Robust FEM-based extraction of finite-time coherent sets using scattered, sparse, and incomplete trajectories

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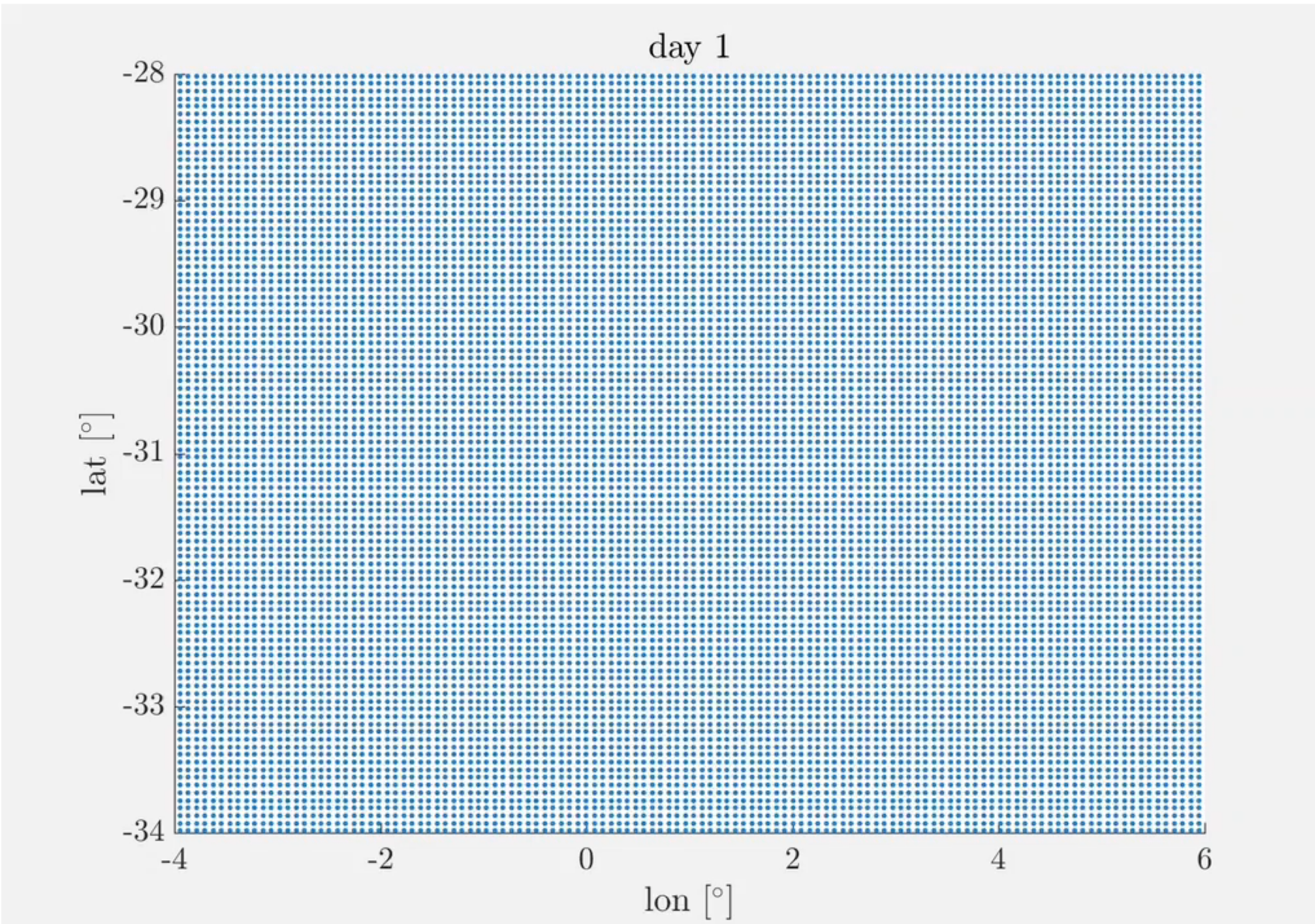
joint work with Gary Froyland, UNSW

Finite Time Coherent Sets

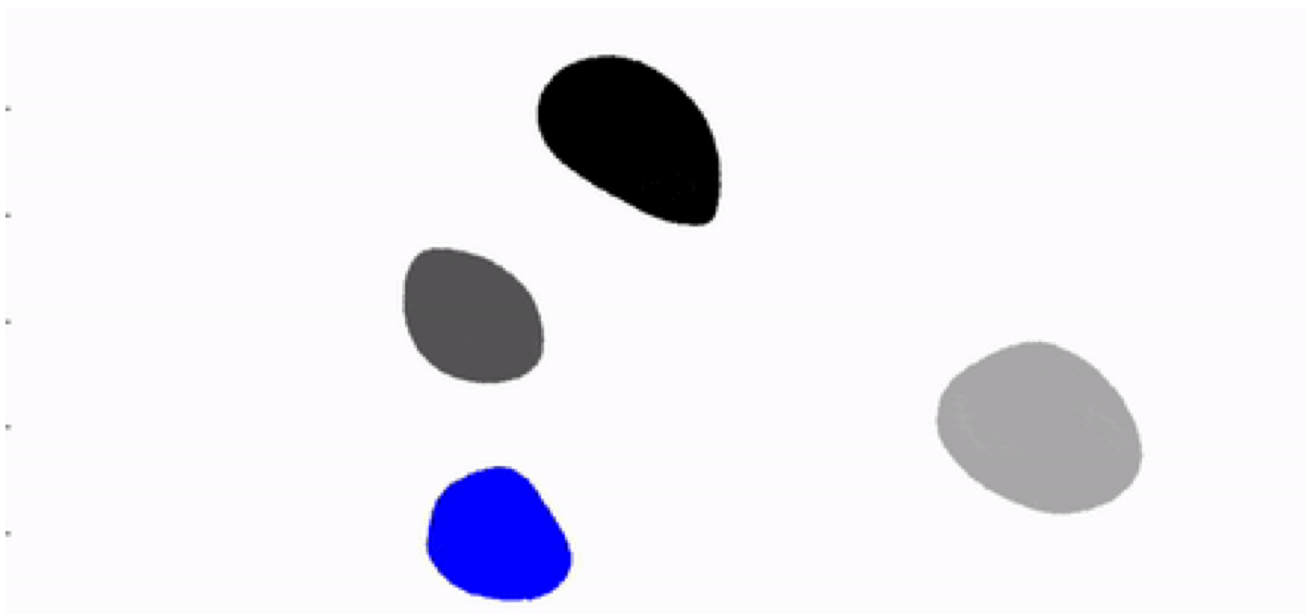
$$\dot{x} = v(t, x)$$



FINITE TIME COHERENT SETS

$\dot{x} = v(t,x)$, $\varphi^{t_0,t}$ flow

coherent sets



non-coherent set

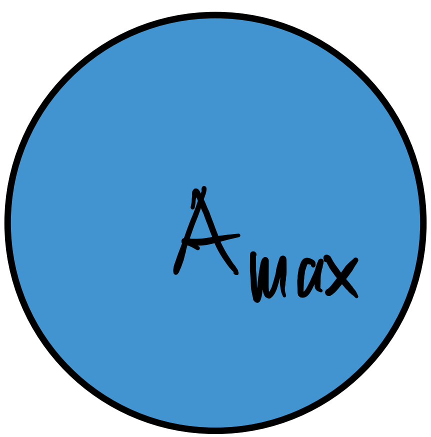
THE ISOPERIMETRIC PROBLEM

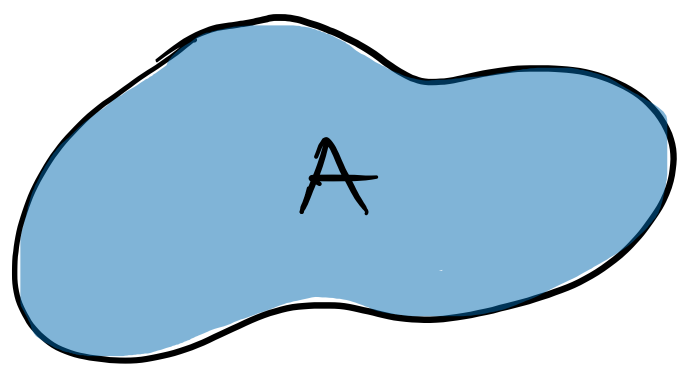
Find a set $A \subset \mathbb{R}^2$ with

$$|\partial A| = \text{fixed}$$

such that $|A| = \max \iff \dfrac{|\partial A|}{|A|} \overset{!}{=} \min$

solution: $A =$ disk





THE STATIC CASE [Federer, Fleming, 60]

$$\inf_A \frac{|\partial A|}{\min\{|A|, |A^c|\}}$$

isoperimetric problem

$A = \{u < 0\}$

$$= \inf_u \frac{\|\nabla u\|_1}{\inf_a \|u - a\|_1}$$

$$\approx \min_{u \perp 1} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}$$

Rayleigh quotient

$$= \lambda_2 \text{ of } \Delta u = \lambda u$$

$+ \text{b.c.}$

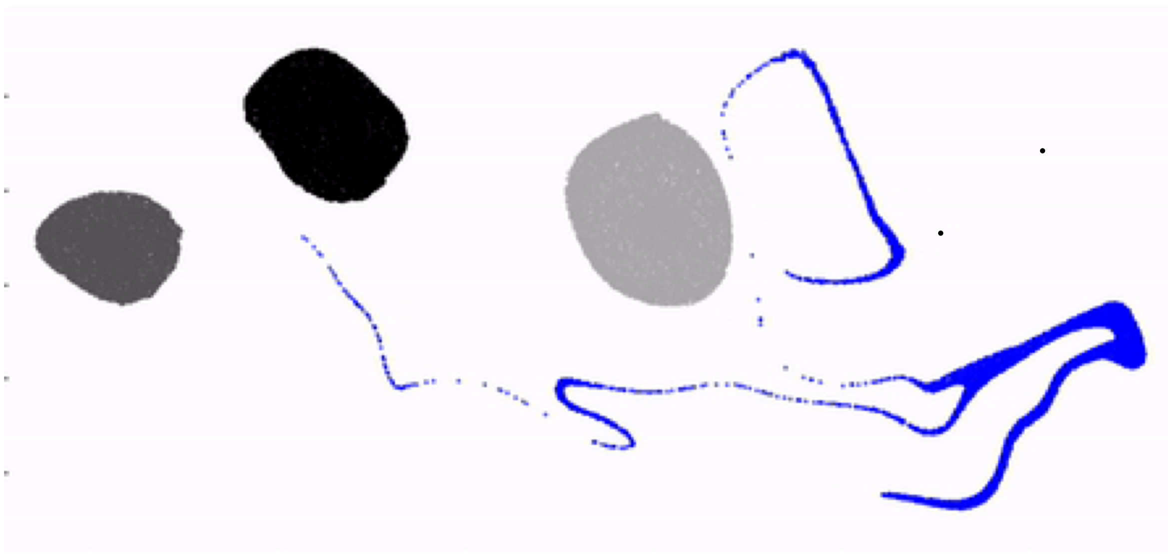
Laplace eigenproblem

Dynamic Isoperimetry

[Froyland, 15]

$$A^t = \varphi^{t_0,t}(A)$$

coherent set: $\displaystyle \frac{1}{T}\int \frac{|\partial A^t|}{|A^t|}\,dt \overset{!}{=} \min$



