Koopman Performance Analysis of Nonlinear Networks

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Joint Work With:

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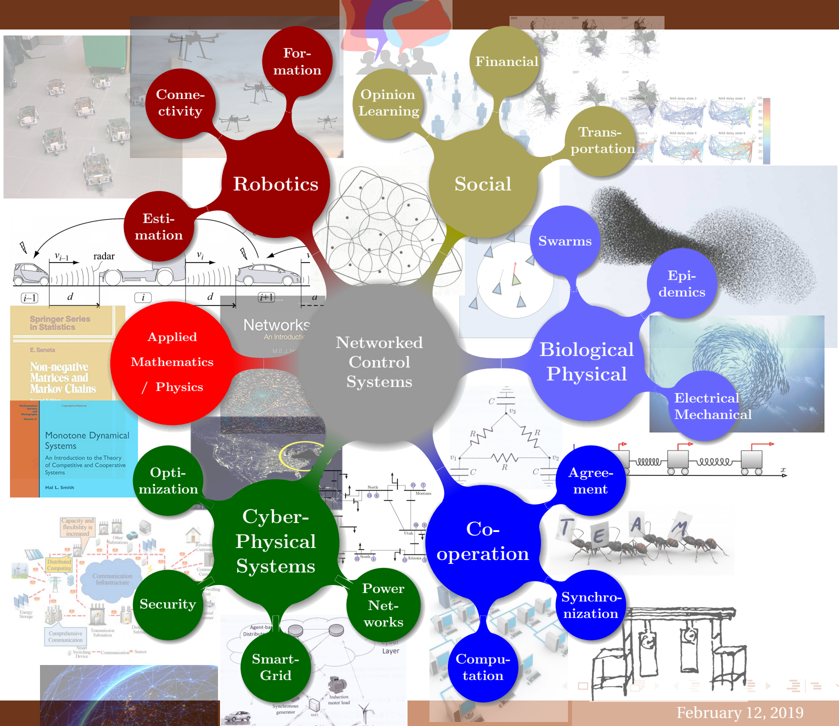

Networked Control Systems
Robotics
Formation
Connectivity
Estimation
Social
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Transportation
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Swarms
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Prototypical Cooperative Protocol : *Agreement*

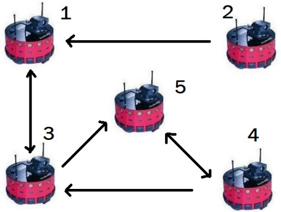
Network of n agents.

- Agent $i \in \{1, \ldots, n\}$ has state $x_i \in \mathbb{R}$.
- Restricted sharing and interacting capabilities (local / nearest neighbors).
- Autonomous computation / state update.

$\dot{x}_i = u_i$

Design feedback control so that

$$x_i(t) \to c^* \text{ as } t \to +\infty$$



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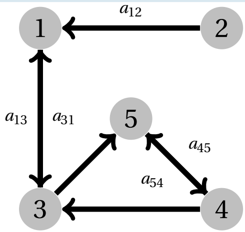

$N_1 = \{3\}, N_2 = \{1\}, N_3 = \{1, 5\},$
$N_4 = \{3, 5\}, N_5 = \{4\},$

$a_{ij} \geq 0, a_{ii} = -\sum_{j \neq i} a_{ij}$

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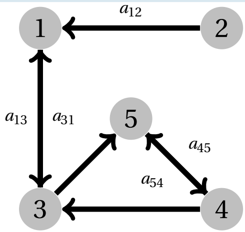
$$\dot{x}(t) = -L\, x(t)$$



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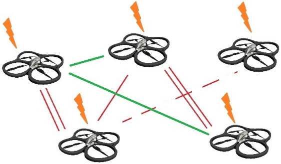
$$x = e^{-Lt} x_0 \;\Rightarrow\; \lim_{t \rightarrow +\infty} x_i(t) = \frac{1}{n} \sum_{j=1}^{n} x_0^{(j)}$$

if and only if Graph Laplacian matrix L, symmetric and connected graph [36].

Network + Uncertainty = ?

- Reaction in presence of noise.
- Notions / Metrics of Systemic Performance.
- Tractability / Scalability.
- Metric-Based Network Synthesis.



w/o noise $\overline{x} = (\overline{x}^{(1)}, \dots, \overline{x}^{(5)})$

w noise $x_t = (x_t^{(1)}, \dots, x_t^{(5)})$

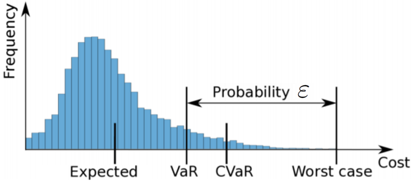
I/O $\mathcal{H}_2$ - norm [aggregate variability]

$$\mathcal{H}_2 = \lim_{t \to +\infty} \mathbb{E}\left[\left\| (x_t - \overline{x}) \right\|_2^2 \right]$$

Risk and its measures [50, 58, 59, 60, 61, 62]

Definition

A functional that assigns to a variable, an index of the level of encapsulated uncertainty.

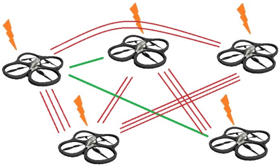


Problem: High values of $z(\omega)$ are undesirable.

How “dangerous" is z ?

- Expectation $\mathbb{E}\left[z(\omega)\right]$
- Worst-case $\sup_{\omega} z(\omega)$
- Variance $\mathbb{E}\left[\left(z(\omega) - \mathbb{E}\left[z(\omega)\right]\right)^2\right]$
- Value-at-Risk $\mathcal{R}_{\varepsilon}(z) = \inf\left\{\delta \in \mathbb{R} \,:\, \mathbb{P}(z > \delta) < \varepsilon\right\}$

The effect of information flow on network resilience.



Mainstream Connectivity Principle

The Stronger ... The Better !

$$\dot{x}_t = -L\, x_t + \xi_t$$

Output : $y_t^{(i)} = x_t^{(i)} - \frac{1}{n}\sum_i x_t^{(i)}$

Theorem

$$\mathcal{H}_2 = \lim_{t\to\infty} E\left[\|y_t\|_2^2\right] \propto \Xi_{\mathcal{G}} = n\sum_{i=2}^{n}\frac{1}{\lambda_i(L)}$$

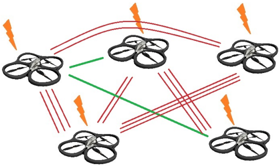
Increase weights/links...make laplacian eigenvalues larger...take Ξ_G (i.e. $\mathcal{H}_2$) down....

