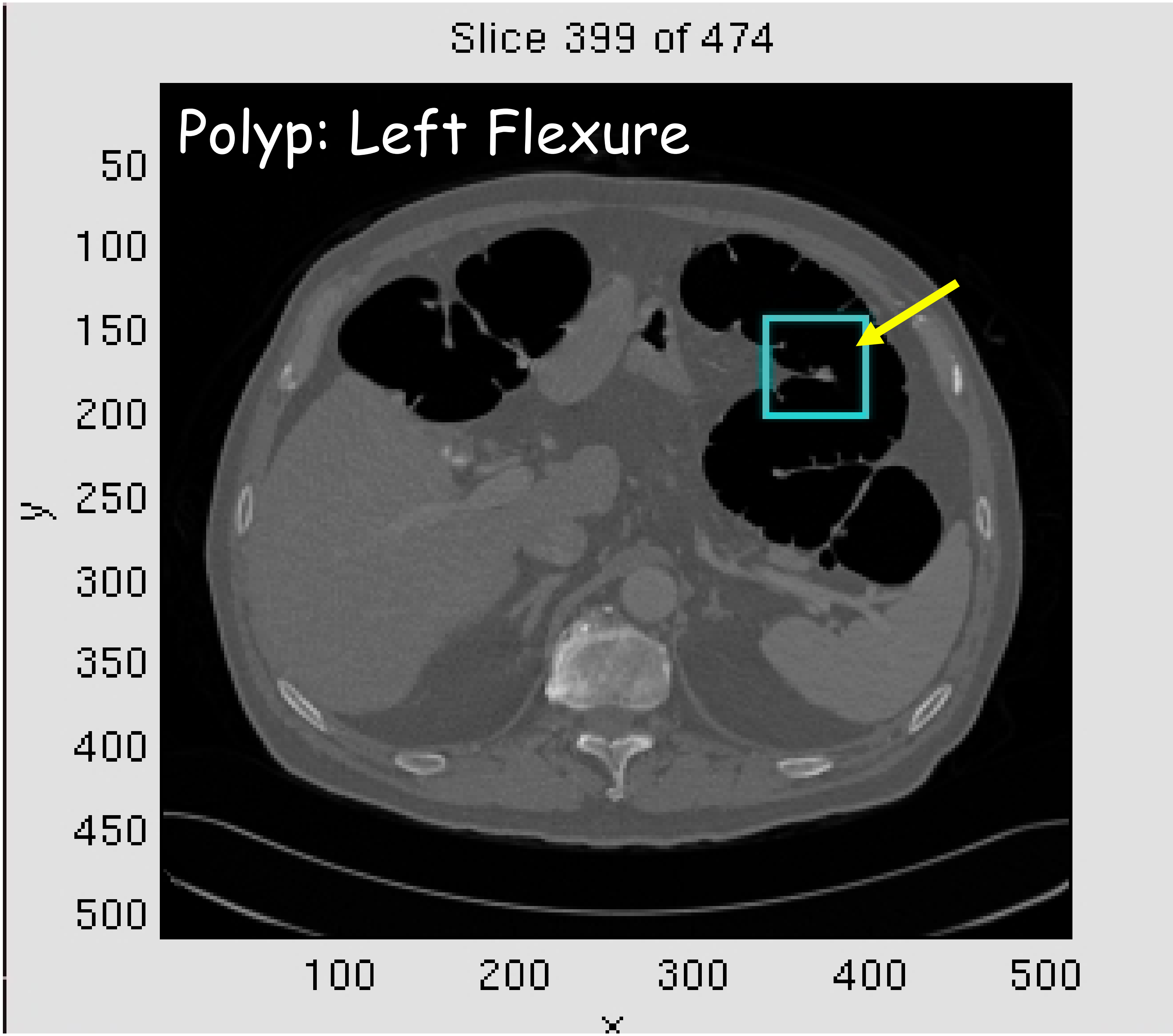
Polyp Highlighted

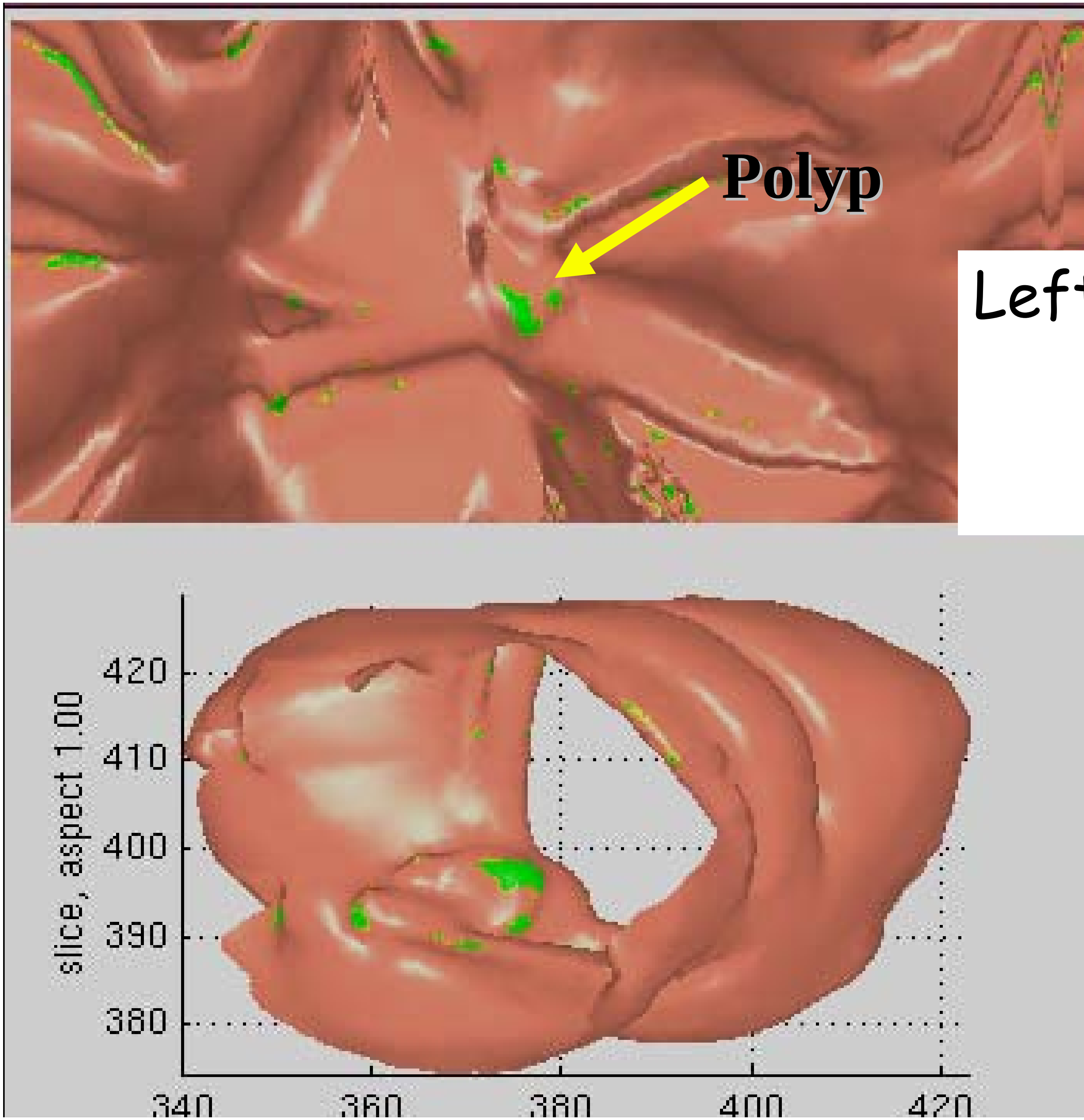



Slice 263 of 474
Cancer: descending colon
Colon cancer
50
100
150
200
250
300
350
400
450
500
100
200
300
400
500



