Continuum models for complex systems

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Summary

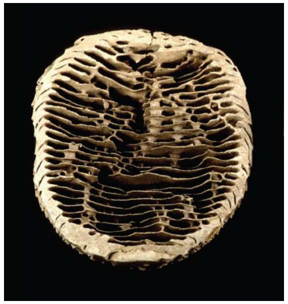
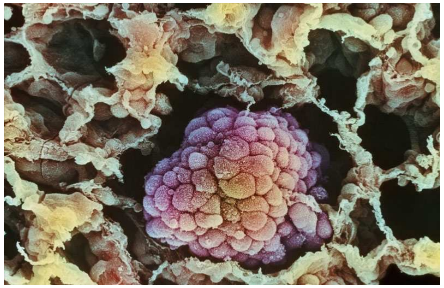
1. Examples
2. Hydrodynamics of the Vicsek system
3. Chaos property for Vicsek-like dynamics
4. Conclusion

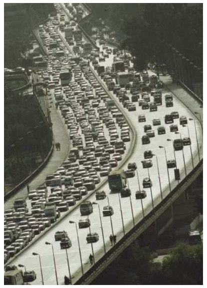
1. Examples

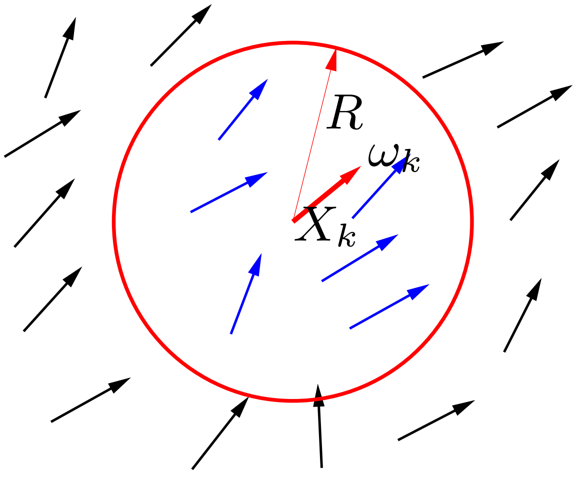
Complex system

- System with locally interacting agents
  - emergence of spatio-temporal coordination
  - patterns, structures, correlations, synchronization
  - No leader / only local interactions



- Time-discrete model:
  - $t^n = n\Delta t$
  - k-th individual
  - X_k^n: position at t^n
  - ω_k^n: velocity with $|\omega_k^n| = 1$

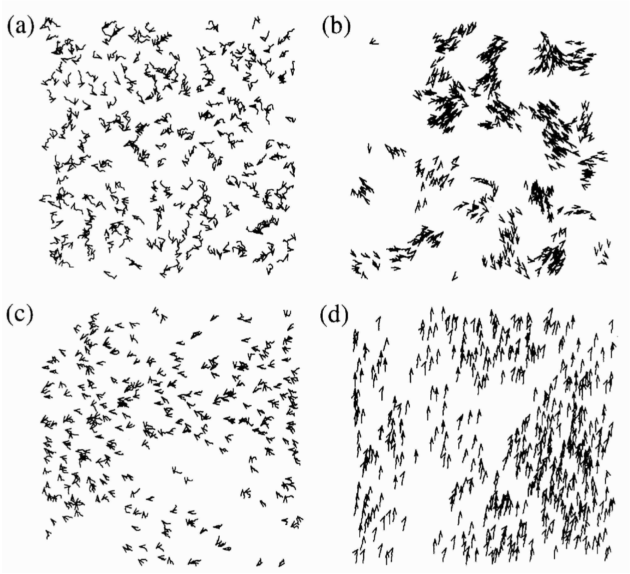
- $$X_k^{n+1} = X_k^n + \omega_k^n \, \Delta t$$

  $$\omega_k^{n+1} = \bar{\omega}_k^n + \text{ noise (uniform in small angle interval)}$$

  $$\bar{\omega}_k^n = \frac{J_k^n}{|J_k^n|}, \quad J_k^n = \sum_{j,\, |X_j^n - X_k^n| \leq R} \omega_j^n$$

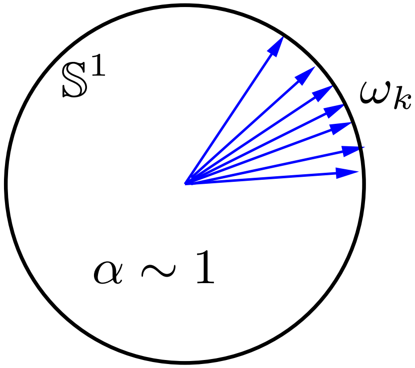
- Alignment to neighbours' mean velocity plus noise

- Phase transition to disorder

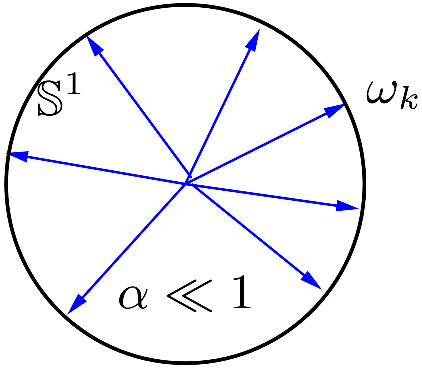


- Order parameter
  - $\alpha = \left| N^{-1} \sum_j \omega_j \right|^2$
  - N = particle number
  - $0 \leq \alpha \leq 1$