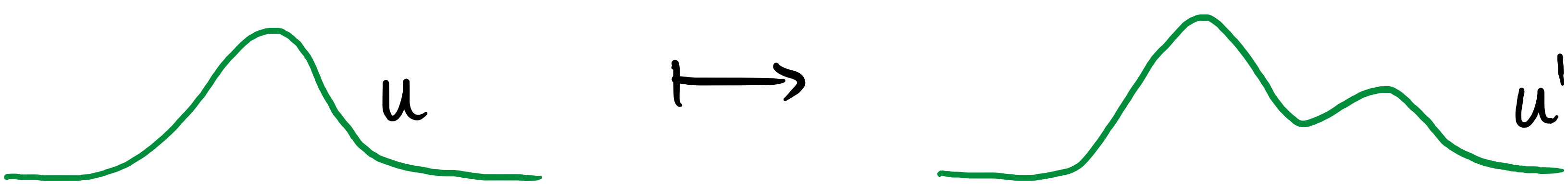
THE DYNAMIC CASE

$$|\partial A| \;\hat{=}\; \| \nabla u \|_2^2$$

$$|\partial A^t| \;\hat{=}\; \| \nabla (\varphi_*^{t_0,t} u) \|_2^2$$

with the transfer operator

$$\varphi_*^{t_0,t} : u \longmapsto u' = u \circ \varphi^{t_1 t_0}$$



(for volume preserving flow maps)

DYNAMIC LAPLACE EIGENPROBLEM

[Froyland, 15]

dynamic isoperimetric problem

$$\min \frac{\frac{1}{2}\left(|\partial A| + |\partial A^t|\right)}{\min\{|A|, |A^c|\}}$$

$$\approx \min_{u \perp 1} \frac{\frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla \varphi_*^t u\|_2^2\right)}{\|u\|_2^2}$$

$$= \lambda_2 \quad \text{of} \quad \underbrace{\frac{1}{2}\left(\Delta + \varphi_*^{t\,\prime} \Delta \varphi_*^t\right)}_{=: \Delta^d} u = \lambda u$$

dynamic Laplace

$$\Leftrightarrow \quad \Delta_{g(t)} u = \lambda u \quad \text{[Karrasch, Keller, 16]}$$

DISCRETIZATION

$$\Delta^d u = \lambda u \quad + \quad \text{Neumann or Dirichlet b.c.}$$

$\downarrow$ weak form

$$a(u,v) = \lambda \langle u,v \rangle_{L^2} \quad \text{for all } v \in H^1 \text{ or } H_0^1$$

$\downarrow$ Ritz-Galerkin

$$\cup$$

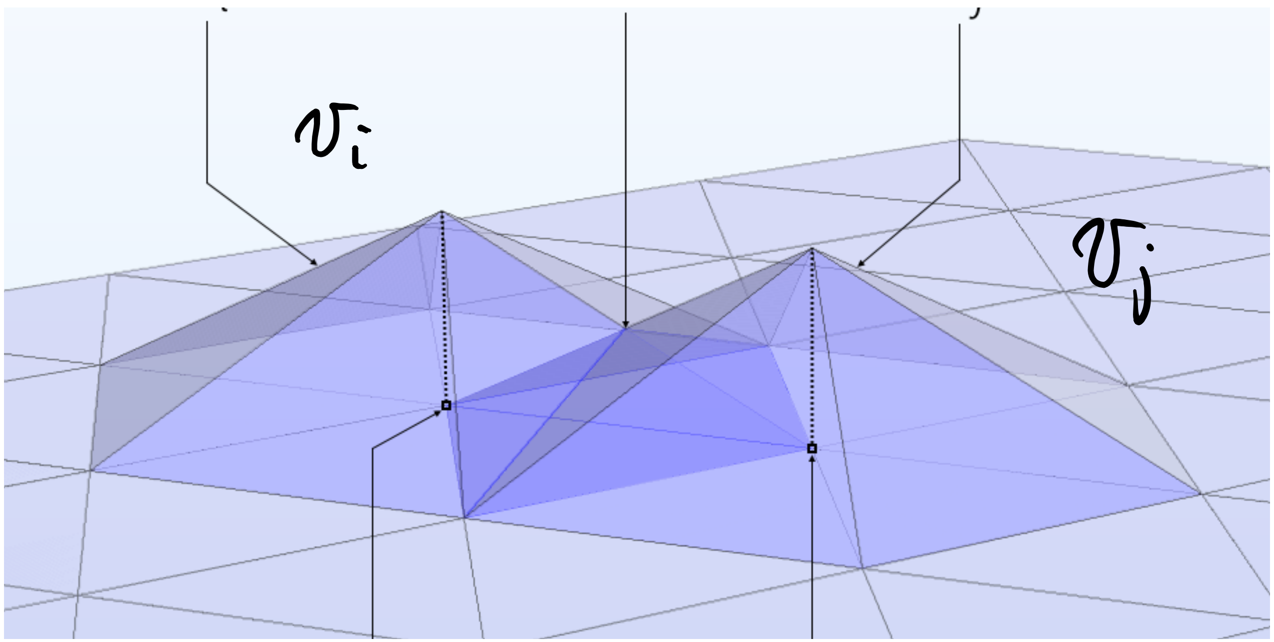
$$a(u,v) = \lambda \langle u,v \rangle_{L^2} \quad \text{for all } v \in V_N \text{ , } \dim < \infty$$

$\downarrow$ choose basis

$$Au = \lambda Mu$$

FINITE ELEMENTS

$$A_{ij} = \frac{1}{T} \sum_t \int_{\varphi^t(M)} \nabla\left(\varphi_*^t v_i\right) \cdot \nabla\left(\varphi_*^t v_j\right) \, d\mu$$



COMPUTING A_{ij}

$$\int_{\varphi^t(M)} \nabla(\varphi^t_* v_i) \cdot \nabla(\varphi^t_* v_j) \, dm$$

$$= \int_M \nabla(\varphi^t_* v_i) \circ \varphi^t \cdot \nabla(\varphi^t_* v_j) \circ \varphi^t \, dm$$

$$= \int_M (D\varphi^t)^{-T} \nabla v_i \cdot (D\varphi^t)^{-T} \nabla v_j \, dm$$

$$= \int_M \nabla v_i^T C_t^{-1} \nabla v_j \, dm$$

COMPUTING A_{ij}

i.e.