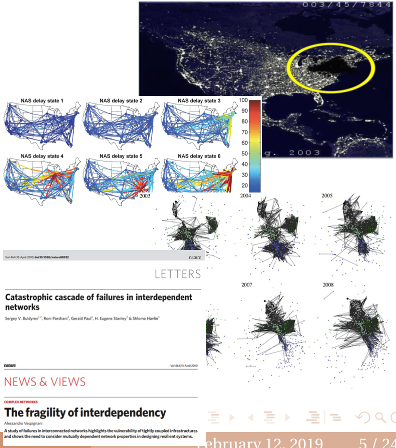
The effect of information flow on network resilience.

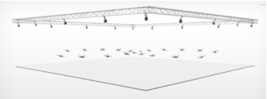
Mainstream Connectivity Principle

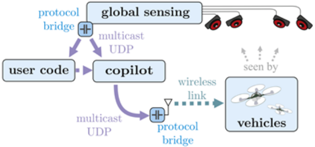
The Stronger ... The Better !

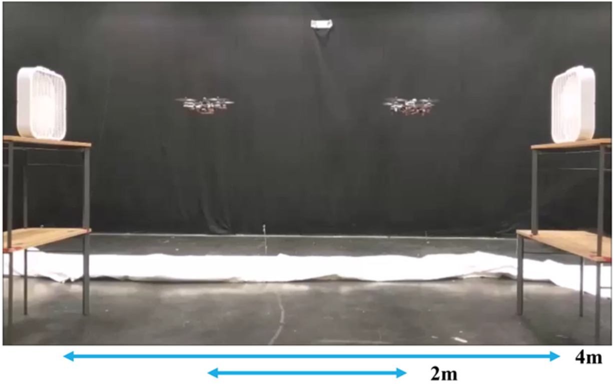
- Valid...in Principle!
- Real-world cases (and research) demonstrate otherwise.
- Types of inter-dependencies (dynamic scale/complexity).
- Inherent limitations/deficiencies.
- Sources of perturbations.



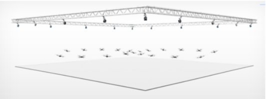




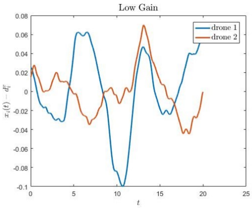


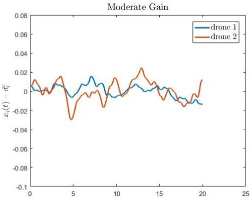


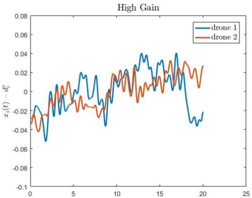
$$\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^6$$



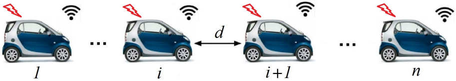
low a | moderate a | high a







The Platooning Problem [IEEE-TAC'19-c, ecc'19, cdc'18-b]

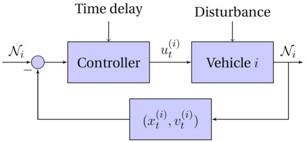


Problem: Finite number of vehicles aim to align their velocities so as to travel along a line within prescribed distance.

Network: Vehicles to utilize information in a decentralized manner.

Working Hypotheses: Network prone to time-delay and exogenous perturbations.

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = u_i(t) + \xi_i(t)$$



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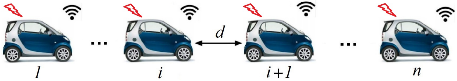


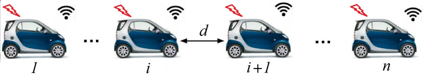
Design a decentralized feedback control law that guarantees formation, under minimum "possibility" of systemic events.

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = u_i(t) + \xi_i(t)$$

$$u_i(t) = \sum_{j=1}^{n} a_{ij}\big(v_j(t-\tau) - v_i(t-\tau)\big) + \sum_{j=1}^{n} \beta\, a_{ij}\big(x_j(t-\tau) - x_i(t-\tau) - (j-i)d\big)$$

- $a_{ij} = a_{ji} \geq 0$ feedback gain between vehicles i and j.
- $\beta > 0$: position/velocity ratio parameter
- $\tau > 0$: time-delay

Platooning



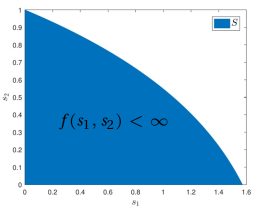
$$dx_t = v_t\,dt \qquad dv_t = -L\,v_{t-\tau}\,dt - \beta\,L\left(x_{t-\tau} - \mathbb{1}d\right) + d\xi_t$$

Systemic Event *Successive Vehicle Collision* : Monitor $x^{i+1} - x^i$

$$\lim_t \left(x^{(i+1)} - x^{(i)}\right) = \Delta_{i+1,i}$$

Proposition

For any $i = 1, \ldots, n-1$, $\Delta_{i+1,i} \sim \mathcal{N}\left(d, \sigma_i^2\right)$
$\sigma_i < \infty$ if and only if $(\lambda_k\tau, \beta\tau) \in S, \quad k = 2, \ldots, n.$

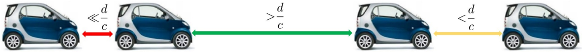


$$\sigma_i^2 = |\xi|^2 \frac{\tau^3}{2\pi} \sum_{k=2}^{n} \left(\left(\mathbf{e}_{i+1} - \mathbf{e}_i\right)^T \mathbf{q}_k\right)^2 f\left(\lambda_k\tau, \beta\tau\right)$$

$$f(s_1, s_2) = \int_{\mathbb{R}} \frac{dr}{(s_1 s_2 - r^2\cos(r))^2 + r^2(s_1 - r\sin(r))^2}$$

