Optimal Subsets of Grassmann Spaces and Applications

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Outline

- Motivations
- Grassmann spaces, geometry and harmonic analysis
- Grassmann codes
- Grassmann designs

Motivations

- Lattices: When the configuration of the lines supporting the shortest vectors is “nice”, the associated sphere packing is also “nice”. Extends to the *m*-dimensional minimal sections of the lattice.
- Related to the notion of designs (Delsarte, Seidel, Bannai, Sloane, Venkov,...).
- Coding theory: In multi-antenna systems of communication, the information is carried by matrices $X \in \mathbb{R}^{m \times n}$ (m is the number of antennas) and the channel changes X to $Y = HX + W$. If H is unknown, Y and X may be identified with the linear span of their rows. It leads to the study of “codes” in Grassmann space, in which the linear subspaces are “fare away one from each other”.
- Chordal distance and related Grassmann packings (Sloane, Hardin, Calderbank, Shor, Barg, Ashikhmin,..)

Motivations

- Signal processing: Frames and fusion frames. Work in progress with Martin Ehler, University of Maryland.
- Question (Optimal 2010, Nashville): what is the minimum of

  $$\sum_{i,j=1}^{N} |\langle P_{V_i}, P_{V_j} \rangle|^p$$

  when $\{V_1, \ldots, V_N\}$ is a collection of linear subspaces of $\mathbb{R}^n$?
- Frames are sets $\{e_1, \ldots, e_N\}$ of unit vectors such that a signal x can be efficiently recovered from $\{\langle x, e_i \rangle, 1 \leq i \leq N\}$.
- Tight frames correspond to the minimization of

  $$\sum_{i,j=1}^{N} |\langle e_i, e_j \rangle|^2$$

Motivations

- If $\{e_1, \dots, e_N\}$ is a tight frame, then

$$x = \frac{n}{N} \sum_{i=1}^{N} \langle x, e_i \rangle e_i$$

$$\langle x, x \rangle = \frac{n}{N} \sum_{i=1}^{N} \langle x, e_i \rangle^2$$

$$\langle x, y \rangle = \frac{n}{N} \sum_{i=1}^{N} \langle x, e_i \rangle \langle y, e_i \rangle$$

- Similar notions when $\{e_1, \dots, e_N\}$ is replaced by $\{V_1, \dots V_N\}$ a collection of linear subspaces and x is projected on each V_i (fusion frames, tight fusion frames).
- Tight = 2-designs in Grassmann space!

Grassmann spaces

- $\mathcal{G}_{m,n}$ is the set of m-dimensional subspaces of $\mathbb{R}^n$, $m \leq n/2$.
- If $m = 1$, $\mathcal{G}_{1,n}$ is the projective space of dimension $n - 1$.

$$\mathcal{G}_{1,n} = \mathcal{P}(\mathbb{R}^n) = S^{n-1}/\{\pm 1\}$$

- In general, $\mathcal{G}_{m,n}$ is a compact manifold of dimension $mn - m^2$.
- The orthogonal group $O(\mathbb{R}^n)$ acts homogeneously on $\mathcal{G}_{m,n}$ and

$$\mathcal{G}_{m,n} \simeq O(\mathbb{R}^n)/O(\mathbb{R}^m) \times O(\mathbb{R}^{n-m}).$$

Principal angles

- If $m = 1$, $p = \mathbb{R}e$, $e \cdot e = 1$, $p' = \mathbb{R}e'$, $e' \cdot e' = 1$, the principal angle $\theta(p, p') \in [0, \pi/2]$ is defined by:

$$\cos\theta(p, p') = |e \cdot e'|.$$

- In general, m principal angles $\theta_i(p, p') \in [0, \pi/2]$, $1 \leq i \leq m$ are defined the following way (all vectors are unit vectors):

$$\begin{aligned}
\cos\theta_1(p, p') \quad &= \max\Big\{|e \cdot e'| : \quad e \in p,\ e' \in p',\Big\} \\
&=: |e_1 \cdot e'_1| \\
\cos\theta_2(p, p') \quad &= \max\Big\{|e \cdot e'| : \quad e \in p,\ e' \in p', \\
&\qquad\qquad\qquad\qquad e \perp e_1,\ e' \perp e'_1\Big\} \\
&=: |e_2 \cdot e'_2| \\
\ldots \quad & \ldots \qquad\qquad\qquad\qquad \ldots \\
\cos\theta_m(p, p') \quad &= \max\Big\{|e \cdot e'| : \quad e \in p,\ e' \in p', \\
&\qquad\qquad\qquad\qquad e \perp e_i,\ e' \perp e'_i,\ 1 \leq i \leq m-1\Big\} \\
&=: |e_m \cdot e'_m|
\end{aligned}$$

