On the Solution of the GPS Localization and Circle Fitting Problems

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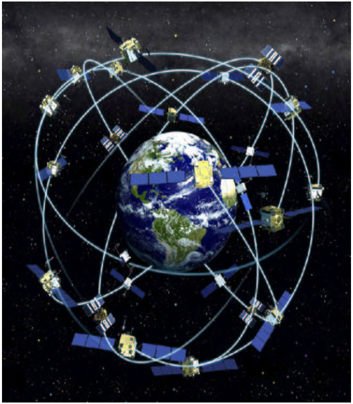
The Global Positioning System

- a system of 31 transmitting satellites (originally 24).
- objective: provide reliable location and time information anywhere on Earth where there is unobstructed line of sight to four or more satellites.

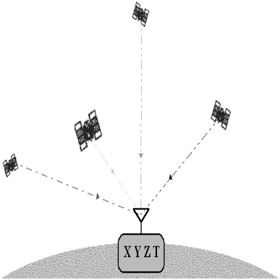


Localization via GPS

- Each satellite transmits its location (x_i, y_i, z_i) and a time stamp t_i.
- The GPS receiver estimates the distances to at least 4 satellites via

$$d_i = c(T - t_i)$$

  c - speed of light.
  T - GPS receiver's clock.
- The receiver's clock is inaccurate (an error of one microsecond corresponds to an error or 300 meters).
- The measured distances are called pesudoranges and include an unknown "clock bias".



The GPS Localization Problem

$$d_i \approx \|\mathbf{x} - \mathbf{a}_i\| - r, \quad i = 1, \ldots, m$$

- $\mathbf{a}_i$ - satellite's location.
- r - unknown bias.
- $\mathbf{x}$ - unknown user's location.
- d_i - i-th pseudorange (can even be negative)
- $m \geq n + 1$.

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A Least Squares Problem:

$$\min_{\mathbf{x}, r} \left\{ \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i - r)^2 \right\}.$$

The GPS Localization Problem:

- without loss of generality: $d_i \geq 0$ (otherwise make a shift and redefine r).
- a mild assumption: $r \geq 0$.

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$$\min_{\mathbf{x},r} \left\{ \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i - r)^2 : r \geq 0 \right\}.$$

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Problem Reduction (minimizing with respect to r):

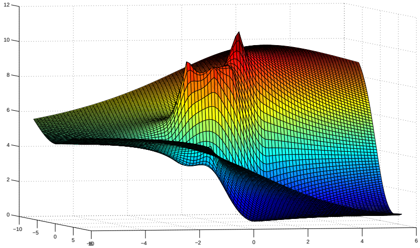
The GPS Least Squares Problem-reduced form

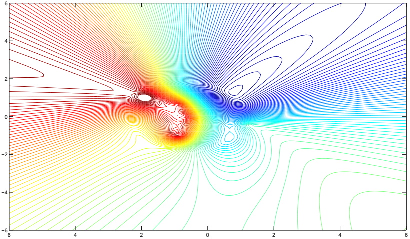
$$\text{(GPS-LS)} : \min_{\mathbf{x}} \left\{ \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i - r(\mathbf{x}))^2 \right\}$$

$$r(\mathbf{x}) := \left[\frac{1}{m} \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i) \right]_+$$

The Bad News...

The GPS-LS problem is nonsmooth and nonconvex





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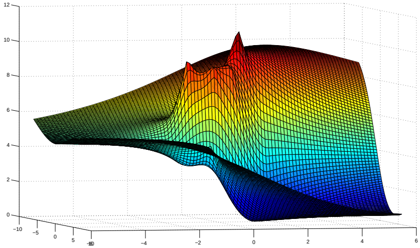
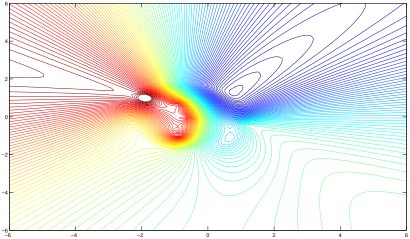