Systemic Events and the associated Risk

Scenarios on vehicle collision



(i.) ***unlikely***

$$d > x_t^{(i+1)} - x_t^{(i)} > \frac{d}{c},\ c > 1$$

(ii.) ***likely***

$$0 < x_t^{(i+1)} - x_t^{(i)} < \frac{d}{c}$$

(iii.) ***very likely***

$$x_t^{(i+1)} - x_t^{(i)} << \frac{d}{c}$$

$$\mathcal{R}_\varepsilon^i = \inf\left\{\delta > 0 \ :\ \mathbb{P}\left(\Delta_{i+1,i} \in\ U_\delta\right) < \varepsilon\right\}$$

$$\{U_\delta\}_{\delta \geq 0}\ :\ U_\delta = \left(-\infty, \frac{d}{\delta + c}\right)$$

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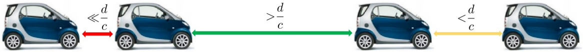

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Network Risk Profile

$$\mathfrak{R}_\varepsilon = \left(\mathcal{R}_\varepsilon^1, \mathcal{R}_\varepsilon^2, \mathcal{R}_\varepsilon^3, \ldots, \mathcal{R}_\varepsilon^{n-1}\right)$$

Systemic Events and the associated Risk

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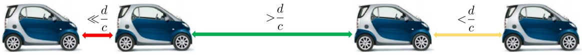

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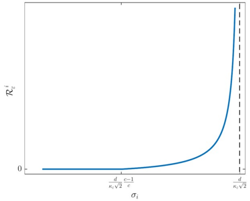
Risk of Successive Vehicle Collision

$$\mathcal{R}_\varepsilon^i = \begin{cases} 0\,, & \sigma_i \leq \frac{d}{\kappa_\varepsilon\sqrt{2}}\frac{c-1}{c} \\ \frac{d}{d-\kappa_\varepsilon\sigma_i\sqrt{2}} - c\,, & \frac{d}{\kappa_\varepsilon\sqrt{2}}\frac{c-1}{c} < \sigma_i < \frac{d}{\kappa_\varepsilon\sqrt{2}} \\ \infty\,, & \sigma_i \geq \frac{d}{\kappa_\varepsilon\sqrt{2}} \end{cases}$$



Risk - Based Network Design

Which topology (if any) minimizes risk of collision ?

<u>Fundamental Limits</u> : $\sigma_i \geq \sigma^*(\tau, |\xi|), \;\; i = 1, \dots, n-1$

$$\mathcal{R}^i_\varepsilon \geq \frac{d}{d - 4.02\,\kappa_\varepsilon |\xi| \tau^{3/2}} - c$$

if $\sigma^* \in \left(\frac{d}{\kappa_\varepsilon \sqrt{2}} \frac{c-1}{c}, \frac{d}{\kappa_\varepsilon \sqrt{2}}\right)$

1. Minimal Risk Graph:

$$(\lambda_k \tau, \beta \tau) = \mathrm{argmin}_{(s_1, s_2) \in S} f(s_1, s_2) \approx (1.111,\ 0.220), \quad k > 1$$

2. Inevitability condition: $|\xi| \cdot \tau^{\frac{3}{2}} \geq \frac{d}{1.12 \cdot \kappa_\varepsilon}$ $\quad \varepsilon \in (0, 1/2)$

The Stronger ...the Better ? No !

Effective Resistance

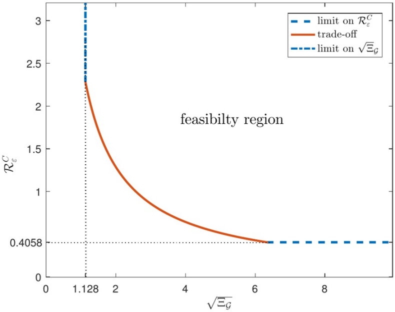
$$\Xi_{\mathcal{G}} = n \sum_{k=2}^{n} \frac{1}{\lambda_k} > \Xi^*(\tau)$$

Number of Spanning-Trees in a Graph.

$\Xi_{\mathcal{G}}$ monotonically decreases with "graph connectedness".

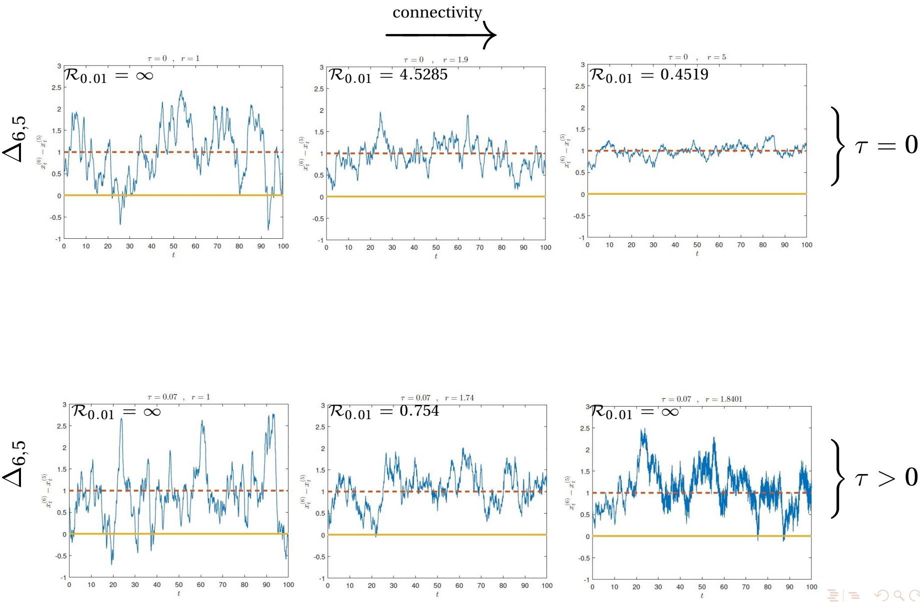
Trade-Off : Risk vs. Connectivity

$$\mathcal{R}_{\varepsilon}^{i} \cdot \sqrt{\Xi_{\mathcal{G}}} > \sqrt{n\tau K_{|\xi|}}$$



Improving connectivity, $\Xi_G \downarrow$, may increase risk of collision, $\mathcal{R}_{\varepsilon}^{(i)} \uparrow$.