$$A_{ij} = \int_M \nabla v_i^T C^{-1} \nabla v_j \, dm$$

with

$$C = \frac{1}{T} \sum_t D\varphi^t (D\varphi^t)^T$$

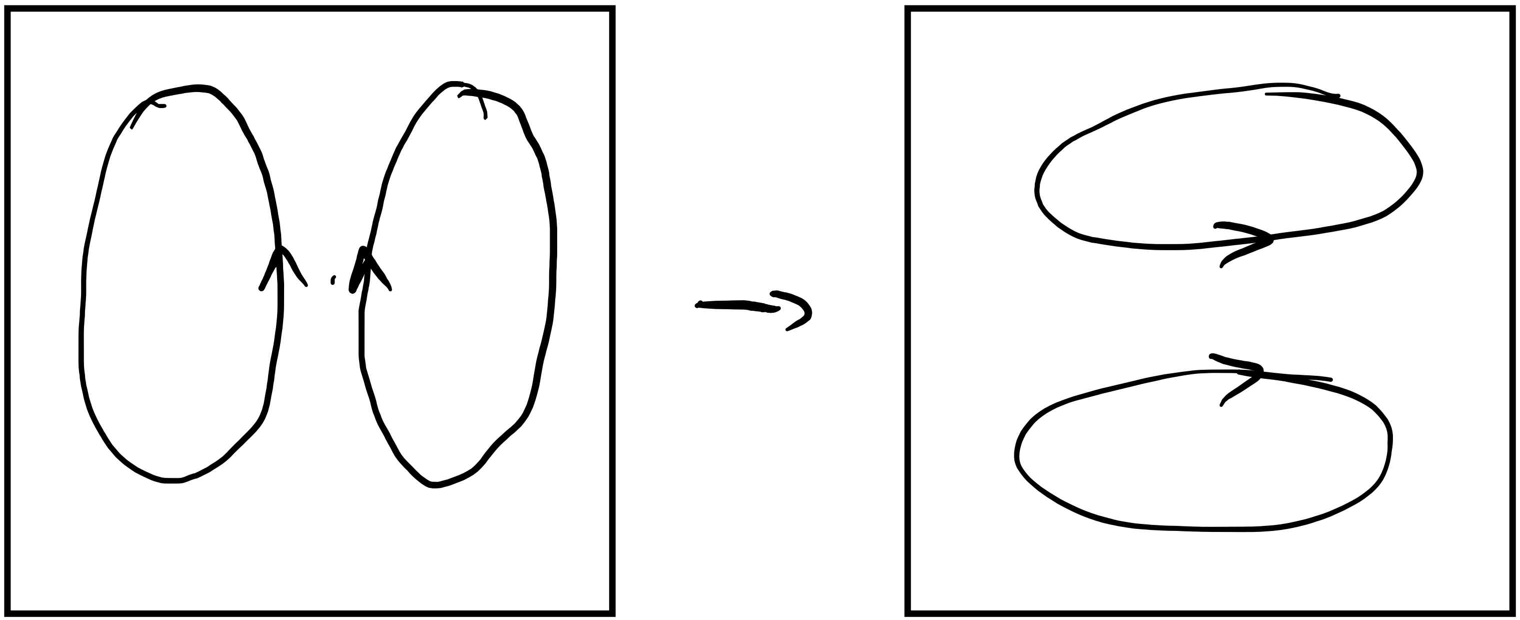
averaged diffusion tensor

→ standard FEM quadrature

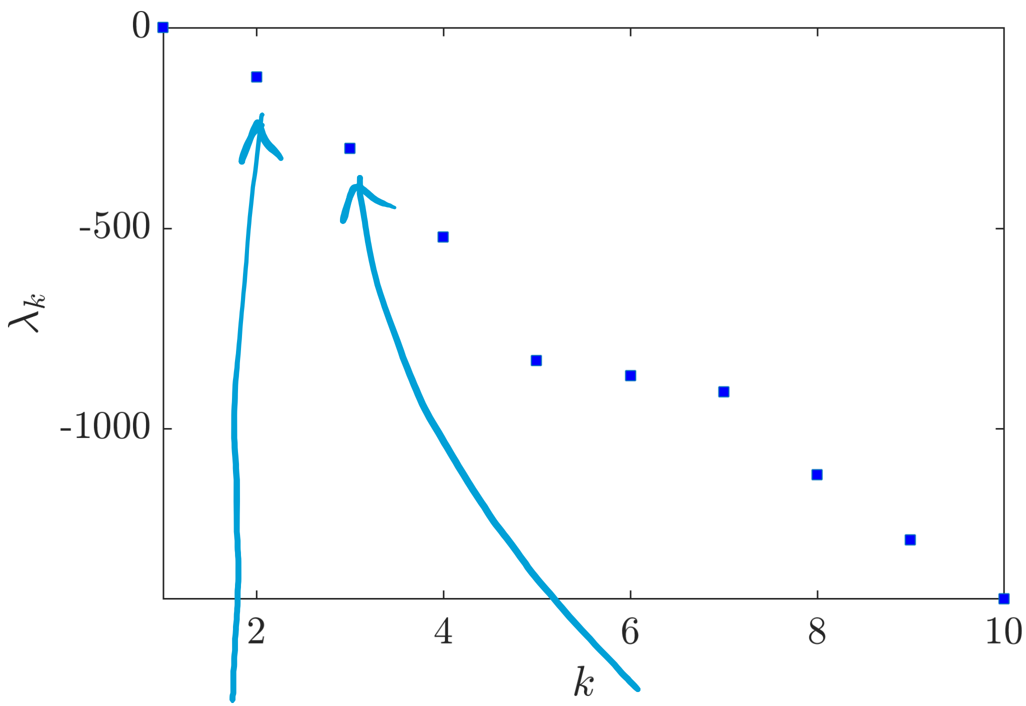
→ ∇v_i constant for linear elements

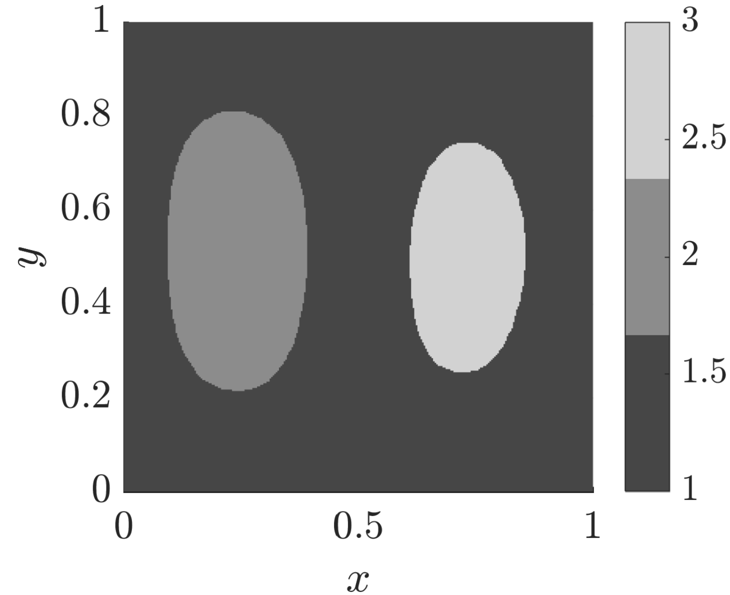
→ often, low order quad suffices

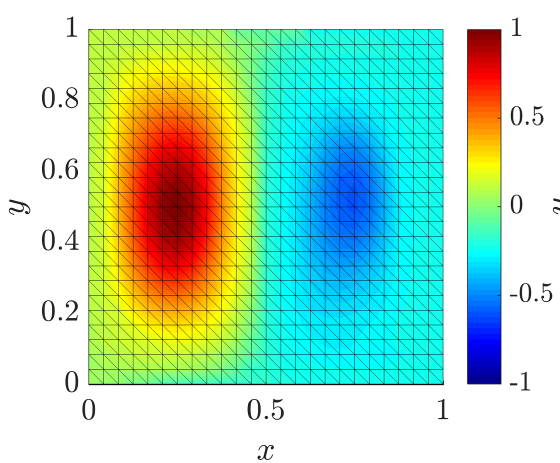
A ROTATING DOUBLE-GYRE

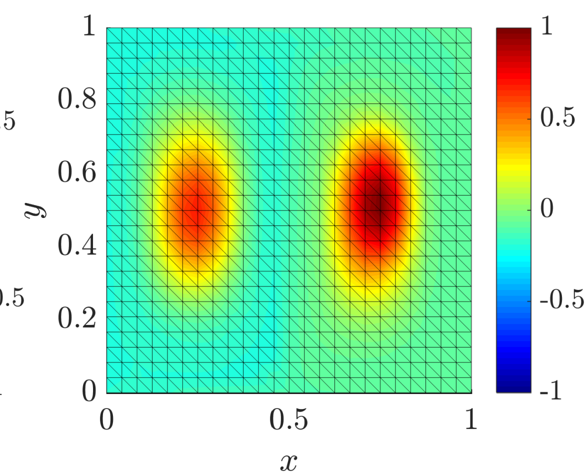


A ROTATING DOUBLE-GYRE