- Measures alignment



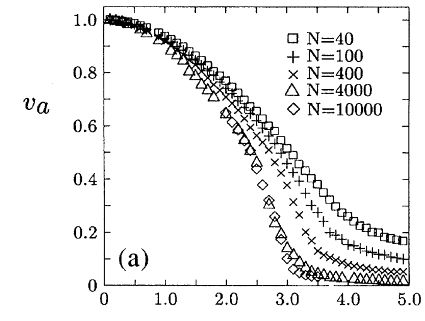
$\alpha \sim 1$: ω aligned



$\alpha \ll 1$: ω random

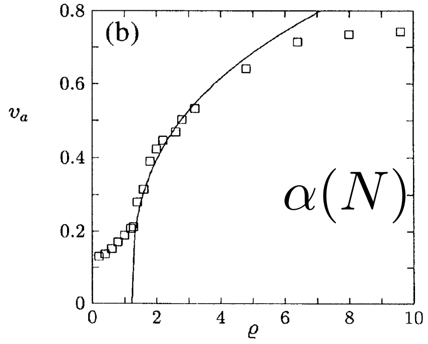
Phase transition to aligned state

- As noise decreases [Vicsek et al, PRL 95]



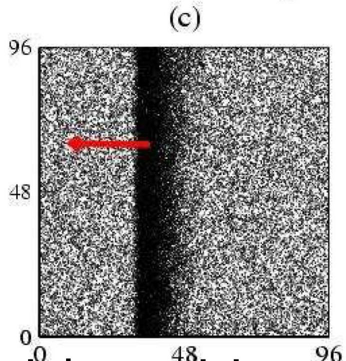
α(noise level)

- As density increases [Vicsek et al, PRL 95]

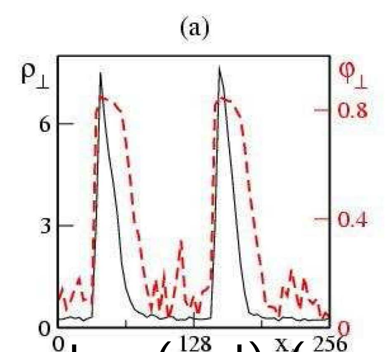


$\alpha(N)$

- Band formation [Chaté et al]



Particle positions



ρ (black) and α (red) (cross section)

Self-organization

- Vicsek dynamics exhibits
  - self-organization & emergence of coherent structures
  - supposes the build-up of correlations between particles
- Kinetic and Hydrodynamic models rely on the chaos assumption
  - When N is large, particles are statistically independent
- Question: are kinetic and hydrodynamic models relevant for Complex Systems ?
  - Goal: provide illustrative examples

2. Hydrodynamics of the Vicsek system

Plan

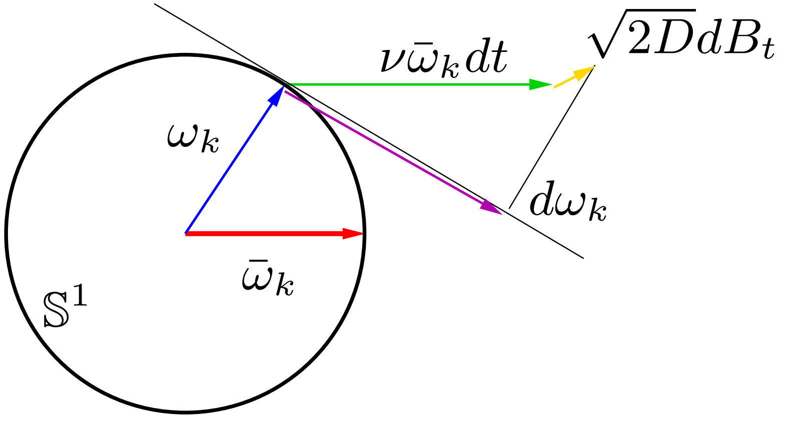
- Leave justification of chaos assumption aside
- Three steps
  - time-continuous particle (IBM) model
  - Mean-field kinetic limit
  - Hydrodynamic limit
- Difficulty
  - dimension of invariants $<$ dimension of equilibria
  - New concept of 'Generalized Collisions Invariants'
  - 1st derivation of non-conservative model from kinetics

Time continuous Vicsek algorithm

- Time continuous dynamics:

$$\dot{X}_k(t) = \omega_k(t)$$
$$d\omega_k(t) = (\mathrm{Id} - \omega_k \otimes \omega_k)(\nu \bar{\omega}_k dt + \sqrt{2D} dB_t)$$
$$\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \sum_{j, |X_j - X_k| \le R} \omega_j$$

- Recover original Vicsek by:
  - Time discretization Δt
  - Gaussian noise $\to$ uniform
  - $\nu \Delta t = 1$



