Lecture Notes

Intro to Monge-Ampère and Optimal Transportation.

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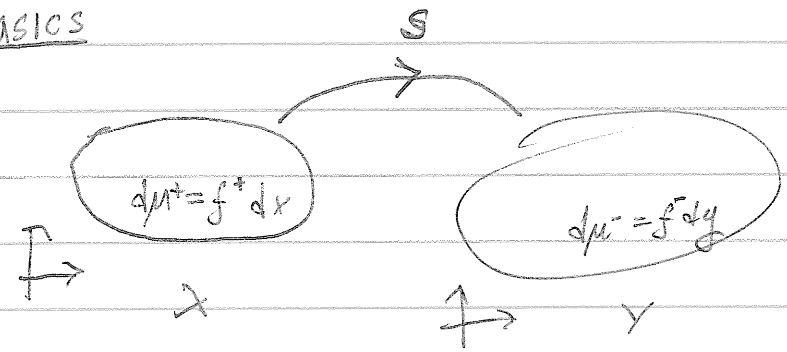
Calculus Push ①

Basics



Calculus change of vars

$$\boxed{\int_X g(s(x))\,dx = \int_Y g(y=s(x))\det(Ds(x))\,dy}$$

Rearrangements

If s rearranges $d\mu^+$ into $d\mu^+$

means

$$\int_X h(s(x))\,d\mu^+(x) = \int_Y h(y)\,d\mu^-(y)$$

for all cts h.

Combine these

PDE

$$\boxed{f^+(x) = f^-(\overset{s(x)}{y})\det(Ds(x))} \quad \text{a.e.}$$

with $s: X \to Y$

[Image holes when non diff'ble / lots of rearrangements.]

Note If $s = \nabla u(x)$, with u convex, Monge-Ampère

$\Rightarrow \det(D^2 u(x)) = f^+(x)/f^-(\nabla u(x))$

Optimal Rearrangements

Given $d\mu^+$, $d\mu^-$
$\quad\quad\quad\; X \quad\;\; Y$

Choose among all suitable rearrangements, A
the one which is minimal w.r.t.
some cost functional:

Given work function

$$c(x,y) : X \times Y \to \mathbb{R}$$

define work of rearrangement plan:

$$I[s] := \int_X c(x, s(x))\, d\mu^+(x)$$

Goal find optimal mass transfer $s^* \in A$

$$I[s^*] = \min_{s \in A} I[s].$$

Not Amenable to Analysis in General.
→ Kantorovich weak version instead
When $c(x,y) = \frac{\|x-y\|^2}{2}$ $\quad d\mu^+ = f^+ dx$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad d\mu^- = f^- dy$

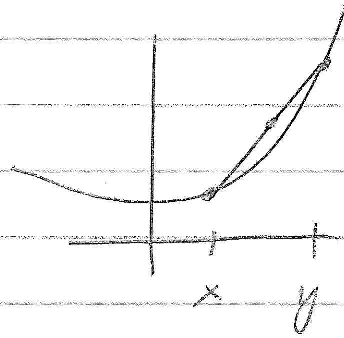
can sometimes solve via PDE.

Convex Analysis 1

<u>Defn</u> $u:\mathbb{R}^n \to \mathbb{R}$ convex if

$$u(tx+(1-t)y) \le t\,u(x) + (1-t)u(y)$$

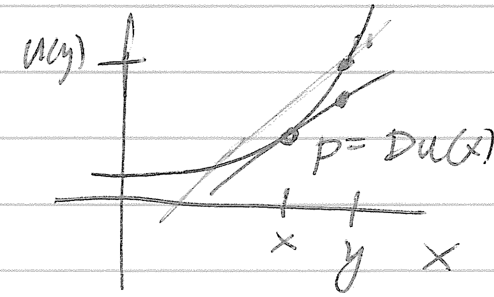
for all $0 \le t \le 1$, x, y



Supporting hyperplane

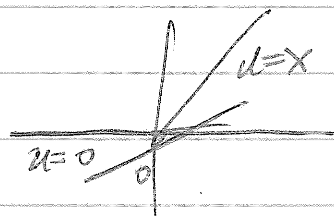
If u is convex, then for any x there is $p \in \mathbb{R}^n$ s.t.

$$u(y) \ge u(x) + p\cdot(y-x)$$



<u>Existence?</u> drop secant down

Subdifferential $p \in \partial^{+}u = [0,1]$

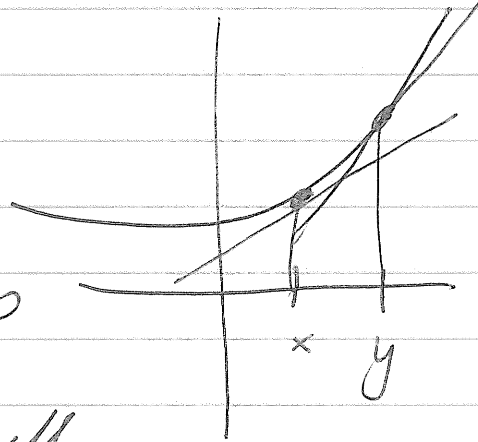


Convex Analysis ②

Graph of convex fn is "monotone"