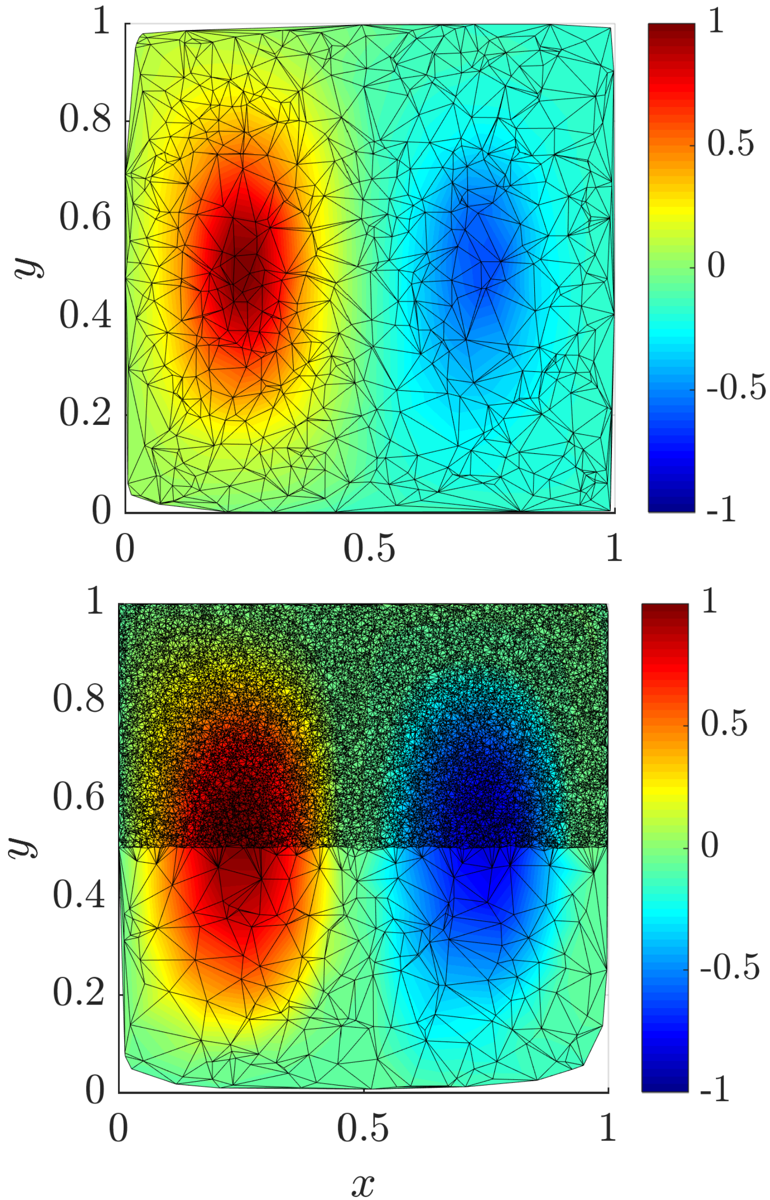




625 nodes

integration : 2 s
assembly : 20 ms
eigenproblem : 50 ms

A ROTATING DOUBLE-GYRE

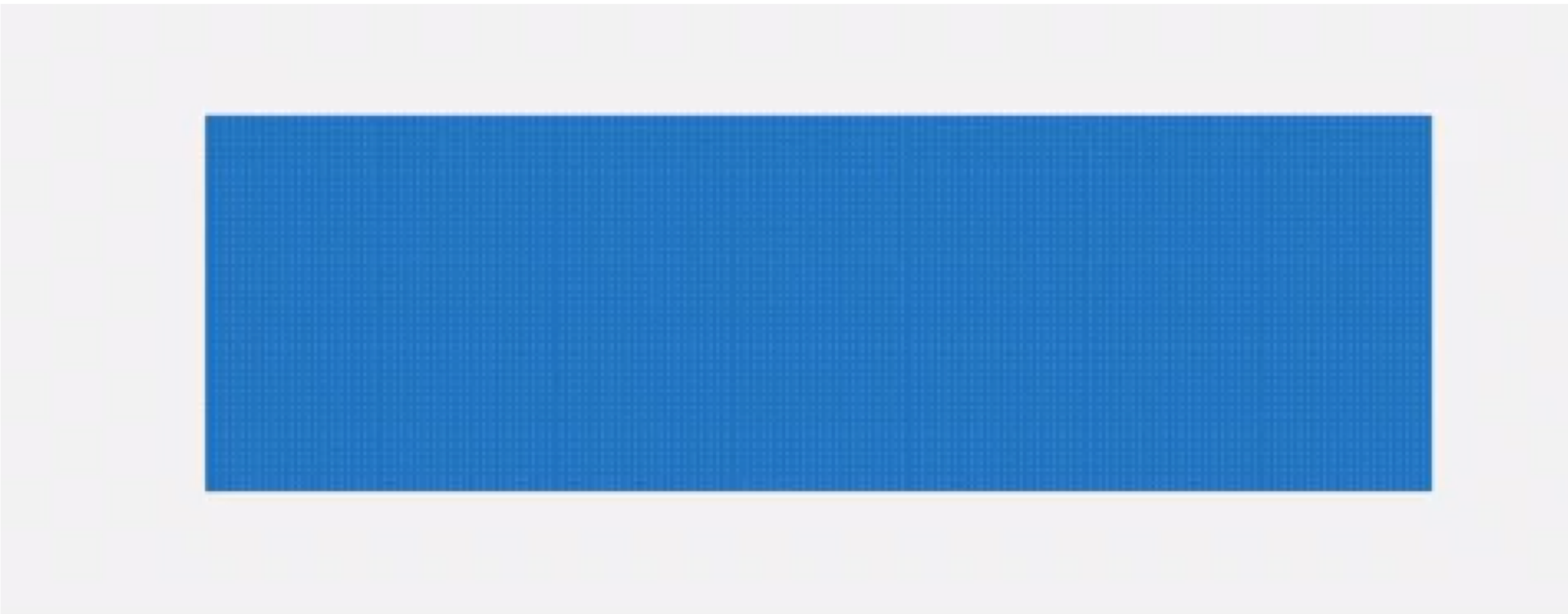


← 625 scattered nodes

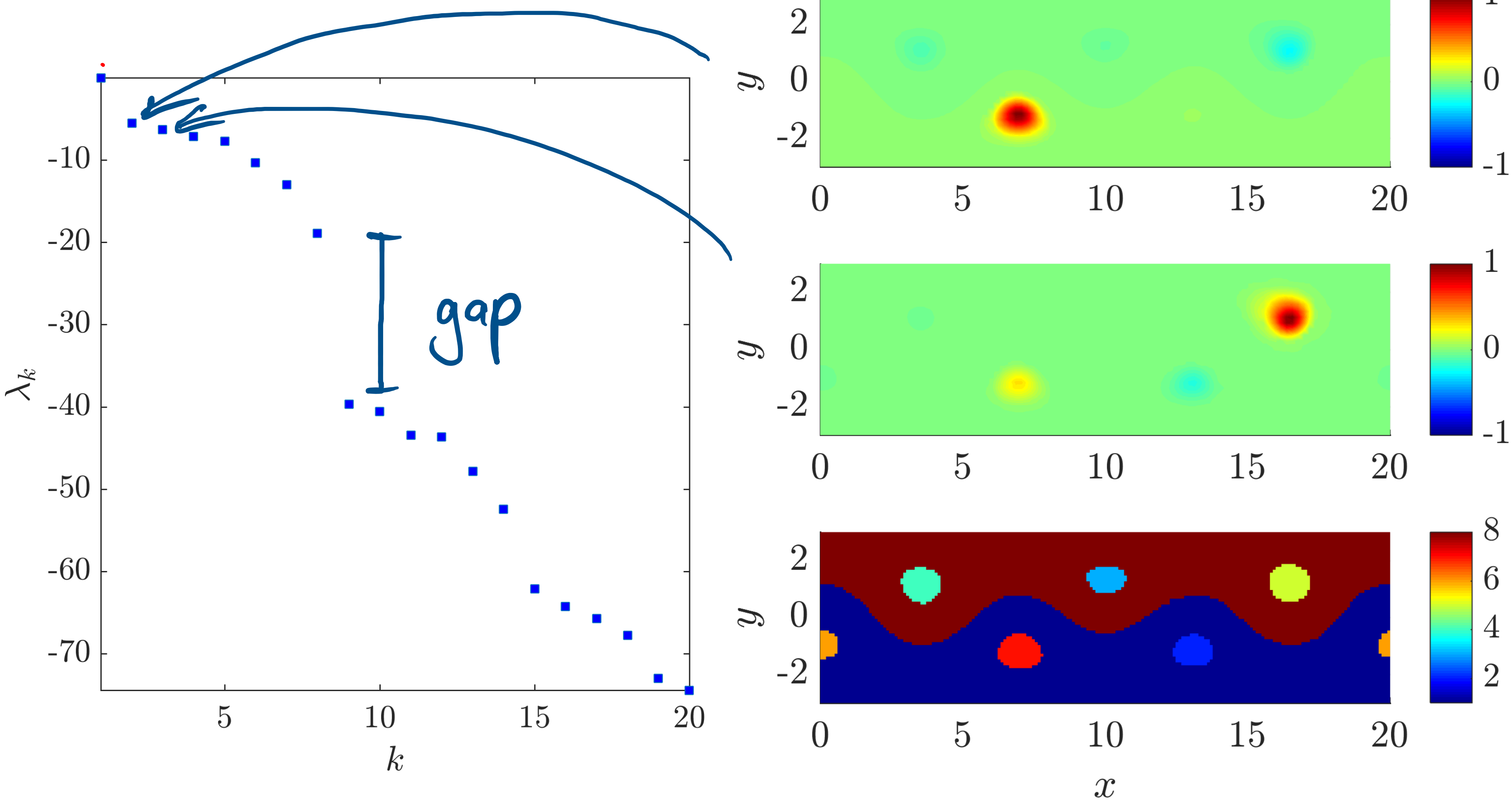
← 20000 nodes

← 200 nodes

The Bickley Jet



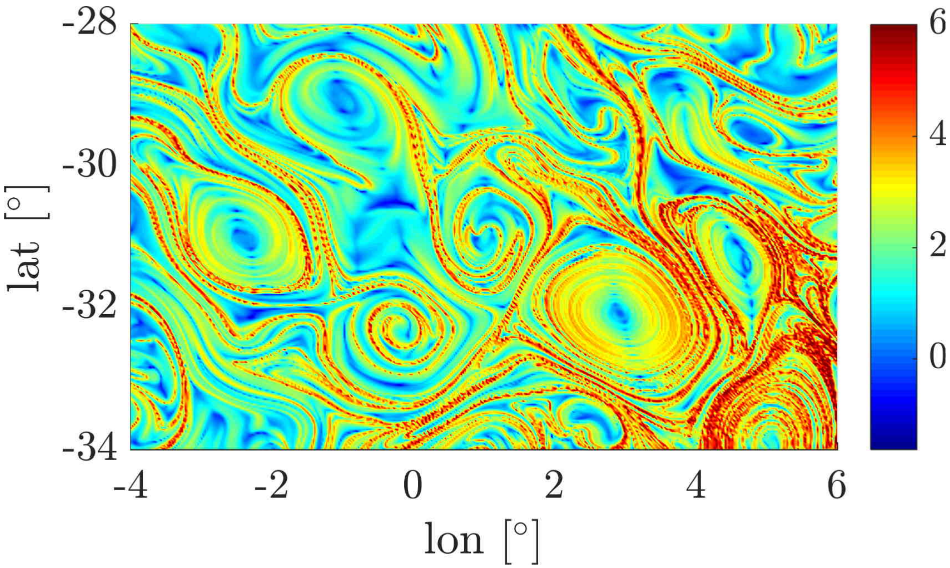
The Bickley Jet

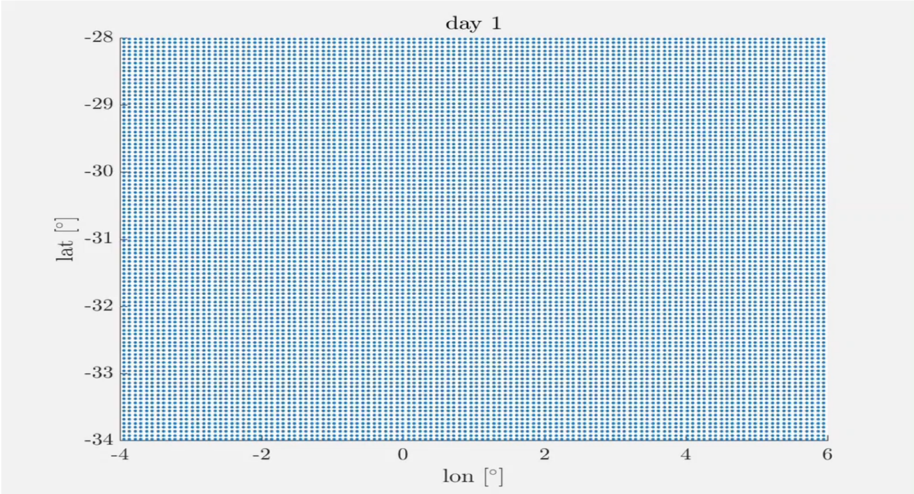


EXAMPLE: OCEAN FLOW

AVISO data / domain of Agulahs leakage

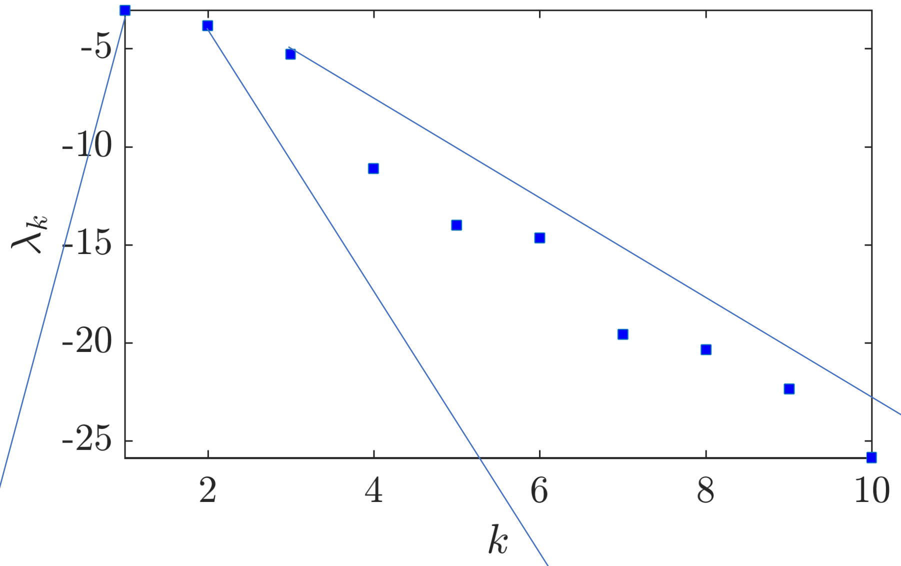
parameters from [Hadjighasem, Haller, 16]