Mean-field model

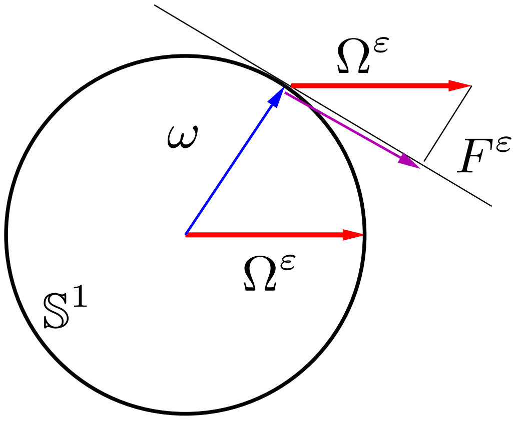
- $f(x, \omega, t) =$ 1-particle proba distr.
  - satisfies a Fokker-Planck equation
  - Scaling to macro variables $\tilde{x} = \varepsilon x, \quad \tilde{t} = \varepsilon t, \quad \varepsilon \ll 1$
  - local interaction: $\tilde{R} = \varepsilon R$

- Fokker-Planck eq. in scaled variables

$$\varepsilon(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon) + \nabla_\omega \cdot (F^\varepsilon f^\varepsilon) = D \Delta_\omega f^\varepsilon$$

$$F^\varepsilon = (\mathrm{Id} - \omega \otimes \omega) \Omega^\varepsilon$$

$$\Omega^\varepsilon = \frac{j^\varepsilon}{|j^\varepsilon|}, \quad j^\varepsilon = \int_{|v|=1} v f^\varepsilon(x, v, t) \, dv$$



  - Ω^ε is the direction of the local flux

- Model can be written

$$\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

  - with collision operator

$$Q(f) = -\nabla_\omega \cdot (F_f\, f) + D \Delta_\omega f$$

$$F_f = (\mathsf{Id} - \omega \otimes \omega) \Omega_f$$

$$\Omega_f = \frac{j_f}{|j_f|}, \quad j_f = \int_{|v|=1} v f(x,v,t)\, dv$$

- Problem: find the limit $\varepsilon \to 0$

Equilibria

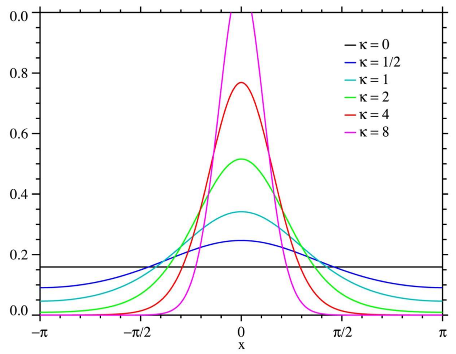
- Equilibrium manifold: $\mathcal{E} = \{ f \,|\, Q(f) = 0 \}$

$$\mathcal{E} = \{ \rho M_\Omega(\omega) \text{ for arbitrary } \rho \in \mathbb{R}_+, \Omega \in \mathbb{S}^2 \}$$

with M_Ω Von-Mises distribution:

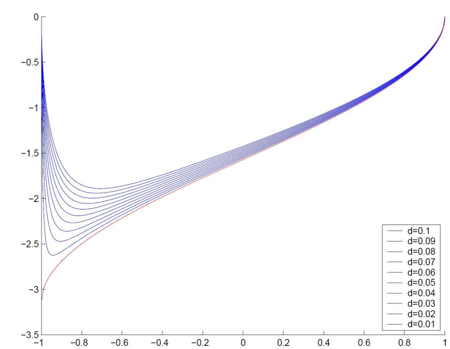
$$M_\Omega(\omega) = Z^{-1} \, \exp \beta(\omega \cdot \Omega), \quad \beta = D^{-1}$$

- dim $\mathcal{E} = 3$ (dim $= 3$) or 2 (dim $= 3$)
- $Q(f) = D \, \nabla_\omega \cdot \left[M_{\Omega_f} \nabla_\omega \left(\frac{f}{M_{\Omega_f}} \right) \right]$
- Entropy dissipation inequality



- $\mathcal{C} = \{\psi(\omega)$ such that $\int Q(f)\psi \, d\omega = 0, \quad \forall f\}$
  - dim $\mathcal{C} = 1 <$ dim $\mathcal{E}$ because $\mathcal{C} =$ Span$\{1\}$
- ψ_Ω is a GCI iff $\int Q(f)\psi_\Omega \, d\omega = 0, \forall f$ s.t. $\Omega_f = \Omega$
- Given Ω, the GCI form a 3-dim vector space spanned by 1 and $\vec{\psi}_\Omega(\omega)$
- $\vec{\psi}_\Omega(\omega) = \frac{\Omega \times \omega}{|\Omega \times \omega|} g(\Omega \cdot \omega)$

  $g(\mu)$ sol. of elliptic eq:



$$-(1-\mu^2)\partial_\mu(e^{\mu/D}(1-\mu^2)\partial_\mu g) + e^{\mu/D} g = -(1-\mu^2)^{3/2} e^{\mu/D}$$

