Bandwidth extension for wave-based imaging

Laurent Demanet

Department of Mathematics, MIT

Nam Nguyen, IBM
Elita Li, NUS

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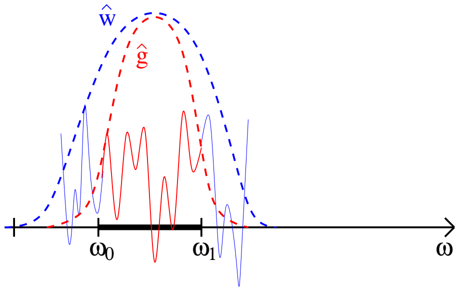
Bandwidth extension...



- Which signal model?
- How far can we hope to extrapolate?
- What level of confidence?

... for wave-based imaging

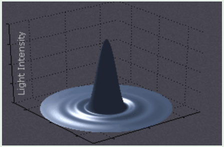
$$\widehat{u}(\omega) = \widehat{g}(\omega)\widehat{v}(\omega)$$

$\widehat{g}(\omega)$ is

- PSF (image processing)
- pulse (radar)
- wavelet (seismic)

Blur at scale $2\pi/\omega_1$

Rayleigh diffraction limit



For illustration, reflection model

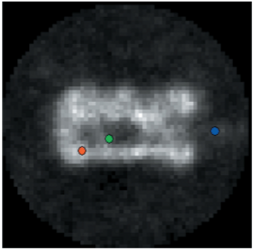
$$\widehat{v}(\omega) = \int G_c(x_S, x; \omega)\rho(x)G_c(x, x_R; \omega)dx + h.o.t.,$$

background velocity c, reflectivity ρ

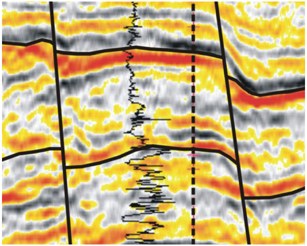
High frequencies

Beyond ω_1: super-resolution

Find small-scale details that seem absent from data



Credit: Papson & Narayanan



Credit: CO2CRC

Super-resolved microscopy: W. E. Moerner.

Low frequencies

Below ω_0: inverse scattering

The dependence on c is "least nonlinear" at low frequencies:

$$G_c(x, y; \omega) \sim e^{i\omega\tau(x,y)} \quad \Rightarrow \quad \frac{\delta^2 \widehat{v}}{\delta c^2} \sim \omega^2$$

FWI: solve for (c, ρ) starting with low frequencies only, slowly grow the window.
EFWI: seed FWI on synthesized low frequencies

$$\text{Extrapolation to low } \omega \quad \Leftrightarrow \quad \text{Inversion of } c, \rho$$

Outline

$$\widehat{v}(\omega) = \int G_c(x_S, x; \omega)\rho(x) G_c(x, x_R; \omega) dx + h.o.t.$$

but $c, \rho, \widehat{g}(\omega)$ are unknown.

1. Sparse superposition of complex exponentials (signal processing)
2. Wave-inspired model reduction (seismology)

1. Sparse model

In place of

$$\widehat{v}(\omega) = \int G_c(x_S, x; \omega)\rho(x) G_c(x, x_R; \omega) dx + h.o.t.,$$

use instead

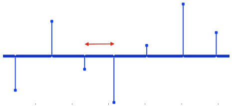
$$\widehat{v}(\omega) = \sum_{j=1}^{k} a_j e^{i\omega\theta_j} + e(\omega)$$

... because wave recordings can often be decomposed into coherent events:

- $G_c(x, y, \omega) \sim e^{i\omega\tau(x,y)}$
- the singularities of $\rho(x)$ matter most

In time: a spike train

$$v(t) = \sum_{j=1}^{k} a_j \delta(t - \theta_j) + \text{error}$$



Super-resolution question

Minimum recoverable distance $\tau = \min_{i \neq j} |\theta_i - \theta_j|$ vs.

- band limit $\Omega = \omega_1 - \omega_0$,
- sparsity k,
- noise level $\sigma = \|e\|_2$.

Extrapolation: determine a_j, θ_j, then evaluate $\widehat{v}(\omega)$ for new ω

Practical indications, ca. 1980

Safe, pessimistic answer: Nyquist/Rayleigh sampling $\tau \geq \frac{2\pi}{\Omega}$.