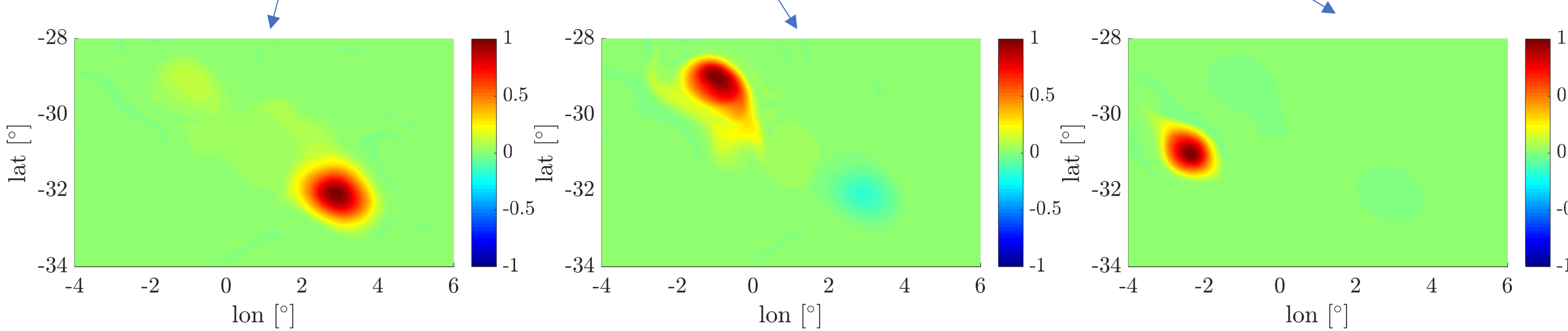




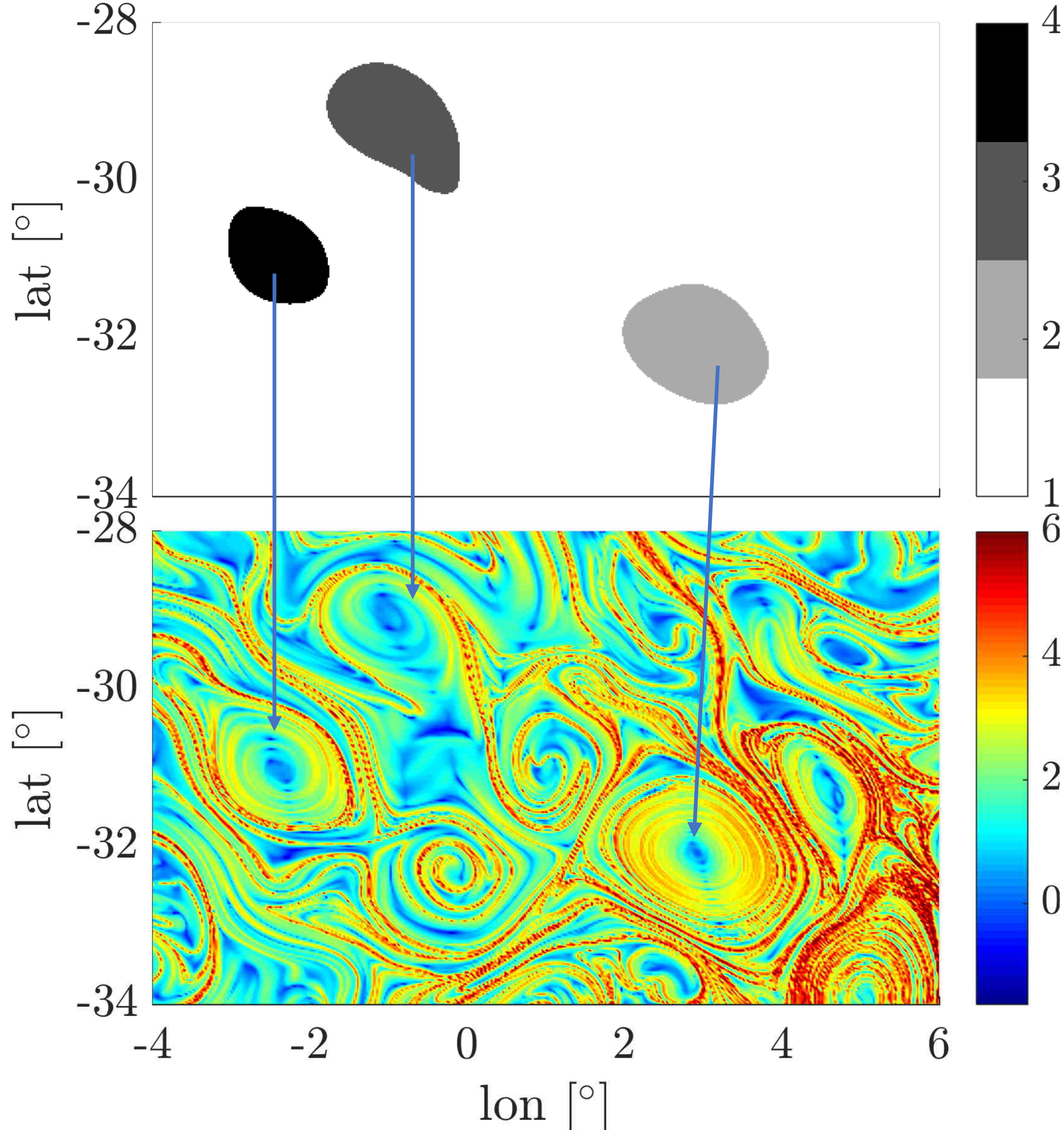
EXAMPLE: OCEAN FLOW

here: zero Dirichlet boundary condition



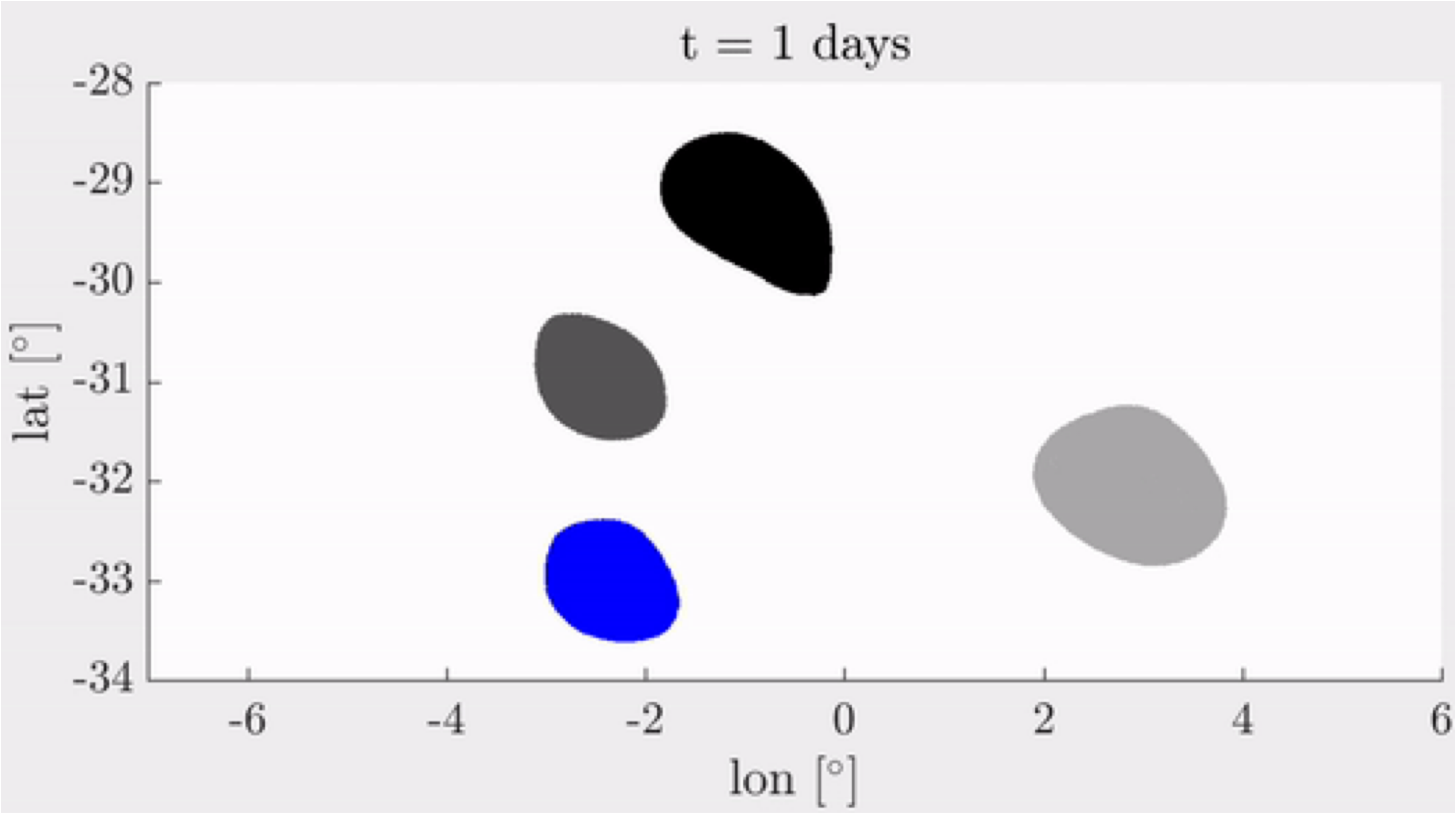


OCEAN FLOW: COHERENT SETS



← obtained by kmeans with u_1, u_2, u_3

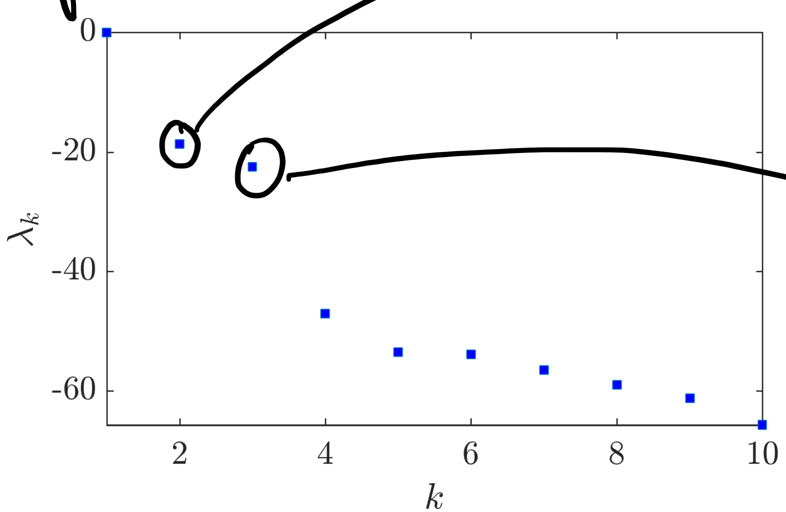
Ocean Flow: Coherent Sets

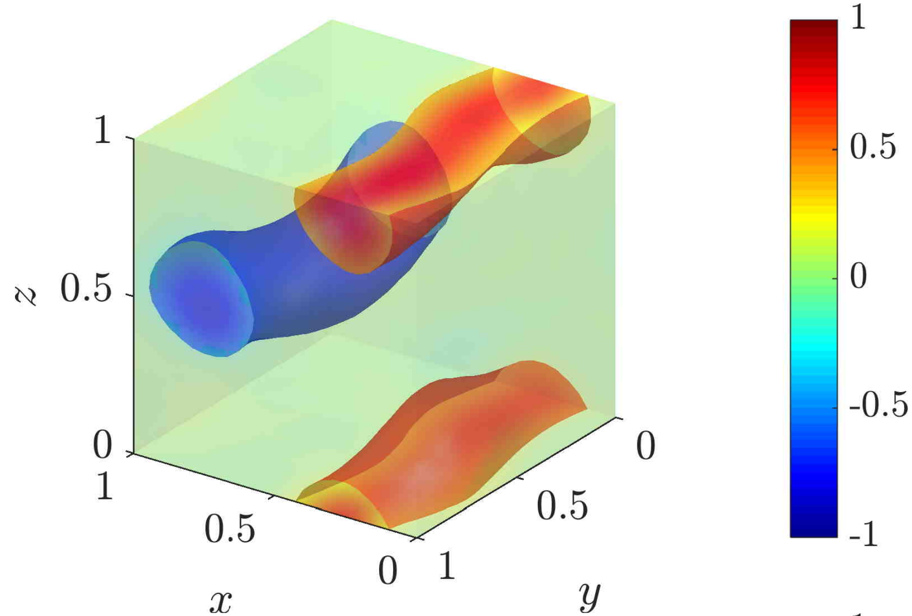


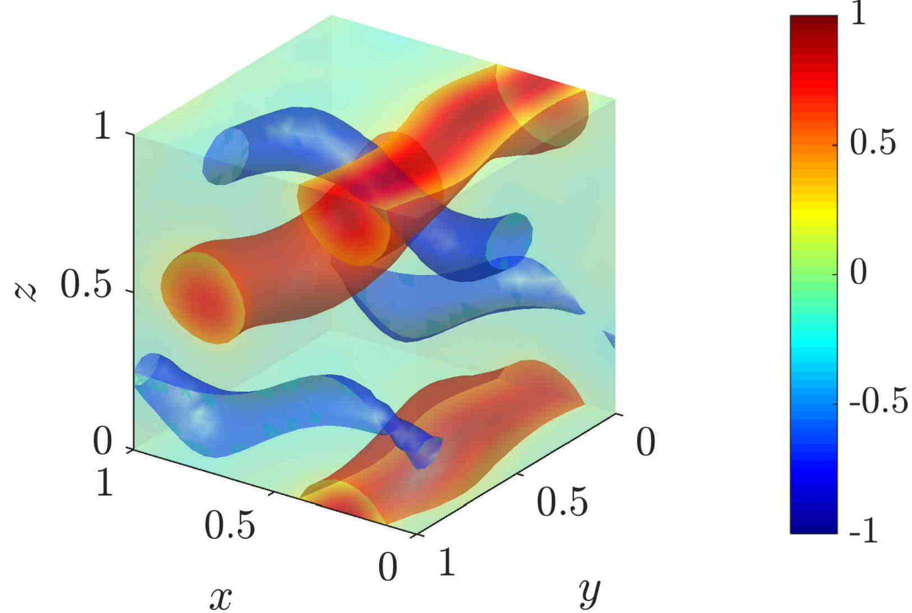
3D EXAMPLE: UNSTEADY ABC-FLOW

$$\dot{x} = (A + \tfrac{1}{2} t \sin(\pi t)) \sin z + C \cos y$$
$$\dot{y} = B \sin x + (A + \tfrac{1}{2} t \sin(\pi t)) \cos z$$
$$\dot{z} = C \sin y + B \cos x$$