Principal angles

- Alternatively, the numbers $\cos\theta_i(p, p')$ are the singular values of the matrix
$$\left(f_i \cdot f'_j\right)_{1\leq i,j\leq m}$$
where $(f_1, \ldots, f_m)$ and $(f'_1, \ldots, f'_m)$ are any orthonormal basis of respectively p and p', and the vectors e_i, e'_i are the corresponding singular vectors.
- If P_p denotes the orthogonal projection on p, the numbers
$$y_i(p, p') := \cos^2\theta_i(p, p')$$
are the m largest eigenvalues of $P_p \circ P_{p'}$.
- We have
$$\langle P_p, P_{p'}\rangle = \sum_{i=1}^{m} y_i(p, p').$$

Principal angles and the action of $O(\mathbb{R}^n)$

- If $g \in O(\mathbb{R}^n)$, (p, p') and $(g(p), g(p'))$ have the same principal angles.
- Conversely, if $\theta_i(p, p') = \theta_i(p_1, p'_1)$ for all $1 \leq i \leq m$ then there exists $g \in O(\mathbb{R}^n)$ such that $(p_1, p'_1) = (g(p), g(p'))$.
- The spaces $\mathcal{G}_{m,n}$ are two-point homogeneous only if $m = 1$ (projective space).
- In general, $\mathcal{G}_{m,n}$ is a symmetric space (i.e. $O(\mathbb{R}^n)$ exchanges any pair (p, p')).

Measures

- $\mathcal{G}_{m,n}$ inherits a measure μ from the Haar measure on $O(\mathbb{R}^n)$, normalized by $\mu(\mathcal{G}_{m,n}) = 1$.
- We have the Hilbert space

$$L^2(\mathcal{G}_{m,n}) = \left\{ f : \int_{\mathcal{G}_{m,n}} |f(p)|^2 d\mu(p) < +\infty \right\}.$$

- The measure μ induces a measure on the y_i:

$$d\mu(y_1, \ldots, y_m) = \lambda \prod_{1 \le i < j \le m} |y_i - y_j| \prod_{1 \le i \le m} y_i^{-1/2} (1 - y_i)^{n/2 - m - 1/2} dy_i$$

supported by $[0, 1]^m$.

The decomposition of $L^2(\mathcal{G}_{m,n})$

- $L^2(\mathcal{G}_{m,n})$ is a unitary representation of $O(\mathbb{R}^n)$ with the standard action:
$$(gf)(p) := f(g^{-1}(p)).$$
- It decomposes into the orthogonal sum of $O(\mathbb{R}^n)$-irreducible subspaces:
$$L^2(\mathcal{G}_{m,n}) = \bigoplus_{\mu} H^{2\mu}_{m,n}$$
where the sum is over the "partitions" $\mu = \mu_1 \geq \cdots \geq \mu_m \geq 0$.
- $H^{2\mu}_{m,n} \simeq V^{2\mu}_n$ where $V^{2\mu}_n$ is an irreducible representation of $O(\mathbb{R}^n)$ canonically associated with 2μ.
- Example: $\mu = (k)$, $V^{(k)}_n \simeq \mathrm{Harm}_k$ are the representations occuring in $L^2(S^{n-1})$.

Zonal spherical functions

- Computed by James and Constantine, 1974.
- To each $H_{m,n}^{2\mu}$ is canonically associated a symmetric polynomial $P_\mu^{m,n} = P_\mu(y_1, \dots, y_m)$, such that

$$p \mapsto P_\mu(\underline{y}(p, p')) \in H_{m,n}^{2\mu}.$$

- The P_μ are pairwise orthogonal for the induced inner product $d\mu(y_1, \dots, y_m)$ (multivariate Jacobi polynomials) and have degree $|\mu| = \sum_{i=1}^{m} \mu_i$.
- They can be computed explicitely as eigenvalues of the Laplace-Beltrami operator.