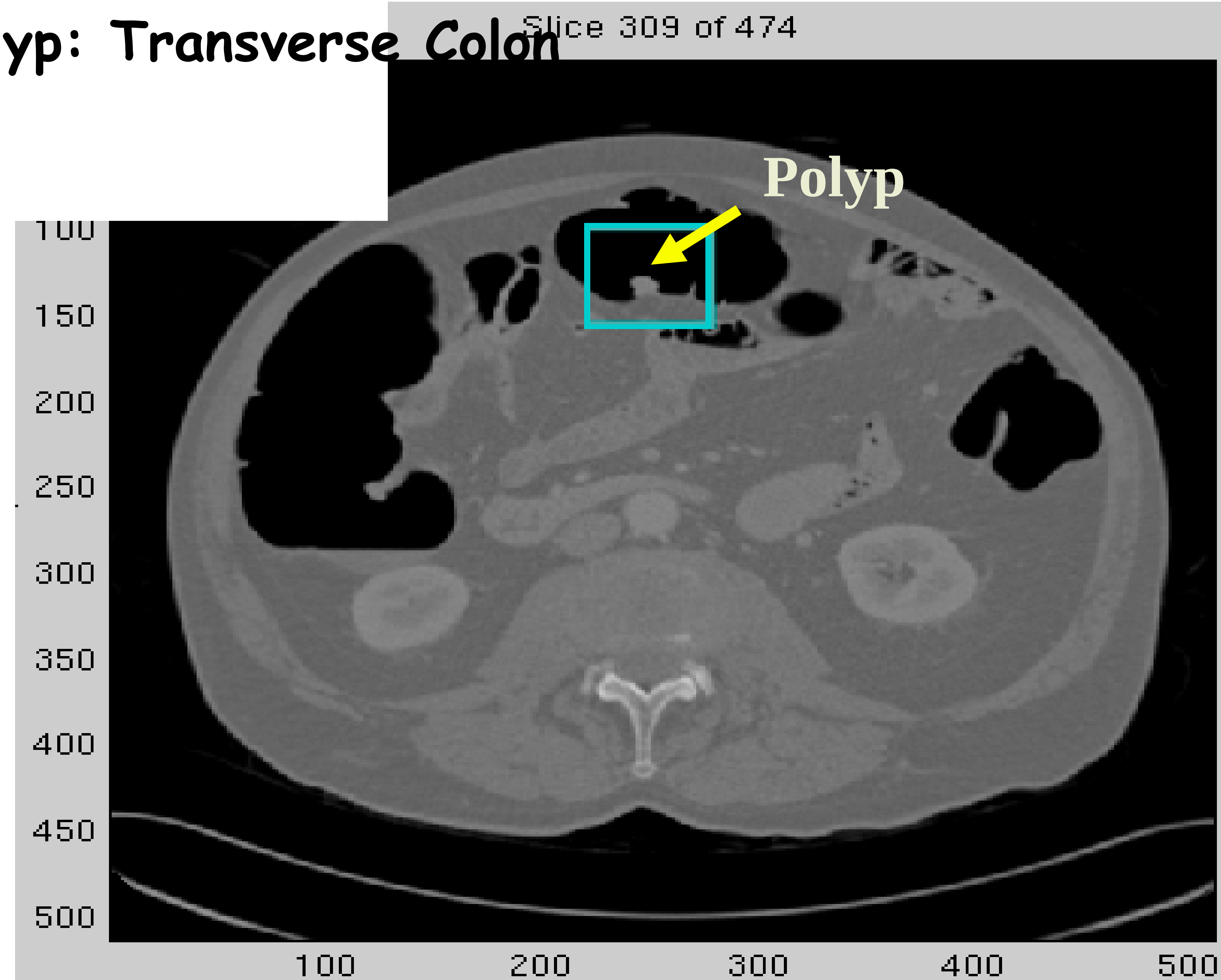
Cancer: Descending Colon


Slice 399 of 474
Polyp: Left Flexure
50
100
150
200
250
300
350
400
450
500
y
100
200
300
400
500
x

