Wave equation imaging by the Kaczmarz method

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Kaczmarz' method for linear problems

$$R_s f = g_s, s = 1\ ,\ldots, p$$

R_s linear bounded operators $H \to H_s$,

H, H_s Hilbert spaces.

Update:

$$f \leftarrow f - \alpha R_s^* C_s^{-1}(R_s f - g_s), s = 1, \ldots, p$$

Convergence:

$$0 < \alpha < 2,\ C_s \geq R_s R_s^* > 0$$

The Kaczmarz method and SOR

$$(L/D/U) = RR^*,$$

$$C_\omega = I - \omega(D + \omega L)^{-1} RR^*,$$

$$c_\omega = \omega(D + \omega L)^{-1} g.$$

Theorem: Let u^k be the SOR iterates for $RR^* f = g$, i.e.

$$u^{k+1} = C_\omega u^k + c_\omega.$$

Then $f^k = R^* u^k$ are the Kaczmarz iterates for $Rf = g$.

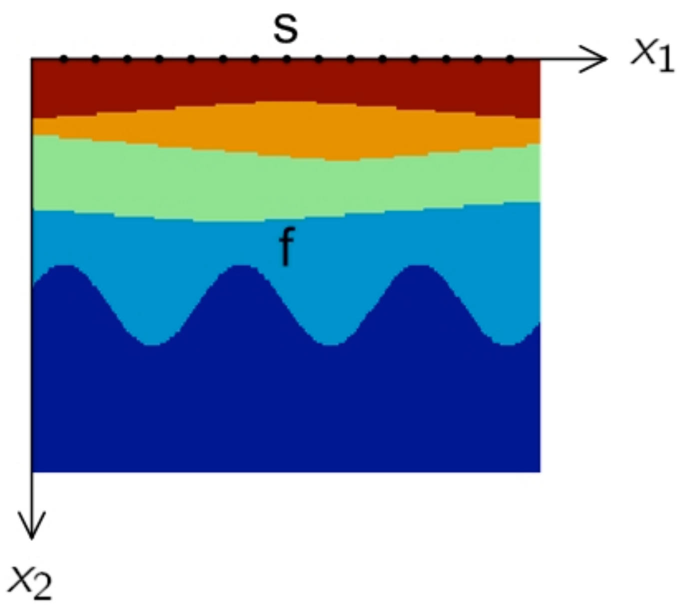
The model problem

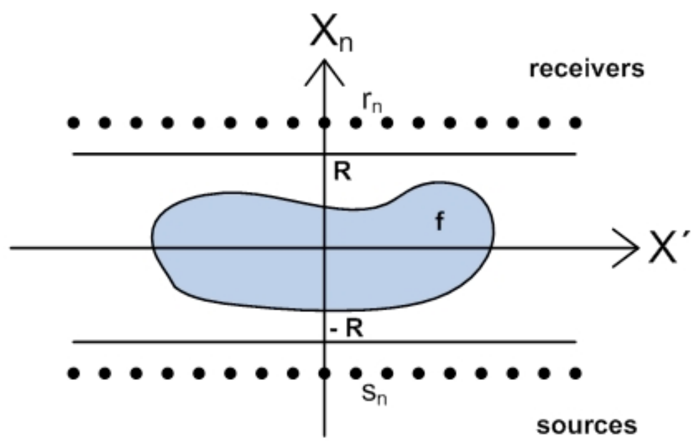
$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2(x)\left(\Delta u(x,t) + q(t)p(x-s)\right), \;\; 0 < t < T,$$

$$u = 0, \;\; t < 0,$$

$$g_s(x',t) = u(x',0,t) = (R_s(f))(x',t) \text{ seismogram for source } s,$$

$$c^2(x) = c_0^2/(1+f(x)).$$





Kaczmarz' method for nonlinear problems (consecutive time reversal)

Solve $R_s(f) = g_s$ for all sources s.

Update:

$$f \longleftarrow f - \alpha (R_s'(f))^* (R_s(f) - g_s)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2(x) \Delta z \text{ for } x_2 > 0,$$

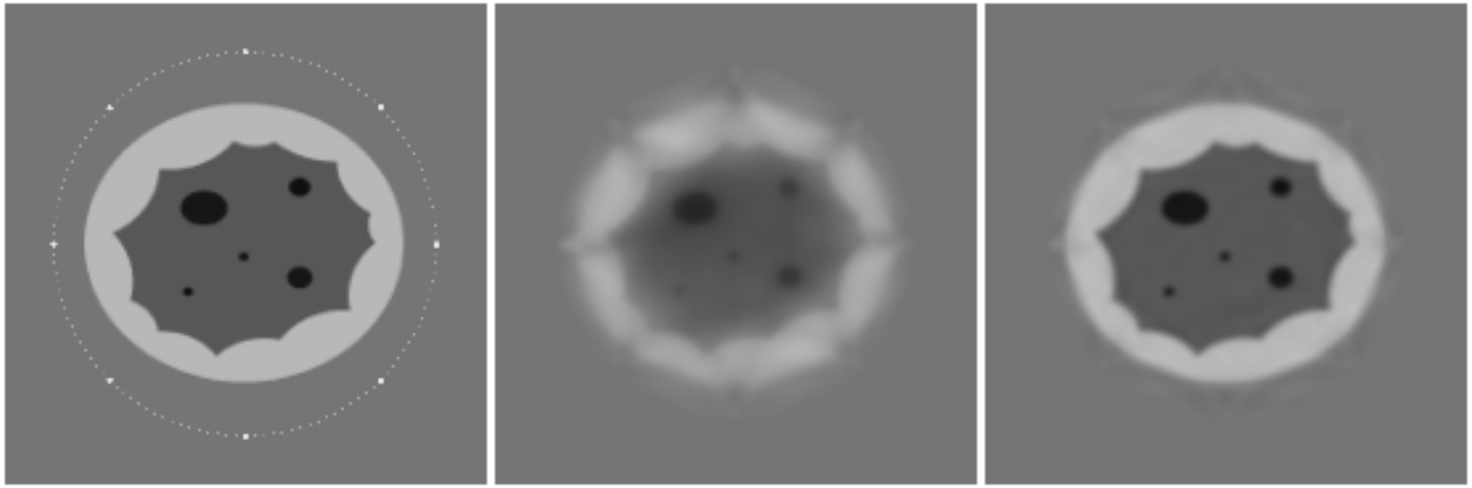
$$\frac{\partial z}{\partial x_2} = r \text{ on } x_2 = 0,$$

$$z = 0 \text{ for } t > T.$$

Compute the adjoint by time reversal:

$$(R_s'(f))^* r)(x) = \int_0^T z(x,t) \frac{\partial^2 u(x,t)}{\partial t^2} dt$$