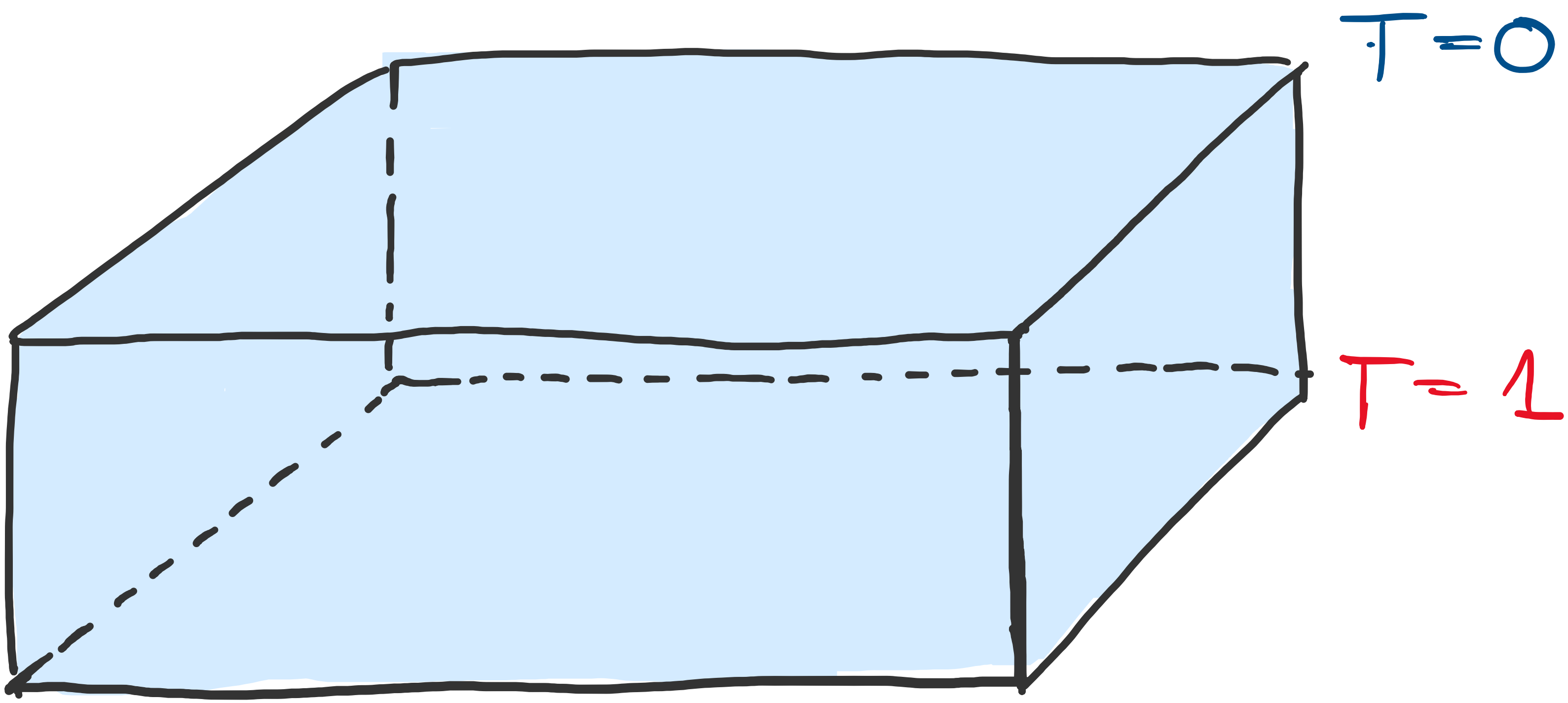
spectrum



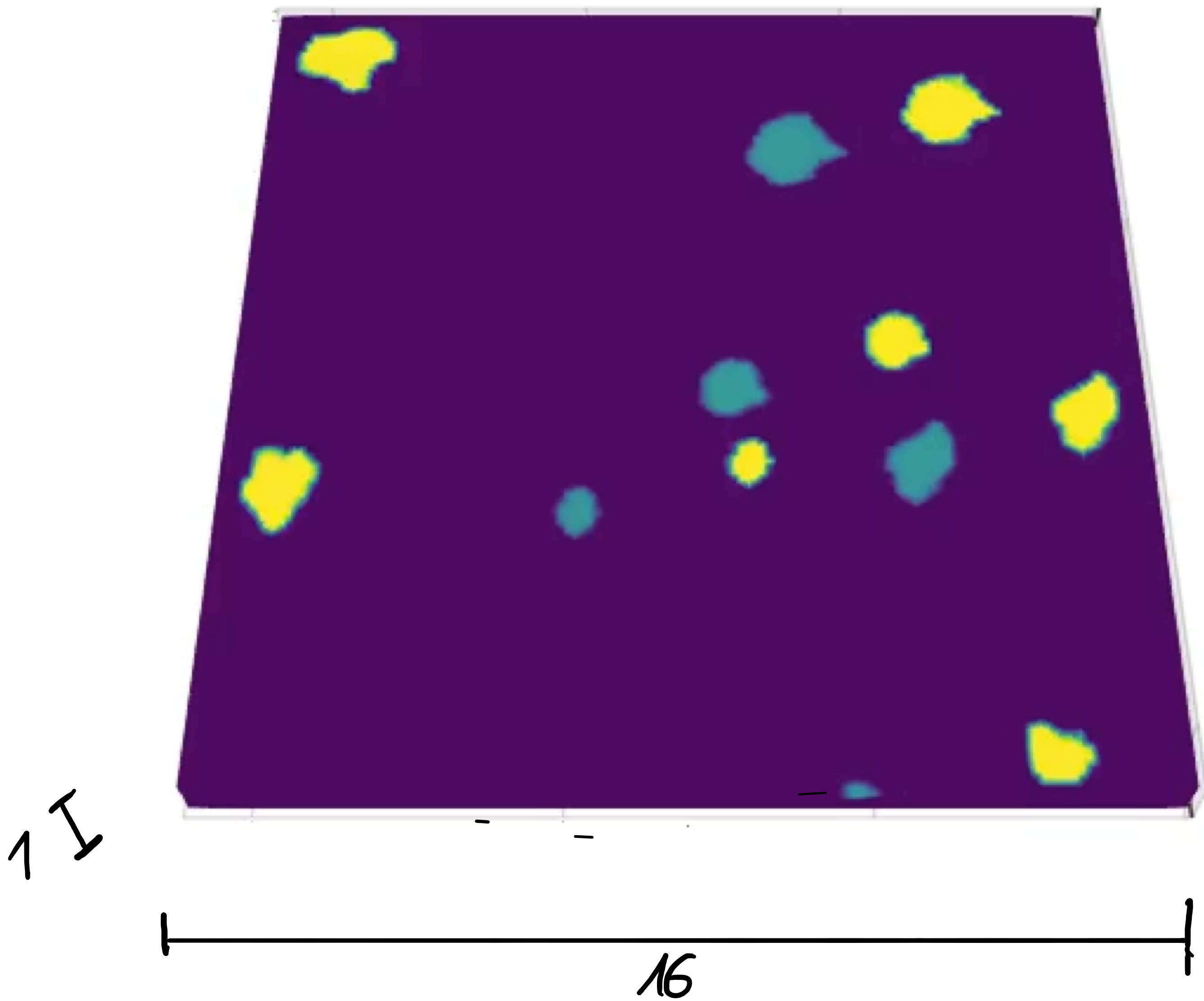




3D EXAMPLE: RAYLEIGH-BÉNARD CONVECTION



3D EXAMPLE: RAYLEIGH-BÉNARD CONVECTION



data by
J. Schumacher,
U Ilmenau

CONVERGENCE ORDER

λ true eigenvalue , λ_h approximation
v true eigenvector , v_h

If $v \in H^{s+1}$, then, using P^s Lagrange elements

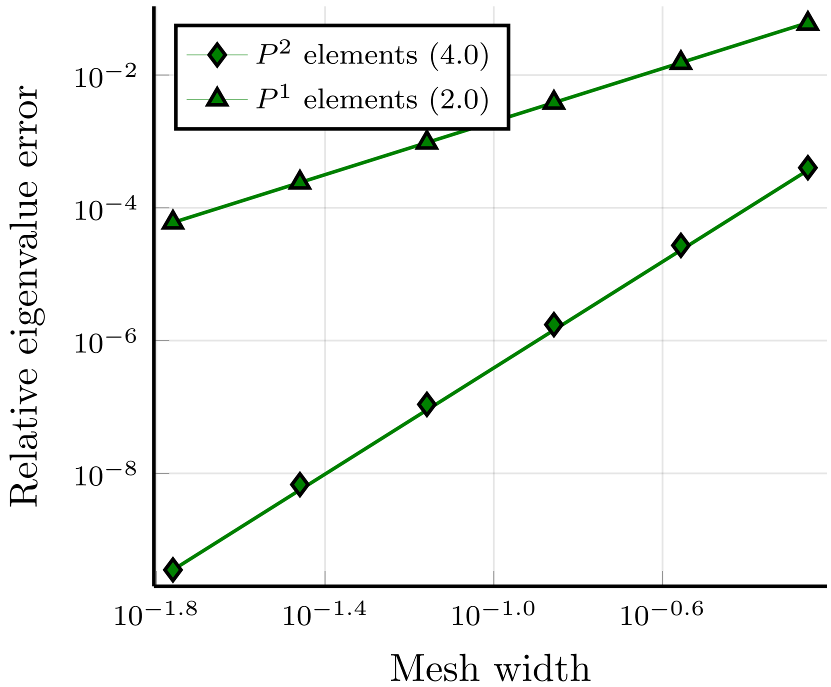
$$\left| \frac{\lambda - \lambda_h}{\lambda} \right| = O(h^{2s})$$