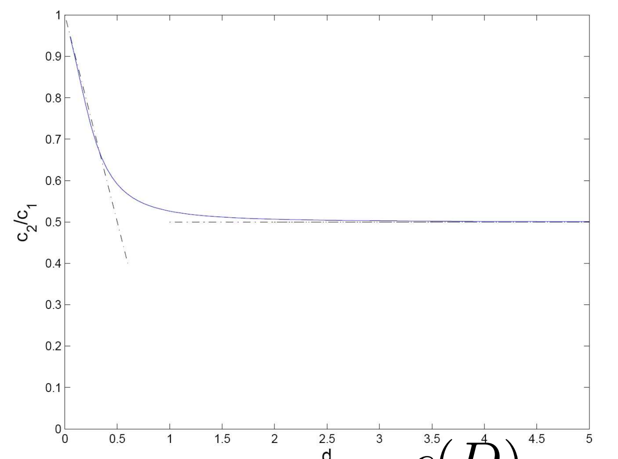
Hydrodynamics of Vicsek model

- Use of GCI leads to hydrodynamic model
- density $\rho(x,t)$; flux director $\Omega(x,t)$:

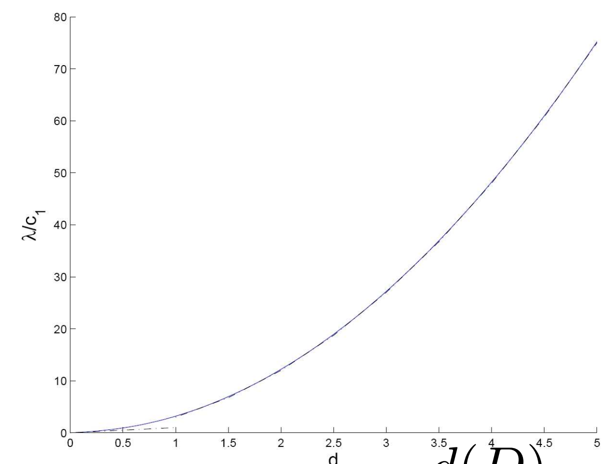
$$\partial_t \rho + \nabla_x \cdot (\rho \Omega) = 0$$

$$\rho \, (\partial_t \Omega + c(\Omega \cdot \nabla)\Omega) + d \, (\mathrm{Id} - \Omega \otimes \Omega) \nabla_x \rho = 0$$

$$|\Omega| = 1$$



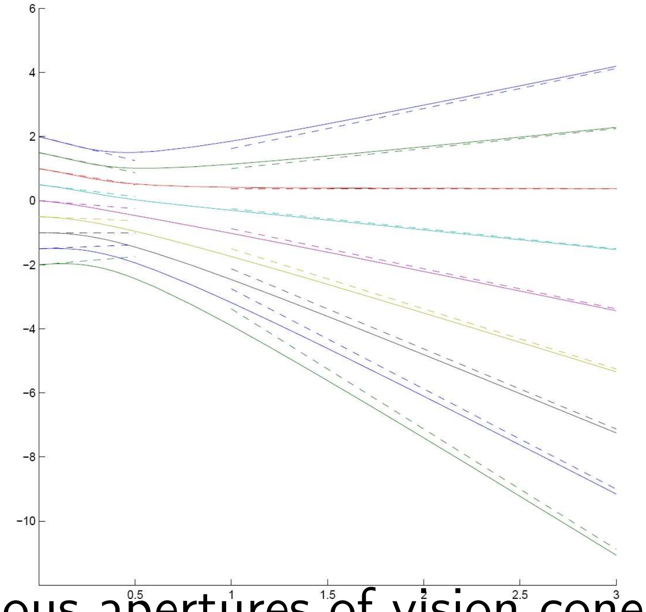
$c(D)$



$d(D)$

Hydrodynamic Vicsek model: comments

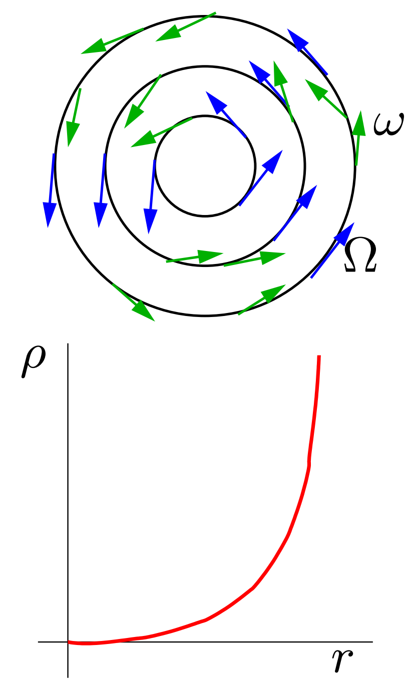
- Hyperbolic model with geometric constraint
  - Non-conservative terms arise from the constraint
  - Hydro & relaxation limits do not commute
- Velocity information travels slower than mass flow
  - like traffic
  - reinforced by forward vision [Frouvelle]



