

# Phase-field modeling of a fluid-driven fracture in a poroelastic medium

Andro Mikelić

Université Lyon 1, Institut Camille Jordan, FRANCE



UNIVERSITÉ  
DE LYON



joint work with M.F. Wheeler (UT Austin), T. Wick (Ecole Polytechnique)

Talk at the IPAM Workshop *Multiphysics, Multiscale, and Coupled Problems in Subsurface Physics*, April 5, 2017

# Corresponding literature

Detailed proofs of the results from the talk are in the article

[\*] **A. Mikelić , M.F. Wheeler, T. Wick** : Phase-field for fluid-filled fractures in poroelasticity, *Computational Geosciences*, Vol. 19(2015), no. 6, 1171-1195.

Related articles are:

A. Mikelić, M. F. Wheeler, T. Wick: A quasistatic phase field approach to fluid filled fractures, *Nonlinearity*, 28 (2015), 1371-1399.

A. Mikelić, M. F. Wheeler, T. Wick: A Phase-Field Method For Propagating Fluid-Filled Fractures Coupled To A Surrounding Porous Medium, *SIAM Multiscale Model. Simul.*, Vol. 13 (2015), no. 1, 367–398.

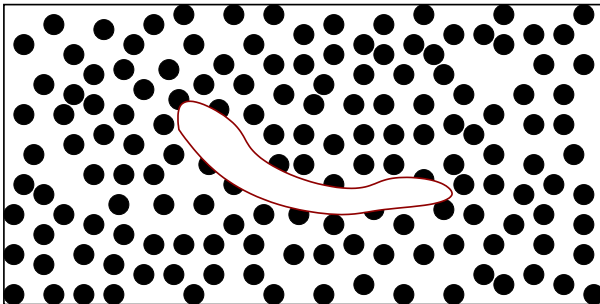
S. Lee, A. Mikelić, M. F. Wheeler, T. Wick: Phase-field modeling of proppant-filled fractures in a poroelastic medium, *Comput. Methods Appl. Mech. Engrg.*, 312 (2016), p. 509-541.

# The hydraulic fracture propagation in a poroelastic medium

- A technique used to release petroleum and natural gas that includes shale gas, tight gas, and coal seam gas for extraction.
- 95% of oil and gas wells drilled today are hydraulically fractured.
- Environmental benefits of producing natural gas, much of which is produced in the United States from fracking.
- From the negative side: environmental impacts such as contamination of ground water, risks to air quality, migration of fracturing chemical and surface contamination . . . .

For this reason, hydraulic fracturing is being heavily scrutinized resulting in the **need for accurate and robust mathematical and computational models** for treating fluid field fractures surrounded by a poroelastic medium.

# The interface fracture/poroelastic medium



**Figure:** Sketch of the geometry.

A crack set  $\mathcal{C}$  is embedded in a poroelastic medium  $\Omega \subset \mathbb{R}^3$ .  
 $B = (0, L)^3 = \Omega \cup \bar{\mathcal{C}}$ .

# Fundamental Equations

The quasi-static Biot equations in  $\Omega$ :

$$\sigma^{por} - \sigma_0 = \mathcal{G}e(\mathbf{u}) - \alpha p_R \mathbf{I}; \quad - \operatorname{div} (\sigma^{por}) = \rho_b \mathbf{g}; \quad (1)$$

$$\partial_t \left( \frac{1}{M} p_R + \operatorname{div} (\alpha \mathbf{u}) \right) + \operatorname{div} \left( \frac{\mathcal{K}_R}{\eta_R} (\rho_R \mathbf{g} - \nabla p_R) \right) = q, \quad (2)$$

(Reservoir Flow Model + Geomechanical Model)

In standard references

(e.g. Dean, R. H., Schmidt, J. H. (2009). Hydraulic-fracture predictions with a fully coupled geomechanical reservoir simulator. SPE Journal, 14(04), 707–714.)

the fracture is 2D.