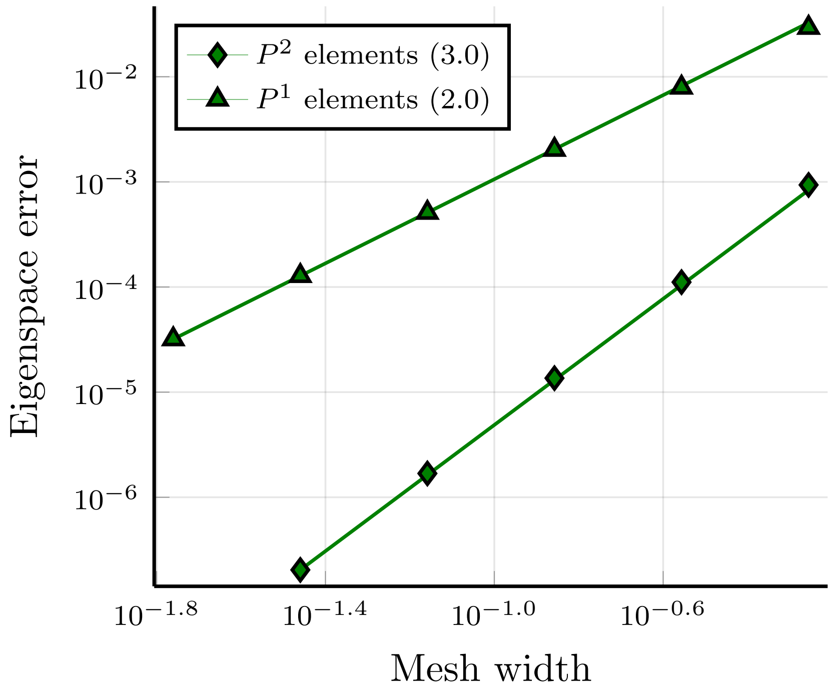
$$\| v - v_h \|_{H^1} = O(h^s)$$

Under certain conditions

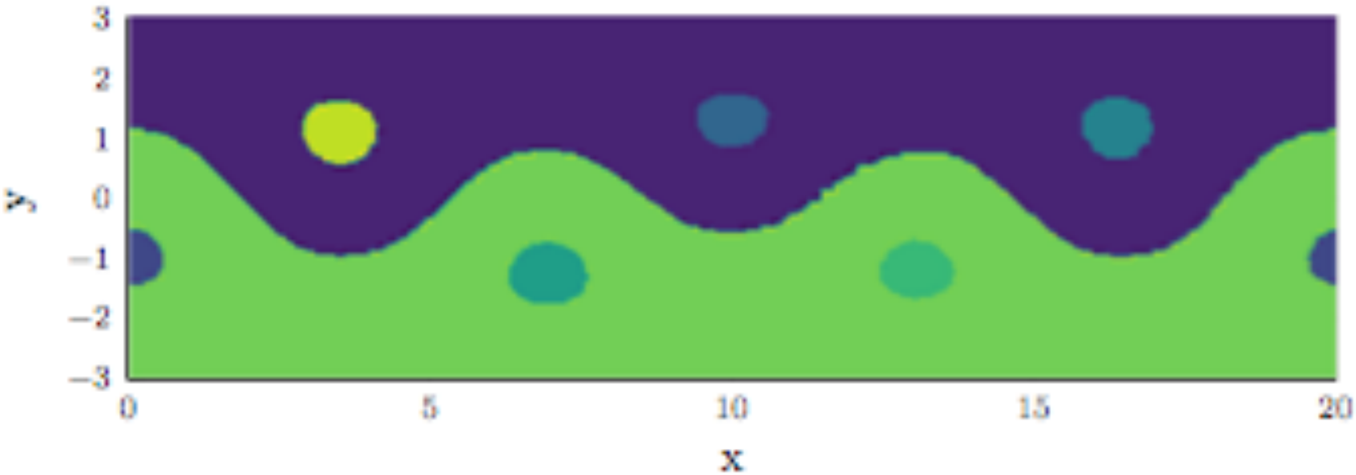
$$\| v - v_h \|_{L^2} = O(h^{s+1})$$

TEST : STANDARD MAP



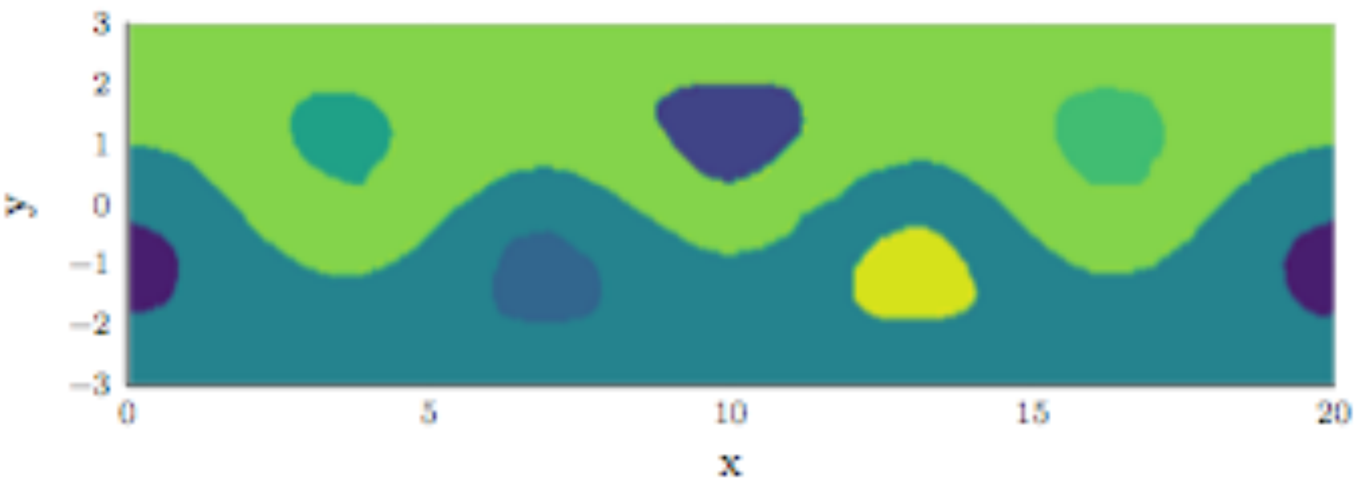


TEST : BICKLEY JET



P^1

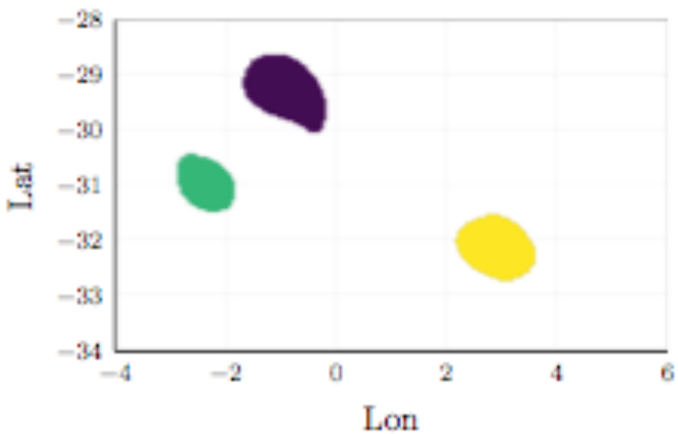
6000 triangles
18000 quad points

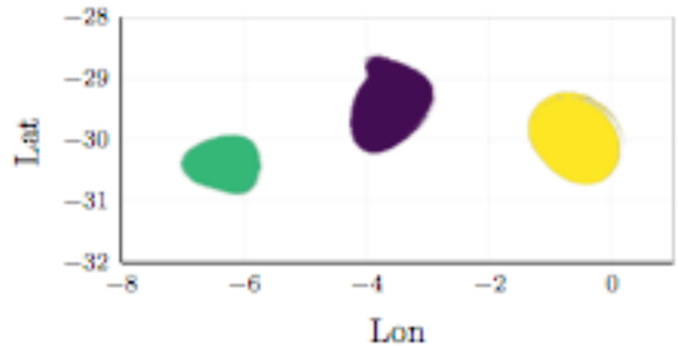


P^2

240 triangles
720 quad points

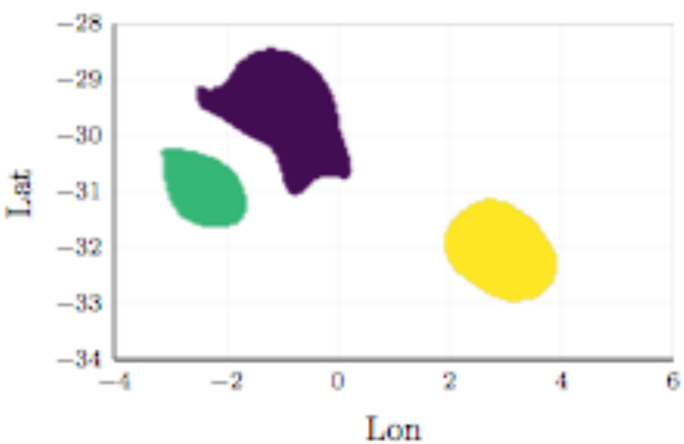
TEST : OCEAN FLOW

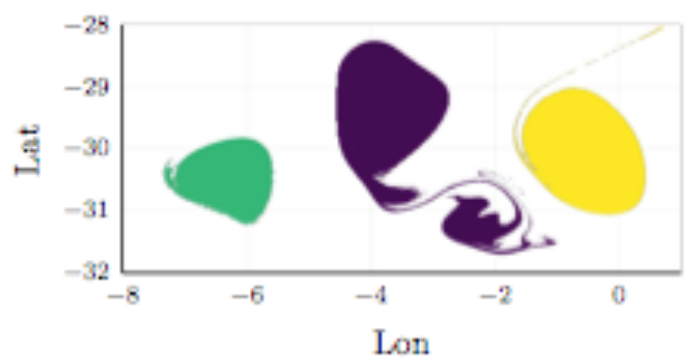




P^1

7000 elements
20 000 quad points





P^2

1000 elements
3000 quad points

COMPUTING A

by approximating the transfer operator:

$$\varphi_*^t v_j^0 \approx \sum_k \alpha_{jk} v_k^t$$

basis at t=0 basis at t

related:

[Froyland, Padberg, 15]
[Hadjighasem et al 16]
[Banisch, Koltai, 17]

COMPUTING A

by approximating the transfer operator using collocation:

$$\varphi_*^t v_j^0(\underset{\substack{\uparrow \\ \text{nodes}}}{x_i^t}) \overset{!}{=} \sum_k \alpha_{jk} v_k^t(x_i^t) \qquad \forall i$$

$$\| \leftarrow \text{nodal basis}$$

$$\sum_k \alpha_{jk} \delta_{ki}$$

$$\|$$

$$v_j^0(\varphi^{-t}(x_i^t)) = \alpha_{ji}$$

COMPUTING A

by approximating the transfer operator using collocation

special case: choose $x_i^t := \varphi^t(x_i)$

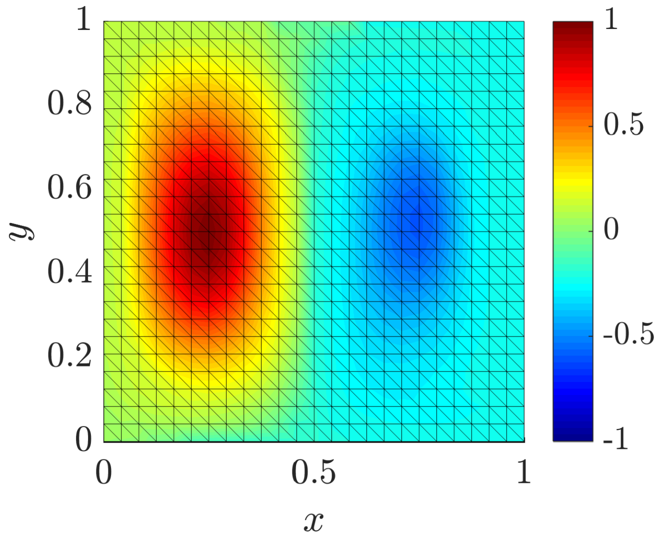
then $\alpha_{ji} = v_j^0(\varphi^{-t}(\varphi^t(x_i)))$

$= v_j^0(x_i)$

$= \delta_{ji}$, (nodal basis again)