Idiosyncratic behavior of Risk : Platooning for $n = 10$ vehicles



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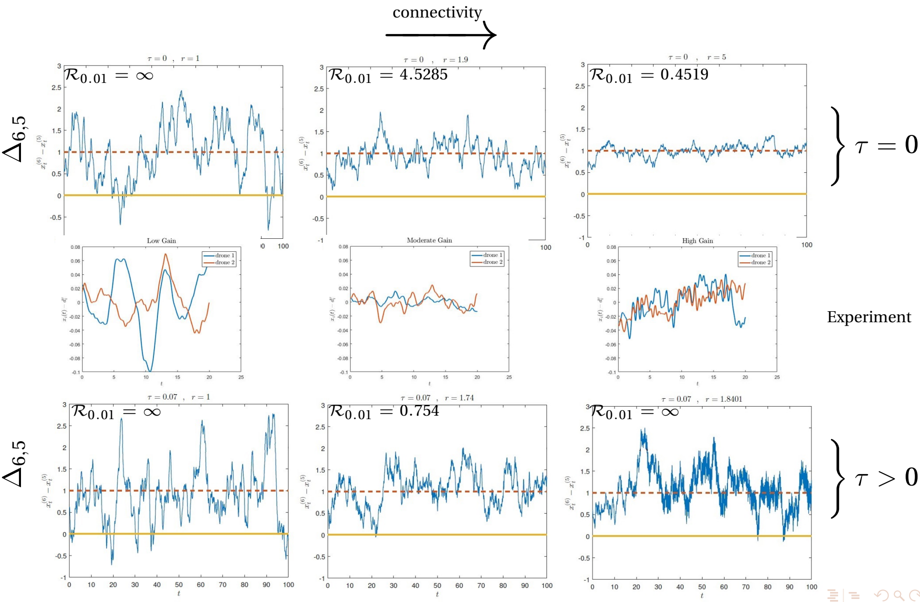


Further Topics on Risk-Aware Networked Control

(i.) **Various / Multiple / Joint Systemic Events** [IEEE-TAC'19-b, IEEE-TAC'19-c]

- Detachment / Loss of Connectivity
- Joint Events (e.g., simultaneous collision on multiple vehicles)

(ii.) **Special Graphs** [IEEE-TAC'19-b, IEEE-TAC'19-c]

- Risk over basic topologies (complete, star, path, cycle, spatially decaying,...)
- Role of topology/defficiencies/uncertainty in vehicles.
- Approximation formulas

(iii.) **Risk-Based Network Synthesis** [IEEE-TAC'19-a, ecc'19, IEEE-TAC'19-c]

- Add, remove, re-weight links w.r.t. risk metrics.
- Scalable algorthmic design methods (multi-objective, non-convex optimization)

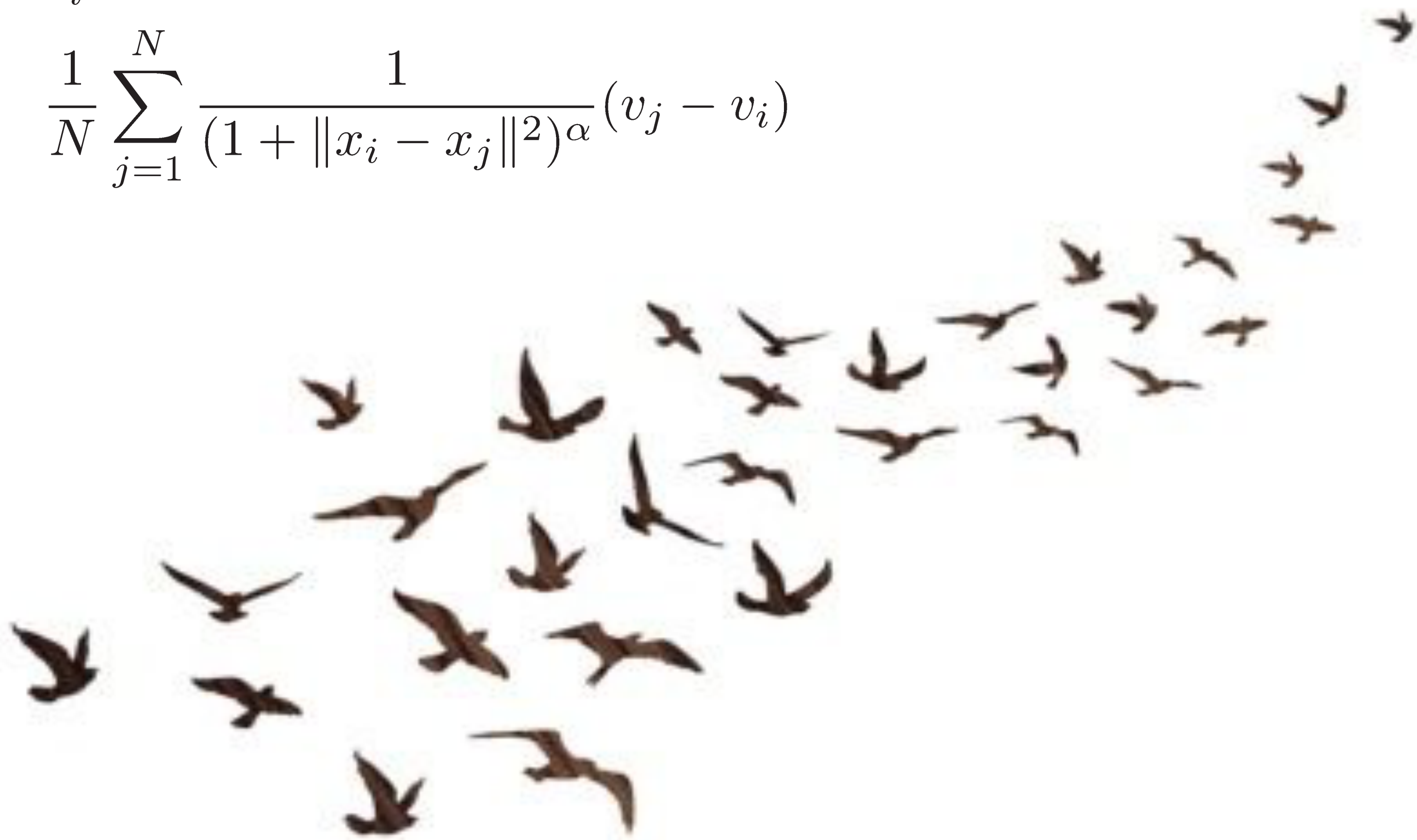
(iv.) **Risk Analysis & Design for MAS** [necsys'18-a, necsys'18-b, cdc'17-b, cdc'17-a, Springer'19]

- Noise fluctuations in time-delayed rendezvous. Structured Noise.
- Phase-Incoherence in wide-area control of power systems.
- State-Dependent communication for alignment.

