

Scattering control and inverse problem for the wave equation with piecewise smooth wave speeds

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analysis of FWI, physics

- uniqueness, partial data
- stability
- conditional Lipschitz stability \rightarrow hierarchy of stable subspaces
- reconstruction through iterative regularization: FWI

time-harmonic:

Sylvester & Uhlmann ('87), Nachman ('88); dH, Scherzer & Qiu ('12), Beretta, dH, Qiu ('13), Datchev & dH ('16), Alessandrini, dH, Gaburro & Sincich ('17)

\rightarrow *hyperbolic* (high-frequency)

analysis of FWI, physics

subspaces tied to a domain partition
(unknown up to perturbations) – “rough”



frequency “domain” (bounded frequency)



time “domain”



“smooth” – wave speed and (completely unknown) interfaces

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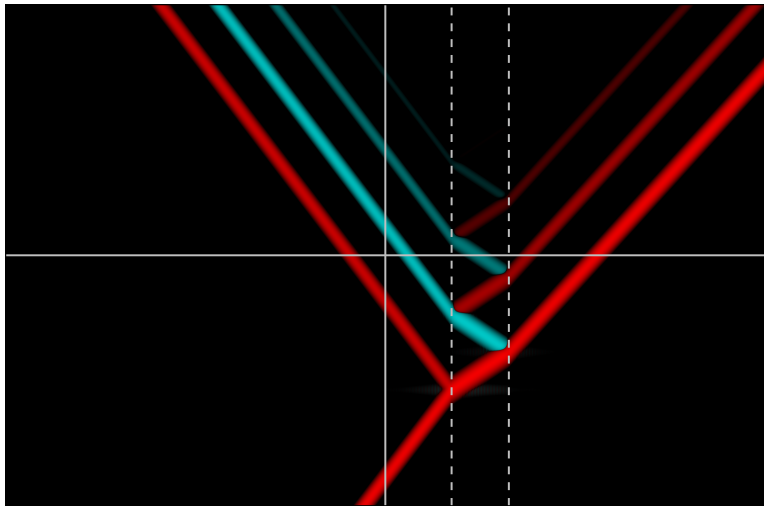
time “domain”



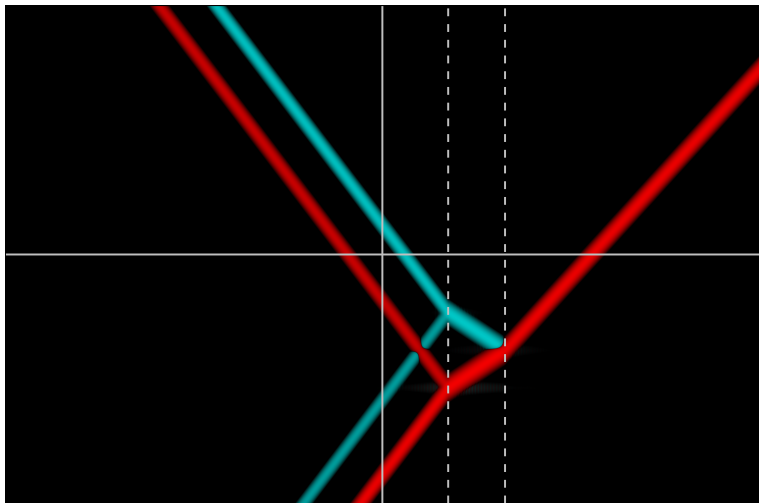
“smooth” – wave speed and (completely unknown) interfaces

- full-wave-based coordinate transformation reconstruction
- physics:
 - instantaneous time mirrors in extended subsurface
 - scattering control and detection of kinetic energy