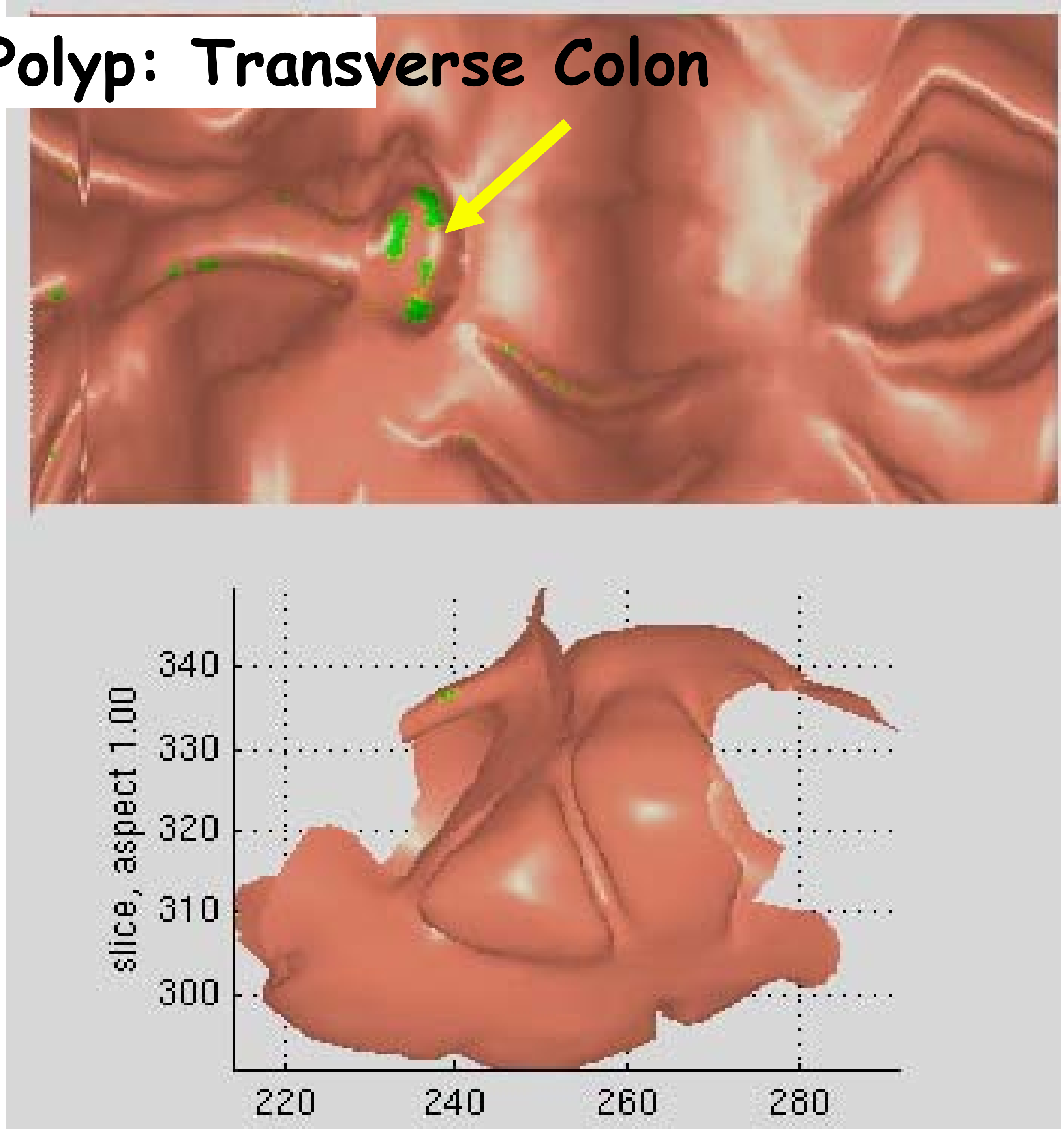


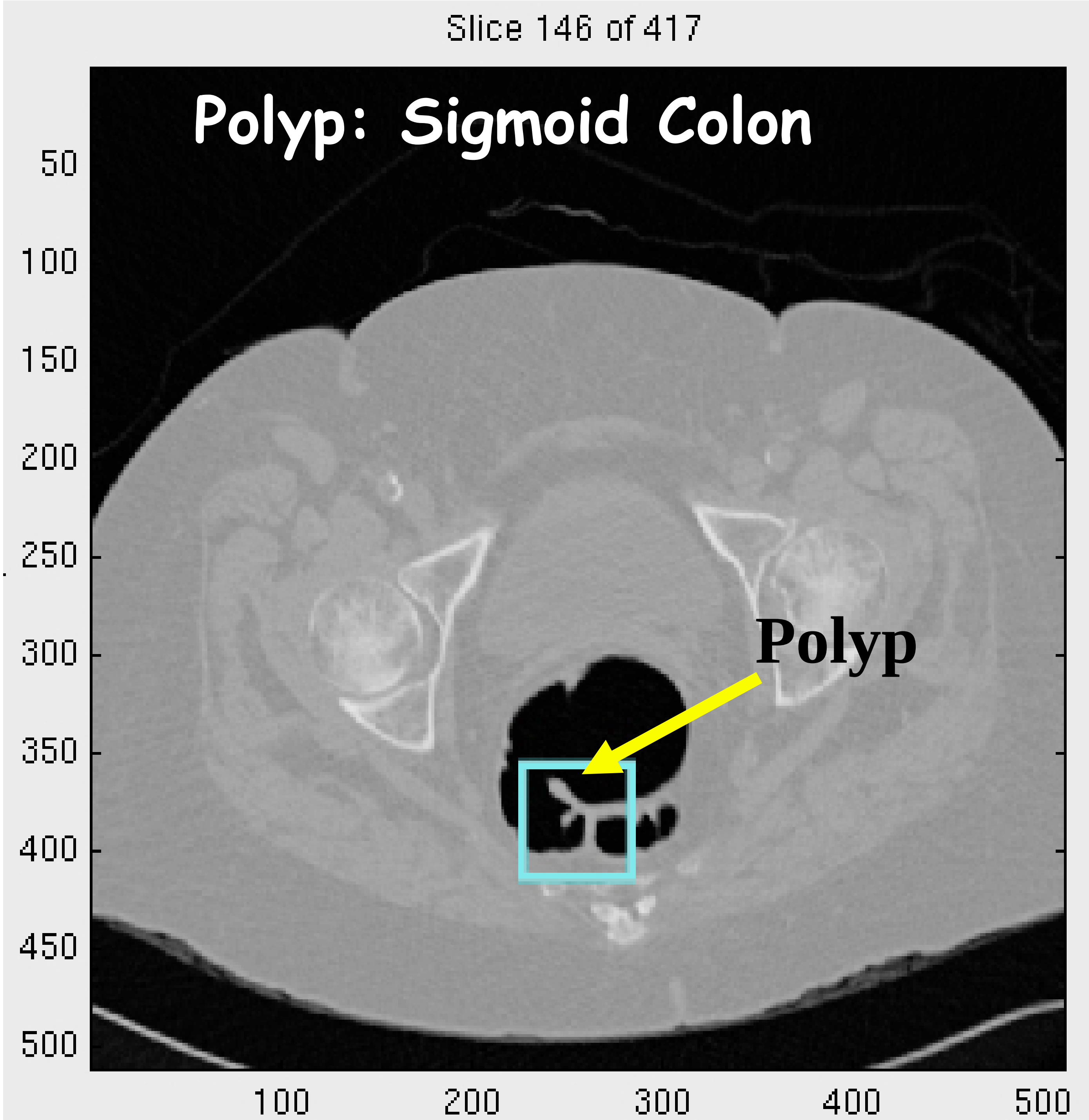
Left Flexur

Polyp: Transverse Colon



Polyp: Transverse Colon




Slice 146 of 417
Polyp: Sigmoid Colon
Polyp
50
100
150
200
250
300
350
400
450
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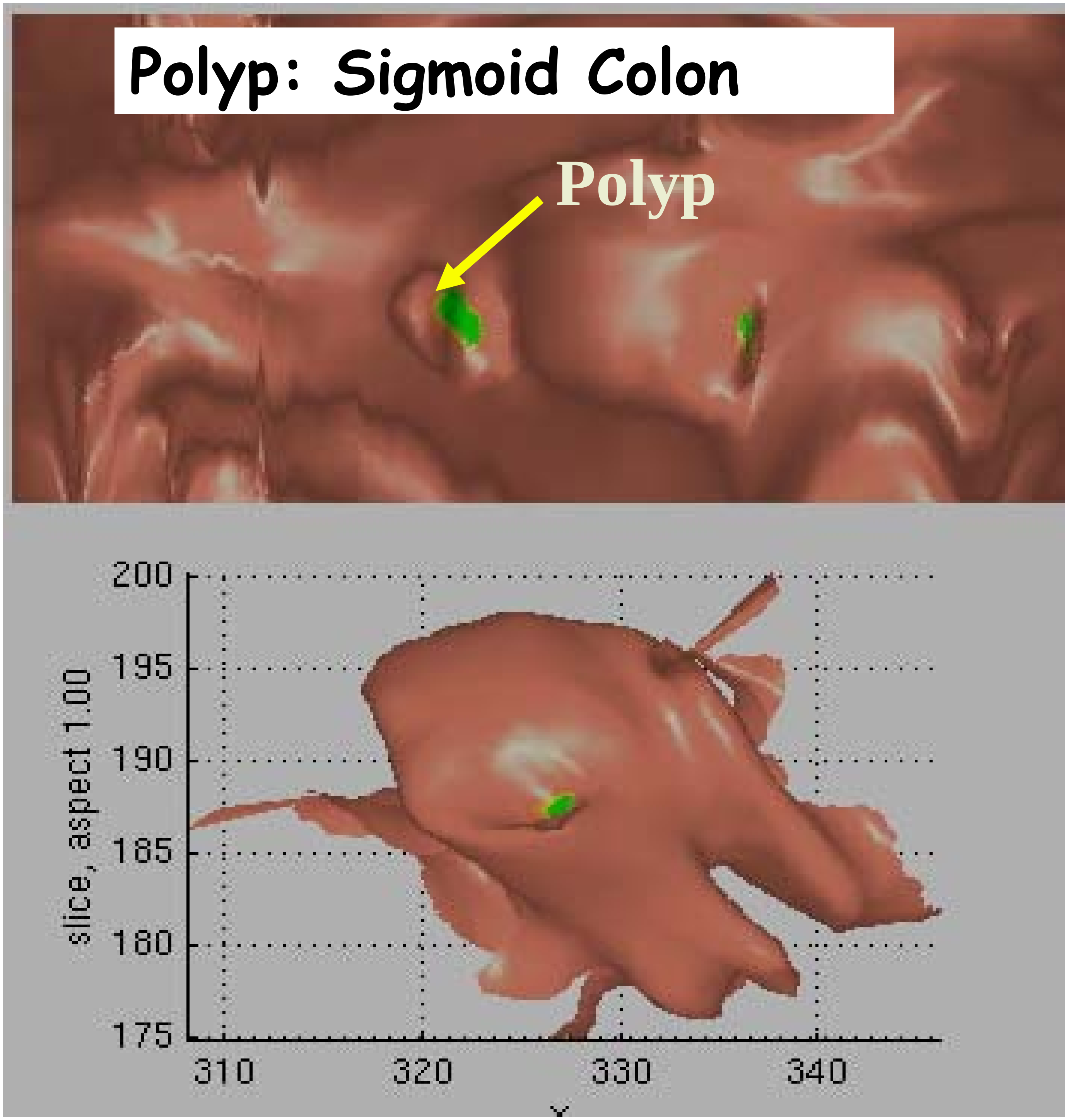
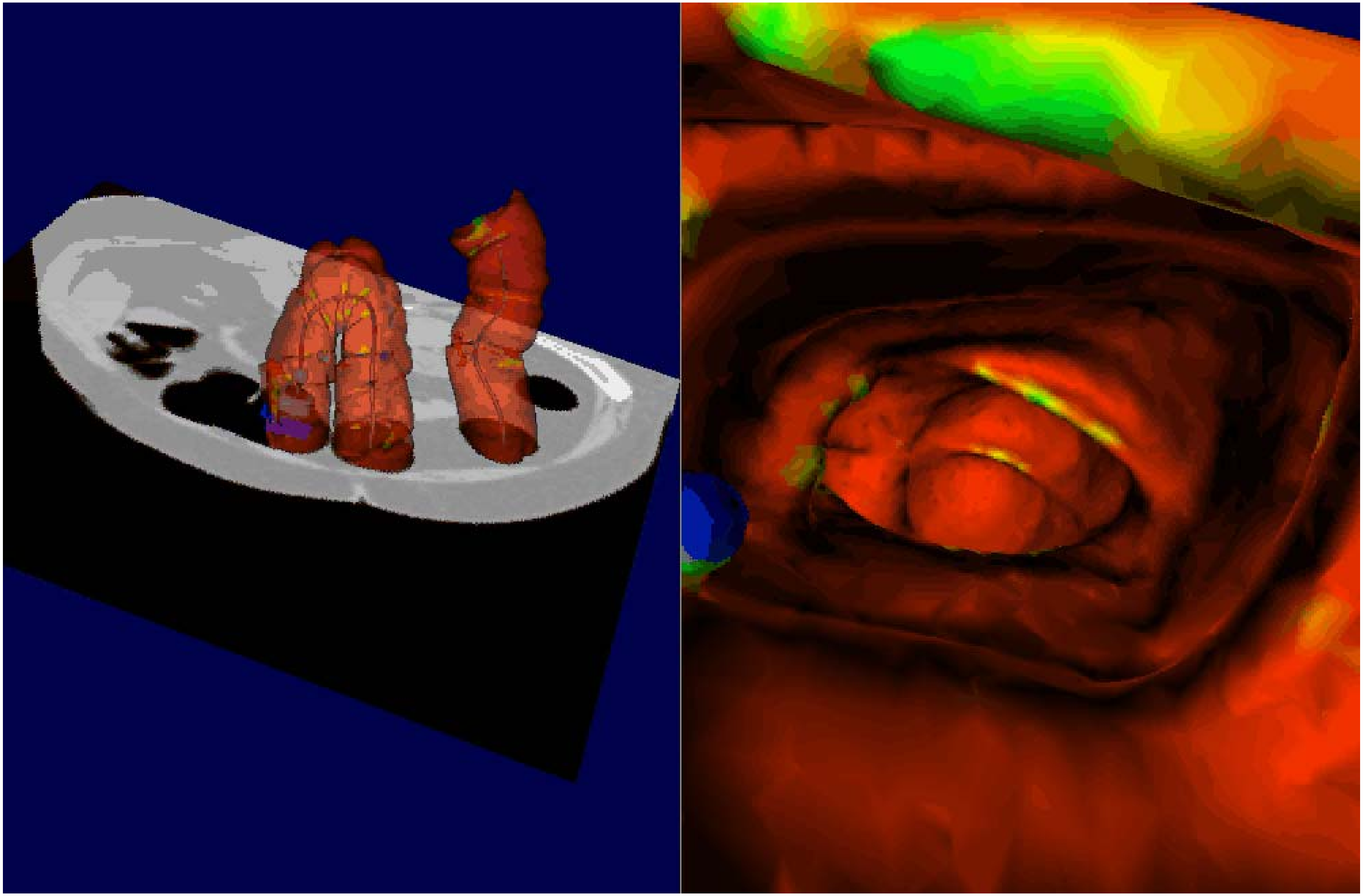
100
200
300
400
500