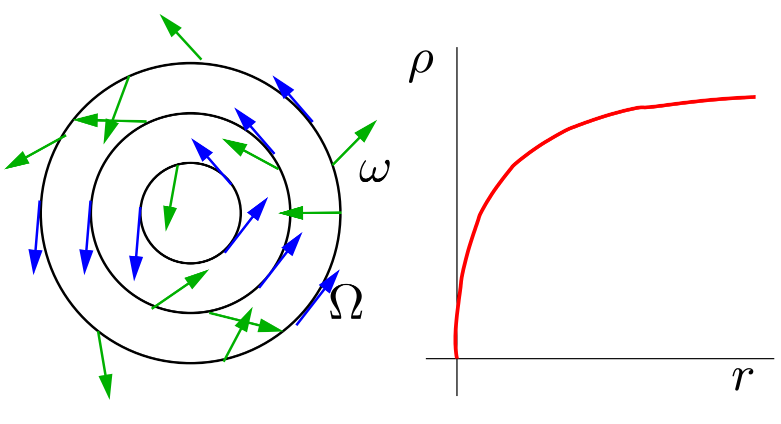
$c(D)$ for various apertures of vision cone

Mills are stationary solutions

- Mills: $\rho(r) = \rho_0 \, (r \, / \, r_0)^{c/d}$, $\Omega = x^{\perp}/r$
- Shape depends on noise level
  - small noise: $\rho(r)$ convex: sharp edged mills
  - large noise: $\rho(r)$ concave: fuzzy edges



small noise

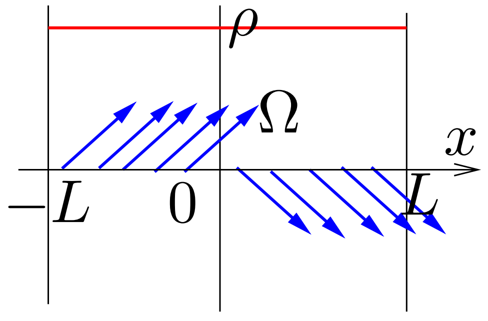


large noise

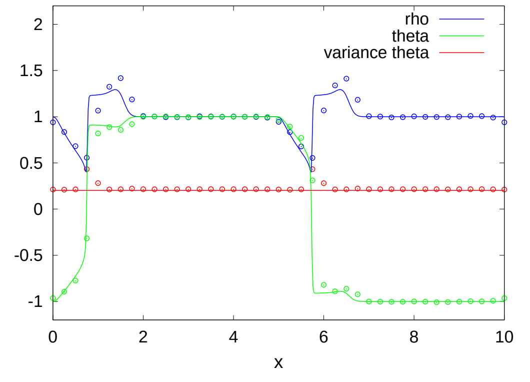
- Stability of mills ?

V. particles vs V. Hydrodynamics

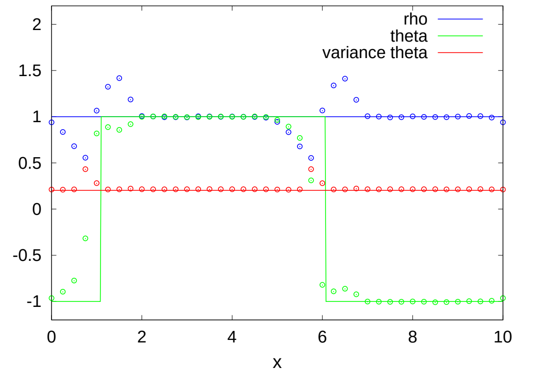
Initial conditions



Relaxation-based



Standard



- Initial contact discontinuity resolved by complex wave pattern
- Reproduced by Relaxation-based scheme
- Not by standard meth.

3. Chaos property in Vicsek-like systems

Chaos property

- Taking the limit $N \to \infty$ 'simplifies' the problem
  - If N large, system is not influenced by the state of one given particle: particles become independent

- More precisely: if at $t = 0$, particles are independent:

$$F^{(j)}(v_1, \ldots, v_j)|_{t=0} = \prod F^{(1)}(v_k)|_{t=0}$$

  - Then: correlations $\to 0$ as $N \to \infty$ for $t \in [0, T]$ with $T = O(1)$
  - see [Lanford], [Kac], ...

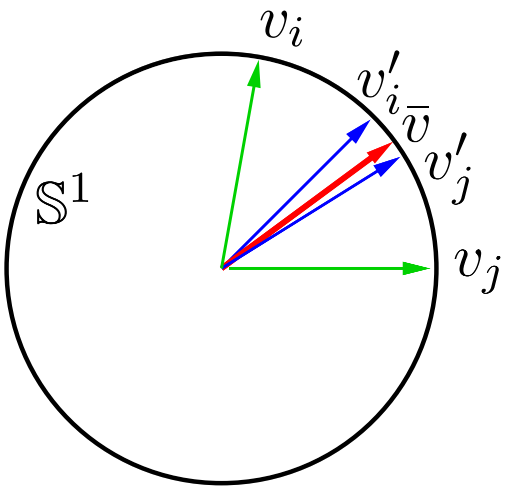
- But: even if N large, a single particle may be enough to probe the full system

Related questions

- As $N \to \infty$:
  - Dynamics becomes irreversible
  - $\exists$ entropy functional H which $\searrow$ in time
  - Dissipation
  - Equilibria $=$ states of maximal disorder
- For classical systems (e.g. rarefied gases)
  - strong relation between these concepts
- Is this still true for self-organization processes ?
  - will some of these concepts survive while others won't ?