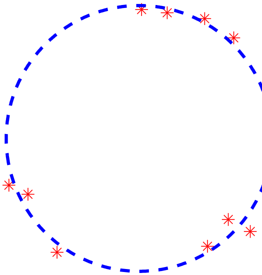
- In principal, it should be difficult to find a global optimal solution

The Circle Fitting Problem - $d_i = 0$

When $d_i = 0$, the problem reduces circle fitting:

Given m points $\mathbf{a}_1, \ldots, \mathbf{a}_m$, find the circle that fits them in the "best way".

$$\|\mathbf{x} - \mathbf{a}_i\| \approx r$$



The Circle Fitting Least Squares Problem

The CF Least Squares Problem

$$\min_{\mathbf{x},r} \left\{ \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - r)^2 : r \geq 0 \right\}.$$

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The CF Least Squares Problem-reduced form

$$\text{(CF-LS)} : \min_{\mathbf{x}} \left\{ \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - r(\mathbf{x}))^2 \right\} \left(r(\mathbf{x}) := \frac{1}{m} \sum_{i=1}^{m} \|\mathbf{x} - \mathbf{a}_i\| \right)$$

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(CF-LS) is the geometric fitting problem: find the circle that minimizes the distances between the circle and the given points

Applications of Circle Fitting

- Archaeology
- Computer graphics
- Coordinate metrology
- Petroleum engineering
- Quality inspection for mechanical parts
- Statistics

Literature

- GPS Localization:
  - Abel (1994) - A variable projection method.
  - Source localization from range-differences (usage of a reference measurement): Huang, Benesty, Elko and Mersereau (2001), Stoica and Li (2006), Beck, Stoica and Li (2008),
  - Sensor network localization from range difference (2009), Yang, Wang ,Luo (usage of *all* differences, SDR approach).
- Circle Fitting:
  - Kasa (1976) - solution of a related squared least squares problem in the 2D case.
  - Gander, Golub and Strebel (1994): algebraic fit + Gauss Newton for (CF-LS).
  - Chernov, Lesort (2005) - Analysis in the 2D case.

The least squares GPS localization problem

Advantage:

- Has a statistical and geometrical meaning.

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- Nonconvex and nonsmooth - seems to be intractable.

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♣ It is therefore important to find a good approximate solution/solution to an approximate problem

The Squared Least Squares Approach

- replace
$$\|\mathbf{x} - \mathbf{a}_i\| \approx r + d_i$$
with
$$\|\mathbf{x} - \mathbf{a}_i\|^2 \approx (r + d_i)^2$$

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The Squared Least Squares GPS problem:

$$\text{(GPS-SLS):} \quad \min \left\{ \sum_{i=1}^{m} \left(\|\mathbf{x} - \mathbf{a}_i\|^2 - (r + d_i)^2 \right)^2 \right\}.$$

Disadvantage: loses the statistical/geomretrical meaning of LS.
Advantage: tractable!! (although quartic)

Equivalence to GTRS

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$$\min_{\mathbf{x}, r, \alpha} \left\{ \sum_{i=1}^{m} \left(-2\mathbf{a}_i^T \mathbf{x} - 2 d_i r + \alpha + \|\mathbf{a}_i\|^2 - d_i^2 \right)^2 : \alpha = \|\mathbf{x}\|^2 - r^2 \right\}.$$

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Lemma

Problem (GPS-SLS) is equivalent to

$$\min_{\mathbf{y} \in \mathbb{R}^{n+2}} \left\{ \|\mathbf{B}\mathbf{y} - \mathbf{b}\|^2 : \mathbf{y}^T \mathbf{D}\mathbf{y} - 2\mathbf{g}^T \mathbf{y} = 0 \right\},$$

$$\mathbf{B} = \begin{pmatrix} 2\mathbf{a}_1^T & -1 & 2d_1 \\ 2\mathbf{a}_2^T & -1 & 2d_2 \\ \vdots & \vdots & \vdots \\ 2\mathbf{a}_m^T & -1 & 2d_m \end{pmatrix}, \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \alpha \\ r \end{pmatrix}, \mathbf{D} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n^T & 0 & 0 \\ \mathbf{0}_n^T & 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \|\mathbf{a}_1\|^2 - d_1^2 \\ \|\mathbf{a}_2\|^2 - d_2^2 \\ \vdots \\ \|\mathbf{a}_m\|^2 - d_m^2 \end{pmatrix}$$