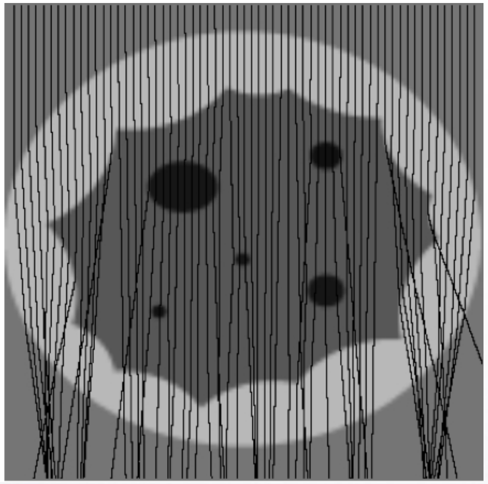
Kaczmarz' method for breast phantom, eight sources

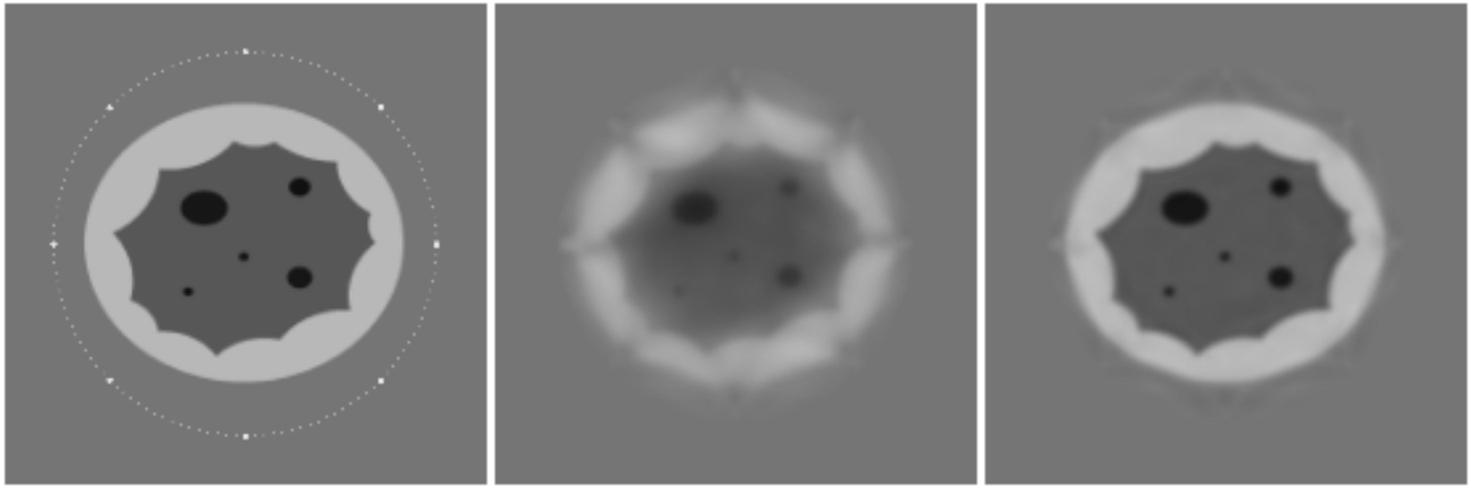


1 sweep 3 sweeps

Rays

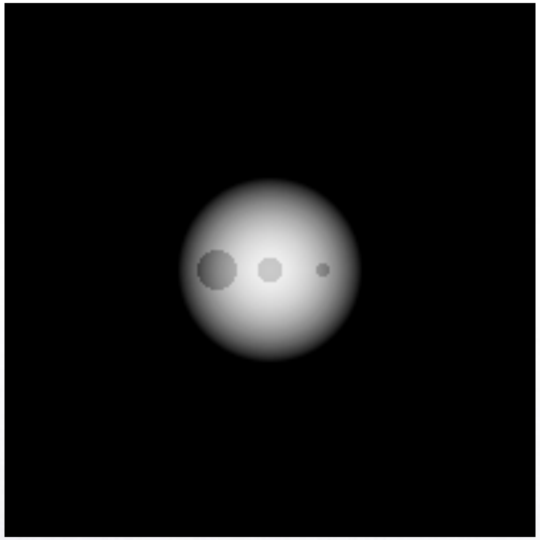
Kaczmarz‘ method for breast phantom, eight sources



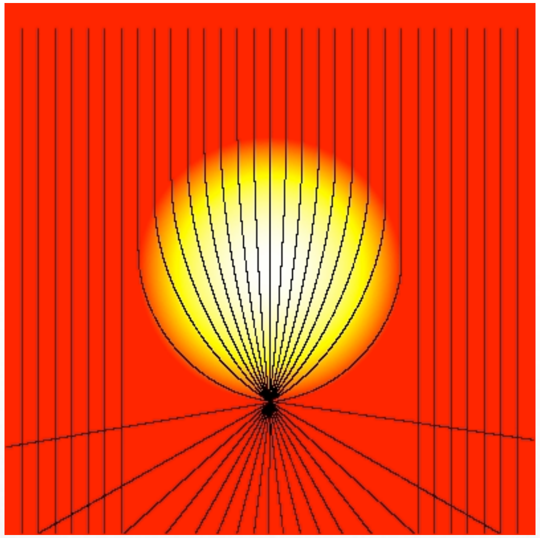


1 sweep 3 sweeps

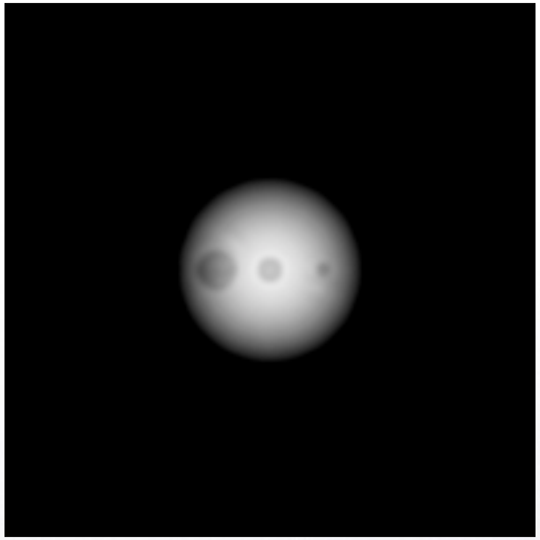
Reconstruction in the presence of focal points