Ex 1. A 'binary' Vicsek dynamics: BDG

- After [Bertin, Droz, Gregoire]
- Pick a pair $\{i, j\}$ at random
  - probability $P_{ij} = 2/N(N-1)$
  - average direction: $v_{ij} = (v_i + v_j)/|v_i + v_j|$
- Add independent noise drawn according to g:
  - $v'_i = v_{ij} w_i \qquad v'_j = v_{ij} w_j$
  - All particles but $\{i, j\}$ unchanged



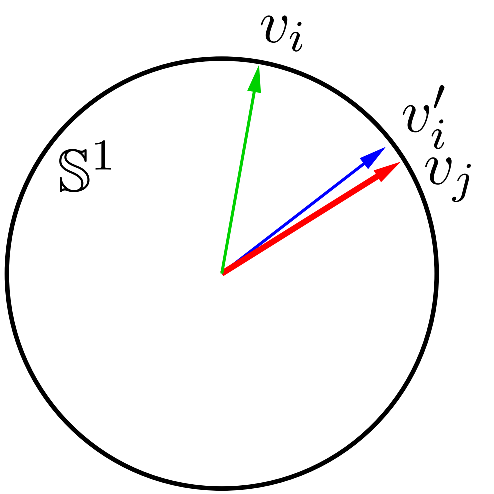
- Variant (acception-rejection)
  - Collision performed with probability $h(v_i v_j^*)$ s.t. $0 \le h \le 1$

Ex 2. 'Choose the Leader' (CLD)

- Pick an ordered pair (i,j) at random
  - Probability $P_{ij} = 1/N(N-1)$
- Then, i joins j plus noise w drawn according to g

$$v_i' = v_j w$$

  - All particles but i unchanged



Noise scaling

- Assumptions on noise distribution as $N \to \infty$:

$$g_N \to \delta(v)$$

$$\text{Var}(g_N) = \frac{\sigma^2}{N} \quad \text{i.e.} \quad \text{MSD}(g_N) = O(\frac{1}{\sqrt{N}})$$

- Goal: find eqs. for the marginals as $N \to \infty$ and $\Delta t = O(\frac{1}{N^2})$ (continuous time limit)

- For BDG: also scale the bias ('grazing collisions')
  - $h_N / \int h_N \to \delta \quad \text{Var}(h_N / \int h_N) = \tau^2 / N$

- CLD: hierarchy closed at all orders

- First marginal:

$$\partial_t f^{(1)} - (\sigma^2/2)\partial^2_{\theta_1} f^{(1)} = 0$$

- Second marginal:

$$\partial_t f^{(2)} - (\sigma^2/2)\Delta_{\theta_1,\theta_2} f^{(2)} + 2f^{(2)} = (f^{(1)}(\theta_1) + f^{(1)}(\theta_2))\delta(\theta_2 - \theta_1)$$

Stationary states as $t \to \infty$

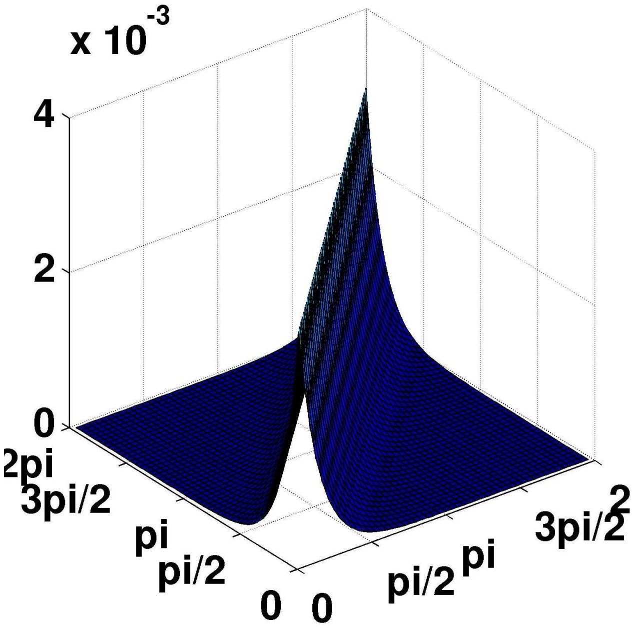
- $f^{(1)} \longrightarrow f^{(1)}_{\text{eq}} = 1$: uniform distribution on $\mathbb{S}^1$

- $f^{(2)} \longrightarrow f^{(2)}_{\text{eq}}$ the unique solution of

$$-(\sigma^2/2)\Delta_{\theta_1,\theta_2} f + 2f = 2\delta(\theta_2 - \theta_1)$$