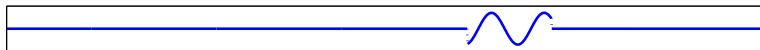
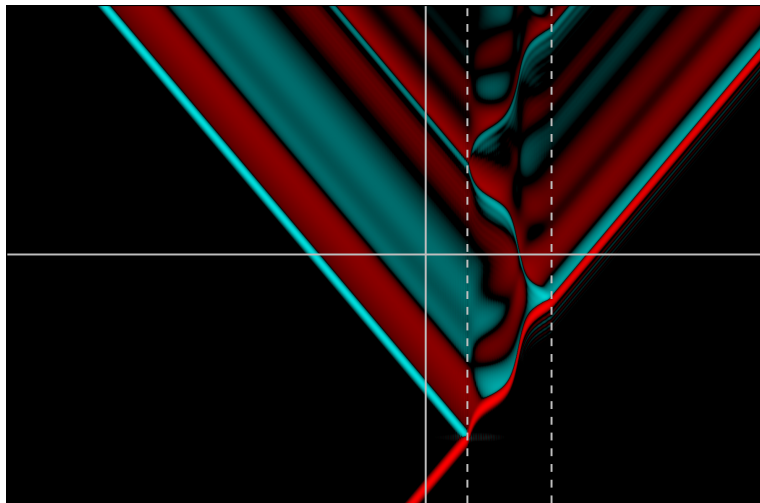
controlling multiple scattering — dimension 1



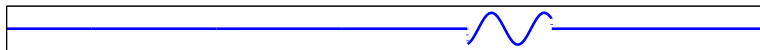
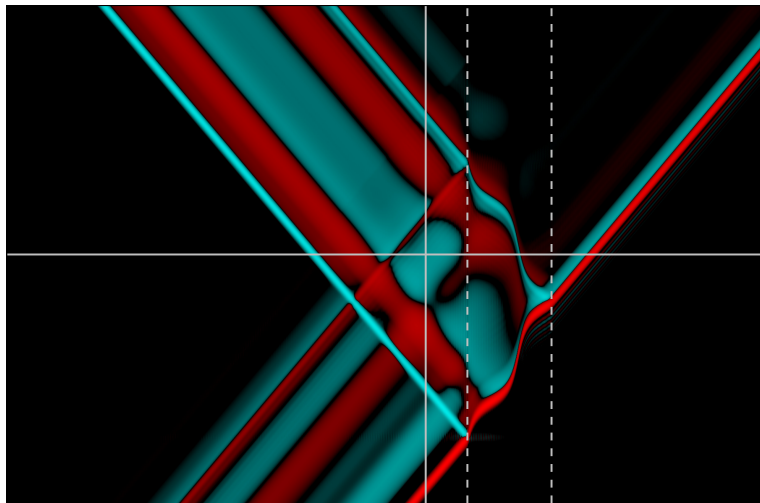
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- Marchenko's classical integral equation solves the inverse scattering problem in dimension one
- Rose (2002) developed an iterative procedure in dimension one, *single-sided autofocusing*, which focuses (geodesic coordinate) and related it to Marchenko's equation

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extend Marchenko approach to higher dimensional inverse problems (related work: Wapenaar *et al.*) using unique continuation (Tataru) and boundary control (Belishev)

assumptions

- $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain
- c is a scalar wave speed:
 - unknown and piecewise smooth (say) on Ω
 - known and smooth on $\Omega^* = \mathbb{R}^n \setminus \overline{\Omega}$

initial value problem and data model

let $h = (h_0, h_1) \in H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$; consider the IVP

$$F: h \mapsto u \text{ s.t. } \begin{cases} \partial_t^2 u - c^2 \Delta u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n \\ u(0, \cdot) = h_0 & \text{in } \mathbb{R}^n \\ \partial_t u(0, \cdot) = h_1 & \text{in } \mathbb{R}^n \end{cases}$$

reflection response after time s

$$R_s: H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \\ h \mapsto (Fh, \partial_t Fh) \Big|_{t=s}$$

known: $R_{2T}h|_{x \in \Omega^*}$ for Cauchy data h supported in Ω^* ,
 $T \in (0, \frac{1}{2} \text{diam } \Theta)$

- let $T > 0$, choose Lipschitz Θ, Υ s.t.

$$\bar{\Omega} \subset \Theta \subset \bar{\Theta} \subset \Upsilon$$

(think of $\Theta \approx \Omega$, and Υ a large ambient space)

