# Semidefinite programming for optimizing convex bodies under width constraints

Térence Bayen[1]
Didier Henrion[2,3]

[1] Math Inst, Univ Montpellier, FR
[2] LAAS-CNRS, Univ Toulouse, FR
[3] Fac Elec Engr, Czech Tech Univ Prague, CZ

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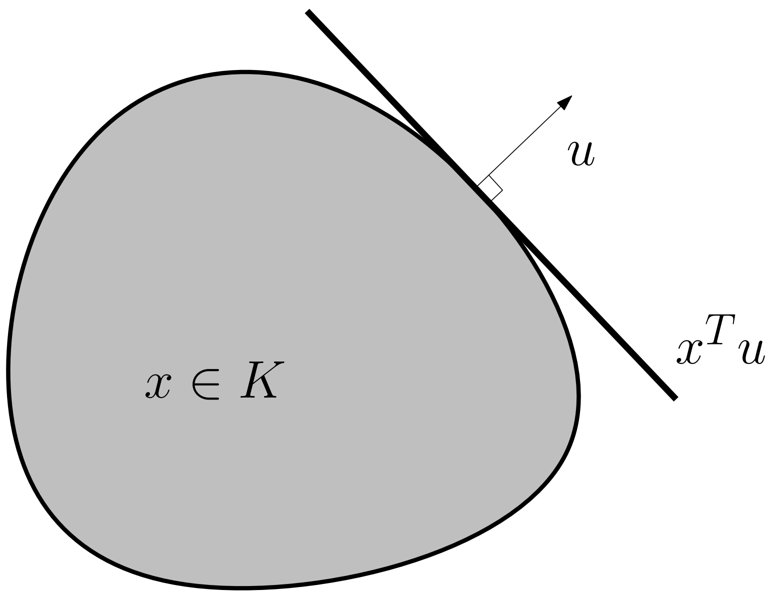
## Support function

Introduced by Hermann Minkowski (1902)

$$u \in \mathbb{R}^n \backslash \{0\} \mapsto p(u) = \begin{array}{ll} \max & x^T u \\ \text{s.t.} & x \in K \subset \mathbb{R}^n \end{array}$$




$$p \in C^1 \iff K \text{ strictly convex}$$

# Convexity

For $n = 2$ and $u(\theta) = (\cos\theta, \sin\theta)$ it holds

$$p(\theta) = p(u(\theta)) = \begin{array}{ll} \max & x_1 \cos\theta + x_2 \sin\theta \\ \text{s.t.} & x \in K. \end{array}$$

If $K$ is strictly convex, its boundary is parametrised as

$$\begin{array}{rcl} x_1(\theta) & = & p(\theta)\cos\theta - p'(\theta)\sin\theta \\ x_2(\theta) & = & p(\theta)\sin\theta + p'(\theta)\cos\theta \end{array}$$

If $p \in C^1$ then $p''$ exists almost everywhere and $K$ is convex if and only if

$$p(\theta) + p''(\theta) \geq 0$$

that is, its boundary has non-negative (radius of) curvature

## Area and perimeter

Using elementary differential geometry, we can express the area

$$\begin{aligned} A(p) &= \frac{1}{2}\int_0^{2\pi} p^2(\theta) - p'^2(\theta) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} p(\theta)(p(\theta) + p''(\theta)) d\theta \end{aligned}$$

and the perimeter

$$P(p) = \int_0^{2\pi} p(\theta) d\theta$$

of convex body $K$ as functions of $p$

## Fourier series

Let us write

$$p(\theta) = \sum_{k \in \mathbb{Z}} p_k e^{ik\theta}$$

with complex Fourier coefficients

$$p_k = \frac{1}{2\pi} \int_0^{2\pi} p(\theta) e^{-ik\theta} d\theta$$

Since $p(\theta) \in \mathbb{R}$ it holds $p_k = (p_{-k})^\star \in \mathbb{C}$

# Trigonometric polynomials

Denoting

$$p_k = a_k - ib_k$$

with $a_k, b_k \in \mathbb{R}$, we have

$$p(\theta) = a_0 + 2 \sum_{k \geq 1} (a_k \cos k\theta + b_k \sin k\theta)$$

Area

$$A(p) = \pi \sum_{k \in \mathbb{Z}} (1 - k^2)(a_k^2 + b_k^2)$$

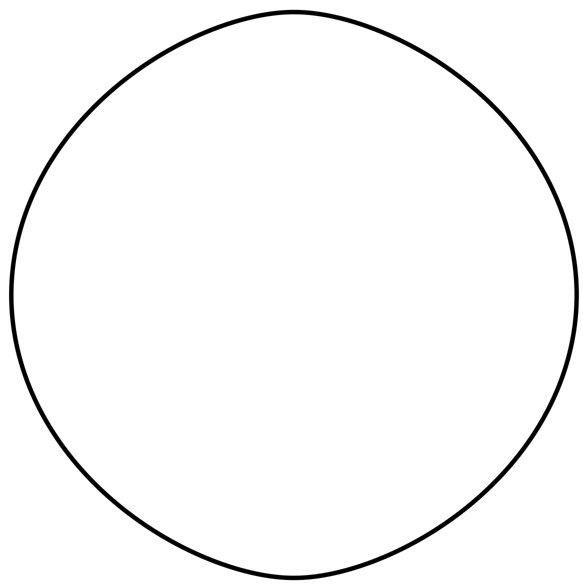
is quadratic and concave in $p$

Perimeter $P(p) = a_0$ is linear and convex in $p$

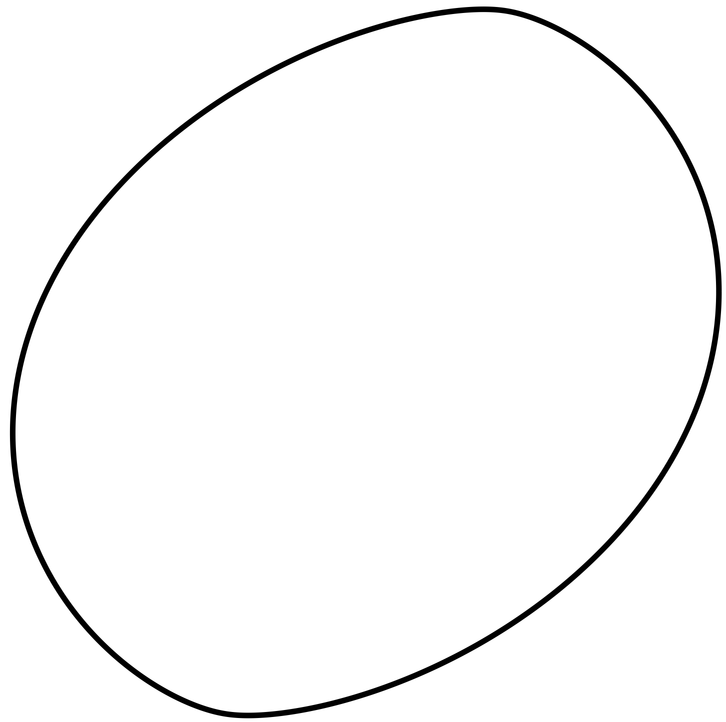
## Fourier coefficients shaping

$p_0 = a_0$ perimeter
$p_1 = a_1 - ib_1$ translation
$p_2, p_3, \ldots$ higher order harmonics