Polyp: Sigmoid Colon
Polyp
200
195
190
185
180
175
slice, aspect 1.00
310
320
330
340

Path-Planning Deluxe



Area-Preserving Flows-I

Let M be a closed, connected n-dimensional manifold. **Volume form:**

$$\tau = g(x)dx, \; dx = dx_1 \wedge \ldots \wedge dx_n,$$
$$g(x) > 0$$

Theorem (Moser):

M, N compact manifolds with volume forms τ and σ. Assume that M and N are diffeomorphic. If

$$\int_M \tau = \int_N \sigma,$$

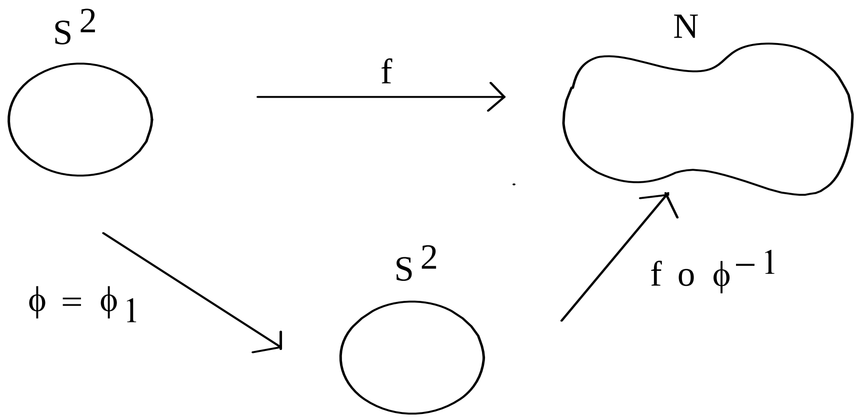
then there exists a diffeomorphism of M into N taking τ into σ.

Area-Preserving Flows-II

- The basic idea of the proof of the theorem is the contruction of an orientation-preserving automorphism homotopic to the identity.

- As a corollary, we get that given M and N any two diffeomorphic surfaces with the same total area, there exists are area-preserving diffeomorphism.
  - This can be constructed explicitly via a PDE.

Area-Preserving Flows for the Sphere-I



Find a one-parameter family of vector fields
$u_t, t \in [0,1]$ **and solve the ODE**

$$\frac{d}{dt}\phi_t = u_t \circ \phi_t$$

to get a family of diffeomorphisms ϕ_t **such that**

$$\phi_0 = id$$

and

$$\det(D\phi_t)((1-t)\det(Df) + t) = \det(Df).$$

Area-Preserving Flows for the Sphere-II

To find u_t **, solve**

$$\Delta\theta = 1 - \det(Df),$$

then

$$u_t = \frac{-\nabla\theta}{(1-t)\det(Df) + t}$$

Registration and Mass Transport

Image registration is the process of establishing a common geometric frame of reference from two or more data sets from the same or different imaging modalities taken at different times.

Multimodal registration proceeds in several steps. First, each image or data set to be matched should be individually calibrated, corrected from imaging distortions, cleaned from noise and imaging artifacts. Next, a measure of dissimilarity between the data sets must be established, so we can quantify how close an image is from another after transformations are applied to them. Similarity measures include the proximity of redefined landmarks, the distance between contours, the difference between pixel intensity values. One can extract individual features that to be matched in each data set. Once features have been extracted from each image, they must be paired to each other. Then, a the similarity measure between the paired features is formulated can be formulated as an optimization problem.

We can use Monge-Kantorovich for the similarity measure in this procedure.

Mass Transportation Problems

- Original transport problem was proposed by Gaspar Monge in 1781, and asks to move a pile of soil or rubble to an excavation with the least amount of work.
- Modern measure-theoretic formulation given by Kantorovich in 1942. Problem is therefore known as *Monge-Kantorovich Problem (MKP).*
- Many problems in various fields can be formulated in term of MKP: statistical physics, functional analysis, astrophysics, reliability theory, quality control, meteorology, transportation, econometrics, expert systems, queuing theory, hybrid systems, and nonlinear control.

Monge-Kantorovich Mass Transfer Problem-I

We consider two density functions

$$\int \mu_0(x)\ \ dx = \int \mu_T(x)\ \ dx$$

We want

$$M: \mathbf{R}^d \to \mathbf{R}^d$$

which for all bounded subsets $A \subset \mathbf{R}^d$

$$\int_{x\in A} \mu_T(x)\ \ dx = \int_{M(x)\in A} \mu_0(x)\ \ dx$$

For M smooth and 1-1, we have (Jacobian equation)

$$\det(\nabla M(x))\, \mu_T(M(x)) = \mu_0(x)$$

We call such a map M ***mass preserving (MP).***

MK Mass Transfer Problem-II

Jacobian problem has many solutions. Want **optimal** one (**Lp-Kantorovich-Wasserstein metric**)

$$d_p(\mu_0, \mu_1)^p := \inf_M \int |M(x) - x|^p \mu_0(x) dx$$

Optimal map (when it exists) chooses a map with preferred geometry (like the Riemann Mapping Theorem) in the plane.

Algorithm for Optimal Transport-I

Subdomains with smooth boundaries and positive densities:

$$\Omega_0, \Omega_1 \subset \mathbf{R}^d$$

$$\int_{\Omega_0} \mu_0 = \int_{\Omega_1} \mu_1$$

We consider diffeomorphisms which map one density to the other:

$$\mu_o = \det(\nabla \tilde{u}) \mu_1 \circ \tilde{u}$$

We call this the ***mass preservation*** *(MP)* property. We let *u* be a initial MP mapping.

Algorithm for Optimal Transport-II

We consider a one-parameter family of MP maps derived as follows:

$$\tilde{u} := u \circ s^{-1},\ s = s(\cdot\ , t),\ \mu_0 = \det(\nabla s)\mu_0 \circ s$$

Notice that from the MP property of the mapping *s*, and definition of the family,

$$\tilde{u}_t = -\frac{1}{\mu_0}\nabla\tilde{u} \cdot \zeta,\ \ \zeta = \mu_0\, s_t \circ s^{-1}$$

$$\mathrm{div}\ \zeta = 0$$

Algorithm for Optimal Transport-III

We consider a functional of the following form which we infimize with respect to the maps $\tilde{u}$:

$$M(t) = \int_{\Omega_0} \Phi(\tilde{u}(x,t) - x)\mu_0(x)\ dx$$

$$= \int \Phi(u(y) - s(y,t))\mu_0(y)\ dy,\ \ x = s(y,t),\ \ s^*(\mu_0(x)dx) = \mu_0(y)dy$$

Taking the first variation:

$$M'(t) = -\int \langle \Phi'(u - s),\ \ s_t \rangle \mu_0 dy$$

$$= -\int \langle \Phi'(\tilde{u}(x,t) - x),\ \ \mu_0 s_t \circ s^{-1} \rangle\ \ dx$$

$$= -\int_{\Omega_0} \langle \Phi'(\tilde{u}(x,t) - x),\ \ \zeta \rangle\ \ dx$$

Algorithm for Optimal Transport-IV

First Choice:

$$\zeta = \Phi'(\tilde{u} - x) + \nabla p$$

$$\mathrm{div}\ \ \zeta = 0$$

$$\zeta\big|_{\partial\Omega_0}\ \ \text{tangential}\ \ \text{to}\ \ \partial\Omega_0$$

This leads to following system of equations:

$$\tilde{u}_t = -1/\mu_0 \nabla\tilde{u}\cdot(\Phi'(\tilde{u} - x) + \nabla p)$$

$$\Delta p + \mathrm{div}\ \ (\Phi'(\tilde{u} - x)) = 0, \ \ \text{on}\ \ \Omega_0$$

$$\frac{\partial p}{\partial \vec{n}} + \vec{n}\cdot\Phi'(\tilde{u} - x) = 0, \ \ \text{on}\ \ \partial\Omega_0$$

Algorithm for Optimal Transport-V

This equation can be written in the *non-local* form:

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{1}{\mu_0} \nabla \tilde{u} \cdot (I - \nabla \Delta^{-1} \nabla \cdot\,) \Phi'(\tilde{u} - x)$$

At optimality, it is known that

$$\Phi'(\tilde{u} - x) = \nabla \alpha$$

where α is a function. Notice therefore for an optimal solution, we have that the non-local equation becomes

$$\frac{\partial \tilde{u}}{\partial t} = 0$$

Solution of L2 M-K and Polar Factorization

For the L2 Monge-Kantorovich problem, we take

$$\Phi(x) = \frac{|x|^2}{2}$$

This leads to the following "non-local" gradient descent equation:

$$\tilde{u}_t = -1/\mu_0 \nabla \tilde{u}(\tilde{u} - \nabla \Delta^{-1} \; \mathrm{div}(\tilde{u}))$$