Cauchy data, spaces

sublevel sets (d_{Θ}^* : signed distance to the boundary $\partial\Theta$)

$$\Theta_t = \{x \in \Upsilon \mid d_{\Theta}^*(x) > t\}$$

$$\Theta_t^* = \{x \in \Upsilon \mid d_{\Theta}^*(x) < t\}$$

(sub)spaces of Cauchy data

$$\tilde{\mathbf{C}} = H_0^1(\Upsilon) \oplus L^2(\Upsilon)$$

$$\mathbf{H}_t = H_0^1(\Theta_t) \oplus L^2(\Theta_t), \quad \mathbf{H} = \mathbf{H}_0$$

$$\tilde{\mathbf{H}}_t^* = H_0^1(\Theta_t^*) \oplus L^2(\Theta_t^*)$$

$\tilde{\mathbf{H}}^* \cap (R_{2T}(H_0^1(\mathbb{R}^n \setminus \bar{\Theta}) \oplus L^2(\mathbb{R}^n \setminus \bar{\Theta})))$: space of Cauchy data in $\tilde{\mathbf{C}}$ whose wave fields vanish on Θ at $t = 0$ and $t = 2T$; $R_{2T} : \mathbf{C} \rightarrow \mathbf{C}$ isometrically

\mathbf{C} : its orthogonal complement inside $\tilde{\mathbf{C}}$

\mathbf{H}_t^* : its orthogonal complement inside $\tilde{\mathbf{H}}_t^*$

inner product on \mathbf{C}

$$\langle (f_0, f_1), (g_0, g_1) \rangle = \int_{\Gamma} (\nabla f_0(x) \cdot \nabla \bar{g}_0(x) + c^{-2} f_1(x) \bar{g}_1(x)) dx$$

energy in open set $W \subseteq \mathbb{R}^n$

$$\mathbf{E}_W(h) = \int_W (|\nabla h_0|^2 + c^{-2} |h_1|^2) dx$$

kinetic energy

$$\mathbf{KE}_W(h) = \int_W c^{-2} |h_1|^2 dx$$

projections inside and outside Θ_t

- orthogonal projections

$$\pi_t : \mathbf{C} \rightarrow \mathbf{H}_t, \quad \pi = \pi_0$$

$$\pi_t^* : \mathbf{C} \rightarrow \mathbf{H}_t^*, \quad \pi^* = \pi_0^*$$

-

$$\bar{\pi}_t = I - \pi_t^*, \quad (\bar{\pi}_t h)(x) = \begin{cases} h(x), & x \in \Theta_t \\ (\phi(x), 0), & x \in \Theta_t^* \end{cases}$$

where ϕ is the harmonic extension of $h|_{\partial\Theta_t}$ to Υ
(with zero trace on $\partial\Upsilon$)

- set $R = v \circ R_{2T}$, where $v : (h_0, h_1) \mapsto (h_0, -h_1)$

$\pi^* R$: reflection response operator

scattering control series:

$$T \in (0, \frac{1}{2} \text{diam } \Theta)$$

given Cauchy data h_0 supported in $\Theta \setminus \overline{\Omega}$, define Neumann series

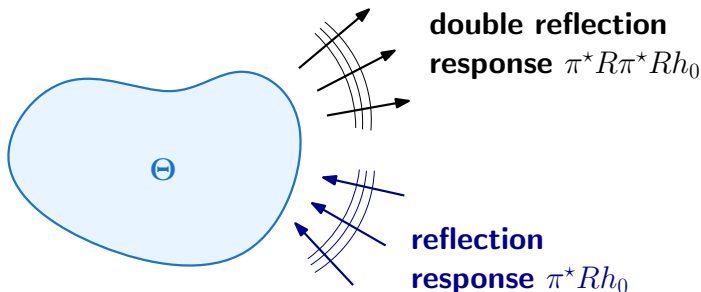
$$h_\infty = (I - \pi^* R \pi^* R)^{-1} h_0 = \sum_{i=0}^{\infty} (\pi^* R \pi^* R)^i h_0$$

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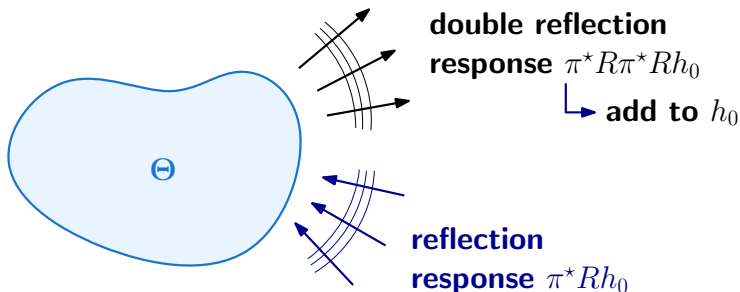