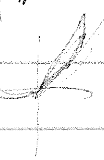
$$u(y) \geq u(x) + Du(x) \cdot (y-x)$$

$$u(x) \geq u(y) + Du(y)(x-y)$$



$$\Longrightarrow \big(Du(x) - Du(y)\big) \cdot (x-y) \geq 0$$

aside / $u''x \geq 0$



True for subgradients as well.

$$\big(\partial^+ u(x) - \partial^+ u(y)\big) \cdot (x-y) \geq 0$$

<u>Defn</u> vector field $h(x)$ is monotone if

$$(h(x) - h(y)) \cdot (x-y) \geq 0 \quad \text{for all } x, y$$

<u>Thm</u> <u>Rockafellar</u>

If $h(x)$ is a monotone vector field

then $h(x)$ lies in the subgradient of a convex function

* cyclically monotone

$$\sum_{k=1}^{m} x_k \cdot (y_{k+1} - y_k) \leq 0$$

Derivation of M-A

Given Map

suppose $y = h(x)$ is optimal map for $c(x,y) = \frac{|x-y|^2}{2}$

Check $(h(x_1) - h(x_2)) \cdot (x_1 - x_2) \geq 0$. for all $x_1, x_2 \in X$

$$x_1 \xrightarrow{h} y_1 = h(x_1)$$

$$x_2 \xrightarrow{h} y_2 = h(x_2)$$

h optimal: put little balls of mass δ around x_i, y_i

Approximate

$$\frac{(y_1 - x_1)^2}{2} + \frac{(y_2 - x_2)^2}{2} \leq \frac{(y_1 - x_2)^2}{2} + \frac{(y_2 - x_1)^2}{2}$$

$$c(x_1, y_1) + c(x_2, y_2) \qquad c(x_1, y_2) + c(x_2, y_1)$$

Simplify $-x_1 y_1 - x_2 y_2 \leq -y_1 x_2 - y_2 x_2$

$$x_2(y_1 - y_2) + (y_2 - y_1) x_1$$

$$(y_1 - y_2) \cdot (x_1 - x_2) \geq 0$$

as desired.

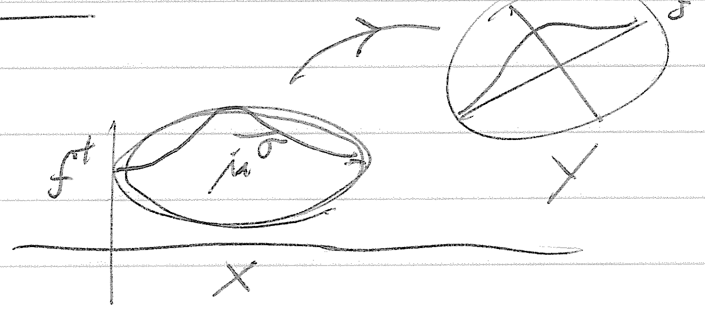
So Want to show $h = \nabla u$ for some convex fn u.

So satisfies

Monge Ampère Eqn
Examples

$$\begin{cases} \det\left(D^2 u(x)\right) = \dfrac{f^+(x)}{f^-(\nabla u(x))} & \text{in } X \\ \nabla u : \underset{\substack{\shortparallel \\ \operatorname{supp} f^+}}{X} \to \underset{\substack{\shortparallel \\ \operatorname{supp} f^-}}{Y}, & \underline{\underline{u \text{ convex}}} \end{cases}$$