Radioastronomy, ultrasound, NMR spectroscopy: super-resolution seems possible when $\tau < \frac{2\pi}{\Omega}$. Sparsity helps.

Classes of algorithms:

1. Subspace / MUSIC (MUltiple SIgnal Classification)
2. Matrix pencil / HSVD / FRI etc.
3. ℓ_1 minimization to recover a sparse x from y
4. Matching pursuits: remove spikes 1-by-1

Donoho, 1992: super-resolution is real

$$\widehat{v}(\omega) = \sum_{j=1}^{k} a_j e^{i\omega\theta_j} + e(\omega), \qquad |\omega| \leq \frac{\Omega}{2}$$

Parameters: $\theta_j \in \tau\mathbb{Z}$, and $\mathrm{SRF} = \frac{2\pi}{\tau\Omega}$. (Nyquist: SRF $= 1$.)

Theorem (Donoho)

Let $S_k = \{k\text{-sparse } x\}$, and consider the minimax error

$$E_{minmax} = \inf_{\widetilde{v}} \sup_{v \in S_k} \sup_{\|e\|_2 \leq \sigma} \|\widetilde{v} - v_0\|_2.$$

Then

$$C_{1,k}(SRF)^{2k-1}\sigma \leq E_{minmax} \leq C_{2,k}(SRF)^{2k+1}\sigma$$

Donoho's original model: sparse clumps of cardinality k.

Minimax theory, 2015

An answer to Donoho's question:

Theorem (D., Nguyen, 2015)

The recoverability scaling is

$$E_{minmax} \asymp C_k \, (SRF)^{2k-1} \, \sigma$$

Additionally, MUSIC reaches this bound.

Consequences for extrapolation:

- Go from band of size Ω to band of size $(\mathrm{SRF})\times\Omega$,
- Conditioning penalty factor $(\mathrm{SRF})^{2k-1}$.

2. Wave-inspired model reduction

In place of

$$\widehat{v}_{R,S}(\omega) = \int G_c(x_S, x; \omega) \rho(x) G_c(x, x_R; \omega) dx + h.o.t.,$$

use a sparse superposition of atomic phases

$$\widehat{v}_{R,S}(\omega) = \sum_{j=1}^{k} a_j(x_R, x_S; \omega) e^{i(\omega \tau_j(x_R, x_S) + \phi_j(x_R, x_S; \omega))}$$

where all functions are as smooth as possible (not oscillatory)

- $a_j(x_R, x_S; \omega)$: geometrical spreading, wavelet inaccuracies
- $\tau_j(x_R, x_S)$: arrival times
- $\phi_j(x_R, x_S; \omega)$: dispersive corrections

Model reduction

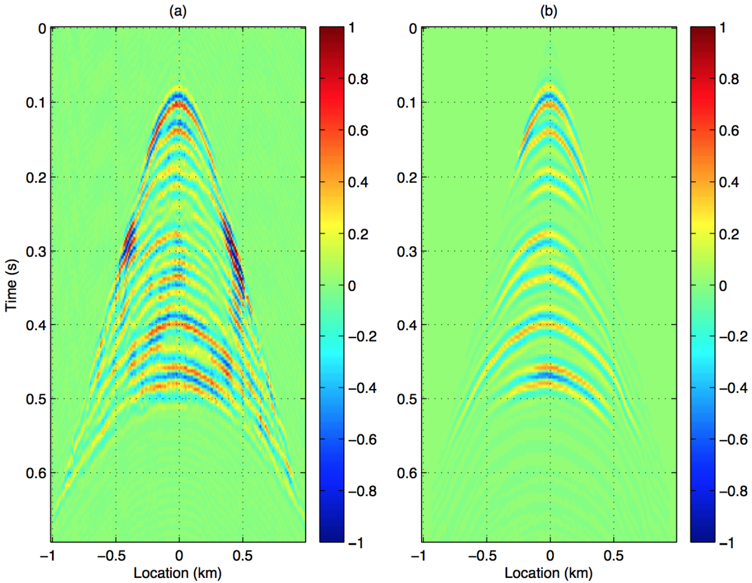
In principle: minimize LS data misfit, while keeping ϕ_j, $\nabla \phi_j$, ∇a_j, $\nabla \tau_j$ small.

Strong nonconvexity!