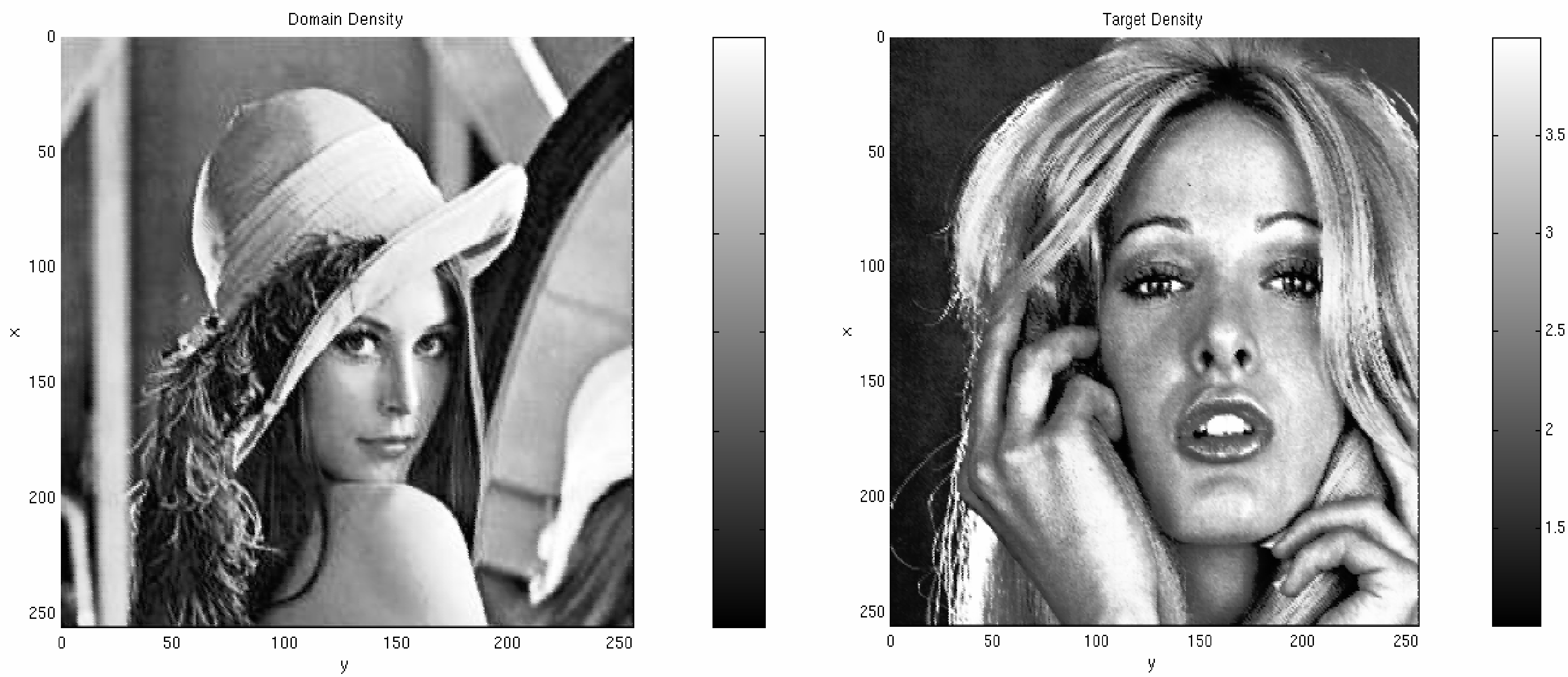
Notice some of the motivation for this approach. We take:

$$\tilde{u} = u \circ s^{-1} = \nabla w + \chi, \;\; \mathrm{div}(\chi) = 0 \;\; \textbf{Helmholtz decomp.}$$

The idea is to push the fixed initial u around (considered as a vector field) using the 1-parameter family of MP maps s(x,t), in such a manner as to remove the divergence free part. Thus we get that at optimality

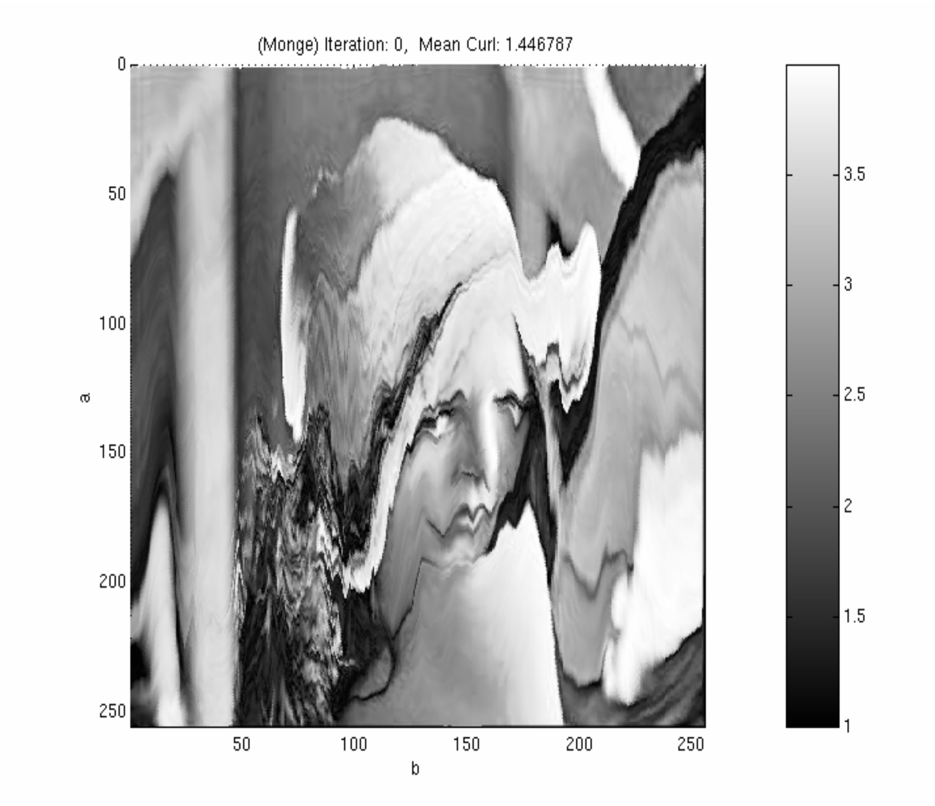
$$u = \nabla w \circ s \;\; \textbf{Polar factorization}$$

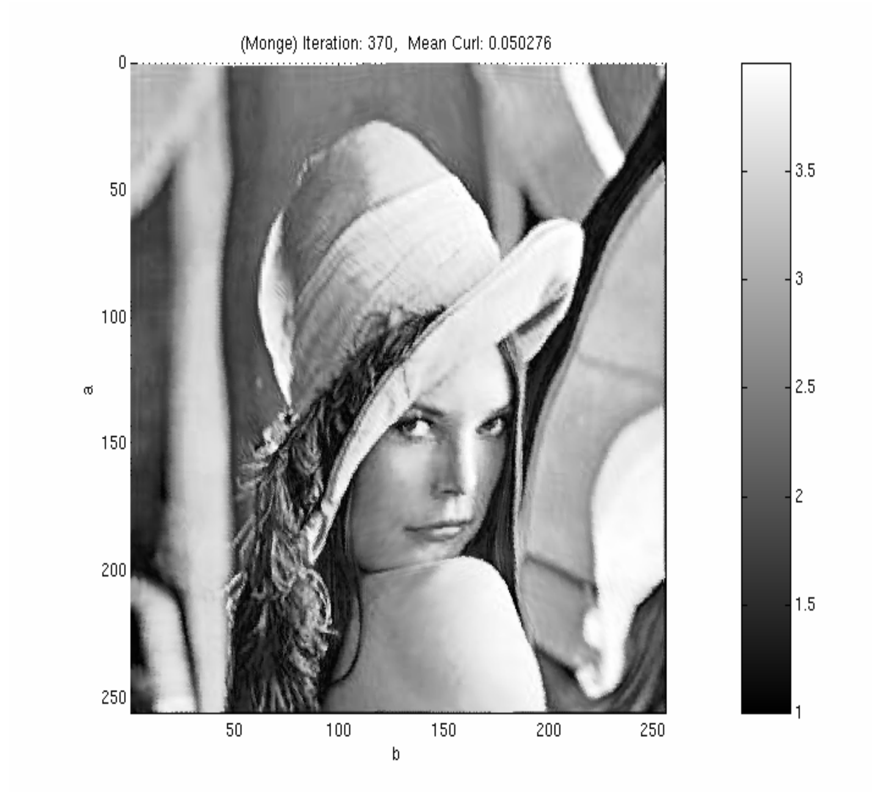
Example of Mass Transfer-I



We want to map the Lena image to the Tiffany one.

