On the Harmonic Mean Curvature Flow

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Introduction

Consider the deformation of a compact surface Σ_t in $\mathbb{R}^3$ by its Harmonic Mean Curvature:

$$\frac{\partial \mathbf{P}}{\partial t} = -\frac{G}{H} \cdot \mathbf{N} \qquad \text{(HMCF)}$$

where G denotes the Gaussian curvature, H the Mean curvature and $\mathbf{N}$ the unit outer normal to the surface, so that

$$\kappa := \frac{G}{H} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} = \frac{1}{\lambda_1^{-1} + \lambda_2^{-1}}.$$

The resulting PDE is fully-nonlinear and remains weakly parabolic without the condition that Σ_t is convex. However, the equation becomes degenerate at points where $G = 0$ and singular when $H \to 0$. In the latter case the flow is not defined.

Known Results

• *Andrews:* Existence of smooth solutions of the HCMF with strictly convex initial data. The solutions exist and remain smooth up to the time T_0 when Σ_t shrinks to a point. It follows from the Gauss-Bonnet formula that $T_0 = \mu_0(\Sigma_0)/4\pi$ with $\mu_0(\Sigma_0)$ the surface area of Σ_0.

• *Andrews:* The HMCF shrinks smooth strictly convex surfaces to round points, i.e. the surface becomes spherical as $t \to T_0$.

• *Diëter:* Convex surfaces with mean curvature $H > 0$ become instantly strictly convex and hence they shrink to round points.

Three Different Cases

• Highly degenerate HMCF: $\Sigma_t = \Sigma_1 \cup \Sigma_2$ with Σ^1 flat & Σ^2 strictly convex.
The flow is highly degenerate
on Σ_1 where $H = 0$.

• HMCF on surfaces of revolution with $H < 0$:
Σ_t is a surface of revolution
with boundary and $H < 0$.
Also, $H \to -\infty$ at $\partial \Sigma_t$.

• HMCF on star-shaped surfaces with $H > 0$:
Σ_0 is compact & star-shaped
without boundary and $H > 0$.
Σ_t remains compact but not
necessarily star-shaped.

We study: short time and long time existence, optimal regularity and final-shape of Σ_t.

Highly degenerate HMCF

Let $\Sigma = \Sigma^1 \cup \Sigma^2$ with Σ^1 *flat* and Σ^2 *strictly* convex. Let $\Gamma = \Sigma^1 \cap \Sigma^2$ denote the interface.

Expressing the lower part of Σ as a graph $z = f(x, y, t)$, we compute:

Evolution of f:

$$f_t = \frac{\det D^2 f}{(1 + f_x^2) f_{yy} - 2\, f_x f_y f_{xy} + (1 + f_y^2) f_{xx}}$$

Non-degeneracy Condition: $\exists p \in (0, 1)$ such that $g = f^p$ satisfies

$$|Dg| \geq \lambda, \quad g_{\tau\tau} \geq \lambda, \qquad \forall P \in \Gamma \qquad (\star)$$

for some number $\lambda > 0$. Here τ denotes the tangential direction to the level sets of g.

STE: Jointly with *M.C. Caputo* we prove that given an initial surface

$$\Sigma = \Sigma^1 \cup \Sigma^2, \text{ of class } C^{k,\alpha}, \; k \geq 1, \; 0 < \alpha \leq 1$$

with Σ^1 flat and Σ^2 strictly convex, which satisfies the non-degeneracy condition $(\star)$, then:

- the HMCF admits a viscosity solution $\Sigma_t = \Sigma_t^1 \cup \Sigma_t^2$ of class $C^{k,\alpha}$ which is C^∞-smooth up to the interface $\Gamma_t = \Sigma_t^1 \cap \Sigma_t^2$.

- the flat side Σ_t^1 persists for some positive time and the interface Γ_t is smooth and evolves by the curve shortening flow.

- LTE: Given any weakly convex $C^{1,1}$ initial surface Σ_0, there exists a $C^{1,1}$ solution Σ_t of the HMCF up to time $T_0 = \sigma_0/4\pi$. In addition, the surface becomes strictly convex at time $\tau < T_0$ and hence it shrinks to a round point.

Open question: Given an initial surface Σ_0 with flat sides of class $C^{k,\alpha}$ show that there exists a solution Σ_t of class $C^{k,\alpha}$ up to the extinction of the flat side.