$$p(\theta) = 1 - \frac{1}{4}\cos 2\theta$$

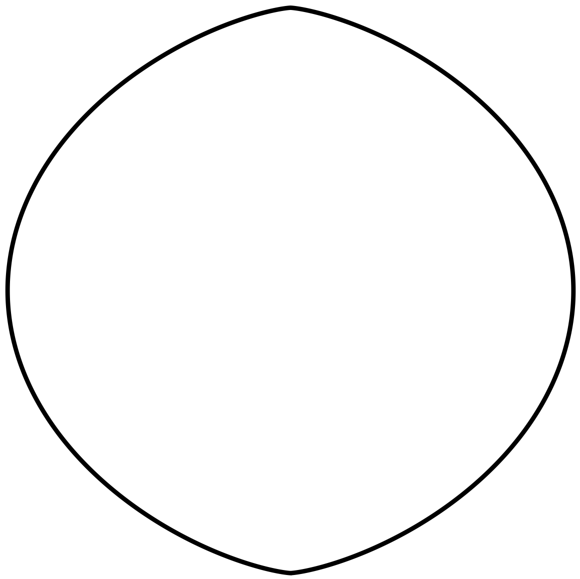


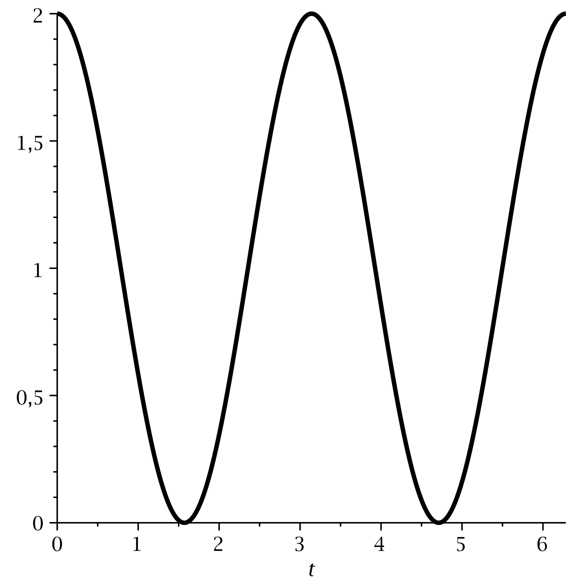
$$p(\theta) = 1 - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 2\theta$$



sine terms add a little twist

$$p(\theta) = 1 - \frac{1}{3}\cos 2\theta$$



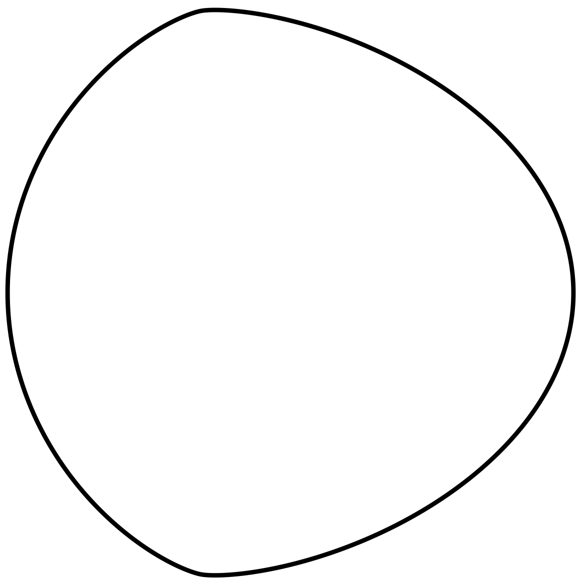


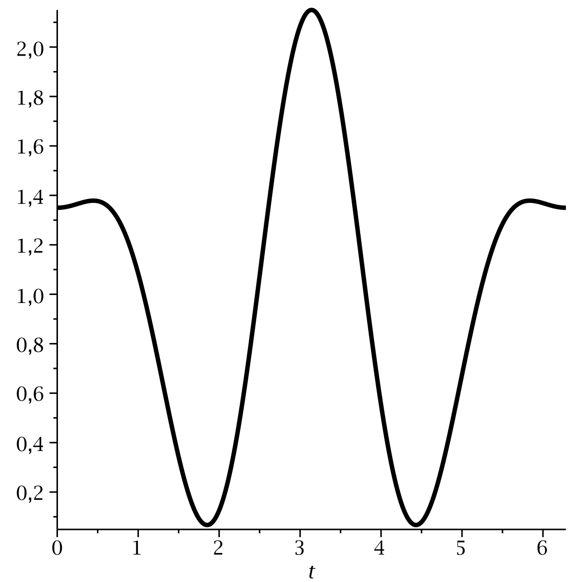

almost nonconvex

since curvature $p + p'' = 1 + \cos 2\theta$

vanishes at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$

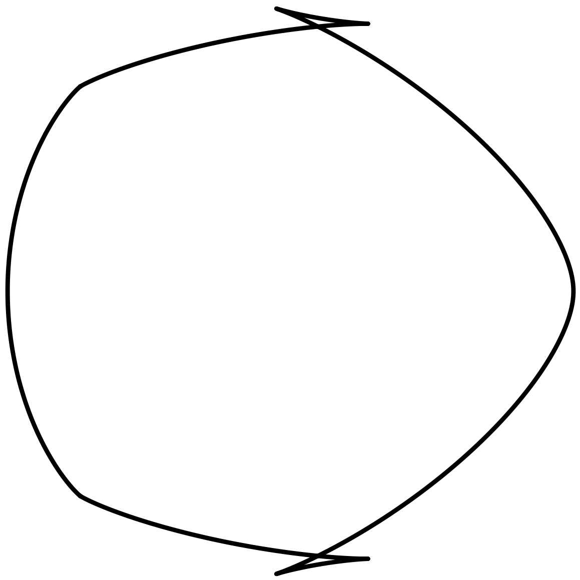
$$p(\theta) = 1 - \frac{1}{4}\cos 2\theta + \frac{1}{20}\cos 3\theta$$

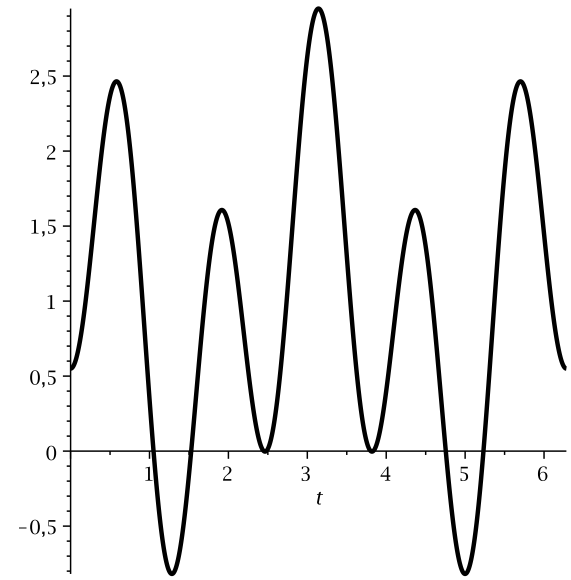





higher order Fourier coefficients
should have small magnitudes
to preserve convexity

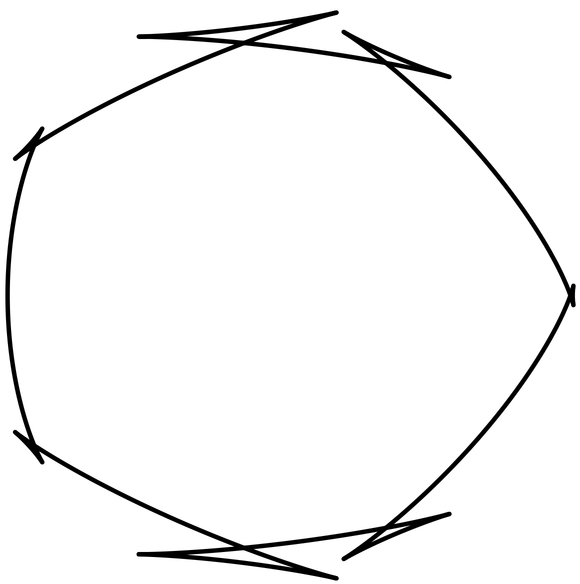
$$p(\theta) = 1 - \frac{1}{4}\cos 2\theta + \frac{1}{20}\cos 5\theta$$

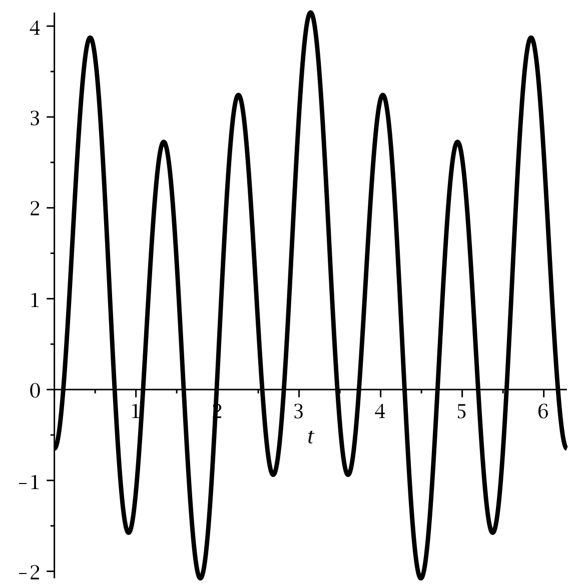





higher harmonics introduce
nasty nonconvex cusps

$$p(\theta) = 1 - \frac{1}{4}\cos 2\theta + \frac{1}{20}\cos 7\theta$$






more nasty nonconvex cusps

## Convexity constraint

$$\begin{aligned} q(\theta) &= p(\theta) + p''(\theta) \\ &= a_0 + 2\textstyle\sum_{k\geq 1}(1-k^2)(a_k \cos k\theta + b_k \sin k\theta) \\ &= \textstyle\sum_{k\in\mathbb{Z}} q_k e^{ik\theta} \end{aligned}$$