Tractability of GTRS Problems

Generalized Trust Region Subproblem (GTRS):

$$\text{(GTRS):} \quad \min\{\mathbf{x}^T\mathbf{A}_1\mathbf{x} + 2\mathbf{b}_1^T\mathbf{x} + c_1 : \mathbf{x}^T\mathbf{A}_2\mathbf{x} + 2\mathbf{b}_2^T\mathbf{x} + c_2 = 0\},$$

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Theorem (More, 93)

Suppose that $\mathbf{A}_2 \neq \mathbf{0}$. Then $\mathbf{x}$ is an optimal solution of (GTRS) if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}
(\mathbf{A}_1 + \lambda\mathbf{A}_2)\mathbf{x} + (\mathbf{b}_1 + \lambda\mathbf{b}_2) &= \mathbf{0}, \\
\mathbf{x}^T\mathbf{A}_2\mathbf{x} + 2\mathbf{b}_2^T\mathbf{x} + c_2 &= 0, \\
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The problem can be solved by a dual approach via a one-dimensional search.

Tractability of the GTRS problem, Cont'd

The global optimal solution of (GPS-SLS) is comprised of the first n components of the vector

$$y(\lambda^*) = (\mathbf{B}^T\mathbf{B} + \lambda^*\mathbf{D})^{-1}(\mathbf{B}^T\mathbf{b} + \lambda^*\mathbf{g}),$$

where λ^* is the root of

$$\phi(\lambda) \equiv \mathbf{y}(\lambda)^T\mathbf{D}\mathbf{y}(\lambda) - 2\mathbf{g}^T\mathbf{y}(\lambda) = 0,$$

over a predefined interval $[\mu_1, \mu_2]$.

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The Basic Assumption: The matrix $\tilde{\mathbf{A}}$ defined by

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has full column rank. That is, $\mathbf{a}_1, \ldots, \mathbf{a}_m$ do not reside in a lower-dimensional space.

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- Further assumptions?

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The minimum of the GTRS problem is attained if at least one of the following conditions is satisfied:

i. $\mathbf{d} \notin Range(\tilde{\mathbf{A}})$.

ii. $\mathbf{d} \in Range(\tilde{\mathbf{A}})$ *and* $\left\| \left[(\tilde{\mathbf{A}}^T\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^T\mathbf{d} \right]_n \right\| \neq \frac{1}{2}$.

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Proof Layout

$$\text{(GTRS)}: \quad \min_{\mathbf{y}\in\mathbb{R}^{n+2}} \left\{ \|\mathbf{B}\mathbf{y}-\mathbf{b}\|^2 : \mathbf{y}^T\mathbf{D}\mathbf{y} - 2\mathbf{g}^T\mathbf{y} = 0 \right\},$$

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$$\Leftrightarrow \begin{pmatrix} \tilde{\mathbf{A}}^T\tilde{\mathbf{A}} + \lambda\mathbf{E} & 2\tilde{\mathbf{A}}^T\mathbf{d} \\ 2\mathbf{d}^T\tilde{\mathbf{A}} & 4\|\mathbf{d}\|^2 - \lambda \end{pmatrix} \succ \mathbf{0}, \quad \mathbf{E} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{0}_n^T & 0 \end{pmatrix}.$$

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Final step:

$$g'(0) \neq 0 \Leftrightarrow \left\| \left[(\tilde{\mathbf{A}}^T\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^T\mathbf{d} \right]_n \right\| \neq 1/2.$$

The Circle Fitting SLS Problem

$$\text{(CF-SLS)}: \quad \min_{\mathbf{x},r} \left\{ \sum_{i=1}^{m} (\|\mathbf{x}-\mathbf{a}_i\|^2 - r^2)^2 : \mathbf{x} \in \mathbb{R}^n, r \in \mathbb{R} \right\}.$$

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A stronger result:

Theorem

(CF-SLS) is equivalent to the linear least squares problem:

$$\min \|\tilde{\mathbf{A}}\mathbf{y} - \mathbf{b}\|^2$$

where $\mathbf{b} = (\|\mathbf{a}_1\|^2, \ldots, \|\mathbf{a}_m\|^2)^T$. *(2D case - Kasa 1976)*

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Proof Idea:

$$\min_{\mathbf{x},r} \left\{ \sum_{i=1}^{m} (-2\mathbf{a}_i^T\mathbf{x} + \underbrace{\|\mathbf{x}\|^2 - r^2}_{R} + \|\mathbf{a}_i\|^2)^2 : \mathbf{x} \in \mathbb{R}^n, r \in \mathbb{R} \right\}.$$

+possible to discard the relation between $R, \mathbf{x}$ and r

So far...

- Main objective: solution of the nonsmooth/nonconvex GPS-LS problem:

$$(\text{GPS-LS}) : \min_{\mathbf{x}} \left\{ \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i - r(\mathbf{x}))^2 \right\}$$

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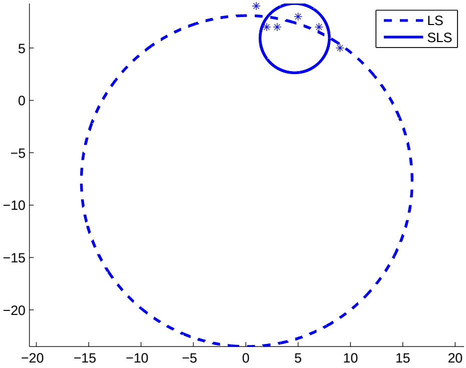
- What about the GPS-LS problem?

Illustration of the superiority of CF-LS over CF-SLS

(CF-LS) can give pretty good results, but...



Analysis of the GPS-LS Problem

$$(\text{GPS-LS}) : \min_{\mathbf{x}} \left\{ f(\mathbf{x}) := \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i - r(\mathbf{x}))^2 \right\}$$

$$r(\mathbf{x}) := \left[\frac{1}{m} \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i) \right]_+$$

Existence of an Optimal Solution

A related question: what is $\liminf_{\|\mathbf{x}\|\to\infty} f(\mathbf{x})$?

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Theorem

$$\liminf_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = \underbrace{\min_{\mathbf{z}} \left\{ (\mathbf{Az}+\mathbf{d})^T \left(\mathbf{I}_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^T \right) (\mathbf{Az}+\mathbf{d}) : \|\mathbf{z}\| = 1 \right\}}_{f_{\liminf}}$$

$$\mathbf{A} := \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{pmatrix}, \mathbf{1}_m = \mathit{ones(m,1)}$$

$f_{\liminf}$ can be efficiently computed via a solution of a GTRS.

Auxiliary Lemmata

Lemma

Let $\mathbf{z}$ be an optimal solution of the liminf problem. Then the sequence defined by $\mathbf{x}_k = k\mathbf{z}$ satisfies $\|\mathbf{x}_k\| \to \infty$ and

$$\lim_{k\to\infty} f(\mathbf{x}_k) = f_{\text{liminf}}.$$

i.e., $\liminf\limits_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) \le f_{\text{liminf}}$.

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$$f(\mathbf{x}) \geq A(\mathbf{x}) f_{\mathrm{liminf}} + C(\mathbf{x}) \quad (A(\mathbf{x}) \to 1, C(\mathbf{x}) \to 0).$$

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$$\Rightarrow \liminf_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = f_{\mathrm{liminf}}$$

Sufficient Conditions for Attainability

[SC1]: there exists $\tilde{\mathbf{x}} \in \mathbb{R}^n$ such that $f(\tilde{\mathbf{x}}) < f_{\mathrm{liminf}}$

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$$[\text{SC2}]: \; f(\mathbf{x}_{\text{sls}}) < f_{\text{liminf}},$$

$\mathbf{x}_{\text{sls}}$ - an optimal solution of (GPS-SLS).

Verifiable.

Is the sufficient condition likely to be satisfied?