Zonal spherical functions

- Examples: up to a positive constant, determined by $P_\mu(1, \dots, 1) = 1$,

$$
\begin{aligned}
P_{(0)} &= 1 \\
P_{(1)} &= \sum y_i - \frac{m^2}{n} \\
P_{(11)} &= \sum y_i y_j - \frac{(m-1)^2}{n-2} \sum y_i + \frac{m^2(m-1)^2}{2(n-1)(n-2)} \\
P_{(2)} &= \sum y_i^2 + \frac{2}{3} \sum y_i y_j + \dots
\end{aligned}
$$

- Main property: they characterize the positive definite functions on $\mathcal{G}_{m,n}$ which are $O(\mathbb{R}^n)$-invariant.

Positive definite functions

- $F \in \mathcal{C}({\mathcal{G}_{m,n}}^2)$ is said to be positive definite ($F \succeq 0$) if for all $k \geq 1$, and all $p_1, \dots, p_k \in \mathcal{G}_{m,n}$,
$$\Big(F(p_i, p_j)\Big)_{1 \leq i,j \leq k} \succeq 0$$
- $F \in \mathcal{C}({\mathcal{G}_{m,n}}^2)$ is said to be $O(\mathbb{R}^n)$-invariant if
$$F(g(p), g(p')) = F(p, p') \text{ for all } g \in O(\mathbb{R}^n).$$
- $F \succeq 0$ and F is $O(\mathbb{R}^n)$-invariant iff
$$F(p, p') = \sum_{\mu} f_\mu P_\mu(\underline{y}(p, p')) \text{ with, for all } \mu,\ f_\mu \geq 0.$$

Application: upper bounds for codes in $\mathcal{G}_{m,n}$

- Let $C \subset \mathcal{G}_{m,n}$ be such that for all $(p, p') \in C^2$, $p \neq p'$, $\underline{y}(p, p') \notin \Omega \subset [0, 1]^m$.
- Example: C avoids small values of the chordal distance

$$d_c(p, p') = \sqrt{\sum_{i=1}^{m} \sin^2 \theta_i(p, p')} = \sqrt{m - \sum_{i=1}^{m} y_i(p, p')}$$

- Example: C avoids small values of the diversity product

$$d_\pi(p, p') = \Big(\prod_{i=1}^{m} \sin \theta_i(p, p') \Big)^{1/m} = \Big(\prod_{i=1}^{m} (1 - y_i(p, p')) \Big)^{1/2m}$$

Application: upper bounds for codes in $\mathcal{G}_{m,n}$

- If C avoids $\Omega \subset [0,1]^m$, then

$$|C| \leq \inf\Big\{\, t : F \succeq 0, \quad \begin{aligned} &F(p,p) \leq t-1, \\ &F(p,p') \leq -1 \text{ for all } (p,p') : \underline{y}(p,p') \notin \Omega \,\Big\}, \end{aligned}$$

- Proof: if (F,t) is feasible, and C avoids Ω,

$$\begin{aligned} 0 &\leq \sum_{(p,p')\in C^2} F(p,p') = \sum_{p\in C} F(p,p) + \sum_{p\neq p'} F(p,p') \\ 0 &\leq (t-1)|C| - (|C|^2 - |C|) \\ 0 &\leq t - |C|. \end{aligned}$$

- Since this program is $O(\mathbb{R}^n)$-invariant, by standard argument we can assume moreover that F itself is $O(\mathbb{R}^n)$-invariant.

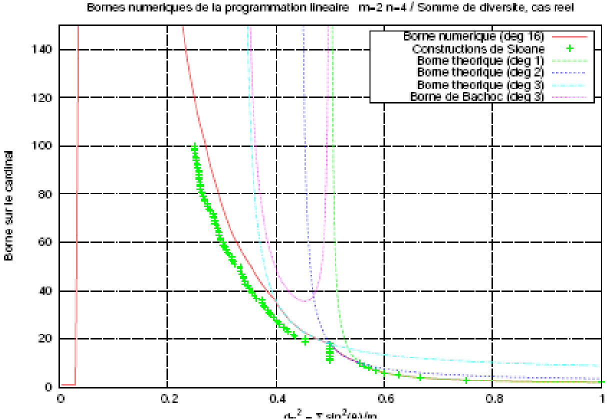
Application: upper bounds for codes in $\mathcal{G}_{m,n}$

- Employing the characterization of invariant positive definite functions, we obtain e.g. in the case of large chordal distance $d_c \geq \sqrt{m-s}$:

$$|C| \leq \inf \Big\{ \sum_\mu f_\mu : \quad \begin{array}{l} f_\mu \geq 0, \ f_0 = 1 \\ \sum_\mu f_\mu P_\mu(y_1, \dots, y_m) \leq 0 \text{ for } \sum y_i \leq s \Big\}, \end{array}$$

- Relaxing to bounded degree polynomials, and to SOS conditions, one obtains good numerical upper bounds for $|C|$. (B. chordal dist 2006, J. Creignou chordal and product, real and complex 2008).
- Clever choice of feasible solutions, i.e. of polynomials $P = \sum_\mu f_\mu P_\mu$ lead to explicit bounds for large distances. (B., J. Creignou for degree up to 3).