Riesz-Féjer theorem

$$q(\theta) \geq 0 \Longleftrightarrow q(\theta) = \sum_j |r_j(\theta)|^2$$

With truncated basis $b(\theta) = (1,\, e^{i\theta},\, e^{i2\theta}, \ldots, e^{iN\theta})$

$$q(\theta) = b^\star(\theta) Q b(\theta), \qquad Q = Q^\star \succeq 0$$

we obtain a semidefinite programming (SDP) problem

Sum-of-squares factors $r_j(\theta)$ obtained by
Schur decomposition of matrix $Q$

# Toeplitz SDP

For truncation order $N = 2$

$$
\begin{aligned}
q(\theta) &= \begin{pmatrix} 1 & e^{-i\theta} & e^{-i2\theta} \end{pmatrix} \overbrace{(Q_R + iQ_I)}^{Q=Q^\star} \begin{pmatrix} 1 & e^{i\theta} & e^{i2\theta} \end{pmatrix}^T \\
&= \text{trace } (Q_R + iQ_I) \begin{pmatrix} 1 & e^{-i\theta} & e^{-i2\theta} \\ e^{i\theta} & 1 & e^{-i\theta} \\ e^{2i\theta} & e^{i\theta} & 1 \end{pmatrix} \\
&= \text{trace}\, Q_R \begin{pmatrix} 1 & \cos\theta & \cos 2\theta \\ \cos\theta & 1 & \cos\theta \\ \cos 2\theta & \cos\theta & 1 \end{pmatrix} + \\
&\quad\ \text{trace}\, Q_I \begin{pmatrix} 0 & \sin\theta & \sin 2\theta \\ -\sin\theta & 0 & \sin\theta \\ -\sin 2\theta & -\sin\theta & 0 \end{pmatrix} \\
&= \alpha_0 + 2\alpha_1 \cos\theta + 2\alpha_2 \cos 2\theta + 2\beta_1 \sin\theta + 2\beta_2 \sin 2\theta
\end{aligned}
$$

Real matrices $Q_R = Q_R'$, $Q_I = -Q_I'$ and coeffs $\alpha_k$, $\beta_k$ are linearly constrained

# Toeplitz SDP

Introduce Toeplitz basis matrices

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

to define linear section

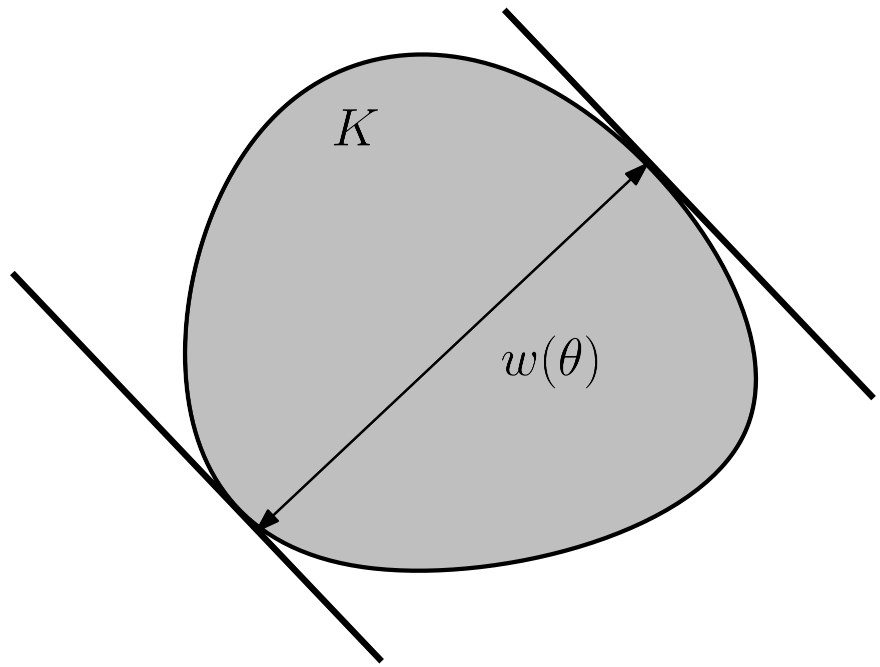
$$\begin{aligned} \alpha_0 &= \text{trace}\, Q_R \\ \alpha_1 &= \text{trace}\, (T_1 + T_1')Q_R \\ \alpha_2 &= \text{trace}\, (T_2 + T_2')Q_R \\ \beta_1 &= \text{trace}\, (T_1 - T_1')Q_I \\ \beta_2 &= \text{trace}\, (T_2 - T_2')Q_I \end{aligned}$$

of the convex cone of positive semidefinite matrices

$$Q_R + iQ_I \succeq 0 \Longleftrightarrow \begin{pmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{pmatrix} \succeq 0$$

## Width

Distance between opposite parallel supporting lines




$$w(\theta) = p(\theta) + p(\theta + \pi)$$

for a body $K$ of support function $p$

## Curves of constant width

Convex planar shape whose width is a constant:

$$p(\theta) + p(\theta + \pi) = 1$$
$$p(\theta) + p''(\theta) \geq 0$$

Linear constraint on support function $p$

From the isoperimetric inequality

$$4\pi \text{ area} \leq \text{length}^2$$

the maximal area curve of constant width is a circle

Less trivial is the minimal area curve of constant width..

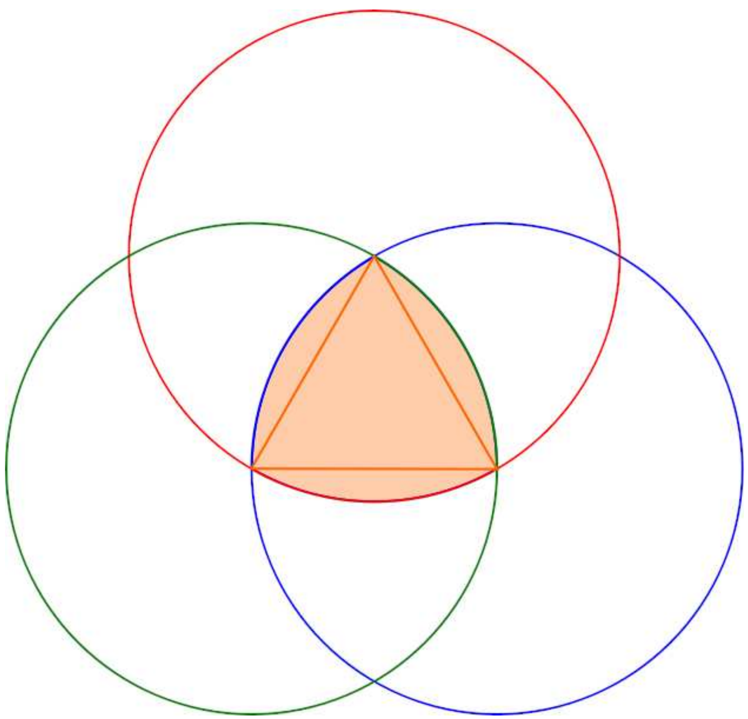
# Examples of curves of constant width

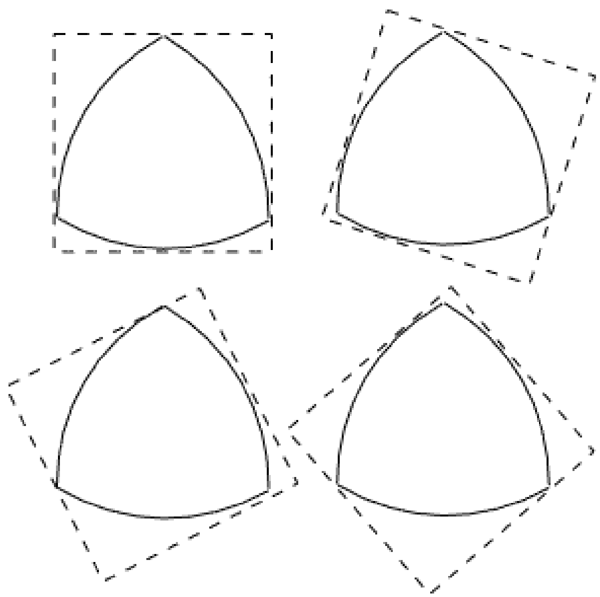
Heptagonal British 20p and 50p coins


Coin machine always measures correct coin diameter independently of coin insertion angle