7.5 5.0 2.5 mm

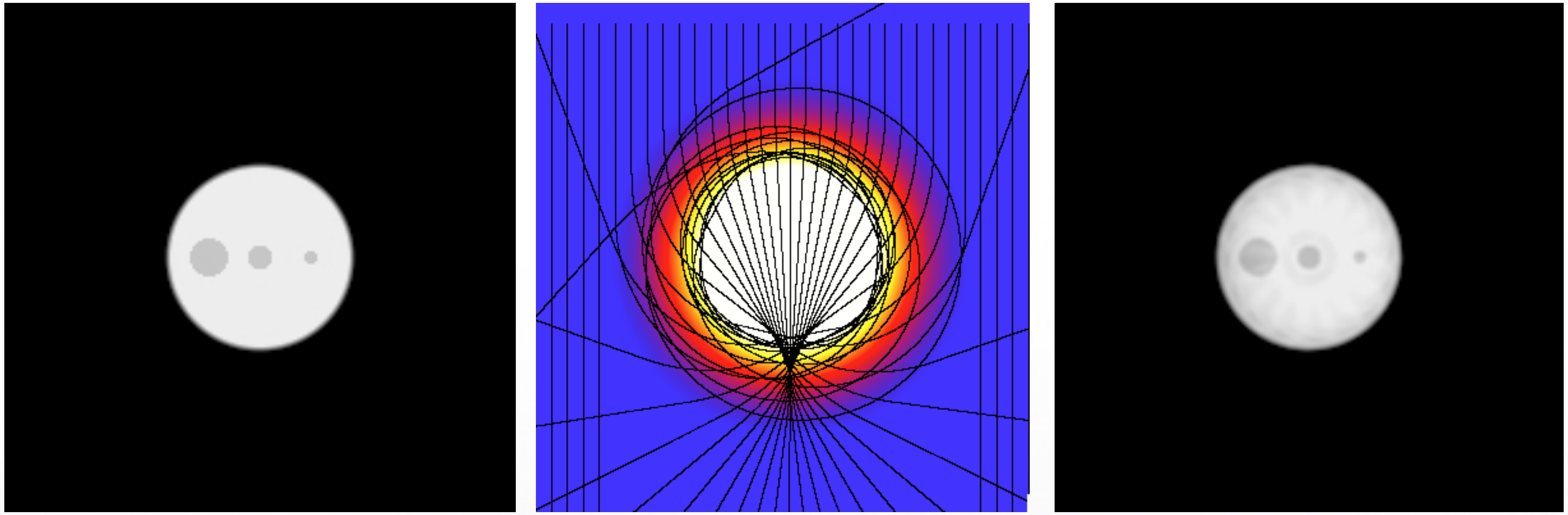


Luneberg lens

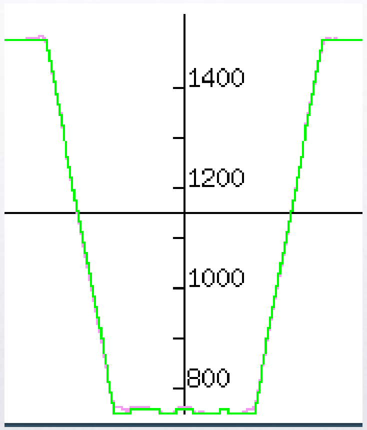


200 kHz
wavelength 7.5 mm

Reconstruction in the presence of trapped rays



crater



200 kHz

Source encoding

$$g_s(x_1, t) = u_s(x_1, 0, t)$$

is the usual seismogram for source s. Let w be a random vector, and let

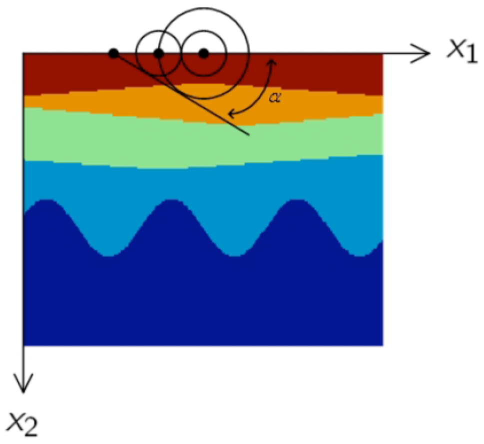
$$g_w(x_1, t) = \sum_s w_s g_s(x_1, t).$$

g_α is the value at $x_2 = 0$ of the solution for the source

$$q_w(x, t) = \sum_s w_s p(x - s) q(t).$$

Anastasio et al. 2014, Haber, Chung, Felix Herrmann 2012

Plane wave stacking



$$g_s(x_1, t) = u_s(x_1, 0, t)$$

is the usual seismogram for source s. Let $\alpha, |\alpha| \leq \pi/2$ be an angle, and let

$$g_\alpha(x_1, t) = \int_{R^1} g_s(x_1, t - \frac{s}{c} \sin\alpha) ds.$$

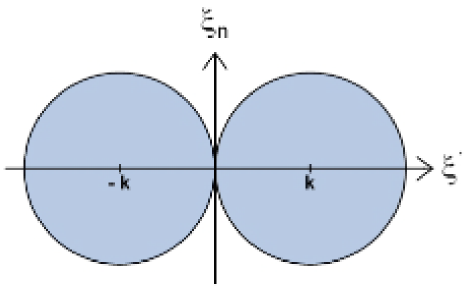
g_α is the value at $x_2 = 0$ of the solution

$$u_\alpha(x, t) = \int_{R^1} u_s(x, t - \frac{s}{c} \sin\alpha) ds$$

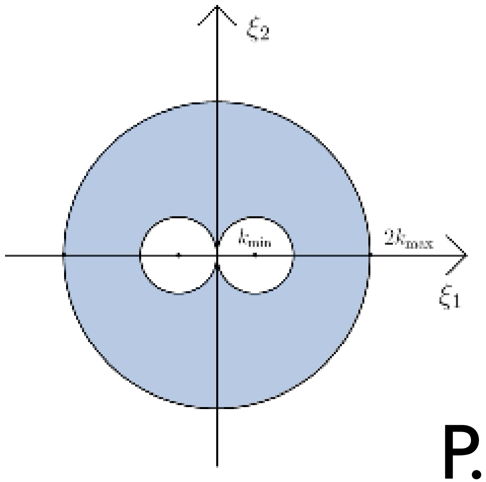
that exhibits a wave front making an angle α with the surface $x_2 = 0$.