More Realistic Models: Flock of Birds

- Cucker-Smale Network Model:

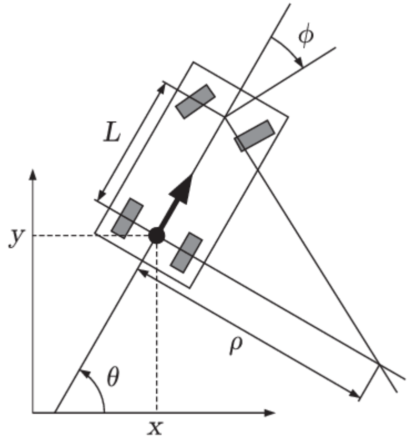
$$
\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N}\sum_{j=1}^{N} \frac{1}{(1+\|x_i - x_j\|^2)^\alpha}(v_j - v_i)
\end{aligned}
$$

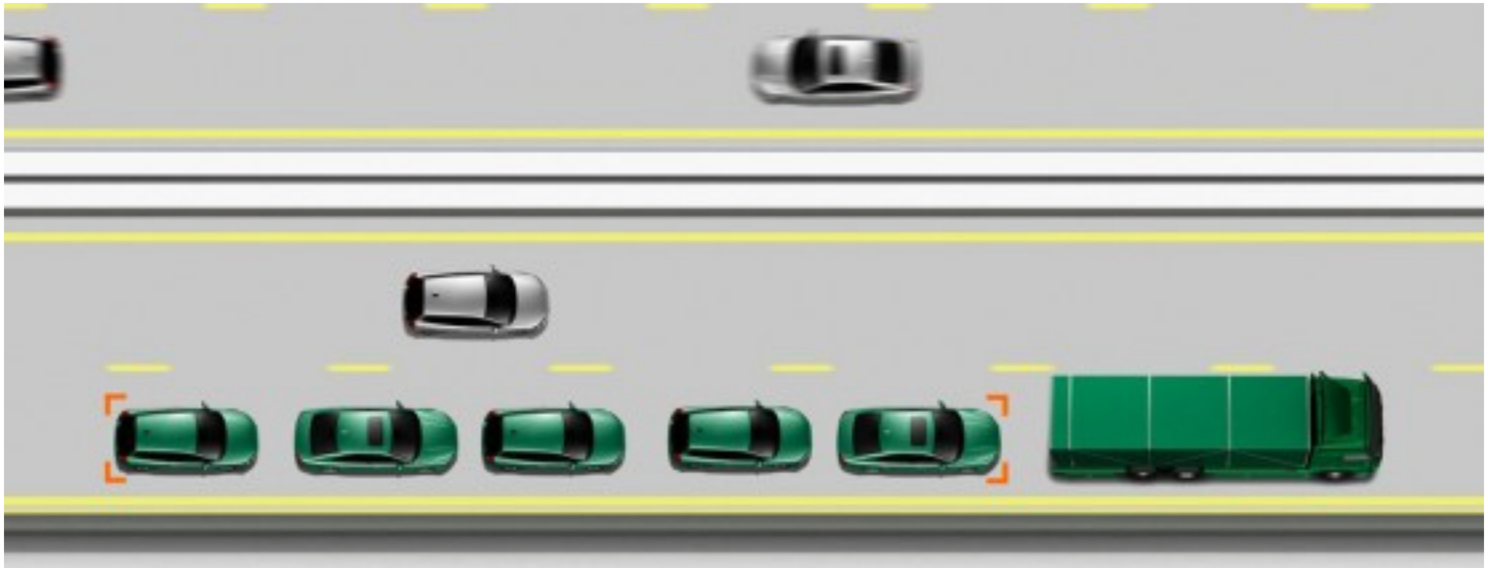


More Realistic Models: Dubins Car Model

- Dubins Car Model:

$$\dot{x}_i = u_s \cos(\theta_i)$$
$$\dot{y}_i = u_s \sin(\theta_i)$$
$$\theta_i = \frac{u_s}{L} \tan(u_\phi)$$





Koopman Operator Theory

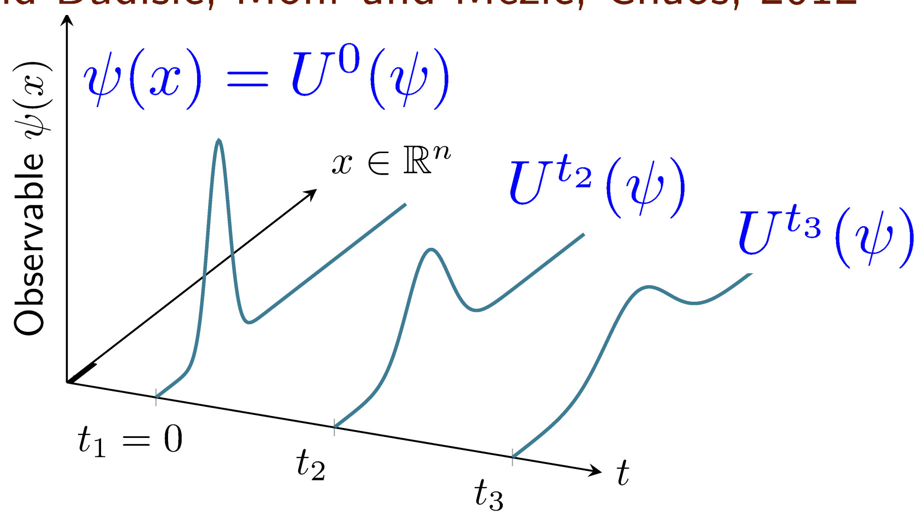
Definition: Let $X \subseteq \mathbb{R}^n$ and $\mathcal{F}$ be a Banach space of observables $\psi : X \to \mathbb{C}$. The Koopman operator U^t for a dynamical system with flow $\mathbf{S}(t, x_0)$ is

$$U^t(\psi) = \psi \circ \mathbf{S}(t, x_0).$$

An eigenfunction of the Koopman operator $\phi(x)$ with an eigenvalue $\lambda \in \mathbb{C}$ is an observable $\phi(x) : X \to \mathbb{C}$ satisfying

$$\phi \circ \mathbf{S}(t, x_0) = e^{\lambda t}\phi(x_0),$$

Koopman, PNAS,1931 and Budišić, Mohr and Mezić, Chaos, 2012



Koopman Operator Theory: Basics

Closedness property of the eigenfunctions and eigenvalues
Budišić, Mohr and Mezić, Chaos, 2012