# Reuleaux triangle

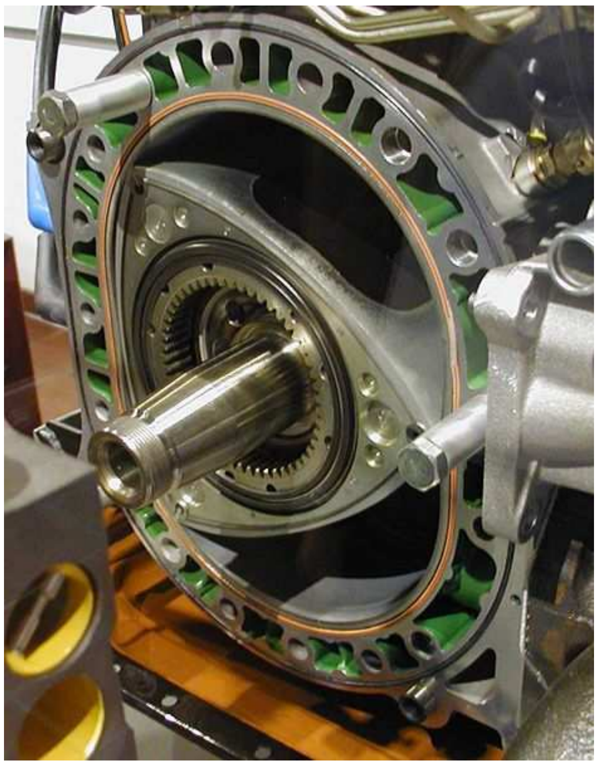
Minimal area curve of constant width is a union of three circular arcs, found by Franz Reuleaux (1829-1905), a German mechanical engineer





## Application in mechanics

Reuleaux triangle used in Wankel engine (converting pressure into rotating motion) equipping Mazda RX8



## SDP formulation of convex curve area minimisation

$$\begin{array}{ll} \min_p & A(p) = \pi \sum_k (1-k^2)|p_k|^2 \\ \text{s.t.} & p(\theta) + p''(\theta) \geq 0 \\ & p(\theta) + p(\theta+\pi) = 1, \quad \forall\theta \in [0,\, 2\pi] \end{array}$$

Convex SDP constraints but..
concave quadratic objective function

Difficult concave minimisation problem,
with potentially many local or global optima

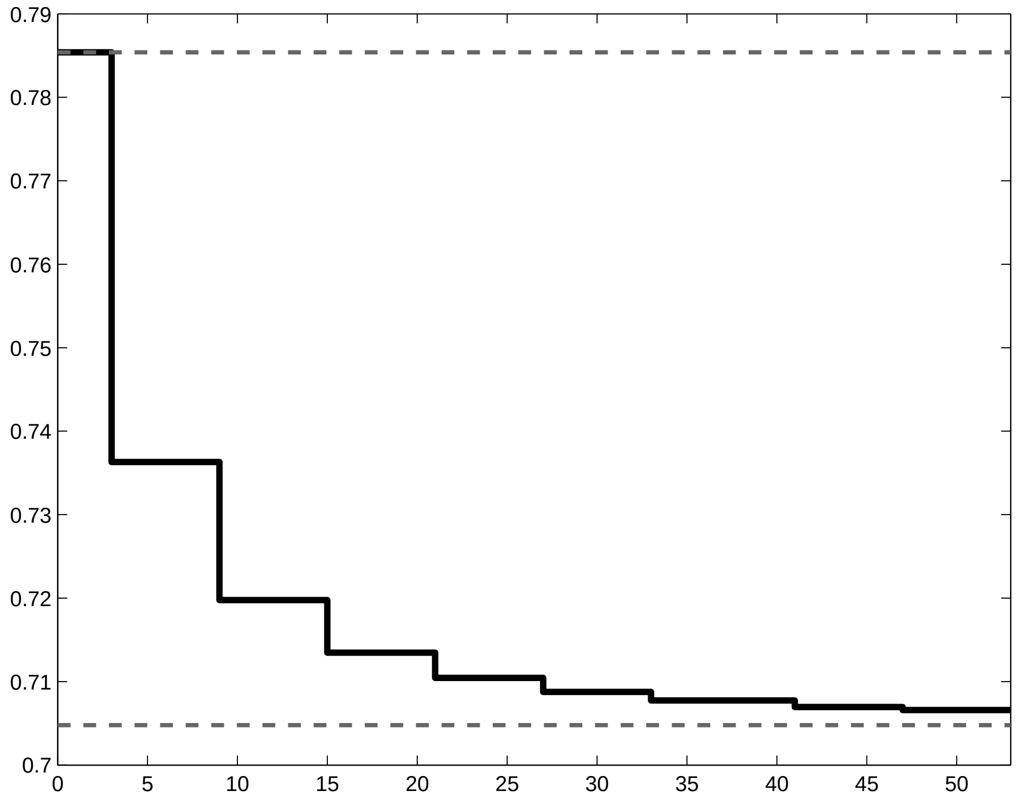
Global optima lie on the boundary of feasible set

Use PENBMI by Michal Kočvara and Michael Stingl,
an implementation of a penalty augmented Lagrangian
method with Matlab YALMIP interface

## Minimal area as function of truncation order

Upper limit = unit width circle $\frac{\pi}{4} \approx 0.78540$

Lower limit = Reuleaux triangle $\frac{\pi-\sqrt{3}}{2} \approx 0.70477$



## $N = 3$

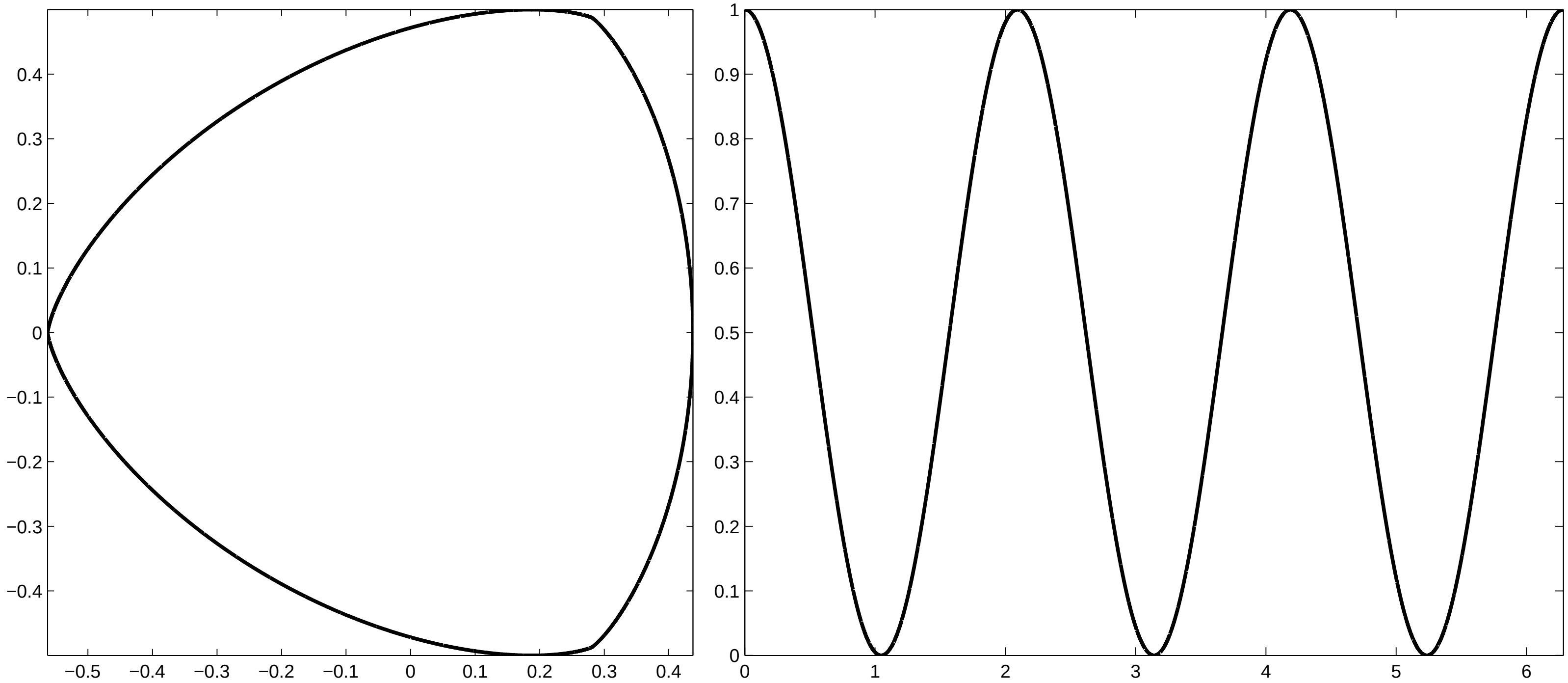
$p(\theta) = \frac{1}{2} - \frac{1}{16} \cos 3\theta$