Example of Mass Transfer-II





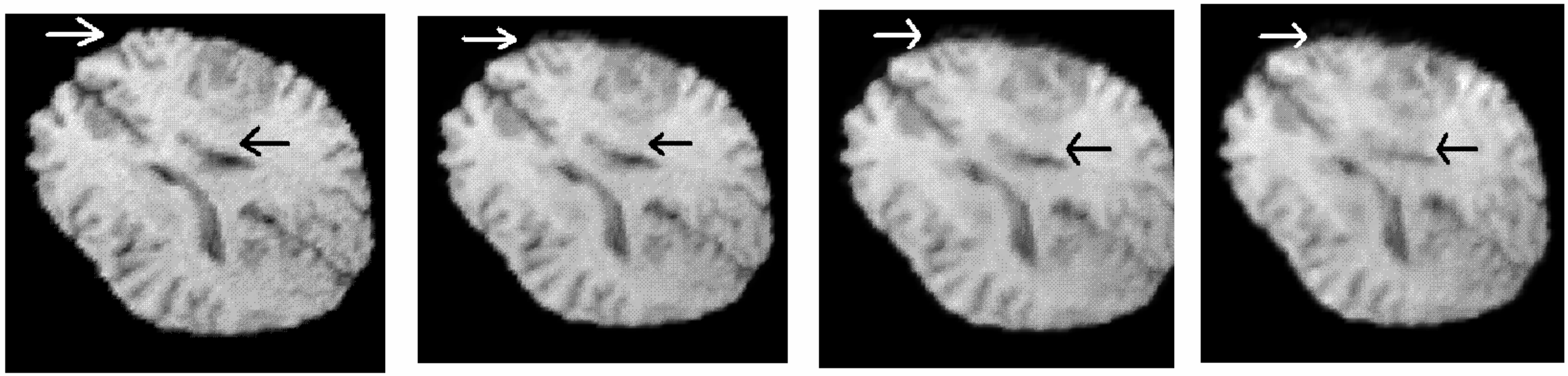
The first image is the initial guess at a mapping. The second is the Monge-Kantorovich improved mapping.

Morphing the Densities-I

$$V(t,x) = x + t(u_{opt}(x) - x)$$

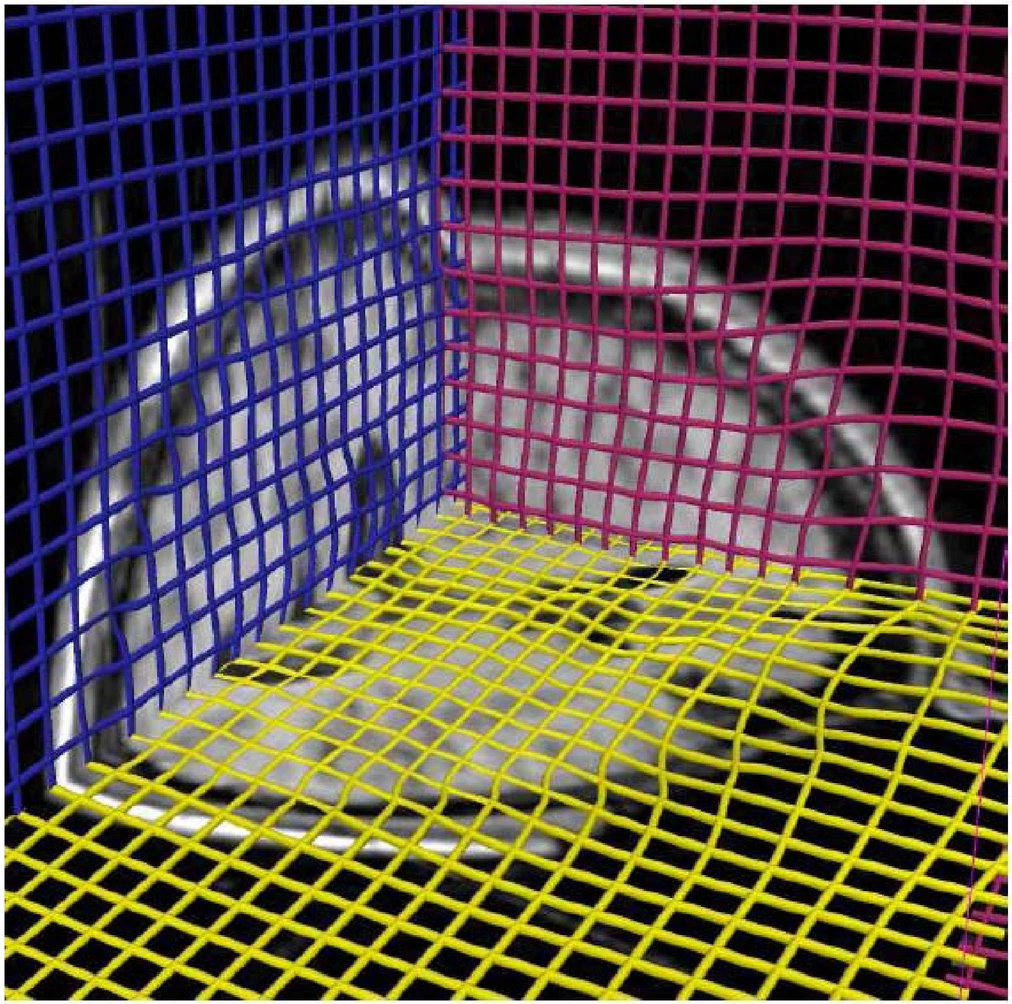
Morphing the Densities-II (Brain Sag)

Deformation Map

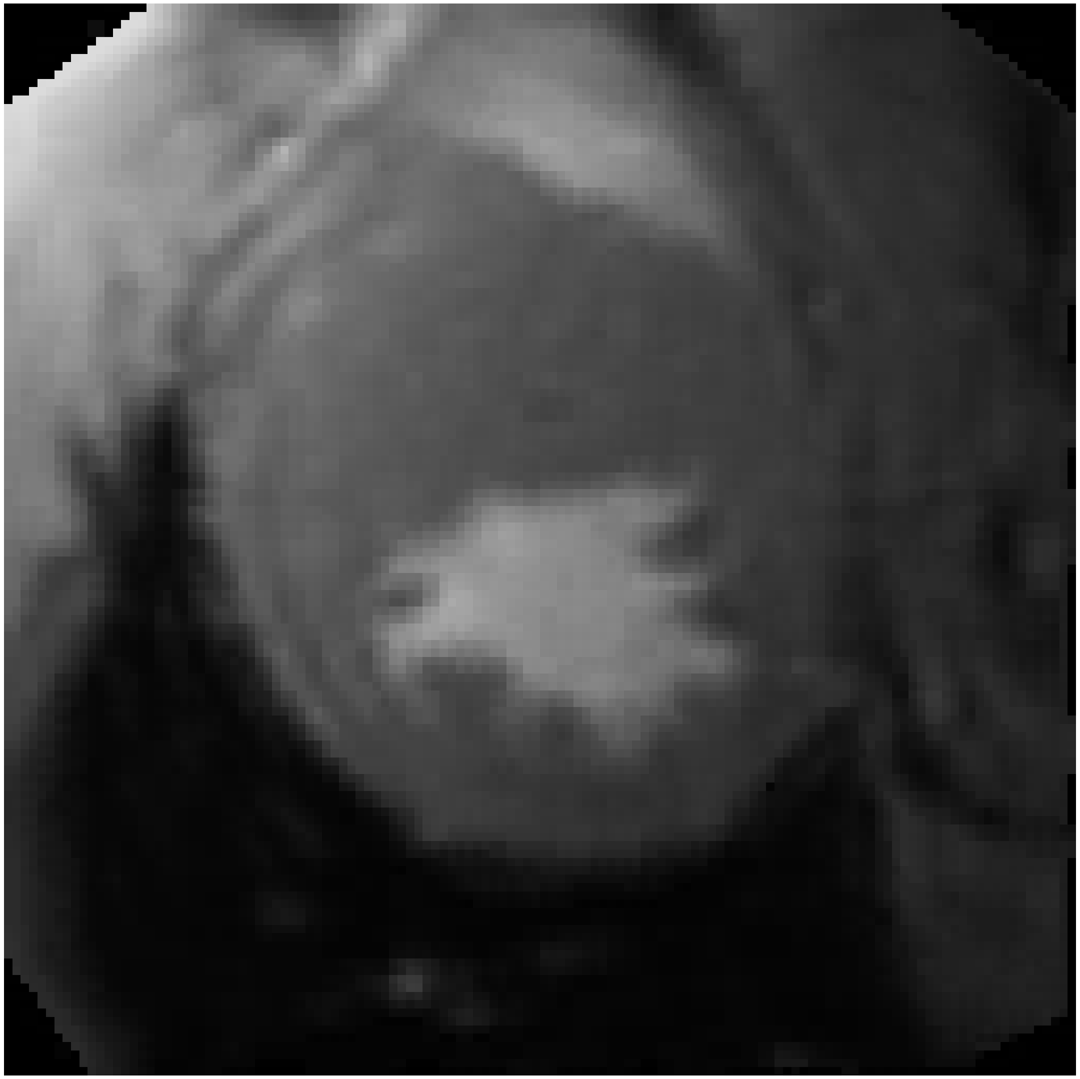


Brain deformation sequence. Two 3D MR data sets were used. First is pre-operative, and second during surgery, after craniotomy and opening of the dura. First image shows planar slice while subsequent images show 2D projections of 3D surfaces which constitute path from original slice. Here time t=0, 0.33, 0.67,and 1.0. Arrows indicate areas of greatest deformation.