Lemma: If $\phi_1, \phi_2 \in \mathcal{F}$ are Koopman eigenfunctions with eigenvalues λ_1 and λ_2, respectively, and $\phi_3 := \phi_1 \phi_2 \in \mathcal{F}$, then ϕ_3 is an eigenfunction of eigenvalue $\lambda_1 + \lambda_2$

Conjugacy Theorems and Koopman Eigenfunctions

Theorem: Let us consider $\dot{x} = F(x)$ with smooth F and the origin being its hyperbolic fixed point. Then there exists a C^1-diffeomorphism H of a neighborhood U of the origin on an open set $\Omega' \subset \Omega$ containing the origin such that for each $x_0 \in \Omega'$, there is an open interval $I(x_0) \subset \mathbb{R}_+$ containing zero such that for all $x_0 \in U$ and $t \in I(x_0)$

$$H \circ \mathbf{S}(t, x_0) = \mathrm{e}^{At} H(x_0)$$

where $A = \frac{\partial}{\partial x} F|_{x=0}$.

- Extension of Hartman diffeomorphism to the whole basin of attraction
  Lan and Mezic, Physica D, 2013

Theorem: If F is C^2 and $A = \frac{\partial}{\partial x} F|_{x=0}$ is Hurwitz, then there exists a diffeomorphism $\alpha : \Omega \to \mathbb{R}^n$ such that

$$\alpha \circ \mathbf{S}(t, x_0) = \mathrm{e}^{At} \alpha(x_0)$$

for all $x_0 \in \Omega$ and $t \geq 0$.

Conjugacy Theorems and Koopman Eigenfunctions

For diagonalizable $A = V\Lambda V^{-1}$, let us define

$$H(x) := V^{-1}\alpha(x)$$

which satisfies

$$H\left(\mathbf{S}(t, x_0)\right) = e^{\Lambda t} H(x_0).$$

Lemma: Let represent map H in terms of its columns

$$H = \left[H_1, H_2, \ldots, H_n\right]^T$$

where each $H_i : \Omega \to \mathbb{C}^n$. If λ_i is the i'th eigenvalue of A, then $\left(\lambda_i, H_i\right)$ is a pair of Koopman eigenvalue and its corresponding eigenfunction.

Koopman Eigenfunctions and the Flow of the System

- Consequently, in the whole domain of attraction of the origin

$$H\left(\mathbf{S}(t,x_0)\right) = e^{\Lambda t} H(x_0) \Rightarrow \mathbf{S}(t,x_0) = H^{-1}\left(e^{\Lambda t} H(x_0)\right).$$

Knowledge of these Koopman eigenfunctions along with eigenvalues of the Jacobian matrix A helps us to characterize the flow of the system.

- Analyticity of the inverse map H^{-1} results in an exact mode decomposition for the system.

Koopman Mode Decomposition

From Stone-Weierstrass Theorem, $H^{-1}(x)$ can be uniformly approximated over the bounded set

$$\tilde{\Omega} := \left\{ \mathrm{e}^{\Lambda t} H(x_0) \;\middle|\; \text{for all } x_0 \in \Omega \text{ and } t \in \mathbb{R}_+ \right\}$$

with maximum error $\epsilon > 0$ by some multivariate polynomials:

$$H^{-1}(x) \overset{\epsilon}{\approx} \sum_{\gamma \in \Gamma_\epsilon} c_\gamma^\epsilon \, x_1^{j_1} \cdots x_n^{j_n}$$

for every x in a rectangular region contained in $\tilde{\Omega}$.

Koopman Mode Decomposition

Let $A = \frac{\partial}{\partial x} F(x)|_{x=0}$ be diagonalizable and Hurwitz with eigenvalues $\lambda_1, \ldots, \lambda_n$. Consider map H^{-1} for every $x_0 \in \Omega$, where the basin of attraction Ω is assumed to be compact. Then, using the polynomial approximation of $H^{-1}(x)$, the flow $\mathbf{S}(\cdot, x_0)$ of the nonlinear system is

$$\mathbf{S}(t, x_0) \overset{\epsilon}{\approx} \sum_{i=1}^{|\Gamma_\epsilon|} c_i \, \mathrm{e}^{\bar{\lambda}_i t} \, \phi_i(x_0)$$

for all $t \geq 0$, where for the ordered vector $\gamma_i = (j_1, \ldots, j_n) \in \Gamma_\epsilon \subset \mathbb{Z}_+^n$ we have

$$\bar{\lambda}_i := \sum_{k=1}^{n} j_k \lambda_k \quad \text{and} \quad \phi_i(x_0) := \prod_{k=1}^{n} H_k^{j_k}(x_0).$$

Consensus Network with State-Dependent Couplings

$$\dot{x}_i = \sum_{\{i,j\}\in\mathcal{E}} w_{ij}\left(x_j - x_i\right)$$

$$y_i = x_i - \frac{1}{n}(x_1 + \cdots + x_n)$$

Coupling function: $w_{ij} = \tilde{w}_{ij} f\left(|x_i - x_j|^2\right)$ where f is analytic

$$\dot{x} = -\mathcal{L}(x)x$$

$$y = M_n x$$

where $M_n = I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ is the centering matrix

KMD for Nonlinear Consensus Network

- The $\dot{x} = -\mathcal{L}(x)x$ does not have a hyperbolic fixed-point at the origin.
- Solution: Disagreement Dynamics: $\mathcal{L}_d(x) = \mathcal{L}(x) + \frac{1}{n}\mathbf{1}\mathbf{1}^T$

Lemma: The output of the nonlinear network follows the dynamics

$$\dot{y} = -\mathcal{L}_d(y)y,$$

where $y = 0$ is an asymptotically stable hyperbolic fixed-point.

$\dot{y} = -\mathcal{L}(y)y \leftrightarrow$ Restriction of $\dot{x} = -\mathcal{L}_d(x)x$ to $\mathbf{1}^\perp$

Observation: We can find the KMD for disagreement dynamics and restrict it to $\mathbf{1}^\perp$ and it is the KMD for the output $y(t)$.

- This observation helps evaluate the performance measure.

Network Model and Objectives

Performance Measure

$$\rho(\mathcal{L}) = \mathbb{E}\left\{\int_0^\infty y^T(t, y_0) Q\, y(t, y_0) dt\right\}$$

This is the average output energy w.r.t. all random inital conditions.