$A(p) = \frac{15\pi}{64} \approx 0.73631$

Algebraic curve of degree $2(N+1) = 8$
with defining polynomial

$$
\begin{aligned}
&-182284263 + 1863254016 x_1 x_2^2 - 4267966464 x_1^2 x_2^2 \\
&+1269789696 x_1^2 - 621084672 x_1^3 - 2133983232 x_1^4 \\
&+12884901888 x_2^6 x_1 - 13589544960 x_2^4 x_1 + 17179869184 x_2^6 x_1^2 \\
&+8254390272 x_2^4 x_1^2 + 21474836480 x_2^4 x_1^3 - 9059696640 x_2^2 x_1^3 \\
&+25769803776 x_2^4 x_1^4 - 7851737088 x_2^2 x_1^4 + 4294967296 x_2^2 x_1^5 \\
&+4529848320 x_1^5 + 17179869184 x_2^2 x_1^6 + 603979776 x_1^6 \\
&-4294967296 x_1^7 + 4294967296 x_1^8 + 1269789696 x_2^2 \\
&-2133983232 x_2^4 + 4294967296 x_2^8 - 469762048 x_2^6
\end{aligned}
$$

$N = 3$



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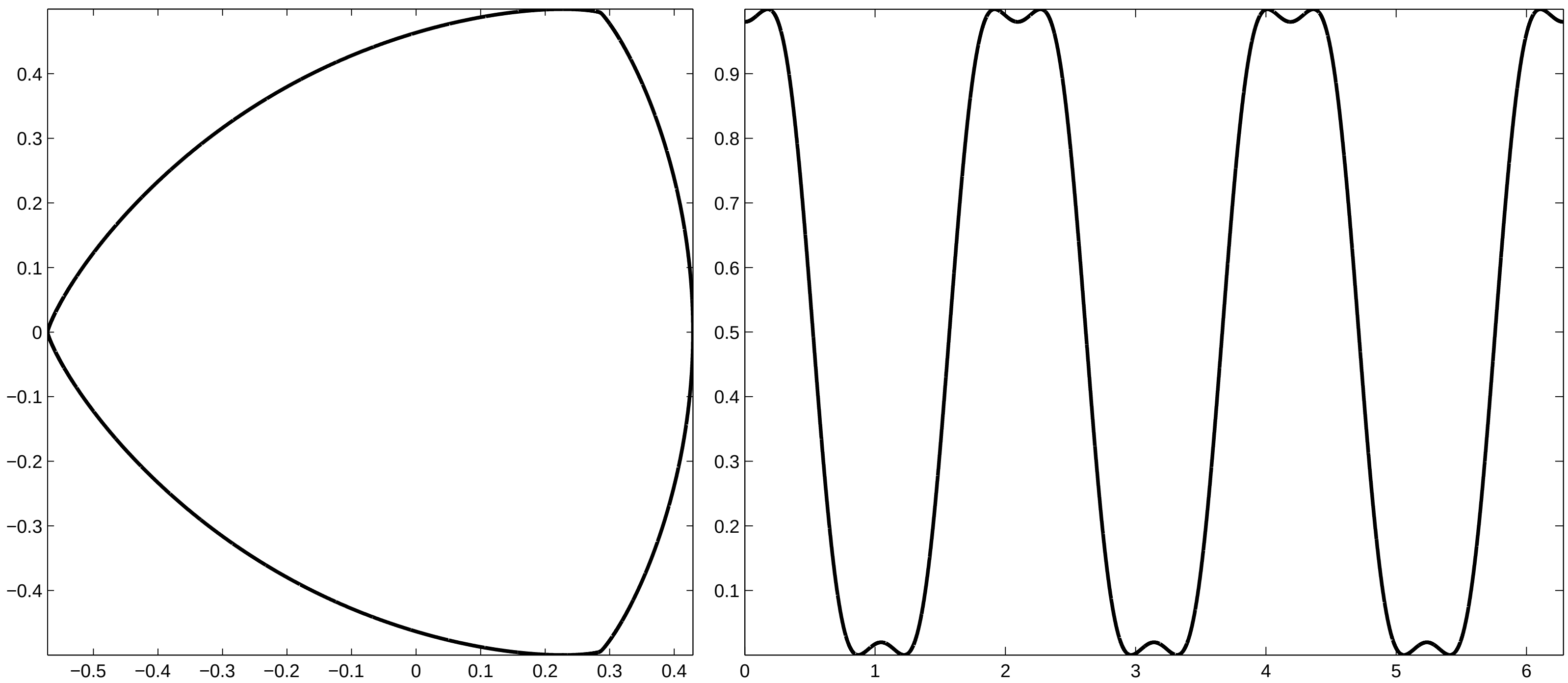
$N = 9$



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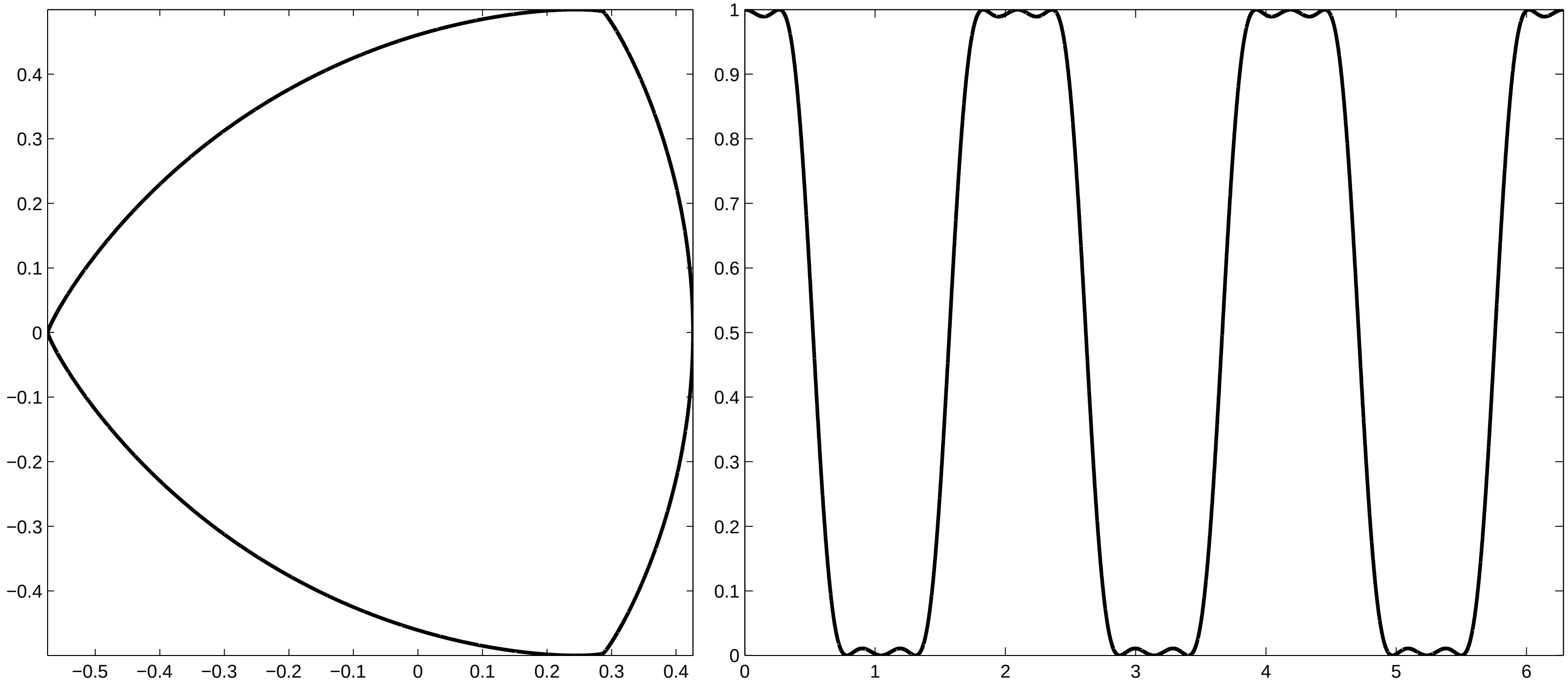
$N = 15$