HMCF on surfaces of revolution with $H < 0$

Assume that $r = f(x,t)$, $0 \le x \le 1$ is a surface of revolution with boundary such that

$$G = \lambda_1 \lambda_2 < 0 \quad \& \quad H = \lambda_1 + \lambda_2 < 0.$$

Since

$$\lambda_1 = -\frac{f_{xx}}{(1+f_x^2)^{\frac{3}{2}}} \quad \& \quad \lambda_2 = \frac{1}{f\,(1+f_x^2)^{\frac{1}{2}}}$$

the HMCF becomes

$$f_t = \frac{f_{xx}}{-f\,f_{xx} + f_x^2 + 1}$$

with $f_{xx} > 0$ and $\tilde{H} := -f\,f_{xx} + f_x^2 + 1 < 0.$

Then, $f_t \le 0$ which makes f to decrease, i.e. the surface of revolution shrinks. The HMCF becomes singular when $\tilde{H} = 0$, i.e. when the mean curvature $H = 0$.

Jointly with R. Hamilton we showed that if the initial surface is as above and satisfies the boundary growth condition

$$c \le x^{2-p} (1-x)^{2-p} f_{xx} \le C, \quad x \in (0,1) \quad (\star)$$

for some numbers $0 < p < 1$, $c > 0$ and $C < \infty$, then $\exists\, T_0 > 0$ where first $\tilde{H} = 0$ at T_0.

In addition, there exists a constant $l_0 > 0$ and an interval I_0 of length $|I_0| \ge l_0$ such that

$$\tilde{H}(\cdot, T_0) \equiv 0, \qquad \text{on } I_0.$$

Under our initial growth conditions $(\star)$ the equation becomes degenerate at the boundary points $x_i = 0, 1$. As a consequence, the boundary of the surface of revolution $z = f(r,t)$ moves by the curve shortening flow.

Our results in particular show that a neck-pinch doesn't occur.

Sketch of proof

• To prove STE we introduce weighted Hölder spaces as before before and establish Schauder estimates in those spaces.

• To prove that a neck pinch doesn't occur we use the inequality $-f\, f_{xx} + f_x^2 + 1 \leq 0$ and compare with the minimal surfaces of revolution $\phi(x) = \theta^{-1} \cosh(\theta\,(x - x_0))$ for appropriate choices of θ and x_0.

• To prove LTE up to when $H = 0$ we show that the boundary condition $(\star)$ is preserved in time and also $f \in C^2$ in the interior.

• To prove that at T_0, H vanishes on I_0 with $|I_0| \geq l_0$, we analyze the PDE for $w := 1/H$ which is modeled on the diffusion

$$w_t = w^2\, w_{xx} + w^3.$$

Such PDE was previously studied by M. Gage and Gage and Hamilton in connection to the curve shortening flow.

HMCF on star-shaped surfaces with $H > 0$

Consider a compact surface Σ_t in $\mathbb{R}^3$ evolving by the (HMCF)

$$\frac{\partial \mathbf{P}}{\partial t} = -\kappa \cdot \mathbf{N} \qquad \text{(HMCF)}$$

with $\kappa(\lambda_1, \lambda_2) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$. Assume that $\lambda_2 \leq \lambda_1$

The resulting PDE is fully-nonlinear, weakly parabolic and it becomes degenerate at points where $\lambda_2 = 0$ and singular if $H = \lambda_1 + \lambda_2 \to 0$. In the latter case the flow is not defined.

The linearized operator $\mathcal{L}$ is given by

$$\mathcal{L}(u) = a^{ik} \nabla_i \nabla_k u, \qquad a^{ik} = \frac{\partial \kappa}{\partial h^i_k}.$$

Notice that in geodesic coordinates around a point at which the second fundamental form matrix $A = \mathrm{diag}(\lambda_1, \lambda_2)$ we have

$$(a^{ik}) = \mathrm{diag}\left(\frac{\lambda_2^2}{(\lambda_1 + \lambda_2)^2}, \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2}\right)$$

Results-Work in progress

Initial assumptions: We assume that Σ_0 is compact of class $C^{2,1}$ and is mean convex i.e. $H > 0$.

Jointly with Natasa Sesum:

- Short time existence: There exists $\tau > 0$, for which the HMCF admits a $C^{2,1}$ solution Σ_t, such that $H > 0$ on $t \in [0, \tau)$.

- Long time existence: Assume in addition that Σ_0 is star-shaped. Let $T_0 = \mu_0(\Sigma_0)/4\pi$. Either, $H \to 0$ at some point $P_0 \in \Sigma_{t_0}$, at time $t_0 < T_0$, or a $C^{1,1}$ solution to the flow exists up to T_0, it becomes strictly convex at time $T < T_0$, and it shrinks to a round sphere at T_0.

- If in addition Σ_0 is a surface of revolution, then $H \geq \delta > 0$ up to T_0. Hence, Σ_t exists up to $T_0 = \mu_0(\Sigma_0)/4\pi$ and shrinks to a round point at T_0. Moreover, $\Sigma_t \in C^\infty$, $0 < t < T_0$.