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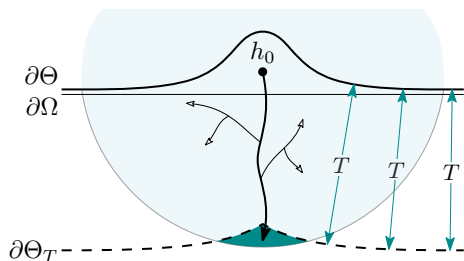
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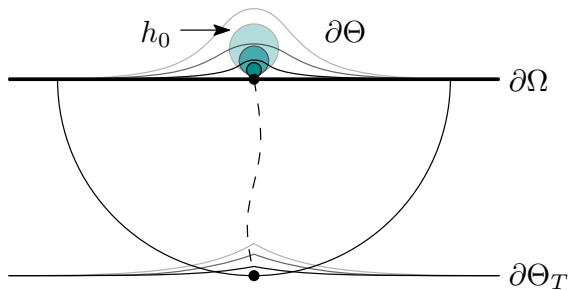
definition

- the T -sublevel set is $\Theta_T = \{x \in \Theta \mid d_\Theta^*(x) > T\}$
- the *almost direct transmission* of h_0 is $R_T h_0|_{\Theta_T}$
- the *harmonic almost direct transmission* is its harmonic extension, $h_{\text{DT}} = \bar{\pi}_T R_T h_0$



- Support of wave field, $t = T$
- Almost direct transmission
- Directly transmitted ray
- Scattered rays

direct transmission, limit



series behavior (theorem I)

Theorem

let $h_0 \in \mathbf{H}$ and $T \in (0, \frac{1}{2} \text{diam } \Theta)$; finding the wave field of the harmonic almost direct transmission of h_0 is equivalent to summing the scattering control series:

$$(I - \pi^* R \pi^* R) h_\infty = h_0 \iff \begin{array}{l} R_{-T} \bar{\pi} R_{2T} h_\infty = h_{DT} \\ \text{and} \\ h_\infty - h_0 \in \mathbf{H}^* \end{array}$$

such an h_∞ , if it exists, is essentially unique

proof

by unique continuation and finite speed of propagation; works for any c with these properties.

series behavior (theorem I)

Theorem

let $h_0 \in \mathbf{H}$ and $T \in (0, \frac{1}{2} \text{diam } \Theta)$; finding the wave field of the harmonic almost direct transmission of h_0 is equivalent to summing the scattering control series:

$$(I - \pi^* R \pi^* R) h_\infty = h_0 \iff R_{-s} \bar{\pi} T_{-s} R_{T+s} h_\infty = h_{DT} \quad \text{and} \quad s \in [0, T]$$
$$h_\infty - h_0 \in \mathbf{H}^*,$$

such an h_∞ , if it exists, is essentially unique

proof

by unique continuation and finite speed of propagation; works for any c with these properties.

series behavior (theorem II)

let h_k be the Neumann series's k^{th} partial sum

$$h_k = \sum_{i=0}^k (\pi^* R \pi^* R)^i h_0$$

Theorem

- *the wave field that h_{DT} generates can be recovered from $\{h_k\}$ regardless of convergence of the scattering control series:*

$$\lim_{k \rightarrow \infty} R_{-T} \bar{\pi} R_{2T} h_k = R_T \chi h_0 = h_{DT}$$

- $\{h_k\}$ converges in energy space on a dense set
- $\{h_k\}$ always converges in a larger weighted space (spectral theorem)

energy recovery (theorem I)

- energy conservation allows us to find the energy of the almost direct transmission using only outside-observable data

Theorem

the energy of the harmonic almost direct transmission (including harmonic extension) is

$$\mathbf{E}(h_{\text{DT}}) = \mathbf{E}(h_{\infty}) - \mathbf{E}(\pi^* R h_{\infty})$$

the kinetic energy (not including harmonic extension) is

$$\mathbf{KE}(h_{\text{DT}}) = \frac{1}{2} \langle h_0, h_0 - R\pi^* R h_{\infty} - R h_{\infty} \rangle$$

energy recovery (theorem II)