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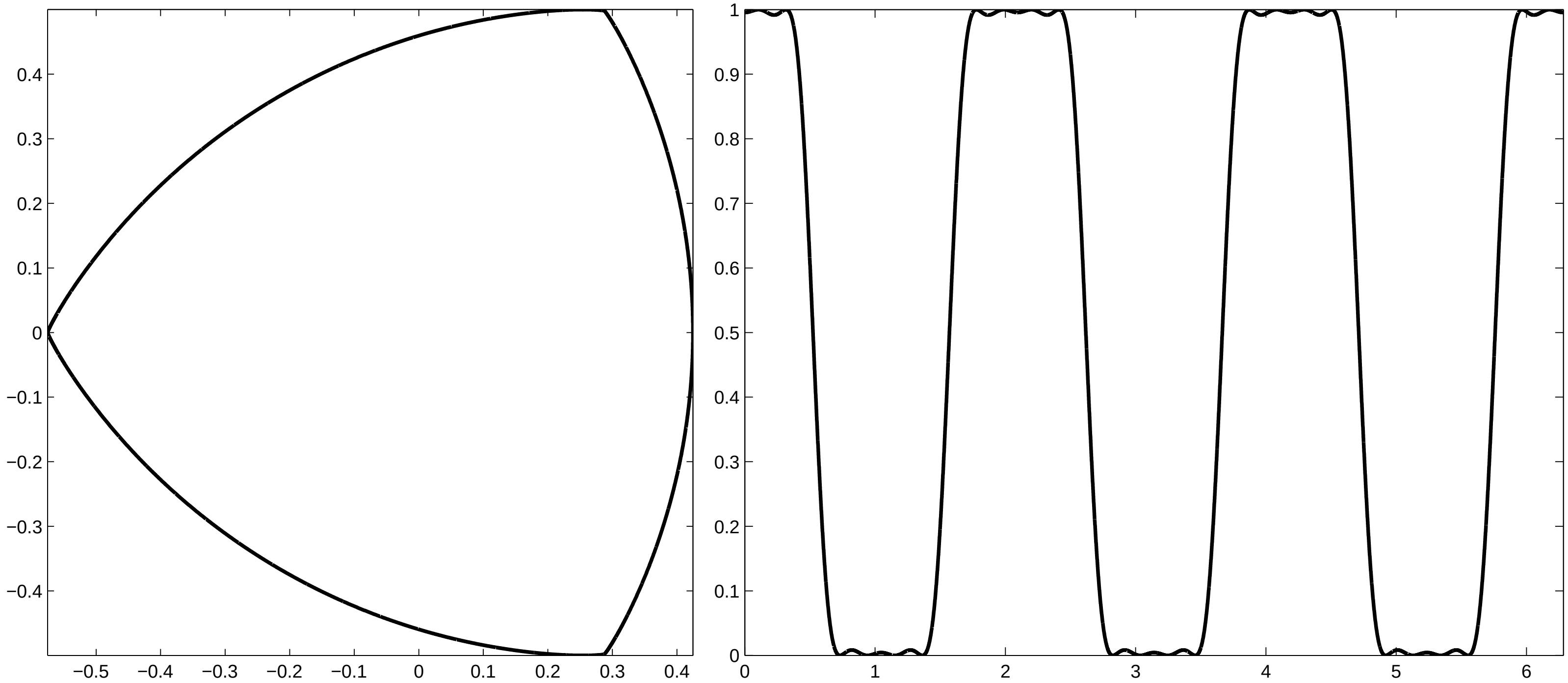
$N = 21$



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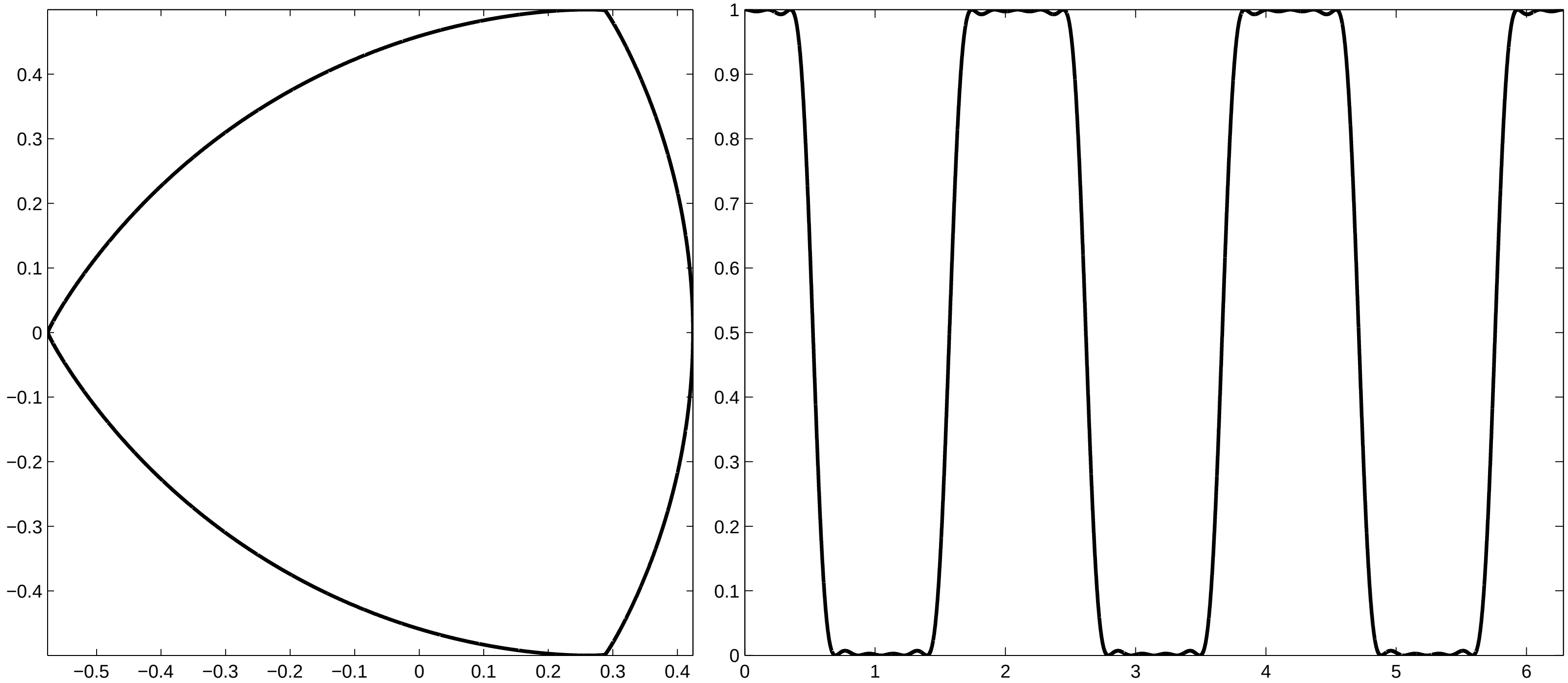
$N = 27$