Schultz-Claerbout 1978, Jun Ji 2001, N. 2005

Coverage in Fourier domain



Transmission



Reflection

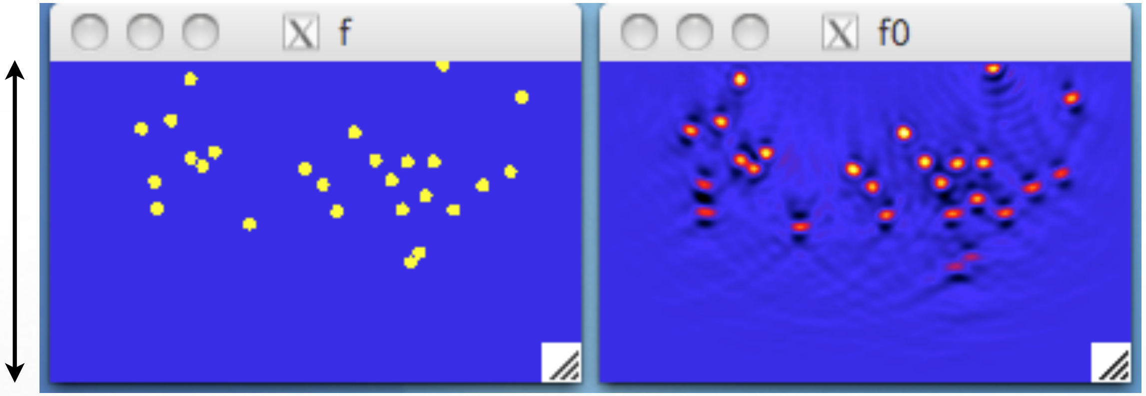
P. Mora , 1989

$$\hat{f}(\sigma+\rho,\kappa(\sigma)-\kappa(\rho)) \quad \hat{f}(\sigma+\rho,\kappa(\sigma)+\kappa(\rho))$$

$$\kappa(\sigma)=\sqrt{k^2-\sigma^2}$$

Easy case Nr. 1: Clutter

Original
12 cm

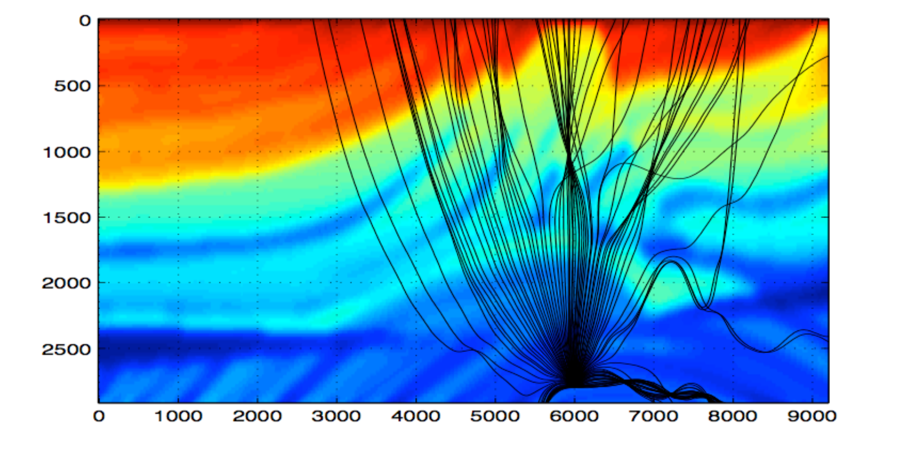


5 sweeps of
Kaczmarz

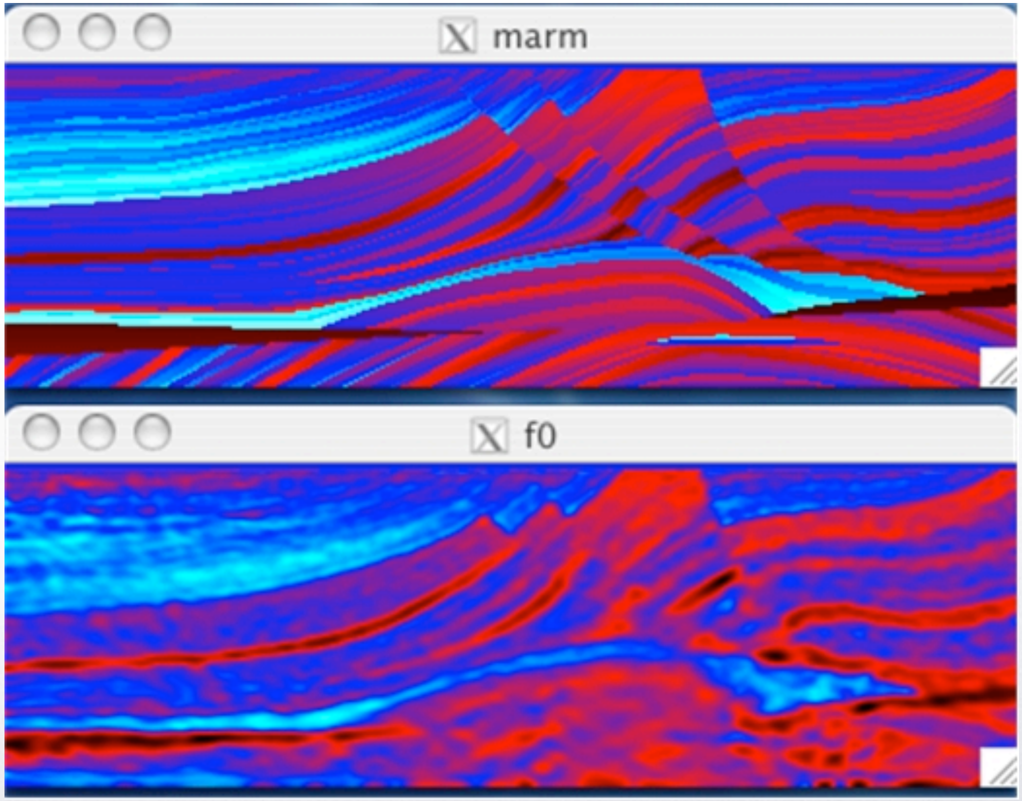
Diameter of
dots 5 mm

Frequency range 50 to 150 kHz

Easy case Nr. 2: Source wavelet q is Gaussian peak.