Best handled by the tracking heuristic:

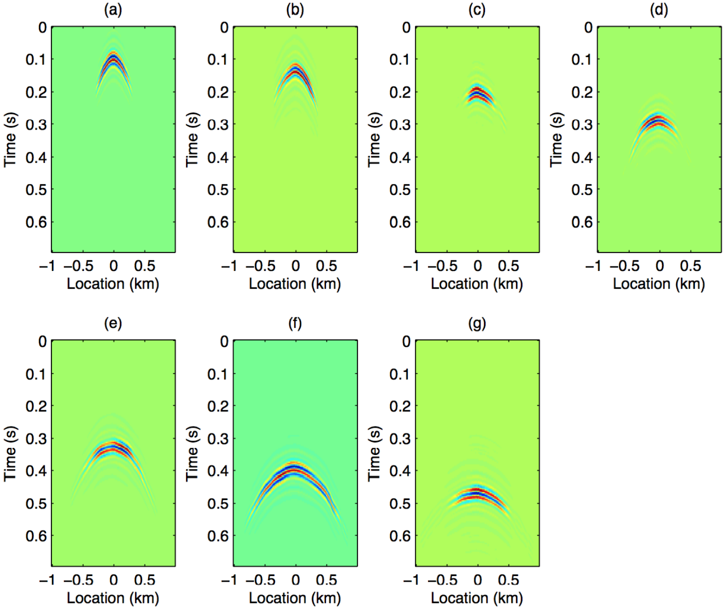
- Initialize with a good linear-phase finder (like MUSIC)
- Iterate to fit the nonlinearities in a_j and ϕ_j.
- Carefully grow a trust region in ω and x_R, x_S.

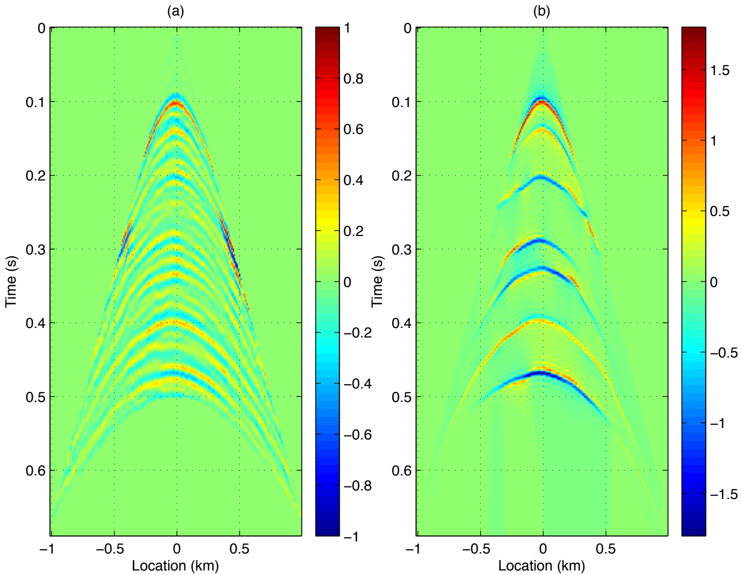
Extrapolation: polynomial fits for a_j, ϕ_j, and evaluate formula for new values of ω.



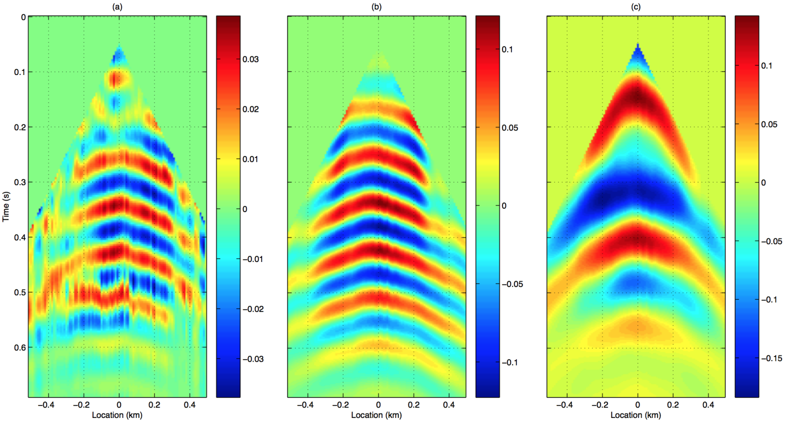
(a)
(b)
Time (s)
Location (km)
Location (km)
0
0.1
0.2
0.3
0.4
0.5
0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

Band: [16, 70] Hz.