- $m = 6, n = 2$.
- $\mathbf{a}_j$ and $\mathbf{x}$ randomly generated from $[-10, 10] \times [-10, 10]$.
- r randomly generated via $N(0, 10^2)$.
- 1000 realizations.
- $d_j = \|\mathbf{x} - \mathbf{a}_j\| - r + \varepsilon_j, \varepsilon_j \sim N(0, \sigma^2)$.
- N_σ - number of iteration in which [SC2] is satisfied.

σ	10^{-2}	10^{-1}	1	10
N_σ	1000	1000	986	513

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- When σ is not large, [SC2] is satisfied.
- $\sigma = 10$ is a huge standard deviation (the pseudoranges are essentially random).

The meaning of f_{liminf} for (CF-SLS)

$$f_{\mathrm{liminf}} = \min_{\mathbf{z}} \left\{ \mathbf{z}^T \mathbf{A}^T \left(\mathbf{I}_m - \frac{1}{m} \mathbf{1}_m \mathbf{1}_m^T \right) \mathbf{A}\mathbf{z} : \|\mathbf{z}\| = 1 \right\}$$

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Question: What is the meaning of this eigenvalue?

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Question: What is the meaning of this eigenvalue?
Answer: It is the optimal value of the Orthogonal Regression Problem.

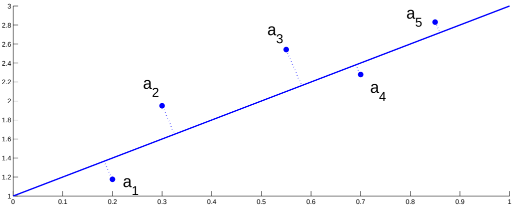
The Orthogonal Regression Problem

Given a set of points $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$, find an hyperplane

$$H_{\mathbf{x},y} := \left\{\mathbf{a} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{a} = y\right\}$$

minimizing the sum of squared Euclidean distances to the set points:

$$f_{\rm OR} = \min_{\mathbf{x},y} \left\{ \sum_{i=1}^m d(\mathbf{a}_i, H_{\mathbf{x},y})^2 : \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n, y \in \mathbb{R} \right\}$$



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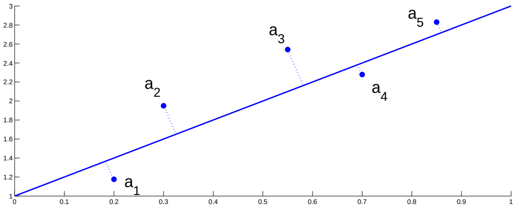
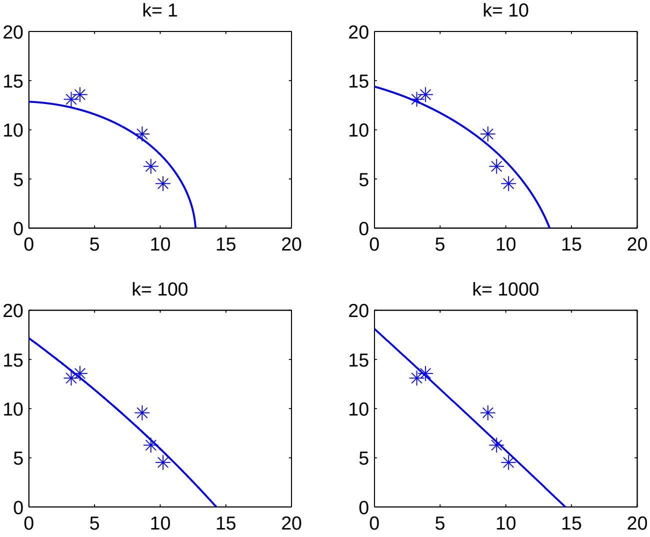
Theorem

$$f_{\mathrm{OR}} = f_{\mathrm{liminf}}$$



Circle Fitting versus Orthogonal Regression

A sequence of circles with corresponding obj. function value converging to the liminf.