Original



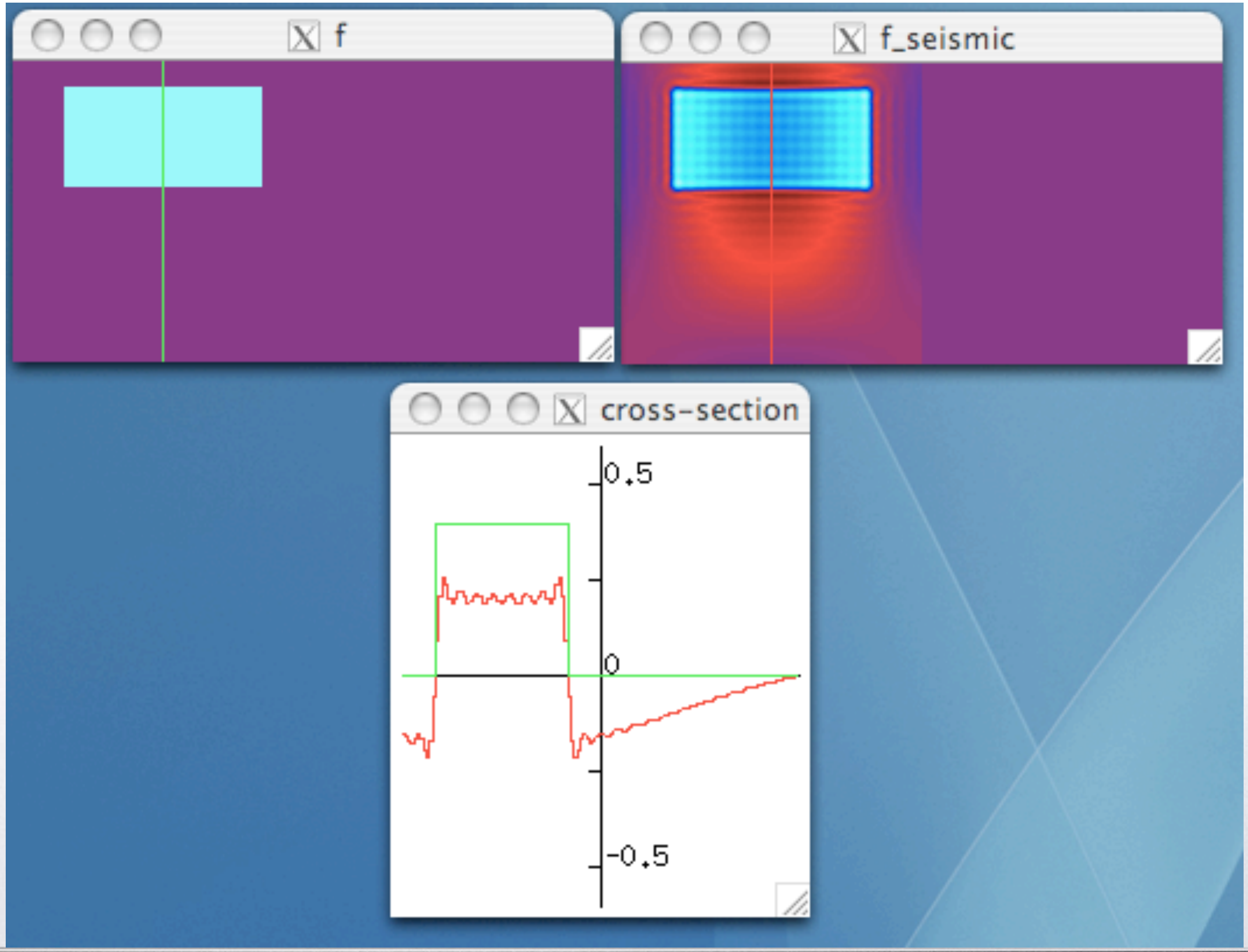
6 sweeps

original

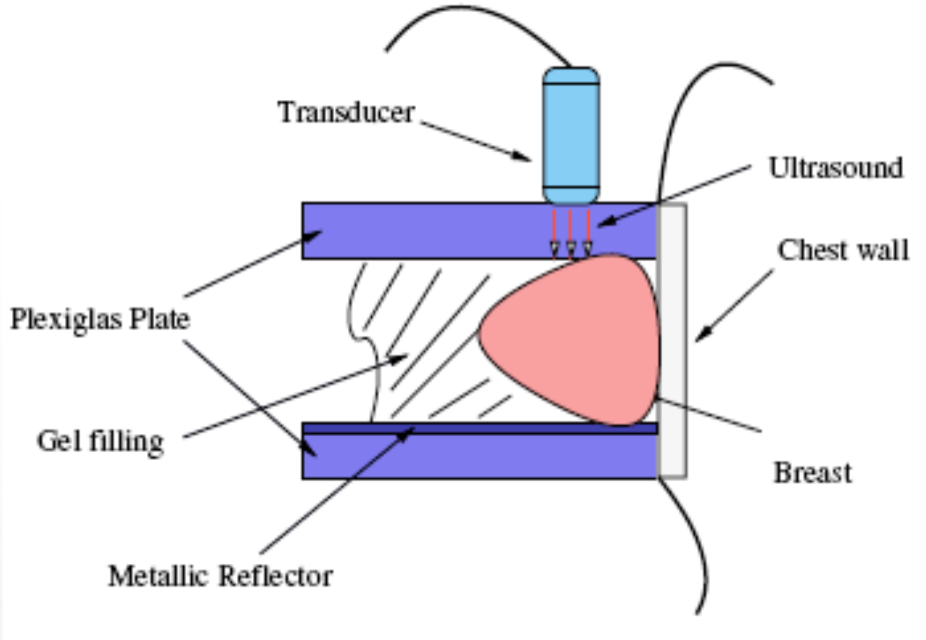
reconstruction

Difficult case

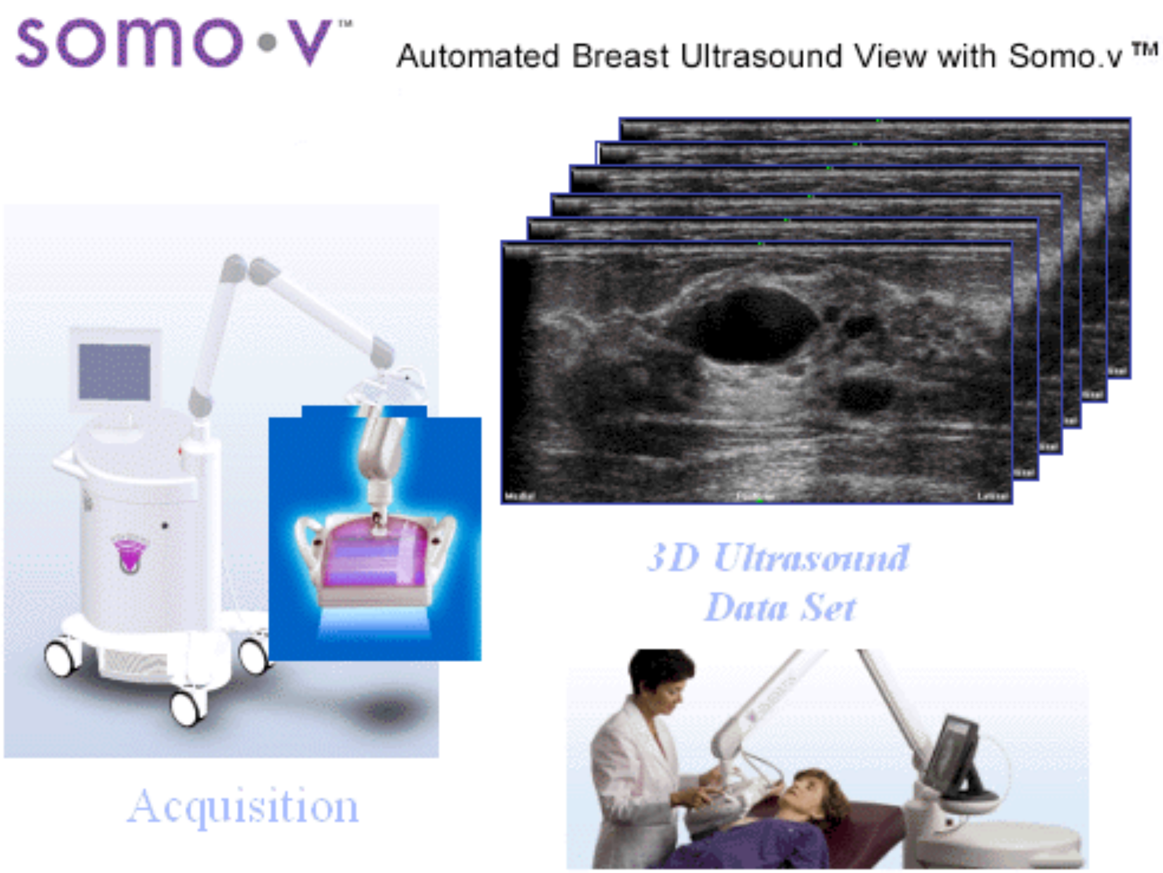
10 kHz - 150 kHz



Suggestion of K. Richter, 1995



Mammography reflection imaging



3D scanner of U-Systems

What can we achieve in reflection mammography?