- even without convergence, one can recover the same energies as monotone limits

Theorem

the energy of the harmonic almost direct transmission (including harmonic extension) is

$$\mathbf{E}(h_{\text{DT}}) = \lim_{k \rightarrow \infty} [\mathbf{E}(h_k) - \mathbf{E}(\pi^* R h_k)]$$

the kinetic energy (not including harmonic extension) is

$$\begin{aligned} \mathbf{KE}(h_{\text{DT}}) = & \frac{1}{4} \lim_{k \rightarrow \infty} \left[\mathbf{E}(h_k) + \mathbf{E}(h_0) - \mathbf{E}(\pi^* R \pi^* R h_k) \right. \\ & \left. + 2 \langle \pi^* R h_k, h_k - R \pi^* R h_k \rangle - 2 \langle h_0, R \pi^* R h_k + R h_k \rangle \right] \end{aligned}$$

broken boundary normal coordinates

- set of disjoint, closed, connected, smooth hypersurfaces:
 $\Gamma = \bigcup \Gamma_i$
- $\{\Omega_j\}$: the connected components of $\mathbb{R}^n \setminus \Gamma$
- we call $x \in \Omega$ *regular* if $x \notin \Gamma$ and the infimum in $d(x, \partial\Omega) = d(\{x\}, \partial\Omega)$ is achieved by a unique purely transmitted broken path that is nowhere tangent to Γ

recovery of interfaces – direct broken transmission

transformation to half wave equations

$$\Phi^{-1} = \frac{1}{2} \begin{bmatrix} I & iB^{-1} \\ I & -iB^{-1} \end{bmatrix}, \quad B^2 = -c^2 \Delta$$

- principal symbol of the directly transmitted component \mathbf{DT}^+ of R^+ at $(p(y), \nu)$, where ν is the inward-pointing normal covector at p : $\mathbf{dt}^+(y)$
- wave packet of “frequency” λ centered at (x, ξ) : $\rho_{\lambda, x, \xi} \phi_{\lambda, x, \xi}$

strategy: *send in a wave packet, vary T , and track ADT energy*

- energy lost at each interface (discontinuity) to reflection
- drop sharper as frequency (λ) increases
- recover depths of interfaces in broken boundary normal coordinates

inverse problem – recovery of interfaces

Theorem

let $y \in \Omega_r$, $p = p(y)$, $T = d(y, p)$, $\varepsilon > 0$ be sufficiently small; then there exists a domain $\Theta \supset \Omega$ and covector $(p^*, \nu^*) \in S^*\Theta$ such that

$$|\mathbf{dt}^+(y)|^2 = \lim_{\lambda \rightarrow \infty} \mathbf{KE}_{\Theta_{T+\varepsilon}} R_{T+\varepsilon} \Phi \begin{bmatrix} -icB^{-1} \rho_{\lambda, p^*, \nu^*} \phi_{\lambda, p^*, \nu^*} \\ 0 \end{bmatrix}$$

because $\mathbf{dt}^+(y)$ is constant along a geodesic except at a discontinuity in c , we can recover the discontinuities of c in boundary normal coordinates:

if γ_y is the broken geodesic connecting y to the surface,

$$\gamma_y^{-1}(\Gamma) = \text{sing supp}(|\mathbf{dt}^+ \circ \gamma_y|)$$

- proof based on microlocal analysis
- energy of ADT governed by principal symbol

recovery of transformation of coordinates

- $f + tg$ is a solution of the wave equation for any f, g harmonic
- this holds for the coordinate functions x^i
-

$$\lim_{k \rightarrow \infty} \bar{\pi} R_{2T} h_k = R_T \bar{\pi}_T R_T h_0$$

it is possible to compute inner products of $\bar{\pi}_T R_T h_0$, for arbitrary h_0 , with arbitrary harmonic Cauchy data (f, g)

- the ADT $\bar{\pi}_T R_T h_0$ may be arbitrarily spatially concentrated, aside from harmonic extensions in the first component
- by taking inner products with $(0, x^i)$ and $(0, 1)$ we recover weighted averages of x^i over this support; take the limit
- by choosing h_0 and a sequence of choices for Θ tending to Ω , Euclidean coordinates for every point $(p, T) \in \Omega_r$ in broken boundary normal coordinates can be obtained
- c is recovered immediately by taking derivatives in T

Theorem

let c be piecewise smooth with discontinuities on a set of disjoint hypersurfaces; assume that c, Ω are such that set of regular points is dense in Ω