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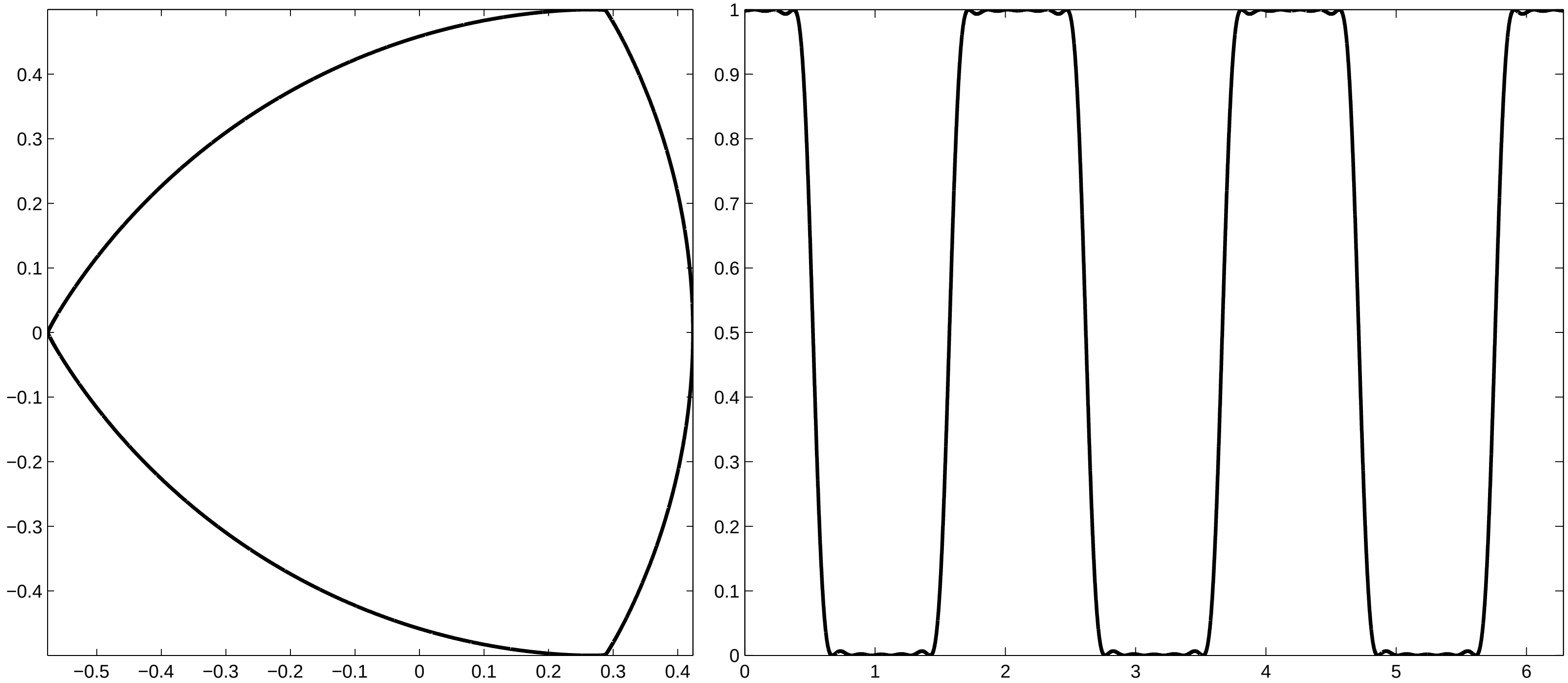
$N = 33$



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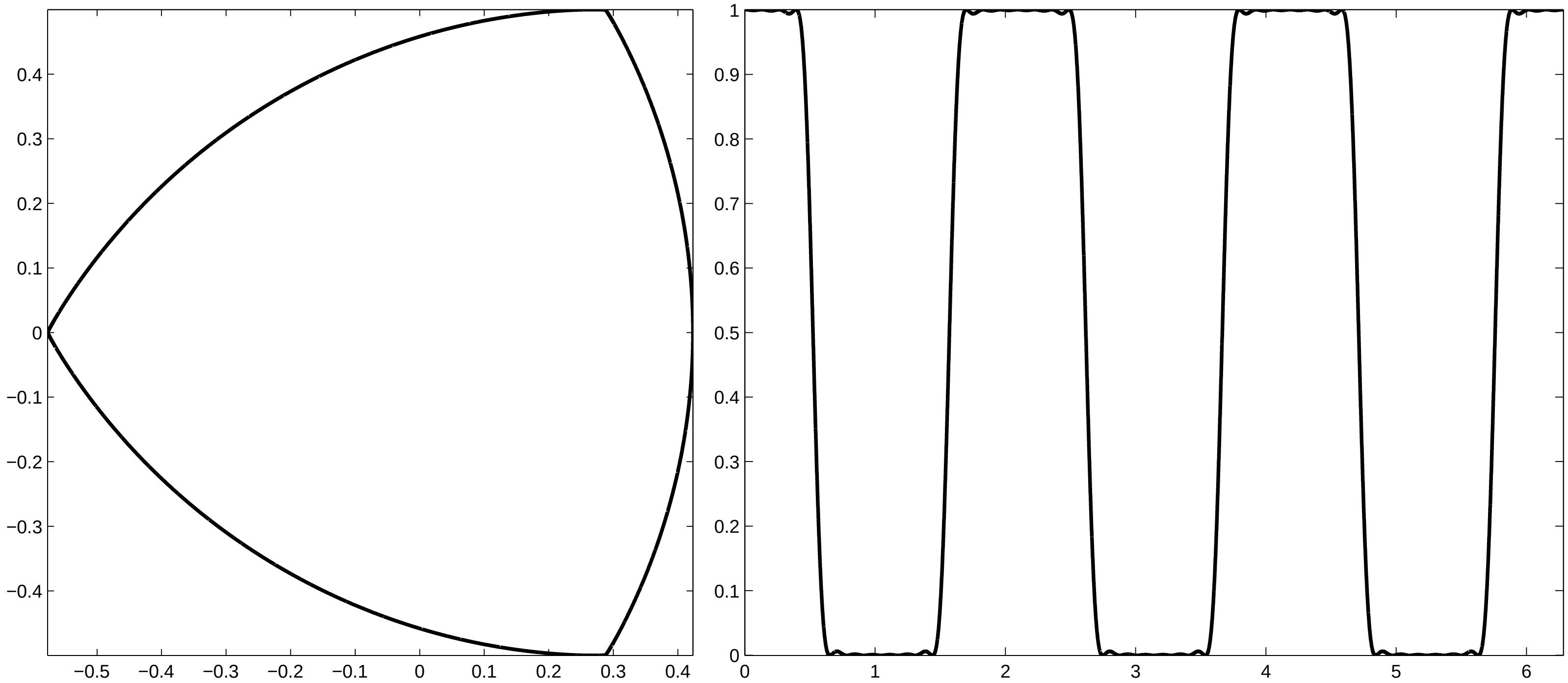
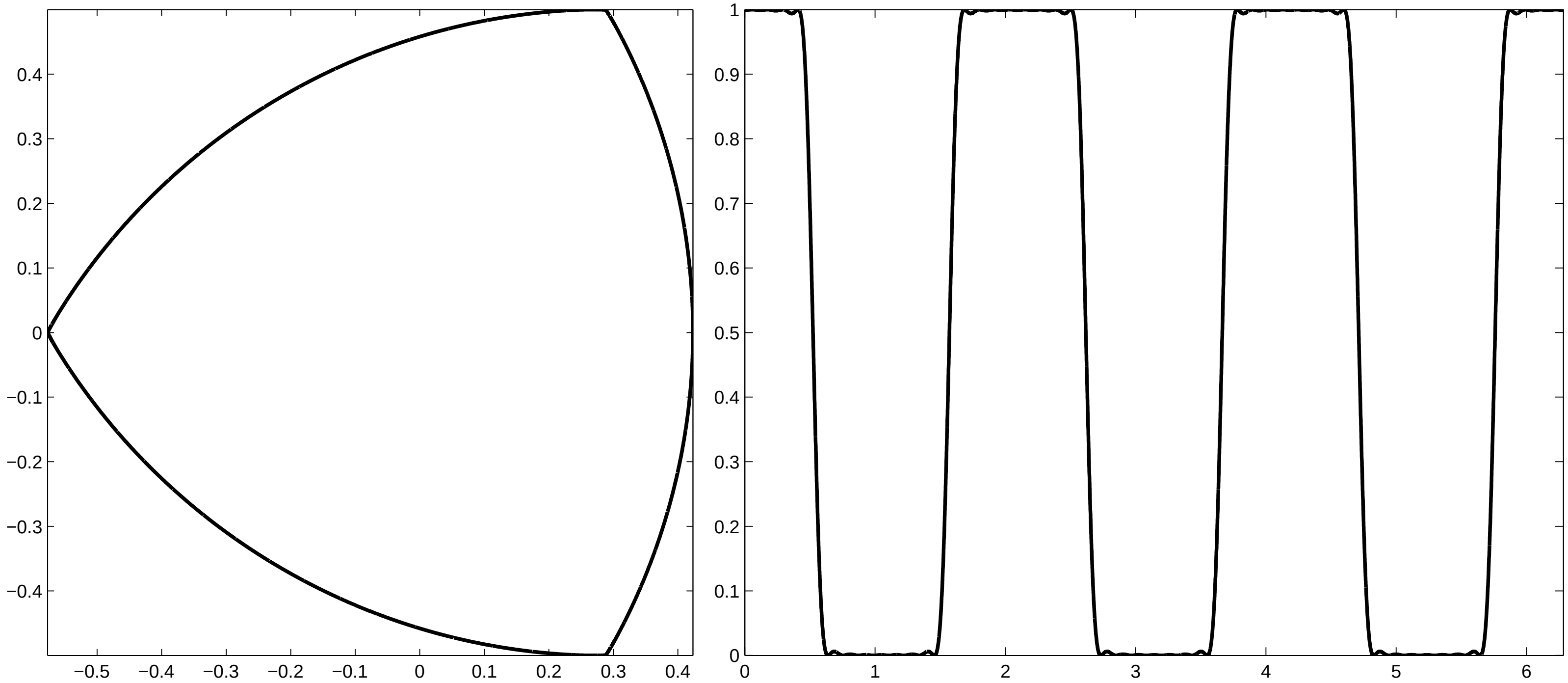