(a)
(b)
(c)
(d)
(e)
(f)
(g)
Time (s)
0
0.1
0.2
0.3
0.4
0.5
0.6
Location (km)
−1
−0.5
0
0.5

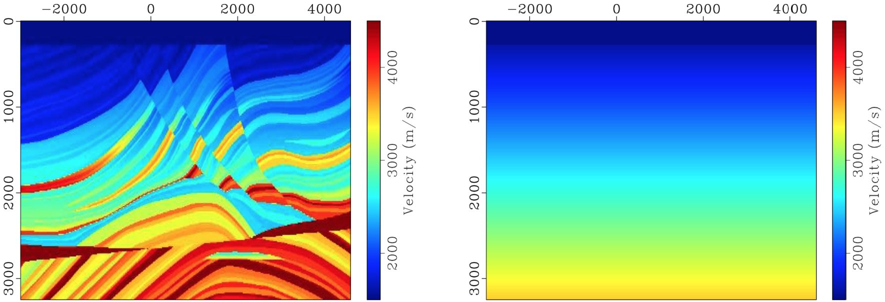


$[10, 85]$ Hz (true) $[0.5, 120]$ Hz (extrap).



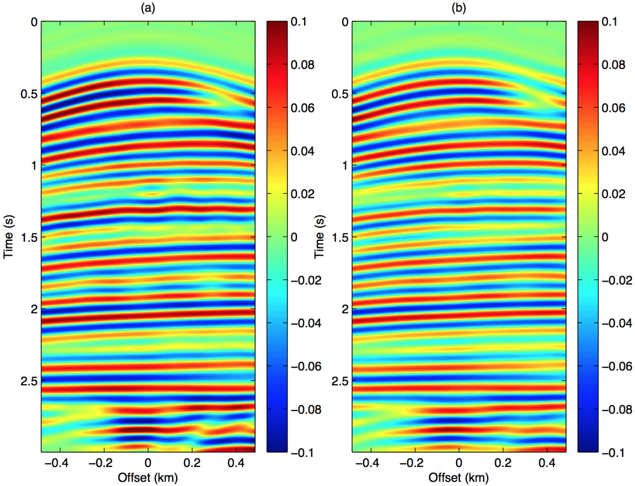
[10, 16] Hz (true) [10, 16] Hz (extrap) [0.5, 10] Hz (extrap)

Example 2: Marmousi at [5, 15] Hz



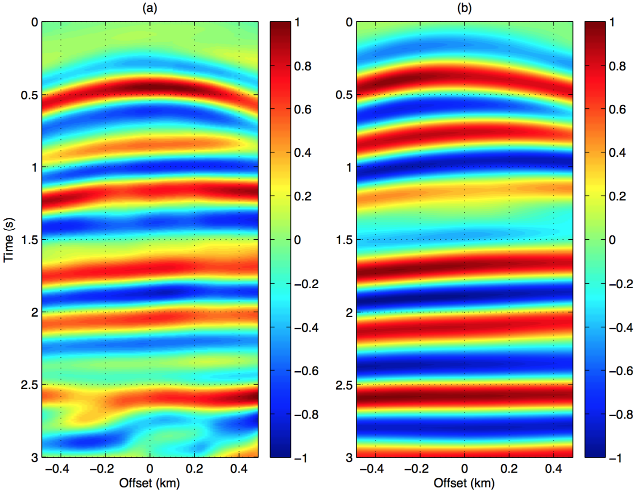
Original vs. initial

Example 2: Marmousi at [5, 15] Hz



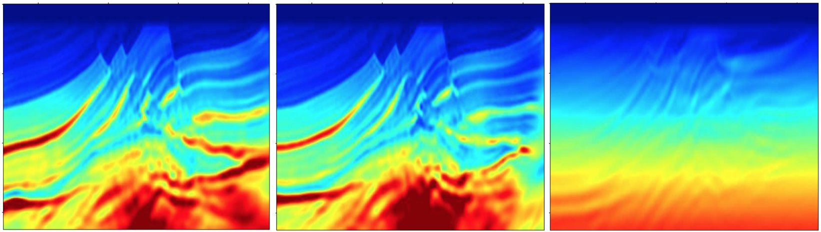
Synthesized data, [5, 15] Hz, vs. phase tracking fit (9 events)

