① It is nice to give this talk, which is based on a joint work with L. Ambrosio, H.K. Kim and T. Pacini

WELL UNDERSTOOD FACTS : Finite dimensional Hamiltonian systems (HS)

Data $h \in C_c^\infty(\mathbb{R}^{2nd})$ $\quad J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$ symplectic matrix $\quad \bar z = (\bar z_1, \dots, \bar z_n) \in \mathbb{R}^{2nd}$ $\quad \bar z_i = (x_i, p_i)$

ODE:
$$\begin{cases} \dot z_i(t) = -J \nabla_{z_i} h(z_1(t), \dots, z_n(t)) \\ z(0) = \bar z \end{cases}$$

- Flow $\quad \bar z \longrightarrow \Phi_h(t, \bar z) := z(t)$ $\quad \Phi_h(t,\cdot)$ hamiltonian diffeomorphism
- Symplectic form $\quad \omega_n : \mathbb{R}^{nD} \times \mathbb{R}^{nD} \longrightarrow \mathbb{R}$ $\quad \omega(\xi, \eta) = \sum_{i=1}^n \langle -J\xi_i, \eta_i \rangle$ $\quad \xi = (\xi_1, \dots, \xi_n)$ $\quad \eta = (\eta_1, \dots, \eta_n)$
- Hamiltonian vector field $\quad X_h = (-J\nabla_{z_1} h, \dots, -J\nabla_{z_n} h)$

  $dH = \omega(X_h, \cdot)$
- Orbit : $\quad O_{\bar z} = \{ \Phi_h(1, \bar z) \quad h \in C_c^\infty(\mathbb{R}^{2d}) \}$ provides a foliation of $\mathbb{R}^{2nd}$

Define
$$\begin{cases} \mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{z_i(t)} \\ \mu_0^n = \frac{1}{n} \sum_{i=1}^n \delta_{\bar z_i^n} \end{cases}$$
$n \longrightarrow +\infty \qquad \mu_t^n \longrightarrow \mu_t$

Question a) what is the equation satisfied by μ_t ? In which sense the equation is an infinite dimensional HS ?

b) $(-J\nabla_{z_1} h, \dots, -J\nabla_{z_n} h) = X_h \longrightarrow X$ which lives in which space ?

Preliminary answers

let $\quad \mathcal{M} = \mathcal{P}_2(\mathbb{R}^{2d}) = \{ \mu$ Borel probability measure on $\mathbb{R}^{2d}$ of bounded second moments $\}$

a) suppose $\quad H : \mathcal{M} \longrightarrow \mathbb{R}$ and

(1) $\quad h(z_1, \dots, z_n) = H\left(\frac{1}{n} \sum_{i=1}^n \delta_{z_i}\right)$

Then we answer question a)

b) when h is of the form (1) then $X_h(z) = (\xi_1(z_1), \dots, \xi_n(z_n))$ $\quad X_h \in L^2(\mathbb{R}^{2d}, \mathbb{R}^{2d}; \mu)$

$\mu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i}$

INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS

* The idea of expressing PDEs as infinite-dimensional HS goes back to W. Pauli (1933) and then Born - Infeld (1935)

* 1950 s Clebsch $\qquad$ * 1970 Arnold

* 1980 s P.J Morrison , J.E Marsden , A Weinstein

②

+ 2007 → J. Lott, P. Lee - B. Khesin

The motivation of Weinstein and Marsden was that Hamiltonian structures can be useful in detecting chaos by the method of Melnikov.
This talk does not address the method of Melnikov but rather focus on getting rigorous results for physical systems where electric and magnetic fields are absent.
In that case, our Poisson structure coincide with the one of the above mentioned works

$\mathbb{R}^D \hookrightarrow \mathcal{P}_2(\mathbb{R}^D) \hookrightarrow (C_c^\infty(\mathbb{R}^D))^*$

<u>Plan of the talk</u>

- Introduce a Poisson structure on $\mathcal{M}$. Define Hamiltonian vector field X_H
- Existence of solutions of $\dot\mu_t = X_H(\mu_t)$
- Green's formula

Def. Let $t \to \mu_t \in \mathcal{M}$. Suppose $Y : \mathcal{M} \to T\mathcal{M}$. Then $\dot\mu_t = Y(\mu_t)$ means

$$\frac{\partial \mu_t}{\partial t} + \operatorname{div}\big(\mu_t Y(\mu_t)\big) = 0$$

Ambrosio - Gigli - Savaré

$$L^2(\mathbb{R}^d, \mathbb{R}^d, \mu) = T_\mu\mathcal{M} \oplus \operatorname{Ker}(\operatorname{div}_\mu)$$

$$Y = X + W \qquad \operatorname{div}_\mu(W) = 0$$

$$X = \pi_\mu(Y) \qquad \pi_\mu : L^2(\mu) \to T_\mu\mathcal{M}$$

$$\dot\mu_t = X(\mu_t)$$

<u>Hamiltonian vector field</u> Let $H : \mathcal{M} \to \mathbb{R}$. Suppose there exists $\xi : \mathcal{M} \to T\mathcal{M}$ such that

$$\frac{d}{dt} H\big((id + t\eta)\#\mu\big)\Big|_{t=0} = \langle \xi, \eta \rangle_{L^2(\mu)} \quad \text{for all } \eta \in \nabla C_c^\infty(\mathbb{R}^D)$$