aperture A= 15cm

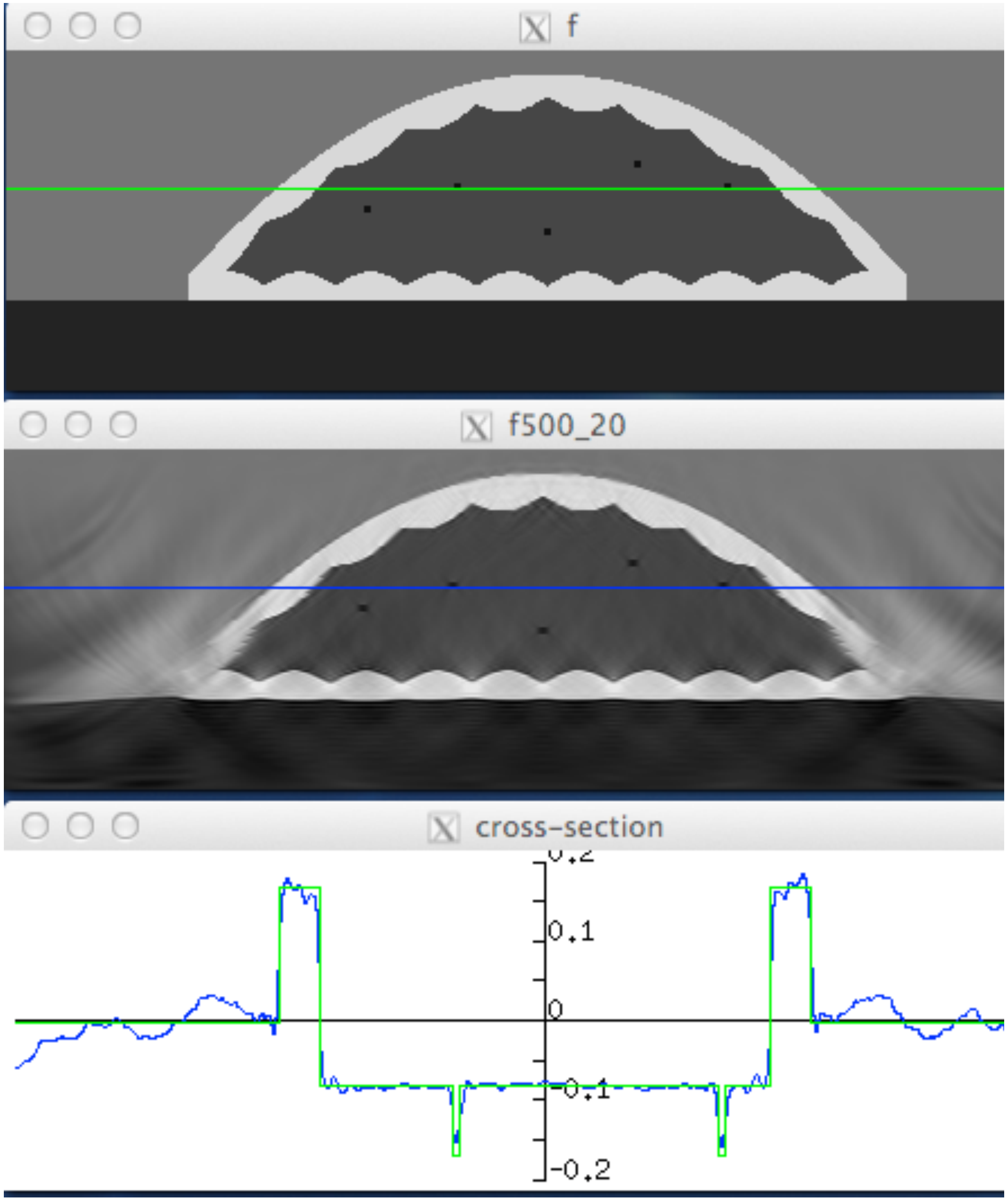
frequencies 15-500 kHz

wavelength 3 mm

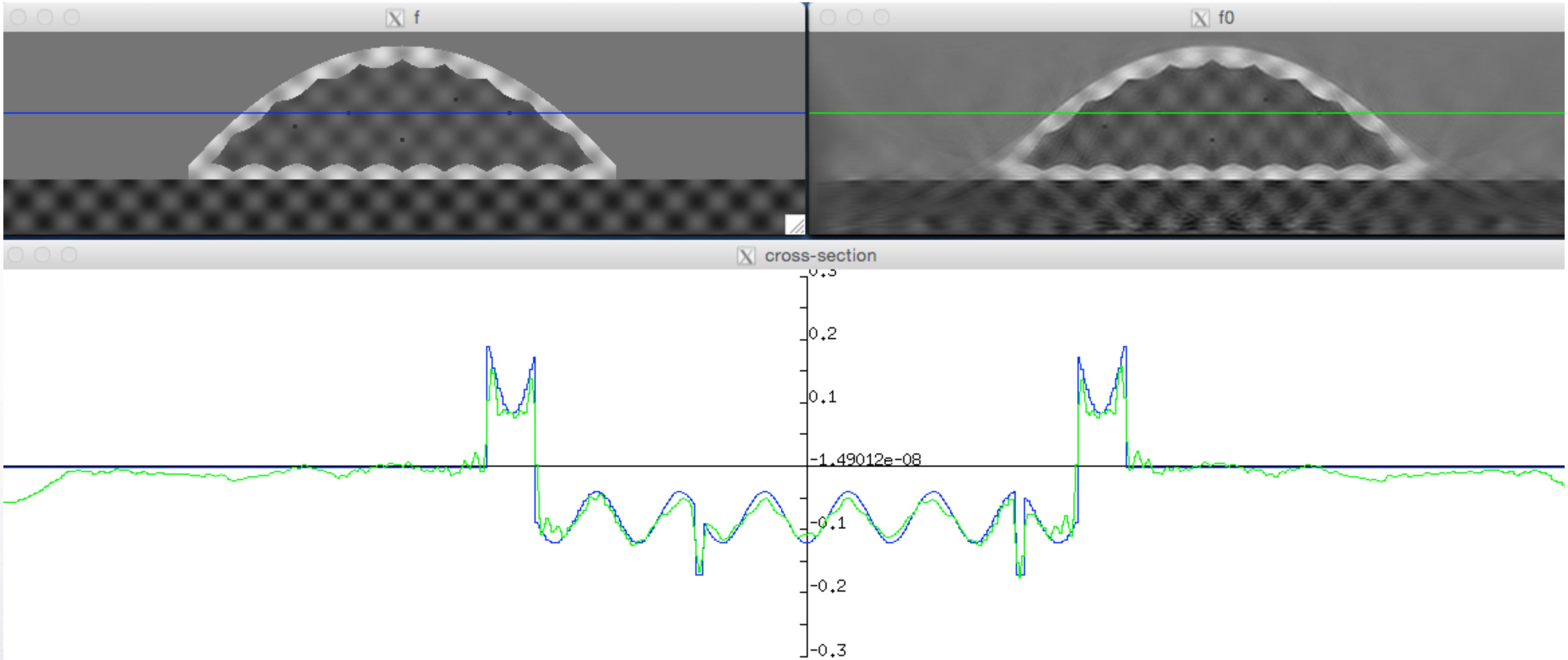
depth 5 cm

tumor diameter 1.5 mm

stepsize 0.5 mm

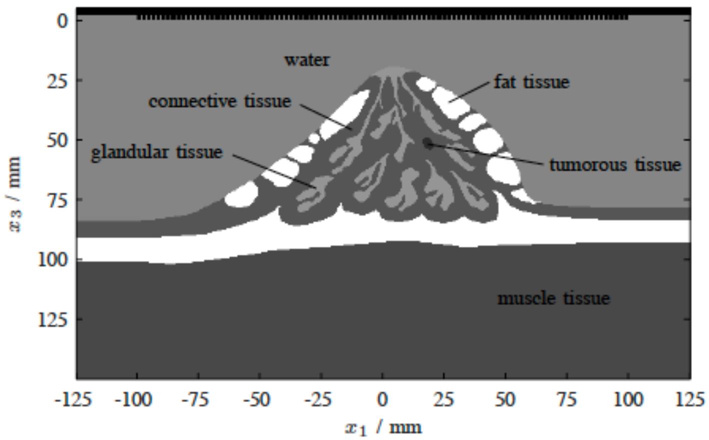


Plane wave stacking in reflection mammography

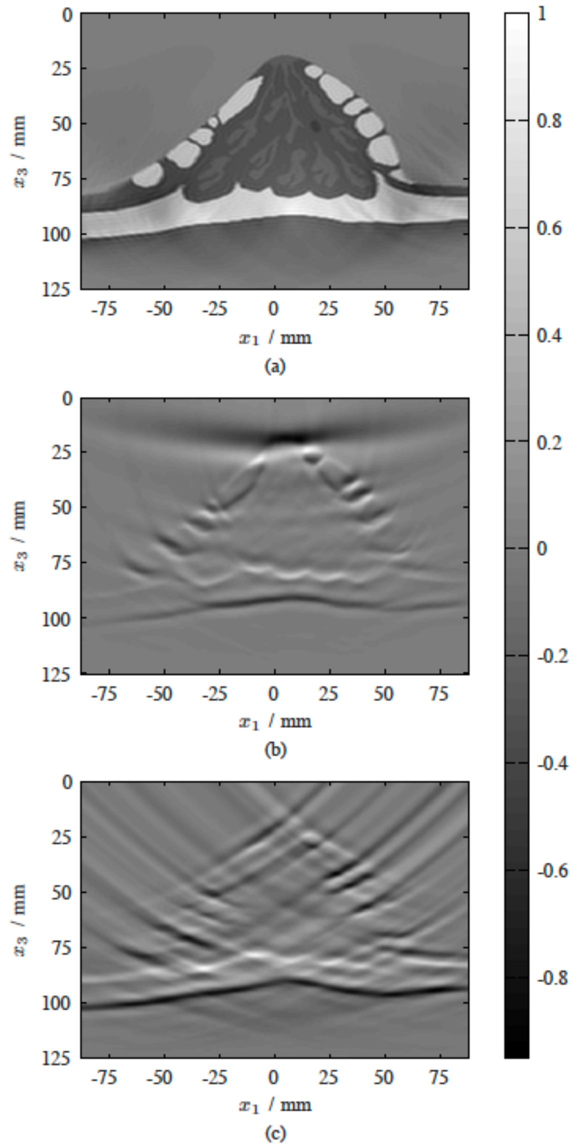


frequencies 15-500 kHz, wavelength 3mm, tumor diameter 1.5 mm, stepsize 0.5 mm, 20 sources, depth 5 cm

Reconstruction with various methods



Hesse & Schmitz 2012



Kaczmarz

SA
synthetic
aperture focusing

DAS
(delay and sum)

Layered medium

$f(x_1, x_2) = f(x_2)$.

Born approximation, one source at $x_1 = 0,\ x_2 = 0$:

$g_k(x) = (2\pi)^{-1/2} \int e^{-ix\xi} \hat{f}(-2\kappa(\xi))d\xi,\ \kappa = \sqrt{k^2 - \xi^2}$.

Finite aperture: Data available for $|x| \leq A$ only.

All we can determine: $\int \delta_A(\eta - \xi) \hat{f}(-2\kappa(\xi)d\xi,\ \delta_A(\xi) = \frac{A}{\pi} sinc(A\xi)$.

Determine $\hat{f}$ from

$$\int \delta_A(\eta - \xi)\hat{f}(-2\kappa(\xi))d\xi,\ \delta_A(\xi) = \frac{A}{\pi} sinc(A\xi),\ \kappa = \sqrt{k^2 - \xi^2}.$$

peaks in η, bandwidth A

bandwidth $2z|\kappa'(\xi)| = 2z|\xi|/\kappa(\xi)$

for line object at depth z:

$f(x) = \delta(x - z),\ \hat{f}(\xi) \sim e^{-iz\xi}$,

$\hat{f}(-2\kappa(\xi)) \sim e^{-2iz\kappa(\xi)}$ for $|\xi| < k$.