Cited from Jean Creignou Phd thesis 2008



Explicit bounds for the chordal distance

- The simplex bound (Conway, Hardin, Sloane 1996) obtained from

$$\begin{aligned} \mathcal{G}_{m,n} &\to S^{(n-1)(n+2)/2} \\ p &\mapsto P_p \in \{S \in S\mathbb{R}^{n\times n} : \mathrm{trace}(S) = m\} \end{aligned}$$

  and the Rankin bound on unit sphere (also LP with degree 1).

$$\text{if } s < \frac{m^2}{n}, \quad |C| < \frac{m-s}{m^2/n - s}.$$

- A spectral bound (B. 2006): if $s \leq \lambda_{k-1}$

$$|C| \leq \frac{4\sum_{|\kappa|=k} d_{2\kappa} a_\kappa}{m - \lambda_k}$$

  where λ_k is the largest eigenvalue of : $P \mapsto (\sum_i y_i)P$ trucated to polynomials of degree at most k.

An asymptotic bound

- (B. Ben-Haim, Litsyn 2008) Obtained from a map

$$\begin{aligned} \phi : \mathcal{G}_{m,n} &\to S^{mn-1} \\ p &\mapsto (e_1|e_2|\ldots|e_m)/\sqrt{m} \end{aligned}$$

and the inequality

$$\langle \phi(p), \phi(p')\rangle^2 \leq \frac{\sum y_i(p,p')}{m} \leq \frac{s}{m}$$

joined with known asymptotic bounds for spherical codes (Kabatiansky, Levenshtein 1978).

$$\frac{1}{n}\log|C| \simeq\leq_{n\to+\infty} m\big((1+\rho)\log(1+\rho) - \rho\log(\rho)\big)$$

where

$$\rho = \frac{1}{2}\big(-1 + (1-\frac{s}{m})^{-1/2}\big).$$

Constructions

- Extensive constructions by Sloane et al. Nice codes are often codes with symmetries.
- Construction of orbit codes: let G be a finite subgroup of $O(\mathbb{R}^n)$ acting irreducibly, let $H < G$ be a subgroup of G an $p \subset \mathbb{R}^n$ a subspace stabilized by H. take $C = G.p$. Then $|C| = [G : H]$.
- Many examples in Sloane et al., in particular from Clifford groups.
- J. Creignou: codes from groups G acting 2-transitively on G/H meeting the simplex bound.

Designs

- A design in a space X is a finite set $D \subset X$ which "approximates well" the space X
- A generic definition: a design $D \subset X$ of strength t (t-design) is one that satisfies the condition:

$$\text{for all } f : X \to \mathbb{C},\ \deg(f) \leq t, \quad \int_X f(x)dx = \frac{1}{|D|} \sum_{x \in D} f(x).$$

- Combinatorial block designs, orthogonal arrays, (t, m, s)-nets, designs in Q-polynomial association schemes, spherical designs, projective designs have been extensively studied (Delsarte, Goethals, Seidel, Sloane, Bannai, etc..).

Designs in Grassmann spaces

- (B., Coulangeon, Nebe, 2002) We say $D \subset \mathcal{G}_{m,n}$ is a $2t$-design if

$$\text{for all } f \in \bigoplus_{|\mu| \le t} H_{m,n}^{2\mu}, \quad \int_{\mathcal{G}_{m,n}} f(p) d\mu(p) = \frac{1}{|D|} \sum_{p \in D} f(p).$$

- A useful criterion: D is a $2t$-design if and only if

$$\text{for all } \mu,\ 1 \le |\mu| \le t, \sum_{(p,p') \in D^2} P_\mu(\underline{y}(p,p')) = 0.$$

- Remark: the above sum is non negative for all $D \subset \mathcal{G}_{m,n}$.