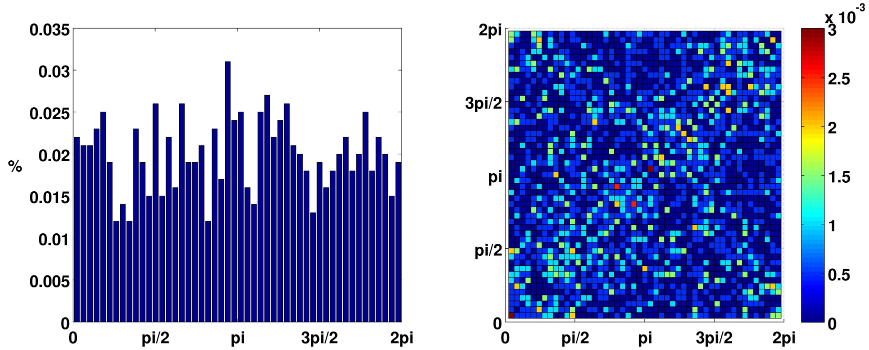
- $f^{(2)}_{\text{eq}}(\theta_1, \theta_2) \neq f^{(1)}_{\text{eq}}(\theta_1)\, f^{(1)}_{\text{eq}}(\theta_2)$
  - Chaos assumption violated

- $f^{(2)}_{\text{eq}}$ peaked at $\theta_1 = \theta_2$
  - coherent motion
  - but no preferred mean direction

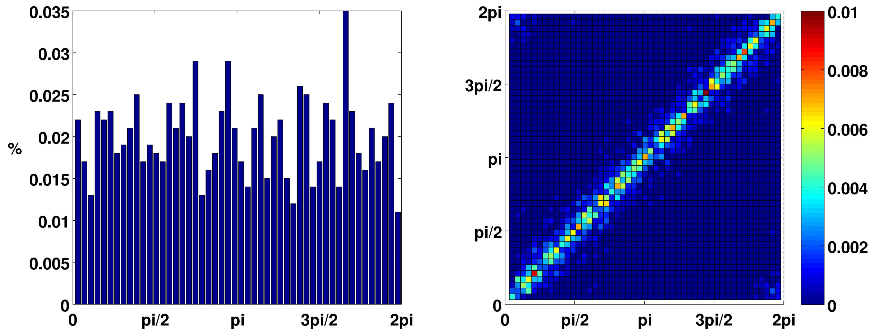


Numerical simulations

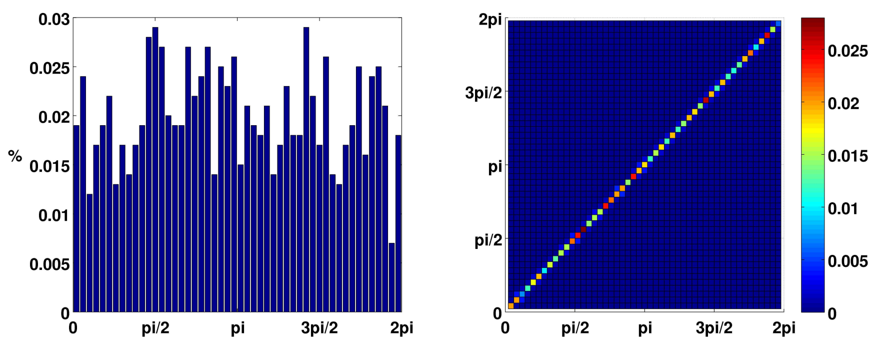
- Experimental protocol
  - simulations with $N = 10^2$, 10^3, 10^4 & 10^5 particles
  - wait until 'stationary state'
  - Pick one i and a pair (i,j) at random
  - Redo the simulation M times to avoid correlations
  - Plot histograms of θ_1 and (θ_1, θ_2) of these M samples
  - Compare with theoretical $f_{\text{eq}}^{(1)}$ and $f_{\text{eq}}^{(2)}$