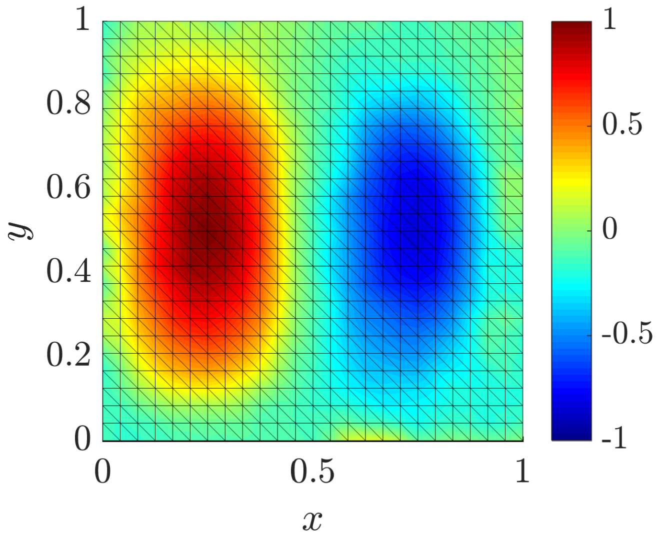
$\rightsquigarrow \quad A = I$ in these bases

Rotating Double Gyre



diffusion tensor

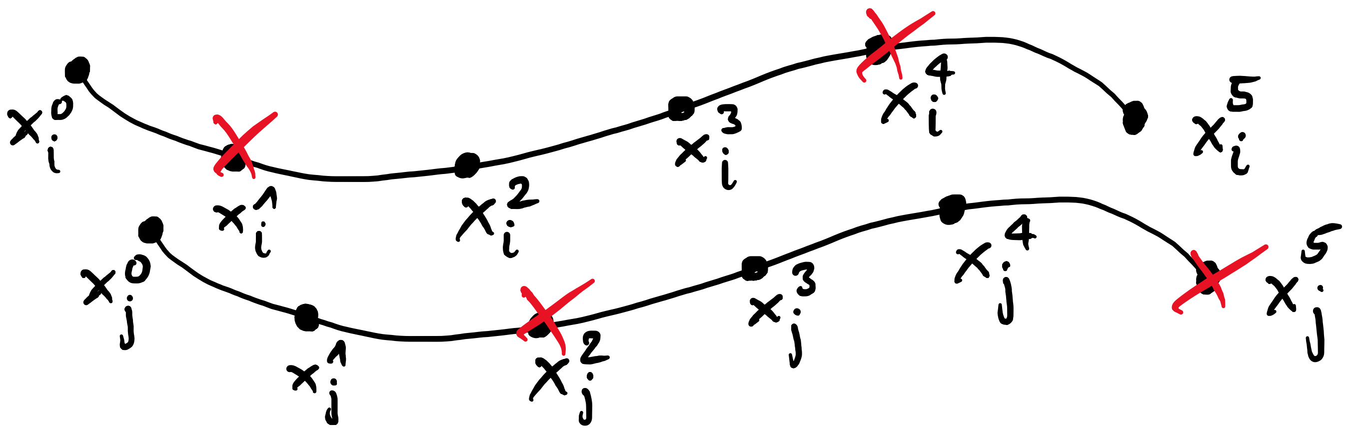
~ 2 secs



transfer operator

~ 0.2 secs

MISSING DATA

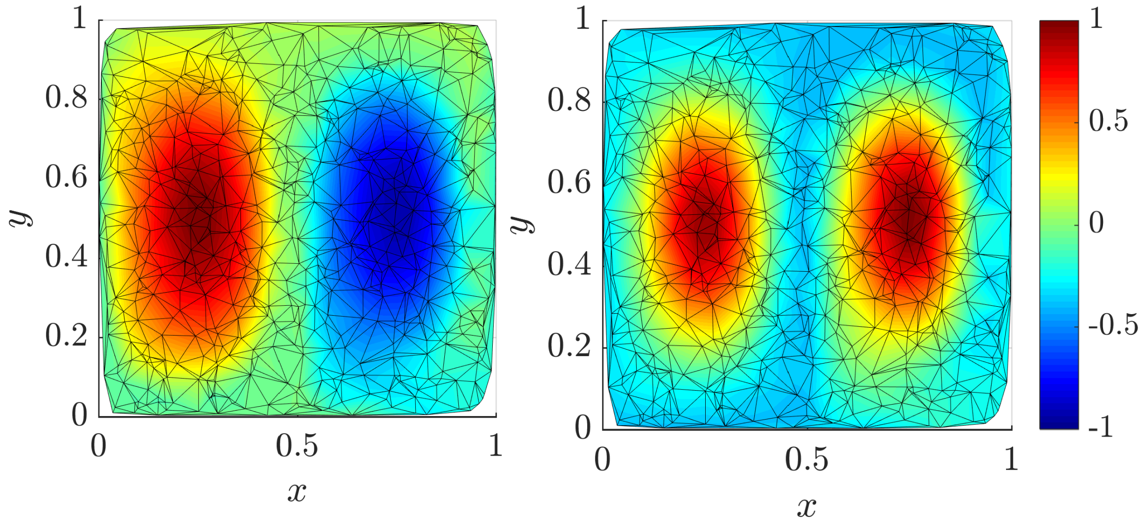


For each time t_k:

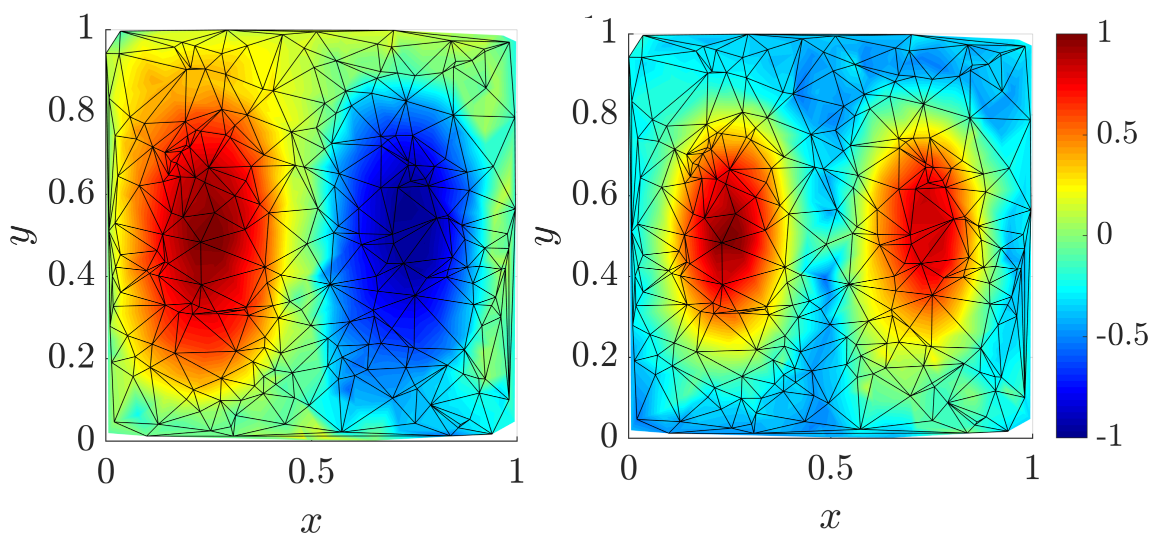
- triangulate remaining points
- compute A, M as usual

Trade missing points for more timesteps.

ROTATING DOUBLE GYRE

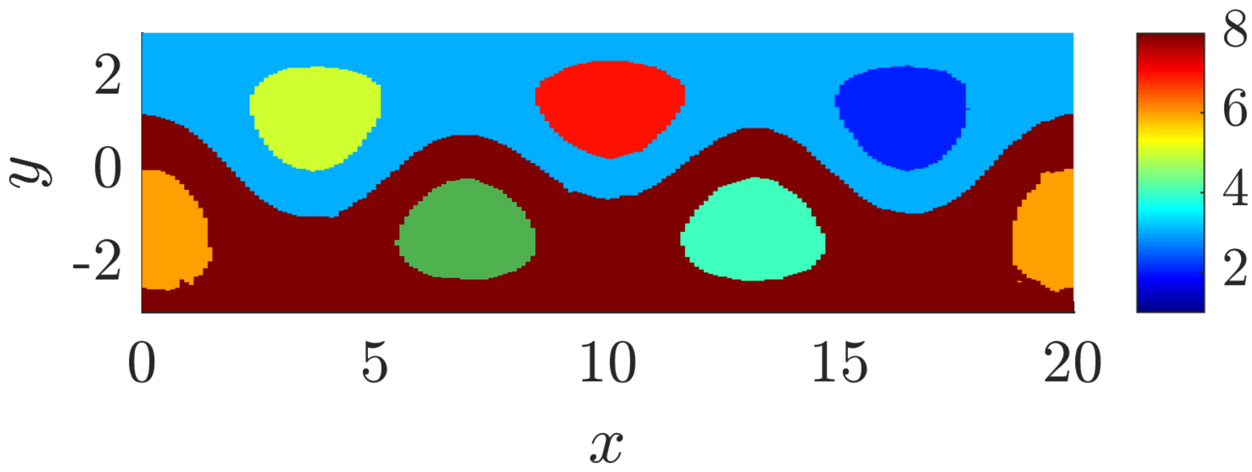


625 points
2 time steps

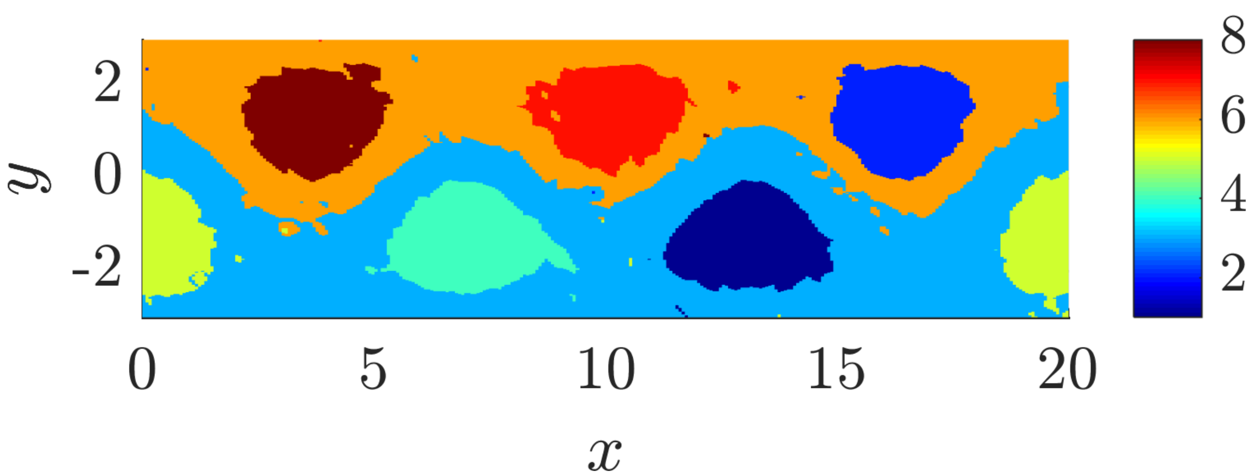


~ 250 points
5 time steps

Bickley Jet



3000 nodes
2 time steps



600 nodes
10 time steps

CONCLUSION

+ works on data (scattered, sparse)
+ robust (data can be incomplete)
+ yields info on entire domain
+ preserves symmetry
+ no free parameter
+ no artificial noise
+ standard convergence theory

- higher order elements → significant mesh size reduction (~ 5x)

References

Froyland, Junge:
Robust FEM-based extraction of finite-time coherent sets using scattered, sparse, and incomplete trajectories,
SIADS, 2018.

FEMDL, Matlab package for Finite Element Methods for Dynamic Laplacians, download on github.

CoherentStructures.jl, Julia package, work in progress.

supported by

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Turbulent Superstructures

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Australian Research Council