Another criterion for designs

- (B. 2005) For all $D \subset \mathcal{G}_{m,n}$, and for all $t \geq 1$ integer,

$$\sum_{(p,p')\in D^2} \langle P_p, P_{p'} \rangle^t \geq \alpha_{t,m} |D|^2$$

with

$$\begin{aligned} \alpha_{t,m} &= \int_{\mathcal{G}_{m,n}{}^2} \langle P_p, P_{p'} \rangle^t d\mu(p,p') \\ &= \int_{[0,1]^m} (y_1 + \cdots + y_m)^t d\mu(y_1, \ldots, y_m). \end{aligned}$$

Moreover, equality holds if and only if D is a $2t$-design.

- Proof: $\langle P_p, P_{p'} \rangle^t - \alpha_{t,m} \succeq 0$ and moreover its coefficients on P_μ, $|\mu| \leq t$, are positive.

Unequal dimensions

- (B. 2010) For all $D \subset \cup_m \mathcal{G}_{m,n}$, and for all t,

$$\sum_{(p,p')\in D^2} \langle P_p, P_{p'} \rangle^t \geq \sum_{1\leq m,q\leq n-1} \alpha_{t,m,q} |C_m||C_q|$$

where

$$\begin{aligned}\alpha_{t,m,q} &= \int_{\mathcal{G}_{m,n}} \int_{\mathcal{G}_{q,n}} \langle P_p, P_{p'} \rangle^t d\mu_m(p) d\mu_q(p') \\ &= \int_{[0,1]^m} (y_1 + \cdots + y_m)^t d\mu_{m,q}(y_1, \ldots, y_m).\end{aligned}$$

$$d\mu_{m,q} = \lambda \prod_{1\leq i<j\leq m} |y_i - y_j| \prod_{1\leq i\leq m} y_i^{(q-m-1)/2} (1-y_i)^{(n-q-m-1)/2} dy_i$$

James, Constantine, 1974: intertwining operators $H_{m,n}^{2\mu} \to H_{q,m}^{2\mu}$ are Jacobi polynomials, orthogonal for $d\mu_{m,q}$.

Designs and codes

- General idea: Designs with large strength and few values of distance are maximal codes.
- For the two-point homogeneous spaces: several results with increasing generality in this direction.
  1. Delsarte 1973, and Delsarte-Goethals-Seidel 1975: The "absolute bounds".
  2. Levenshtein 1992: "Delsarte codes" are maximal codes
  3. Cohn-Kumar 2006: "Sharp arragements" (= Delsarte codes) are "universally optimal", i.e. optimal for any reasonable potential function.
- For $\mathcal{G}_{m,n}$: B-Bannai-Coulangeon 2004, generalization of the absolute bounds.

  Wanted: an analog of Cohn-Kumar result...

Designs and potential minimization

- Cohn-Kumar result for the sphere S^{n-1}: the potential energy of a finite subset $C \subset S^{n-1}$ relative to a continuous, decreasing potential function f equals:

$$P_f(C) := \sum_{(x,y)\in C^2,\ x\neq y} f(\|x-y\|^2).$$

- A sharp configuration is a set C such that $\{\|x-y\|^2, (x,y) \in C^2, x \neq y\}$ has m elements and C is a $(2m-1)$-spherical design.
- If f is completely monotonic, i.e. $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in (0,4]$ and all $k \geq 0$, then C realizes a global minimum of P_f (for fixed $|C|$) and moreover is unique up to isometry.

Designs and groups

- Nice designs have symmetries!
- Let $G < O(\mathbb{R}^n)$ be a finite group, and $m_0 \leq n/2$ be fixed. The following are equivalent:
  1. For all $m \leq m_0$, for all $p \in \mathcal{G}_{m,n}$, $G.p$ is a $2t$-design.
  2. For all μ, $1 \leq |\mu| \leq t$, $l(\mu) \leq m_0$, $(V_n^{2\mu})^G = \{0\}$.
- If $G < O(\mathbb{R}^n)$ is absolutely irreducible then all orbits are 2-designs.
- Weyl groups of root systems A_2, D_4, E_6, E_7 provide 4-designs, of E_8 provide 6-design, Clifford groups $\mathcal{C}_k = 2_+^{1+2k}.O_{2k}^+(2)$ provide 6-designs, $2.Co_1$ leads to 10-designs.
- (Tiep 2006) Complete classification of the groups that provide 4-designs in all $\mathcal{G}_{m,n}$. The maximal strength is 10.