$$\sigma = \pi$$

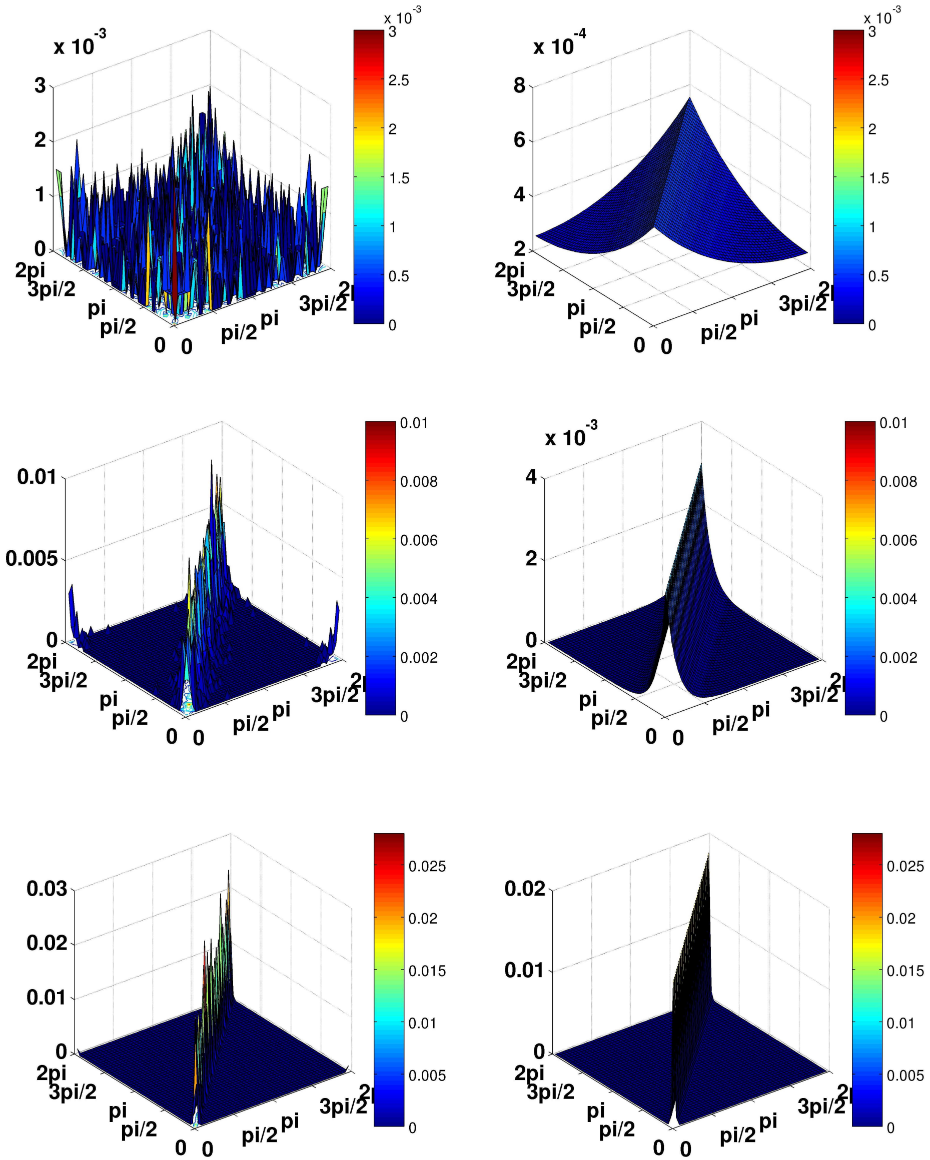


$$\sigma = \pi/10$$



$$\sigma = \pi/100$$

$f^{(2)}_{\text{eq}}$: experiments vs theory $N = 10^3$



$\sigma = \pi$

$\sigma = \pi/10$

$\sigma = \pi/100$

$$\partial_t f^{(1)} = (\sigma^2 - \tau^2)\, \partial_\theta^2 f^{(2)}(\theta,\theta)|_{\theta=\theta_1}$$

$$\partial_t f^{(2)} = (\sigma^2 - \tau^2)(\partial_\theta^2 f^{(3)}(\theta,\theta_2,\theta)|_{\theta=\theta_1} + \partial_\theta^2 f^{(3)}(\theta_1,\theta,\theta)|_{\theta=\theta_2})$$

$$\vdots$$

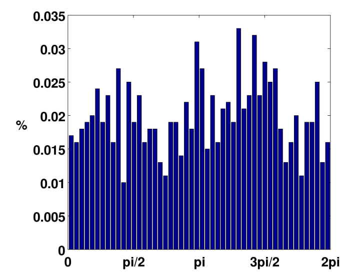
$$\partial_t f^{(j)} = (\sigma^2 - \tau^2) \sum_{k=1}^{j} \partial_\theta^2 f^{(j+1)}(\theta_1,\ldots,\theta_{k-1},\theta,\theta_{k+1},\ldots,\theta_j,\theta)|_{\theta=\theta_k}$$

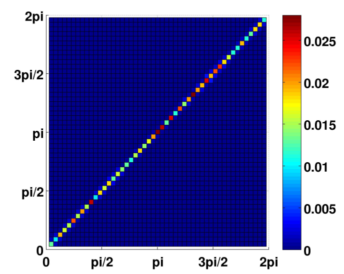
Interpretation

- If chaos assumption holds, $f^{(1)}(\theta)$ satisfies

$$\partial_t f = (\sigma^2 - \tau^2)\,(f^2)'' = 2(\sigma^2 - \tau^2)\,(f\,f')'$$

  - nonlinear heat equation
  - $\sigma > \tau$: well-posed ; noise added wider than initial spread
  - $\sigma < \tau$: ill-posed ; noise added narrower: concentration ?
- BUT: Chaos assumption does not hold





- Existence for hierarchy ?
  - infinitely many stationary states

4. Conclusion

Hydrodynamics of Vicsek particles

- Lack of collision invariants
  - New concept: Generalized Collision Invariant
  - 1st derivation of non-conservative model from kinetic theory
- New features
  - geometrical constraint
  - information velocity $\neq$ particle velocity
- Further studies
  - Phase transition A. Frouvelle
  - Other types of interactions, . . .

Chaos propagation

- 'Simple' dynamics of aggregation do not satisfy chaos assumption
  - How can kinetic theory survive this situation ?
  - Requires rethinking of classical concepts (entroypy, dissipation, irreversibility, equilibria, . . .)
- Spatialization
  - Kinetic & fluid models
  - application to practical systems (swarming, trail formation, construction, . . .)