Example 2: Marmousi at [5, 15] Hz



Unavailable data, [1, 5] Hz, vs. extrapolated data

Example 2: Marmousi at [5, 15] Hz



FWI [1, 15] Hz, vs. EFWI [5, 15] Hz, vs. FWI [5, 15] Hz
(Cheating) (New method) (Old method)

Conclusions

Recent results help understand what is possible with extrapolation:

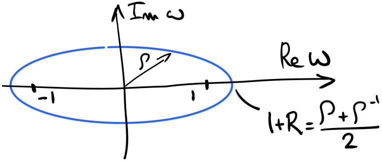
- k-sparsity enables provable bandwidth extension
- model reduction with phases is robust enough to handle seismic shot records

Outlook / other topics:

- smoothness enables extrapolation up to a fraction of the characteristic smoothness length
- model extension can be used in place of model reduction for frequency extrapolation

Bonus slides

1. Analytic model



$$\left\| \frac{d^n}{d\omega^n} \widehat{v}(\omega) \right\|_{L^\infty(-1,1)} \leq Q\, R^{-n}\, n!$$

... because the scattering poles of $G_c(x, y; \cdot)$ are hopefully far from the real axis:

- radiation condition
- no resonant cavity
- attenuation

Main result

Noise level $\sigma = \|e\|_\infty$, N equispaced points, Bernstein radius ρ.
Least-squares fit by a polynomial $p_M(\omega)$ of degree

$$M = \min\{\frac{\sqrt{N}}{2}, \log_\rho(1/\sigma)\}$$

Theorem (D., Townsend, 2016)

Let $r(\omega) = \frac{\omega+\sqrt{\omega^2-1}}{2}$, *so that* $r \to 1^-$ *when reaching the ellipse. Then*

$$|\widehat{v}(\omega) - p_M(\omega)| \leq C_{\rho,\sigma} \frac{1}{1 - r(\omega)} \sigma^{\alpha(\omega)},$$

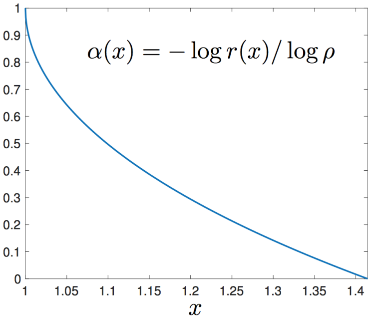
where

$$\alpha(\omega) = -\log_\rho(r(\omega)).$$

This rate is minimax optimal, and N does not enter the rate.

$\alpha(\omega)$

$$|\widehat{v}(\omega) - p_M(\omega)| \leq C_{\rho,\sigma} \frac{1}{1 - r(\omega)} \sigma^{\alpha(\omega)}$$



In short, the news are bad.

Model extension

1. LSERTM: solve in the ULS sense

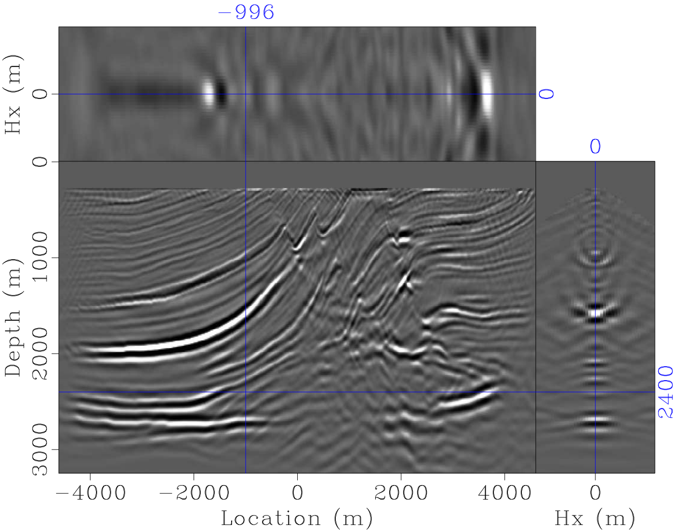
$$\min_{\overline{\rho}} \iint |\overline{\rho}(x,h)|^2 dxdh \qquad \text{s.t.} \qquad \widehat{v} = F_e(\overline{\rho})$$