3D Registration: Brain Sag



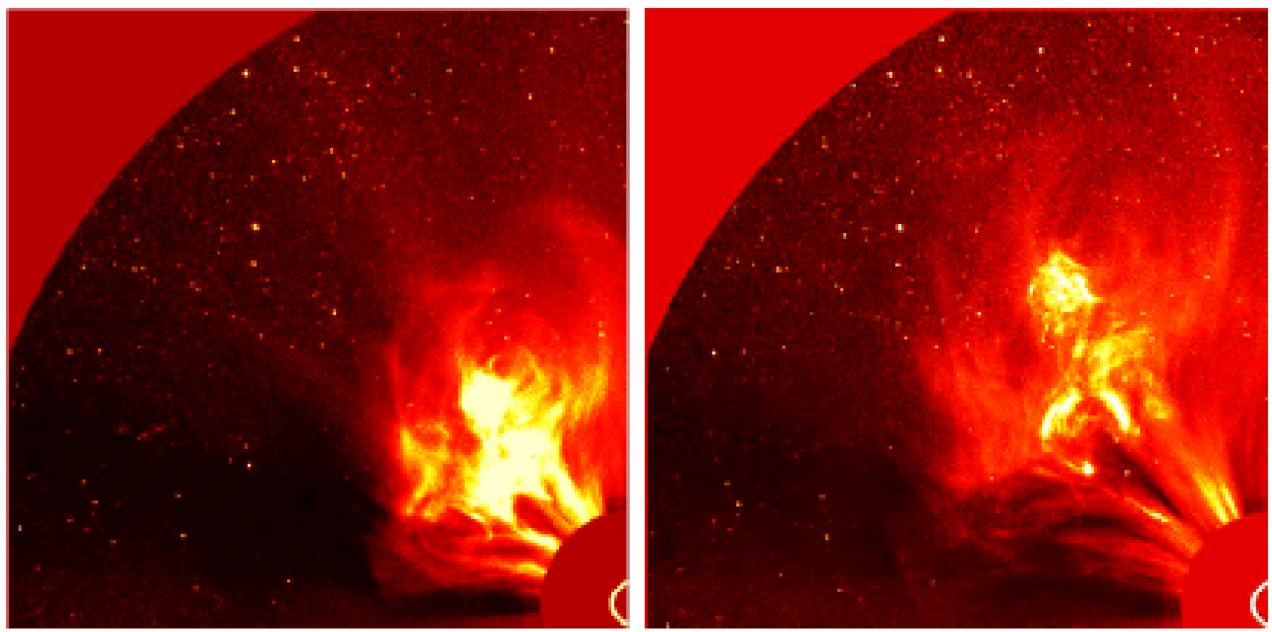
Morphing-III



Morphing-IV

Morphing-V

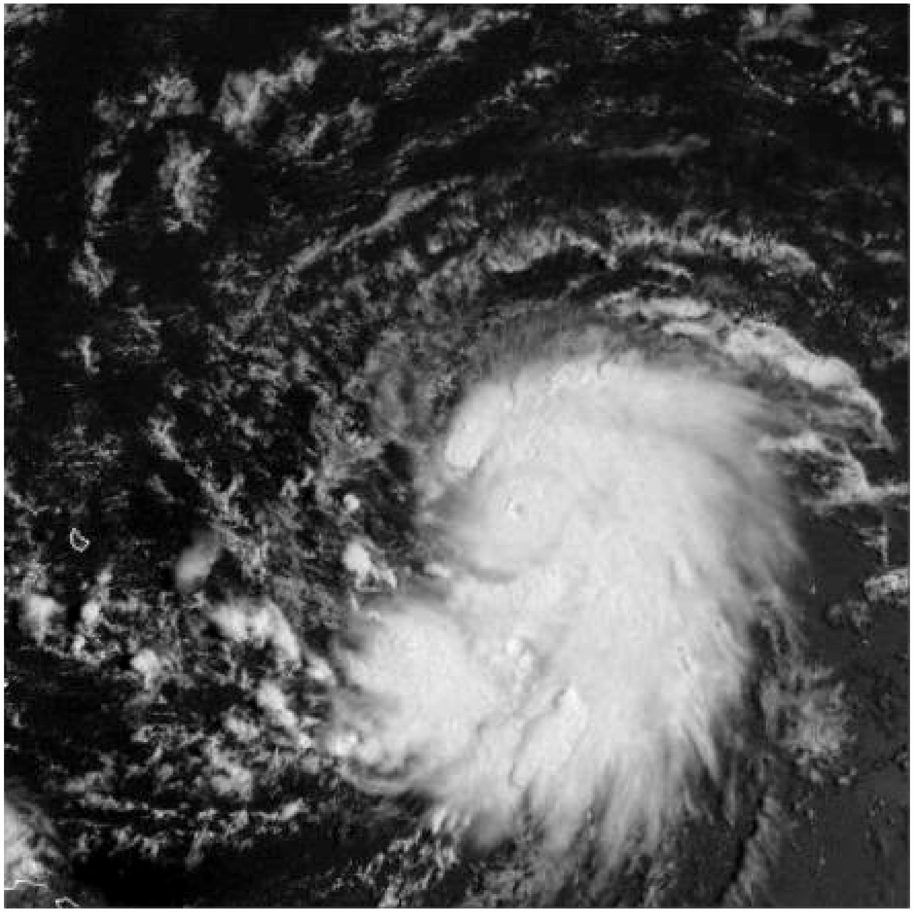
Example: Solar Flare



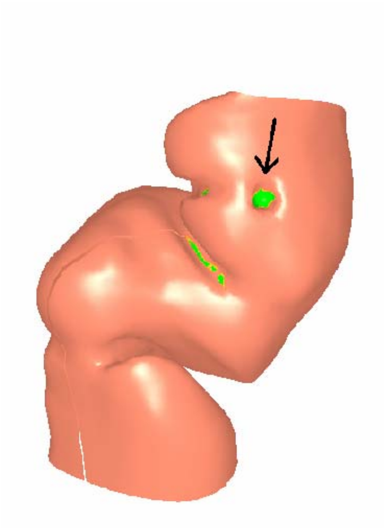
(a) Source Image (b) Target Image

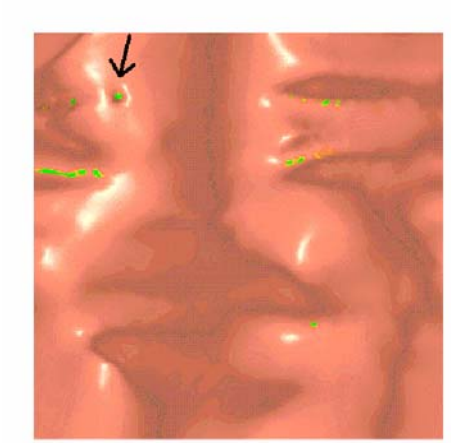
Example: Solar Flare

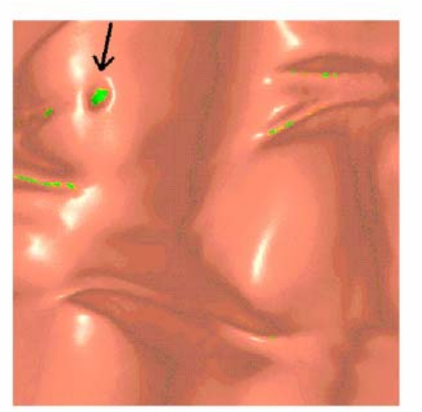
Example: Hurricane Dean



Surface Warping-I

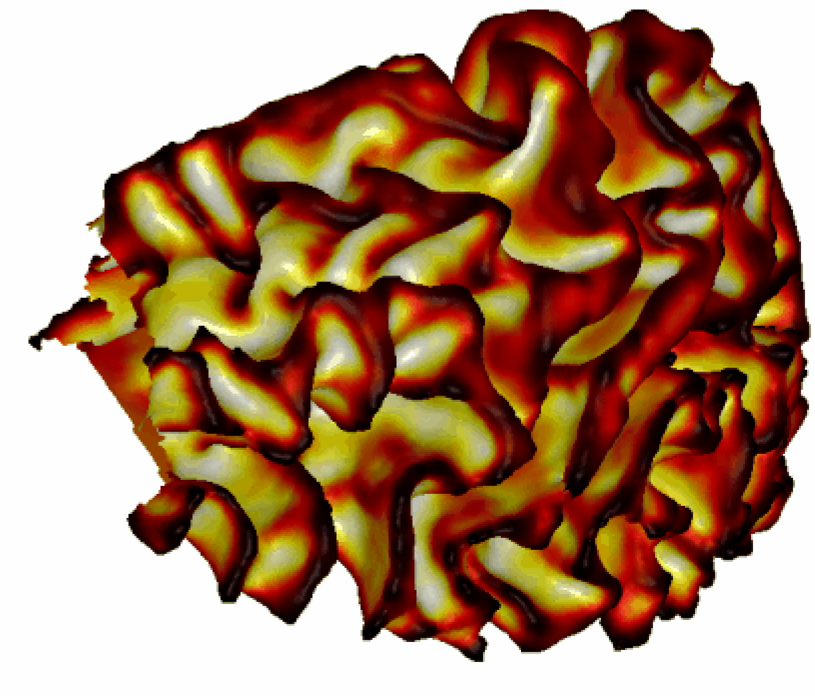


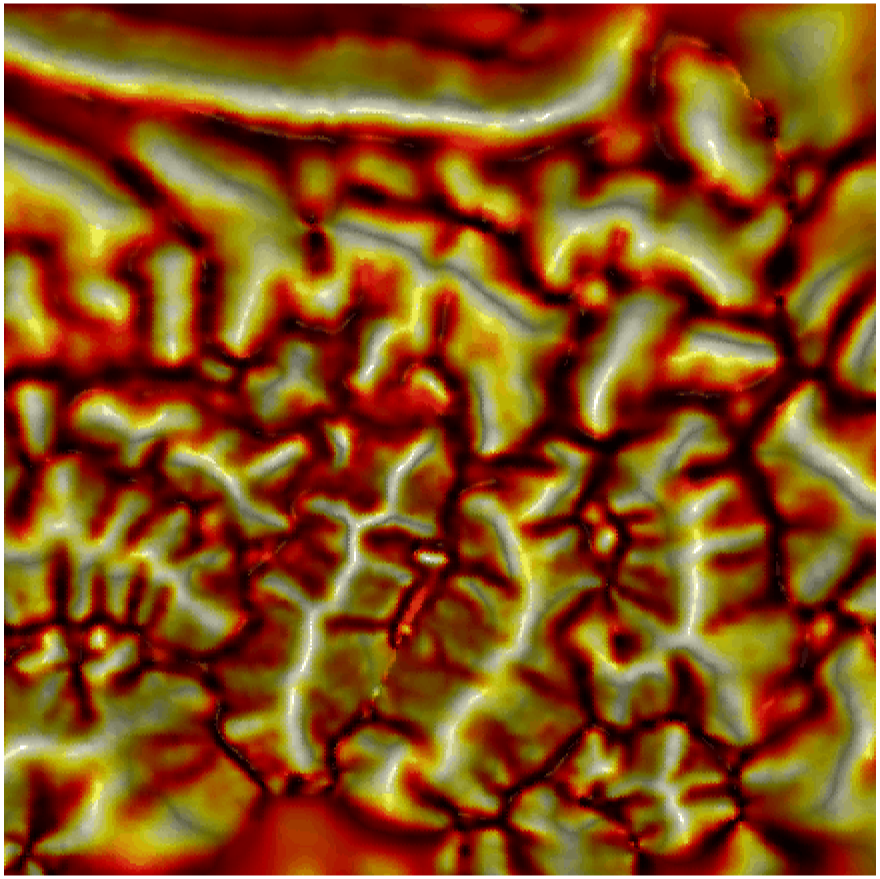




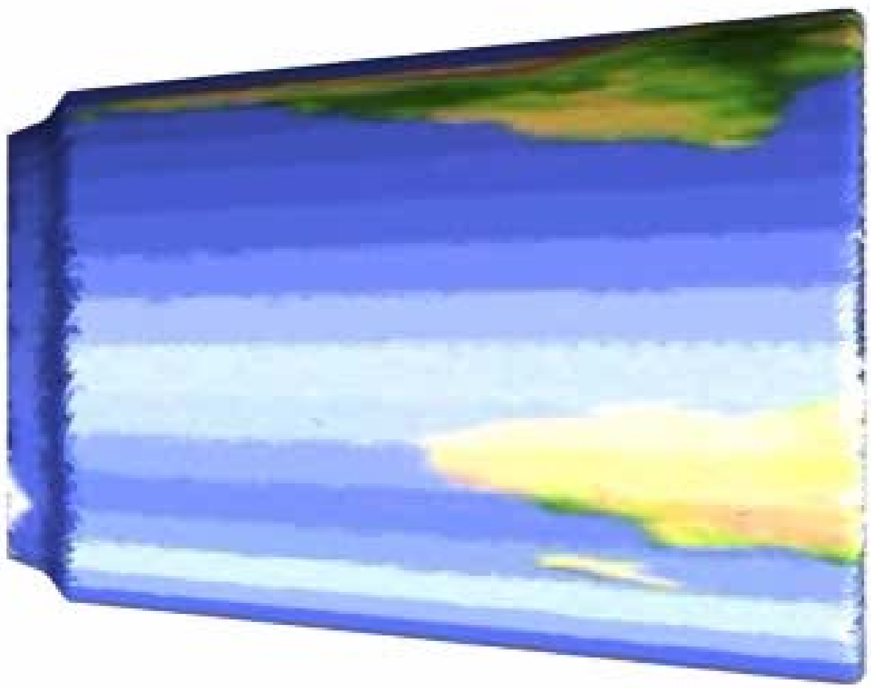