Important evolution equations:

The $H = \lambda_1 + \lambda_2$ and $\kappa = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$ satisfy:

- $\frac{\partial H}{\partial t} = \mathcal{L}(H) + \frac{\partial^2 \kappa}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_j h_m^l + 2\,\kappa^2\,H$

- $\frac{\partial}{\partial t}\kappa = \mathcal{L}(\kappa) + 2\,\kappa^3$

where by direct computation we get

$$\frac{\partial^2 \kappa}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_j h_m^l \cdot H \leq 0$$

because the operator $\kappa(\lambda_1, \lambda_2)$ is concave. Hence, when $H > 0$ this quadratic derivative term drives H down. On the other hand the constant order terms push κ and H up. In particular, they make κ and H blow up in finite time.

Short time existence

We first regularize the flow by considering

$$\frac{\partial \mathbf{P}}{\partial t} = -\kappa_\epsilon \cdot \mathbf{N}$$

with $\kappa_\epsilon = \kappa + \epsilon\, H$. The resulting flow is non-degenerate and has similar properties. To show STE for this flow on $[0, \tau_\epsilon)$ is standard. To pass to the limit we need a'priori estimates independent of ϵ and that $\tau_\epsilon \geq \tau_0 > 0$.

The evolutions of κ_ϵ and the second fundamental form h_i^j combined together imply that a singularity will not develop in short time τ_ϵ with $\tau_\epsilon \geq \tau_0 > 0$, unless $H \to 0$. Since H evolves by

$$\frac{\partial H}{\partial t} = \mathcal{L}(H) + \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_j h_m^l + 2\, \kappa_\epsilon^2\, H$$

to avoid $H \to 0$ we need to control the term $\nabla^i h_q^p \nabla_j \partial h_m^l$ uniformly in ϵ.

We do so by estimating from above the quantity $\frac{1}{H} + |\nabla h_i^j|^2$. This is possible if Σ_0 in $C^{2,1}$.

Long time Existence

We first establish LTE for the regularized flow and then pass to the limit. It is important to establish a'priori estimates which are independent of ϵ.

• An increasing quantity: Let $Q = \langle F, \nu \rangle + 2t\,\kappa_\epsilon$. We have $Q(0) \geq 0$ if Σ_0 is star-shaped. Using the evolution of Q we conclude that

$$\frac{d}{dt} Q_{\text{min}}(t) \geq 0, \quad \text{i.e } Q(t) \geq 0, \ \forall t. \qquad (\star)$$

• Pinching estimate: $\exists\, C_1 > 0, C_2 > 0$ such that:

$$\lambda_{\text{max}} \leq C_1\, \lambda_{\text{min}} + C_2. \qquad (\star\star)$$

We prove this by establishing that the quantity $H/(\langle F, \nu \rangle + 2t\,\kappa_\epsilon)$ is decreasing in time.

Since $|\langle F, \nu \rangle| \leq C$ always, $(\star)$ implies $\kappa_\epsilon \geq -C$ for all t. We then show that $\lambda_{\text{min}} \geq -C$ for all t. Also, if $H \to 0$ both $\lambda_{\text{max}} \to 0$ and $\lambda_{\text{min}} \to 0$.

• By the Gauss-Bonnet theorem the ϵ−flow must extinct at time

$$T_\epsilon = \frac{1}{4\pi}\mu_0(\Sigma_0) - \frac{\epsilon}{4\pi}\int_0^{T_\epsilon}\int_{\Sigma_t^\epsilon} H^2\,\mu_t.$$

• To show that a singularity doesn't occur before T_ϵ, unless $H \to 0$, we assume that $|A| \to \infty$ at some time $T < T_\epsilon$ at which $H > 0$ and use a blow up argument. By blowing up around a point of maximal curvature and passing to the limit we obtain a surface which is convex and because of $(\star)$ it satisfies the pinching estimate $\lambda_{\text{max}} \leq C_1\,\lambda_{\text{min}}$.

• By a result of R. Hamilton such a surface must be compact, which after we re-scale back implies that $\text{diam}\,\Sigma_t \to 0$ as $t \to T$, contradicting that $T < T_\epsilon$. By working a little harder we show that Σ_t becomes strictly convex before T_ϵ and hence by the results of B. Andrews shrinks to a round point at T_ϵ.

• To pass to the limit $\epsilon \to 0$ we use the above blow up argument in a manner which is independent of ϵ.

• We show that the limit is independent of sequences $\epsilon_j \to 0$ by combining our a'priori estimates with a uniqueness result of Chen, Giga and Gotto.

• In the special case of a surface of revolution, we show that $H \geq \delta > 0$ independently of time. Hence, any star-shaped surface of revolution becomes strictly convex and shrinks to a round point. It is maybe possible to remove the "star-shaped" assumption.

Question: In the non-radial case, does $H \to 0$ at some time $t_0 < T_0$?

Question: Consider similar flows in higher dimensions.