$\hat{f}(-2\kappa(\xi))$ can be stably

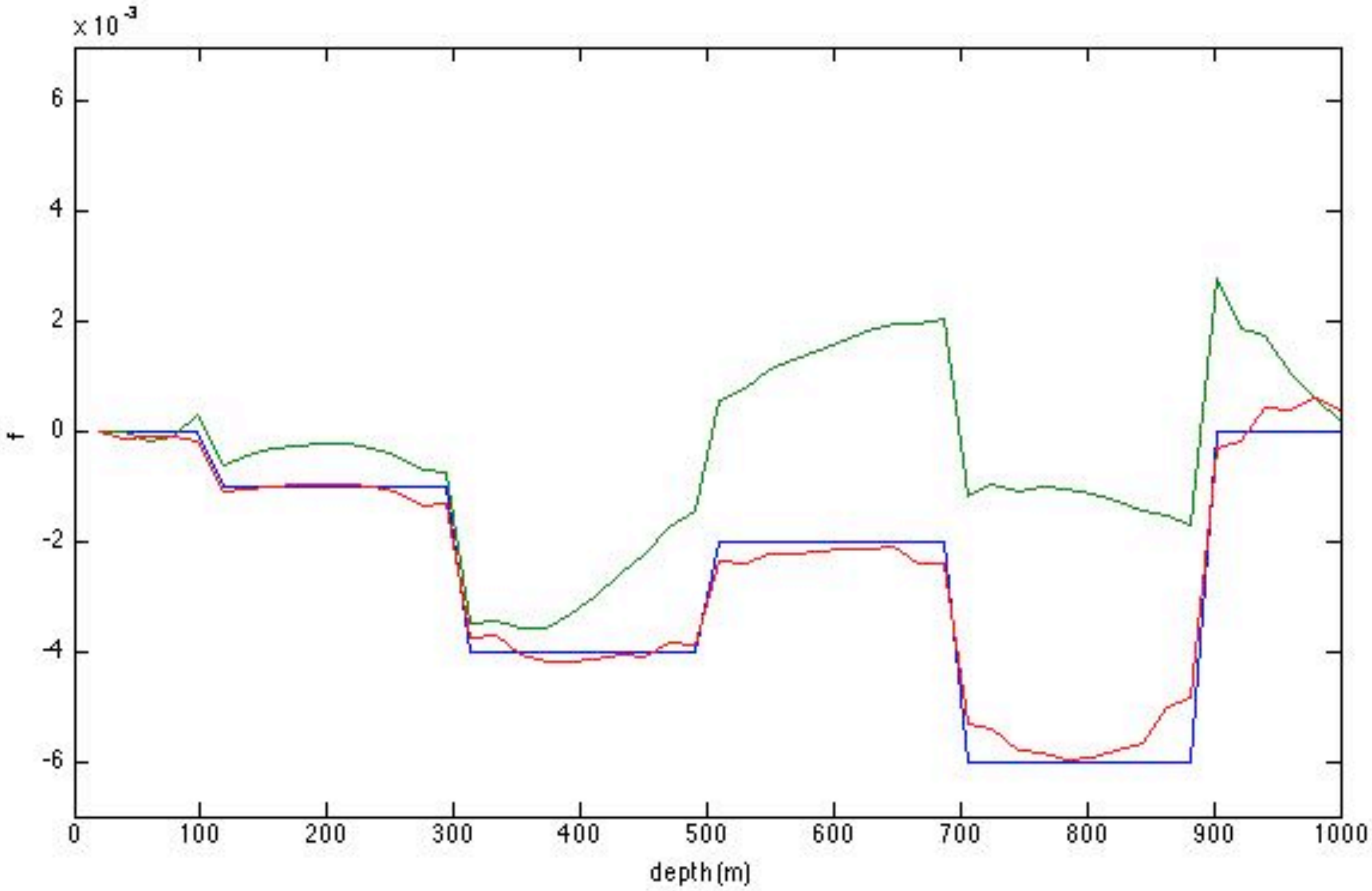
determined for $A > 2z|\xi|/\kappa(\xi)$

i.e. $\frac{2k}{\sqrt{1+A^2/4z^2}} < 2\kappa < 2k.$

Sirgue & Pratt 2004:

$\hat{f}(\xi)$ can be determined for $|\xi| \geq \dfrac{2k}{\sqrt{1 + A^2/4z^2}}$

Kaczmarz‘ method, frequencies 5-25 Hz

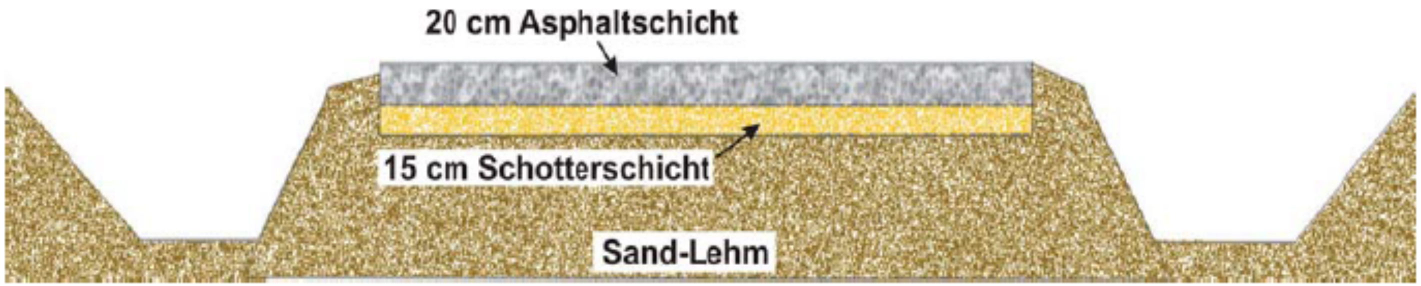


— true profile

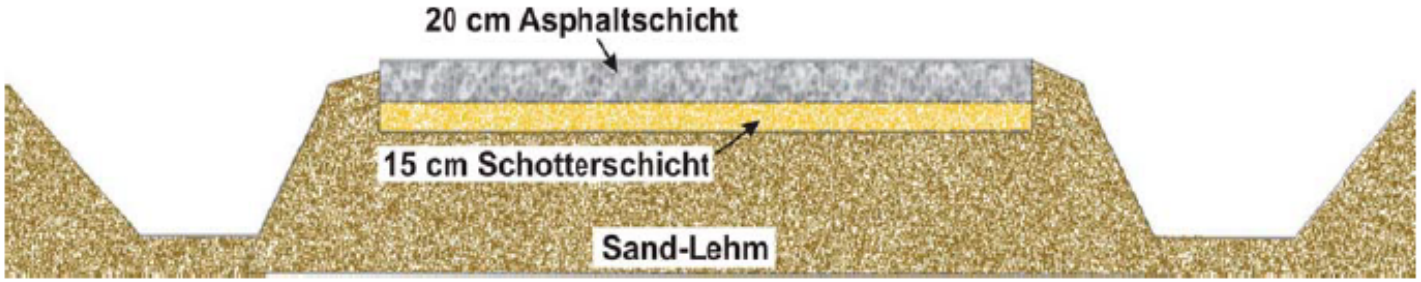
— Kaczmarz starting at f=0

— Kaczmarz with analytic continuation

BASt



BASt



Falling weight deflectometer (FWD)