[SC1] for (CF-LS) Revisited

$$[SC1] f^* < f_{\text{liminf}}$$

[SC1] It is better to fit with a circle than with a line

A Fixed Point Method for Solving (GPS-LS)

First Observation:

$$f(\mathbf{x}) = \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i - r(\mathbf{x}))^2 = \sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i)^2 - mr(x)^2.$$

- $\sum_{i=1}^{m} (\|\mathbf{x} - \mathbf{a}_i\| - d_i)^2$ - obj. function of the source localization problem.
- $mr(x)^2 = m\left[\frac{1}{m}\sum_{i=1}^{m}(\|\mathbf{x} - \mathbf{a}_i\| - d_i)\right]_+^2$ - a convex function.

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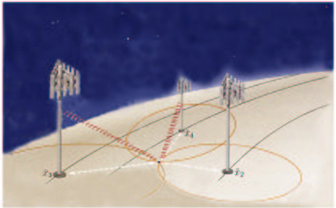
The Source Localization Problem: Given noisy observations of the distances between the source and the sensors $d_i \approx \|\mathbf{x}-\mathbf{a}_i\|$, find a good estimate of $\mathbf{x}$.



A Fixed Point Method for Solving (GPS-LS)

A generalization of a FP method constructed for the source localization problem (Beck, Teboulle, Chikishev, 2008)

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General step.

$$\mathbf{x}_{k+1} = T(\mathbf{x}_k), k = 0, 1, 2, \ldots$$

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- Question I: Can we prove monotonicity/convergence to a stationary point?
- Question II: Can we avoid the nondifferentiability points $\mathcal{A}$?

Convergence Analysis Technique for FP Methods

FP method:

$$\mathbf{x}_{k+1} = T(\mathbf{x}_k), \quad k = 01, 2, \ldots$$

for solving

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Results:

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Another example: Gradient method. $T(\mathbf{y}) = \mathbf{y} - \frac{1}{L}\nabla f(\mathbf{y})$

$$h(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{L}{2}\|\mathbf{x} - \mathbf{y}\|^2$$

The Auxiliary Function

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- $f_{\mathrm{liminf}} > \min\{f(\mathbf{a}_1), \ldots, f(\mathbf{a}_m)\} \Rightarrow \ldots$
  - Pick
    $$p \in \operatorname*{argmin}_{i=1,\ldots,m}\{f(\mathbf{a}_i)\}.$$
  - Find a descent direction $f'(\mathbf{a}_p, d) < 0$.
  - Define $\mathbf{x}_0 = \mathbf{a}_p + \varepsilon \mathbf{d}$.

Result: it is always possible to construct $\mathbf{x}_0$ satisfying $\clubsuit$.

Comparing (GPS-LS) and (GPS-SLS)

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- $\mathbf{a}_j$ and $\mathbf{x}$ randomly generated from $[-10, 10] \times [-10, 10]$.
- 1000 realizations.
- $d_j = \|\mathbf{x} - \mathbf{a}_j\| - r + \varepsilon_j, \varepsilon_j \sim N(0, \sigma^2)$.
- N_σ - number of iteration in which [SC2] is satisfied.

σ	rel. er. SLS	rel. er. LS	I_σ
10^{-2}	0.0033	0.0028	651
10^{-1}	0.0355	0.0279	634
1	0.3193	0.2846	605

Comparing (GPS-LS) and (GPS-SLS)

- $m = 6, n = 2$.
- $\mathbf{a}_j$ and $\mathbf{x}$ randomly generated from $[-10, 10] \times [-10, 10]$.
- 1000 realizations.
- $d_j = \|\mathbf{x} - \mathbf{a}_j\| - r + \varepsilon_j, \varepsilon_j \sim N(0, \sigma^2)$.
- N_σ - number of iteration in which [SC2] is satisfied.

σ	rel. er. SLS	rel. er. LS	I_σ
10^{-2}	0.0033	0.0028	651
10^{-1}	0.0355	0.0279	634
1	0.3193	0.2846	605

- I_σ - no. of runs in which the LS solution is better than the SLS solution
- rel. er. SLS (LS) - average of $\frac{\|\mathbf{x}_{\rm SLS} - \mathbf{x}_{\rm true}\|}{\|\mathbf{x}_{\rm true}\|}$ $\left(\frac{\|\mathbf{x}_{\rm LS} - \mathbf{x}_{\rm true}\|}{\|\mathbf{x}_{\rm true}\|}\right)$.

The End

Thank you!