M-K allows one to find area-correcting flattening. After conformally flattening surface, define density mu_0 to be determinant of Jacobian of inverse of flattening map, and mu_1 to be constant. MK optimal map is then area-correcting.

Surface Warping-II

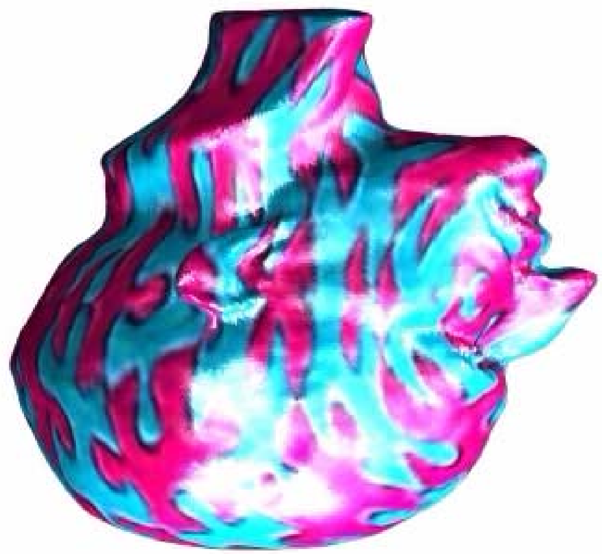




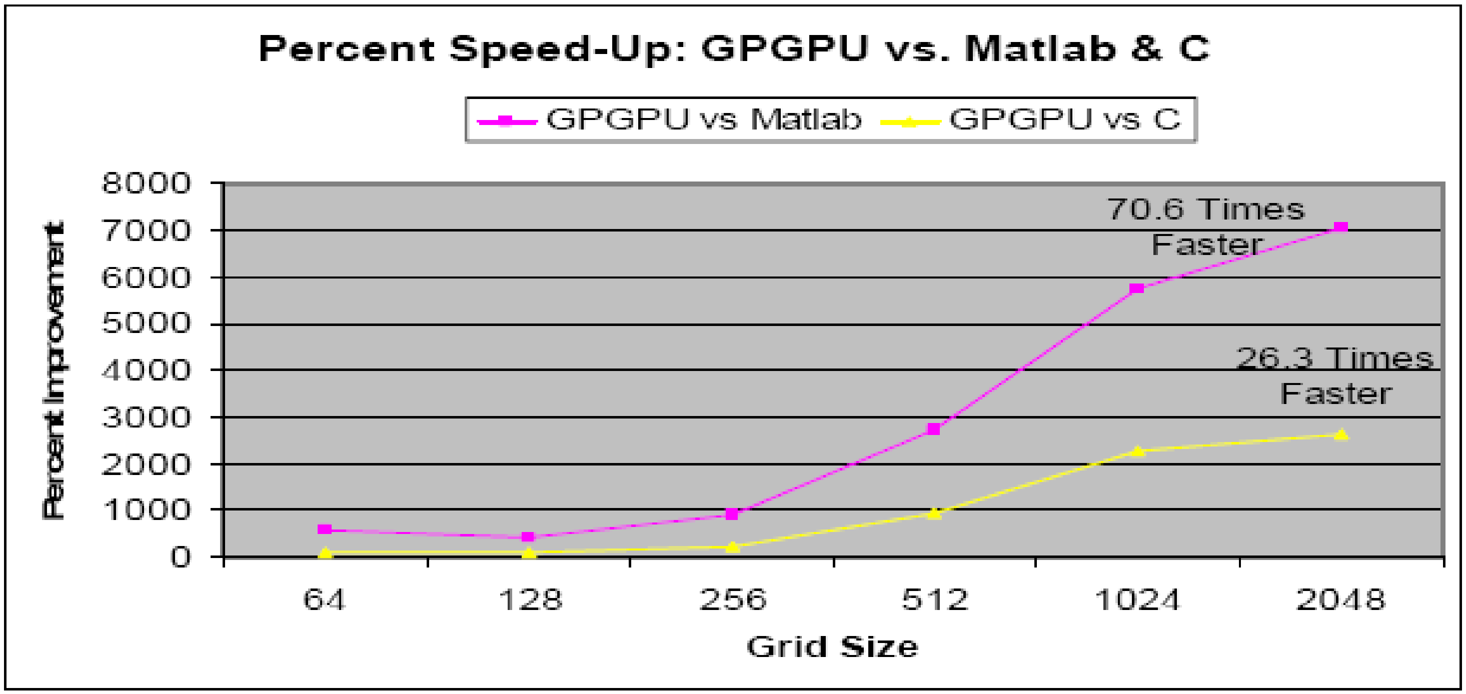
Example of OMT Mapping on Spherical Shape



Some More OMT



Algorithms on the GPU



Conclusions

- Deformations of Surfaces
- Optimal Transport Methods
- Optical Flow