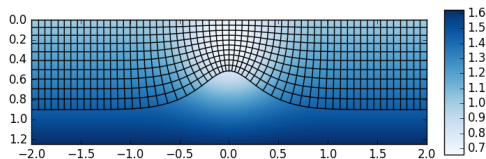
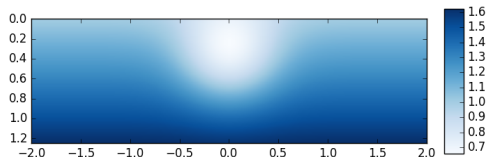
then c is uniquely determined on Ω_T^ by $R_{2T}|_{\Omega^*}$*

proof

- completely constructive
- makes use of behavior of solutions near the boundary of their domain of influence

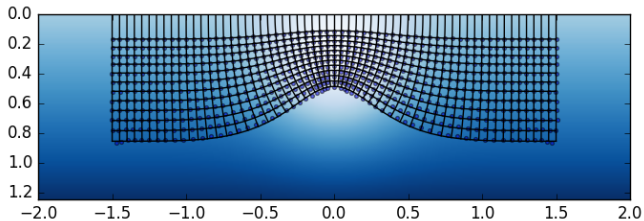
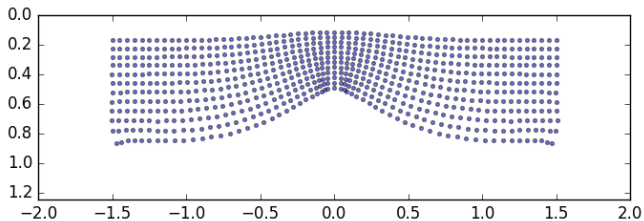
computational experiment – smooth wave speed

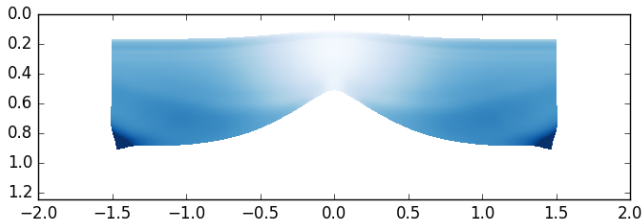
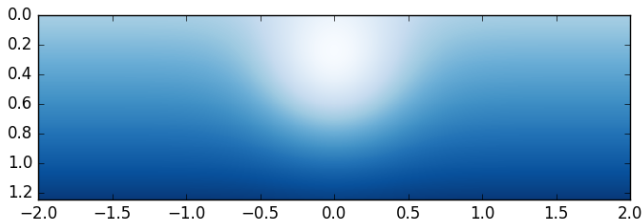
- $T = 1.0$
- $\Gamma = [-3.0, 3.0] \times \{0\}$
- $\mathcal{R} = [-4.5, 4.5] \times \{0\}$



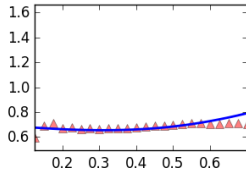
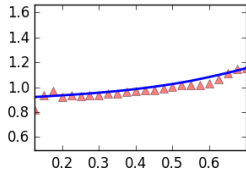
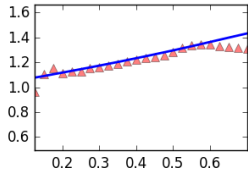
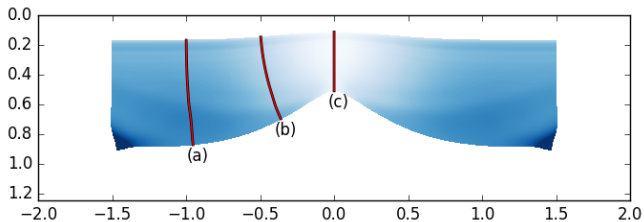
$$c(x_1, x_2) = 1 + \frac{1}{2}x_2 - \frac{1}{2} \exp(-4(x_1^2 + (x_2 - 0.375)^2))$$

coordinate reconstruction from boundary data





wave speed reconstruction (comparison along rays)



“smooth” – wave speed and (completely unknown) interfaces

- instantaneous time mirrors in extended subsurface
- scattering control and detection of kinetic energy
- full-wave-based coordinate transformation reconstruction