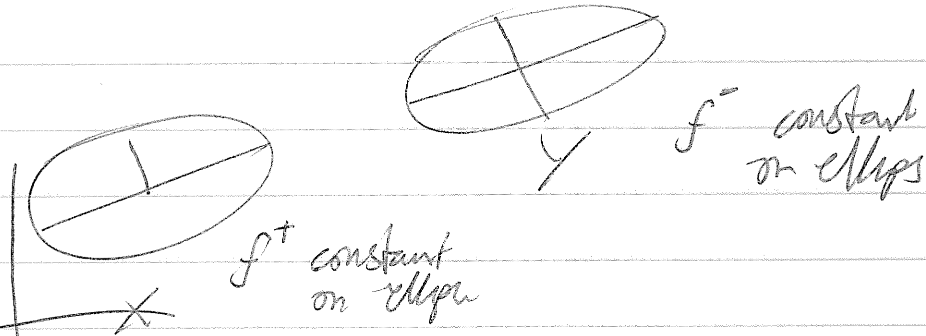
<u>exact solutions?</u>



① When f^+, f^- are gaussians

$S(x) =$ ~~linear~~ affine map, can compute it

② Similarly



then $S(x) =$ <u>affine</u>

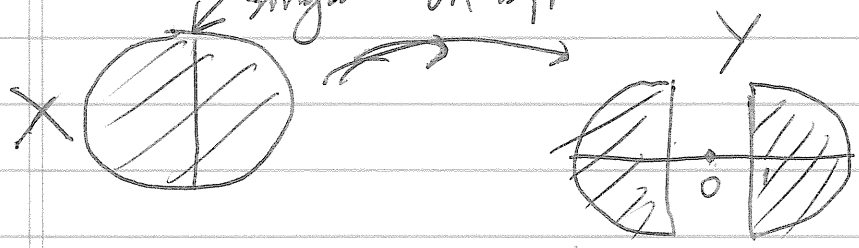
Monge-Ampère Regularity

weak solutions

Alexandrov soln

→ still may fail

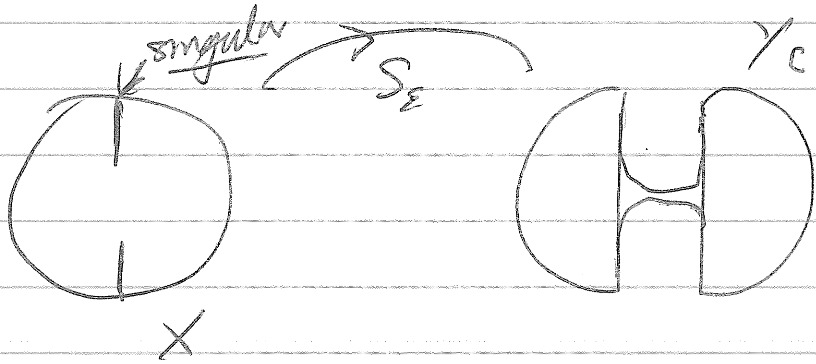
singular on $|x_1| = 0$



$S = D\varphi \quad \varphi = |x_1| + \frac{1}{2}(x_1^2 + x_2^2)$

$$D\varphi = \begin{cases} 1 + x, & x_1 \geq 0 \\ -1 + x, & x_1 < 0 \end{cases}$$

even if we perturb



However if Y convex
then do get a weak soln (Alexandroff)

Existence of Solution

Discrete
① Approximate & solve the ~~Linear Program~~ problem
McCann

OR

② Duality. Then show
compactness of dual problem

(see notes) Evans.

Cts Dual Problem

$$\mathcal{L} = \{(u,v) \mid u(x) + v(y) \leq c(x,y)\}$$

$$K[u,v] = \int u(x)\, d\mu^{+}(x) + \int v(y)\, d\mu^{-}(y)$$

find optimal (u^*, v^*) to max K.

Linear Programming Duality

$$\min\ c\cdot x$$

$$\text{s.t.}\quad Ax = b$$

$$x \geq 0 \qquad (\text{componentwise } x_i \geq 0\ \forall i)$$

$$L(x,y) = c\cdot x + y(Ax-b)$$

<u>Claim</u> $\displaystyle \min_{x\geq 0} \underbrace{\max_{y} L(x,y)}_{\infty \text{ unless } Ax=b} = \max_{y} \underbrace{\min_{x\geq 0} L(x,y)}$

$$\min_{x\geq 0}\ (A^Ty + c)\cdot x + yb$$

$$= \begin{cases} +yb & A^Ty + c \geq 0 \\ -\infty & A^Ty - c \not\geq 0 \end{cases}$$

<u>If can switch</u>

So get D: $\displaystyle \max_{y}\ y\cdot b$

$$\text{s.t.}\quad A^Ty \geq c$$

Relaxation

$$\sum \mu_i^+ = \sum \mu_j^- = 1$$

$$\min \sum c_{ij} \, \mu_{ij}$$

s.j.

$$\left\{ \begin{array}{l} \begin{array}{c} \begin{bmatrix} \longrightarrow \mu_{ij} \longrightarrow \end{bmatrix} \mu_i^+ \\ \mu_j^- \end{array} \qquad \begin{array}{l} \sum_j \mu_{ij} = \mu_i^+ \\ \\ \sum_i \mu_{ij} = \mu_j^- \end{array} \\ \\ \mu_{ij} \geq 0 \end{array} \right.$$

General Form

$$\min \; c \cdot x$$

(P) s.j. $\left\{ \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right.$

(D) $\left\{ \begin{array}{l} \max \; y \cdot b \\ \text{s.j. } \; A^T y \leq C \end{array} \right.$

$$\Longrightarrow \max_{u,v} \sum u_i \mu_i^+ + \sum v_j \mu_j^-$$

$$\text{s.j.} \quad u_i + V_j \leq C$$

0.

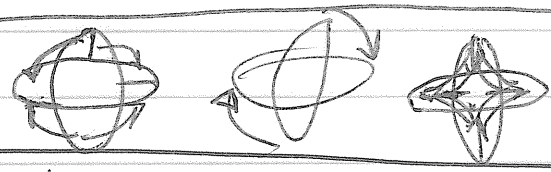
First pictures & weak solutions

From McCann Solutions on Line

Weak Solutions

- Solve ∞-dimensional LP
- Discretize with point masses to get Finite dimensional LP.

Can prove existence this way & compute solutions numerically.

Blobs 

Strong Solutions

More an application of Mass-Transport to PDEs than genuinely of interest to Mass Transp.

$\Longrightarrow$ probably want to use it.