2. Better: TV regularization

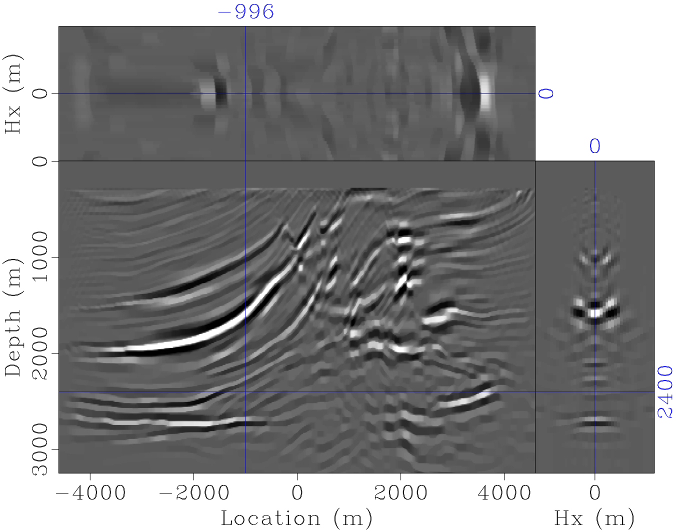
$$\min_{\overline{\rho}} \lambda \iint |\nabla_{x,h}\overline{\rho}(x,h)| dxdh + \|\widehat{v} - F_e(\overline{\rho})\|_2^2$$

Extrapolation: run the F_e model with a different wavelet $\widehat{w}(\omega)$. Results similar to model reduction.

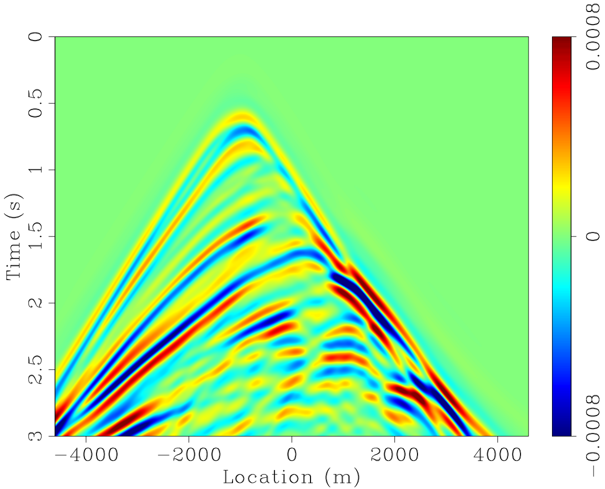
Model extension



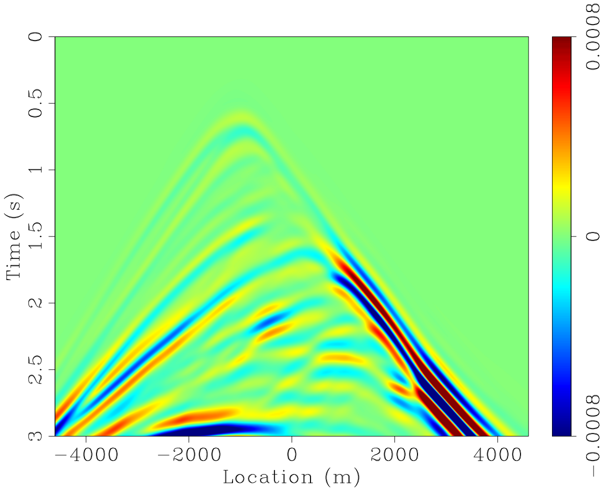
Model extension



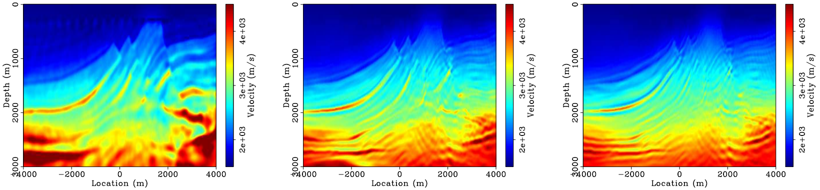
Model extension



Model extension



Model extension



FWI [2, 50] Hz, vs. EFWI [6, 50] Hz, vs. FWI [6, 50] Hz
(Cheating) (New method) (Old method)