$N = 47$



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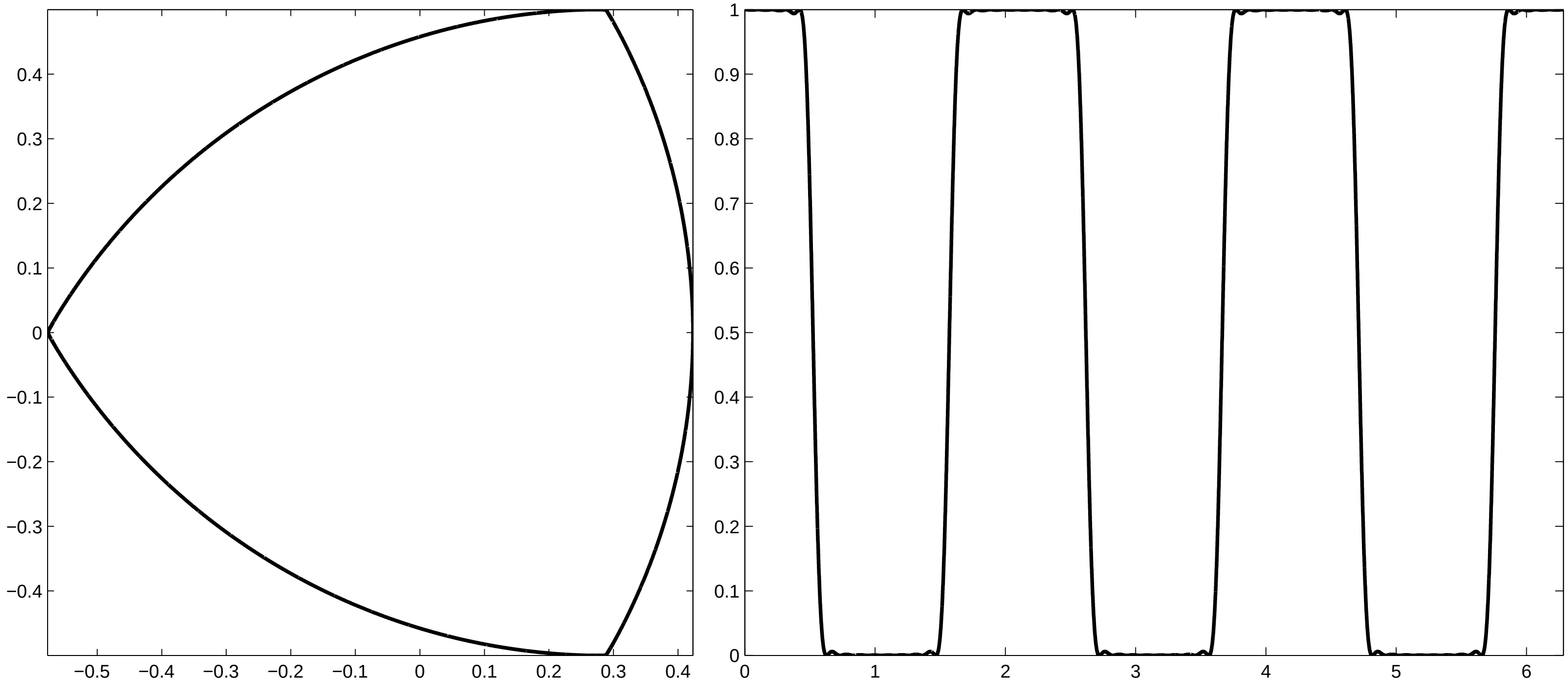
$N = 53$



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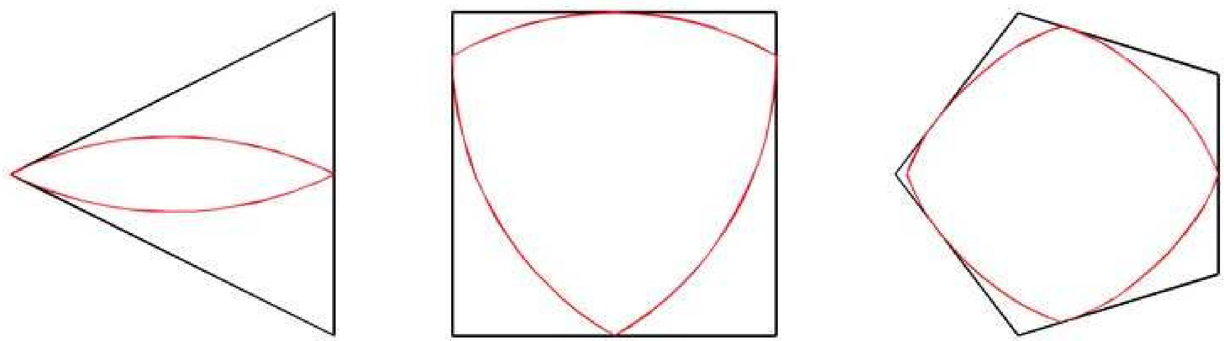
# Rotors

Generalization of constant width body
inscribed in a regular $n$-side polygon

$$p(\theta) - 2\cos\frac{2\pi}{n}p(\theta + \frac{2\pi}{n}) + p(\theta + \frac{4\pi}{n}) = 1$$
$$p(\theta) + p''(\theta) \geq 0$$

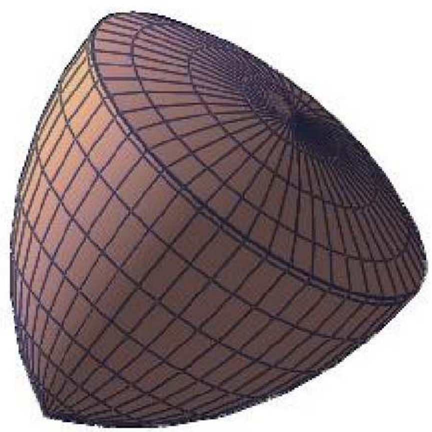
Reuleaux triangle $n = 4$



## Constant width bodies of revolution

$$p(\theta) + p(\theta + \pi) = 1$$
$$p(\theta) - p(\theta - \pi) = 0$$
$$p(\theta) + p''(\theta) \geq 0$$

$$\begin{aligned} x(\theta, \phi) &= (p(\theta)\cos\theta - p'(\theta)\sin\theta)\cos\phi \\ y(\theta, \phi) &= (p(\theta)\cos\theta - p'(\theta)\sin\theta)\sin\phi \\ z(\theta, \phi) &= p(\theta)\sin\theta + p'(\theta)\cos\theta \end{aligned}$$



## Convex bodies in $\mathbb{R}^3$

Condition of convexity more intricate than in $\mathbb{R}^2$..
Spherical harmonics, 2nd fundamental form, Hessian ?

Positive semidefinite bivariate trigonometric matrix =
SDP formulation with a gap vanishing asymptotically

Open problem: minimum volume constant width body