- Quantify the performance in terms of
  - Parameters of the network
  - The graph topology
- Design/Modify the network for a superior performance

It is assumed that the underlying coupling graph is undirected and connected.

Evaluating the Performance Measure

Let us consider flow $\mathbf{S}(\cdot, y_0)$ of the disagreement dynamics for all initial disagreements. Then,

$$\rho(\mathcal{L}) \approx \sum_{i,j\geq 1} \phi_{ij} c_{ij} \frac{1}{\overline{\lambda}_i + \overline{\lambda}_j},$$

where $\{\overline{\lambda}_i\}_{i=1}^{|\Gamma_\epsilon|}$ is the sequence of Koopman eigenvalues in the approximate KMD, enumerated by an arbitrary numbering of $\gamma_i = (j_2, \ldots, j_n) \in \mathbb{Z}_+^{n-1}$ as

$$\overline{\lambda}_i := \sum_{k=2}^{n} j_k \lambda_k \quad \text{and} \quad \phi_i(x_0) := \prod_{k=2}^{n} H_k^{j_k}(x_0).$$

with $\lambda_2, \ldots, \lambda_n$ the nonzero eigenvalues of the linearized Laplacian $\frac{\partial}{\partial x}\mathcal{L}(x)|_{x=0} = \mathcal{L}_0$. Moreover, $\phi_{ij} := \mathbb{E}\{\phi_i(y)\phi_j(y)\}$ and $c_{ij} := c_i^T Q c_j$ are computed in terms of Koopman eigenfunctions and modes, respectively.

Examples: LTI Consensus Network

Example: The map of Koopman eigenfunctions for LTI consensus network with Laplacian $\mathcal{L} = R\Lambda R^T$ and its inverse are (Budišić et al. 2012)

$$H(y) = R^T y, H^{-1}(y) = Ry.$$

Assuming zero mean initial conditions with $\mathbb{E}(xx^T) = I_n$ the coefficients of involved in the performance measure are

$$\phi_{ij} = \delta_{ij}, c_{ij} = \delta_{ij} \text{ if } i, j \in \{2, 3, \ldots, n\}$$
$$c_{ij} = 0 \text{ if } i \text{ or } j \notin \{2, 3, \ldots, n\}$$

which coincides with the well-known result for the LTI networks

$$\rho(\mathcal{L}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i},$$

Governing Equation of Eigenfunctions

- A useful governing equation for the Koopman eigenfunctions.

> **Lemma:** Let $\phi(x)$ be an eigenfunction of Koopman operator with a Koopman eigenvalue λ. The eigenfunction satisfies the following PDE.
>
> $$\nabla^T \phi(x) F(x) = \lambda \phi(x).$$
>
> e.g. see Mauroy and Mezić, TAC 2016

- Computationally advantageous
- May be directly integrated for low-dimensional dynamical systems
- We will leverage it to numerically find the eigenfunctions and Koopman mode decomposition

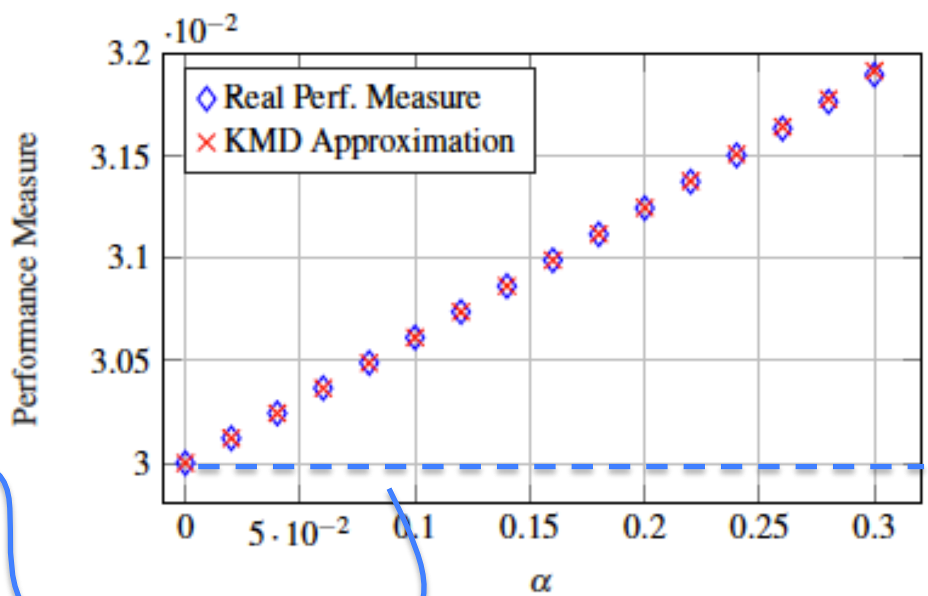
Examples: Network of Two Agents

Example: The nonlinear consensus network of two agents with

$$w_{ij} = \frac{1}{(1+(x_i - x_j)^2)^\alpha}, \quad \text{for some } \alpha > 0$$

Map of Koopman eigenfunctions: $H(x) = (\phi(x), -\phi(x))$, $x = (p, -p)^T$.

Using PDE: $\phi(x) = 2p \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)2^{2n-1}p^{2n}}{n \times n!}\right)$.