Then

$$\xi = \nabla_\mu H \qquad X_H = \pi_\mu(-J\nabla_\mu H)$$

Two-form

$$\Omega(X_H, X_G) = \int_{\mathbb{R}^{2d}} \langle J\nabla_\mu H ; \nabla_\mu G \rangle \, d\mu$$

Ω is alternative ; non degenerate and closed

$$-dH = \Omega(X_H, \cdot)$$

<u>Theorem</u> (Ambrosio, G) Suppose $H : \mathcal{M} \to \mathbb{R}$, $\nabla_\mu H$ exists

- $|\nabla_\mu H(z)| \le C(1 + |z|)$ μ a.e.
- If $\mu_n = \rho_n \mathcal{L}^D$, $\mu = \rho\mathcal{L}^D$ and $\mu_n \xrightarrow{W_2} \mu$ then $\nabla H(\mu_n)\mu_n \longrightarrow \nabla H(\mu)\mu$

Then given $\bar\mu = \bar\rho\mathcal{L}^D$, $\begin{cases} \dot\mu_t = X_H(\mu_t) \\ \mu_0 = \bar\mu \end{cases}$ admits a solution for $t \in [0, T]$

Furthermore $t \to \mu_t$ is L - Lipschitz $\quad L := L(C_0, T, M(\bar\mu))$

If in addition H is λ - convex then $H(\mu_t) = H(\mu_0)$

<u>Example</u> a)

$$H(\mu) = -\frac{1}{2} W_2^2(\mu, \nu_0) \qquad \nu_0 << \mathcal{L}^d$$

b)

$$H(\mu) = \int_{\mathbb{R}^D} V \, d\mu + \int_{\mathbb{R}^D} W * \mu \, d\mu$$

③

a)
$$H(\mu) = \int_{\mathbb{R}^{2d}} h(x,p)\, d\mu(x,p)$$
$$\nabla_\mu H = \nabla h$$

Mean field game

Orbit If $\mu \in \mathcal{M}$
$$\mathcal{O}_\mu = \{ \Phi_h(1,\cdot) \# \mu : h \in C([0,1], C^{1,1}(\mathbb{R}^{2d})) \}$$

If H is smooth and $\dot\mu_t = X_H(\mu_t)$ then $\mu_t \in \mathcal{O}_{\mu_0}$. If H is λ-convex then $\exists$ $\mu_t \in \bar{\mathcal{O}}_{\mu_0}$.

Differential

Def let $\mu \in \mathcal{M}$ and $X \in \nabla C_c^\infty(\mathbb{R}^D)$ and $f : \mathcal{M} \to \mathbb{R}$
$$X \cdot f(\mu) = \lim_{h \to 0} \frac{f((id + tX)\#\mu) - f(\mu)}{t}$$

If $\Lambda : T\mathcal{M} \to \mathbb{R}$ and $X, Y \in \nabla C_c^\infty(\mathbb{R}^D)$ $\quad [X,Y] = \nabla Y X - \nabla X Y$
$$d\Lambda(X,Y) = X \cdot (\Lambda(Y)) - Y \cdot (\Lambda(X)) - \Lambda[X,Y]$$

Def Suppose $C : t \to \mu_t \in \mathcal{M}$ and $\frac{\partial \mu_t}{\partial t} + \operatorname{div}(\mu_t V_t) = 0$ $\quad \|V_t\|_{L^2(\mu_t)} \in L^2(0,T)$

Then
$$\int_C \Lambda := \int_0^T \Lambda_{\mu_t}(V_t)\, dt$$
independent of Lipschitz reparametrization.

Suppose $S : (t,s) \longrightarrow \mu_t^s \in \mathcal{M}$ and $\frac{\partial \mu_t^s}{\partial t} + \operatorname{div}(\mu_t^s V_t^s) = 0$ $\quad \frac{\partial \mu_t^s}{\partial s} + \operatorname{div}(\mu_t^s W_t^s) = 0$

Then if $\Omega : T\mathcal{M} \times T\mathcal{M} \longrightarrow \mathbb{R}$
$$\int_S \Omega = \int_0^T dt \int_0^T ds\, \Omega_{\mu_t^s}(V_t^s, W_t^s)$$

Riesz Representation If $\Lambda : T\mathcal{M} \to \mathbb{R}$ is a one-form then there exists $A_\mu \in L^2(\mu)$ such that $\forall X \in L^2(\mu)$

(2)
$$\Lambda_\mu(X) = \int_{\mathbb{R}^D} \langle X, A_\mu \rangle\, d\mu$$

Remark If $\bar A_\mu = A_\mu + W_\mu$ $\quad \operatorname{div}_\mu(W) = 0$ then (2) holds for $\bar A_\mu$

Example If $A \in C_c^\infty(\mathbb{R}^D, \mathbb{R}^D)$ and
$$\Lambda_\mu(X) = \int_{\mathbb{R}^D} \langle A, X \rangle\, d\mu$$

Theorem Suppose $t \to \mu_t \in \mathcal{M}$ and $\frac{\partial \mu_t}{\partial t} + \operatorname{div}(\mu_t V_t) = 0$ $\quad \|V_t\|_{L^2(\mu_t)} \in L^2(0,T)$

Set $D_s z = s z$ $\quad (D_s : \mathbb{R}^{D} \to \mathbb{R}^D)$ and $\mu_t^s = D_s \# \mu_t$ $\quad (s,t) \in (0,1) \times (0,T) \longrightarrow \mu_t^s$ Then

if A satisfies appropriate condition
$$\int_S d\Lambda = \int_C \Lambda$$