Performance of Linearized Systems

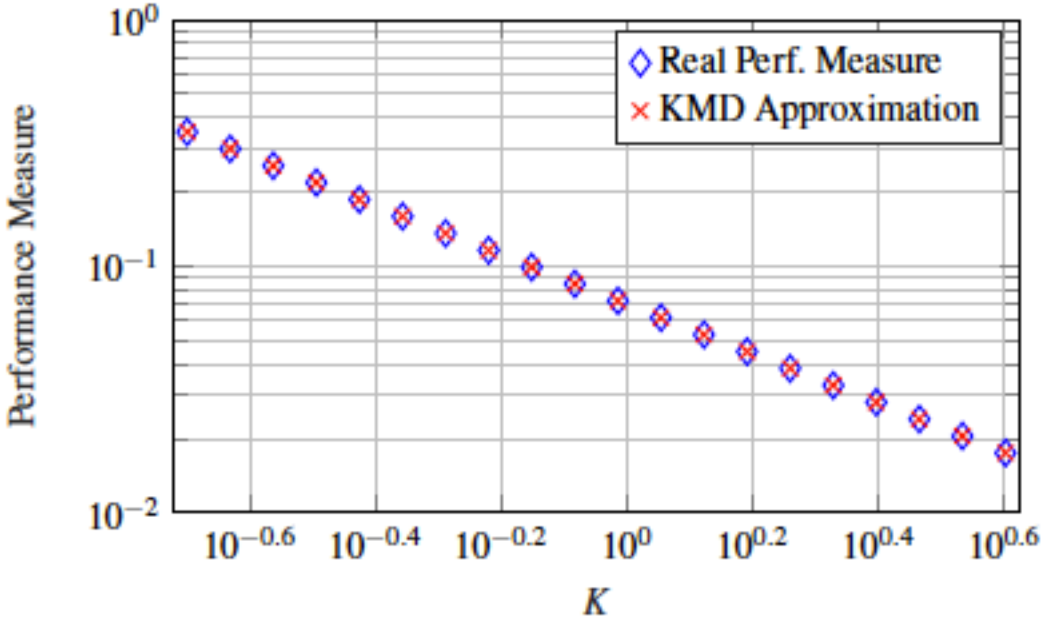
Examples: Two Identical Kuramoto Oscillators

Example: For two identical Kuramoto oscillators

$$w_{ij} = K\frac{\sin(x_i - x_j)}{x_i - x_j}, \text{ for some } K > 0$$

Map of Koopman eigenfunctions: $H(x) = (\phi(x), -\phi(x))^T$, $x = (\theta, -\theta)^T$.

Using PDE: $\phi(x) = \tan(\theta) \Rightarrow H^{-1}(x) = (\arctan(\theta), -\arctan(\theta))^T$.



Sparse Approximation of Eigenfunctions & Decomposition

Lemma: Let $\phi(x)$ be an eigenfunction of Koopman operator with a Koopman eigenvalue λ. The eigenfunction satisfies the following PDE.

$$\nabla^T \phi(x) F(x) = \lambda \phi(x).$$

- We use Smolyak polynomials $\hat{f}(x) = \sum_{i=1}^{M} \Theta_i \mathrm{T}_i(x)$ in a sparse grid

tensor product of Chebyshev polynomials

- Lemma + the gradient of ϕ_λ equal to left eigenvector r

$$\underset{\Theta \in \mathbb{R}^m}{\text{minimize}} \quad \|\mathcal{A}\Theta\|_2^2,$$

SDP: can be solved via convex optimization tools

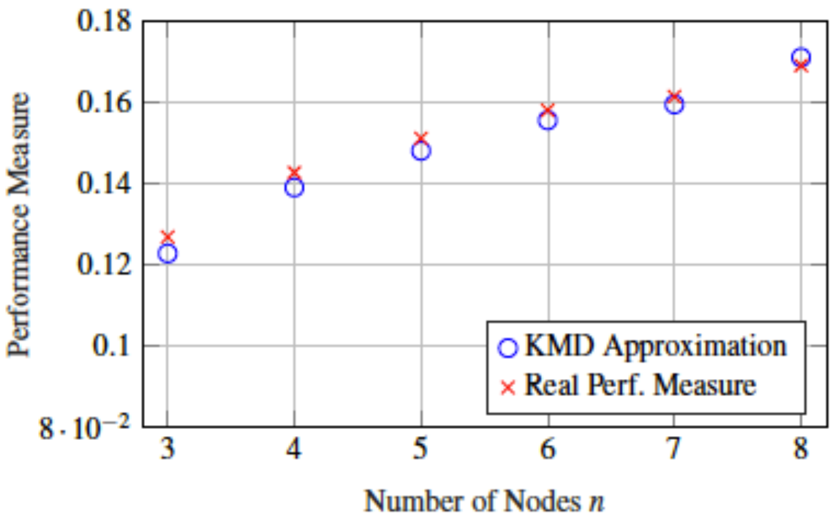
$$\text{subject to} \quad \begin{bmatrix} \mathcal{B} \\ \mathcal{C} \end{bmatrix} \Theta = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

- Similar optimization problem for approximation to components of $H^{-1}(x)$

Examples: Application of Numerical Method

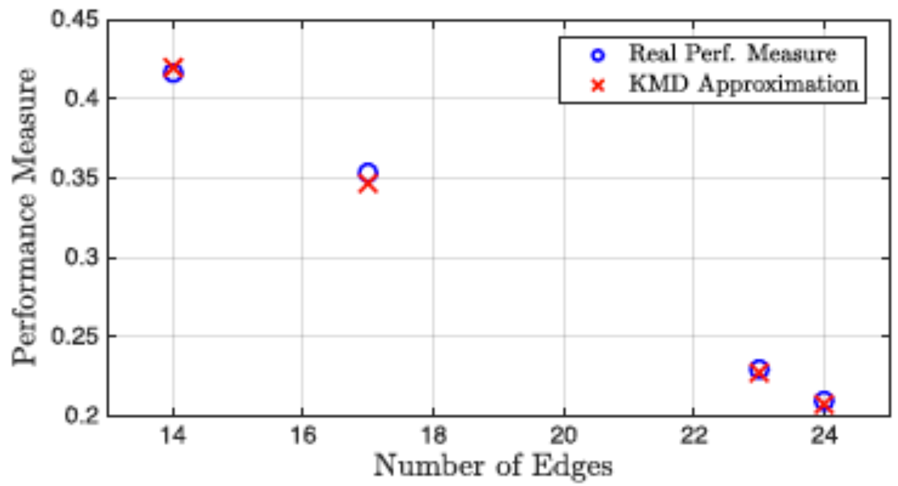
Example: For $\alpha = 0.25$, with the nominal graph the complete graph over n nodes

$$w_{ij} = \frac{1}{(1 + (x_i - x_j)^2)^\alpha},$$



Example: A nonlinear consensus network with $n = 8$, $\alpha = 0.25$ and random graphs with different number of edges

$$w_{ij} = \frac{1}{(1 + (x_i - x_j)^2)^\alpha},$$



Conclusion and Future Directions

- We demonstrated the effectiveness of Koopman spectral analysis using the KMD to (at least locally) approximate the flow of the nonlinear consensus networks.
- The methodology can be applied beyond consensus networks and can be further investigated for other dynamical networks.
- If the linearized spectrum is not resonant, the local analytic approximations do exist, according to Poincare's theorem.
- Role of time delay in spectral analysis of nonlinear systems.
- Related design problems are widely open problems.