– Lubrication approximation for fracture flow:

$$\mathbf{Q} = -\frac{w^3}{12\eta} (\nabla p - \rho g \nabla h) \quad (3)$$

# 2D Fracture Model

- Conservation of mass in the fracture:

$$\frac{\partial}{\partial t}(\rho w) = - \operatorname{div}(\rho \mathbf{Q}) + q_w - q_L, \quad (4)$$

where  $q_w$  accounts for well injection into the fracture cells,  $q_L$  accounts for the leakoff from the fracture face into the reservoir,  $p$  is the fluid pressure in the fracture and  $w$  is the fracture width.

The fluid pressure in the fracture controls the flow rate in the fracture as well as acts as a normal traction condition for the geomechanical calculations.

- Propagation is modeled by releasing nodes at the fracture tip that previously had zero displacement constraints.

# Classical Fracture Propagation Models

Dean and Schmidt use 2 fracture propagation criteria for determining when and where the fracture will grow.

The first propagation option uses a critical fracture opening criteria based upon the stress intensity factor. The  $K_{IC}$  fracture propagation option releases nodes at the fracture tip whenever the fluid pressure at the tip of the fracture is large enough to keep the new fracture area open, and the fracture width at a neighboring node exceeds

$$w_{neighbor} \geq \frac{4(1 - \nu^2)}{E} K_{IC} \sqrt{\frac{2\Delta a_{neighbor}}{\pi}},$$

where  $\Delta a_{neighbor}$  is the distance from a neighboring node to the node at the fracture tip.

The second fracture propagation criterion is based on modeling of a cohesion zone in the fracture. The fracture fluid flows to the tip of the cohesive zone and there are cohesive forces attempting to hold the fracture forces together (poroplastic calculations).

# Classical modeling of the interface conditions poroelastic medium–fracture

Classically it is supposed

- $w = -[\mathbf{u}]_c \cdot \mathbf{n}$ ;
- $\sigma^{por} \mathbf{n}|_c = -p\mathbf{n}|_c$ .

In addition in

A lubrication fracture model in a poro-elastic medium, V Girault, MF Wheeler, B Ganis, ME Mear, *Mathematical Models and Methods in Applied Sciences* 25 (2015), 587–645, it is supposed that  $\mathbf{Q} \cdot \mathbf{n}|_c$  is equal to the leakage. In their model the fracture does not move.

In a series of papers on fracture spreadings in an "impermeable elastic medium", Detournay and collaborators suppose given tip velocity, which is equal to the average fluid velocity.

# Difficulties with modeling the tip between the fracture and the poroelastic medium

In

Existence of solutions for a higher order non-local equation appearing in crack dynamics, C Imbert, A Mellet, Nonlinearity 24(2011) 3487–3514,

give a mathematically rigorous analysis of the 1D crack spreading. At the tip the zero contact angle condition + zero fluid flux + zero toughness are imposed.

It is well known that the **lubrication approximation is not valid at the fracture tip.**

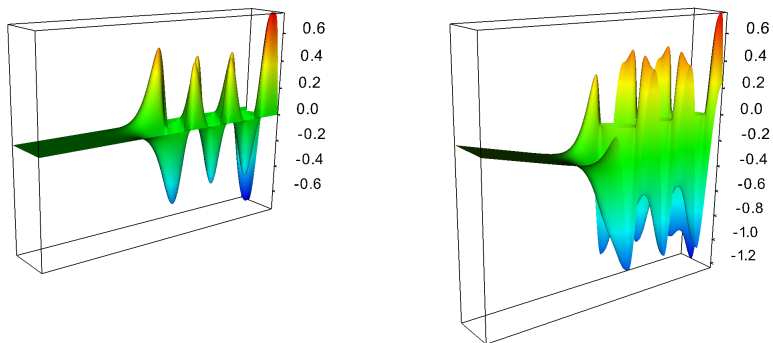
# Difficulties with modeling the interface between the fracture and the poroelastic medium

Is the condition  $\sigma^{por} \mathbf{n}|_c = -p\mathbf{n}|_c$  correct? More generally, what are the conditions to be imposed at the interface porous medium – free flow?

The case of a rigid porous medium. 2 different cases:

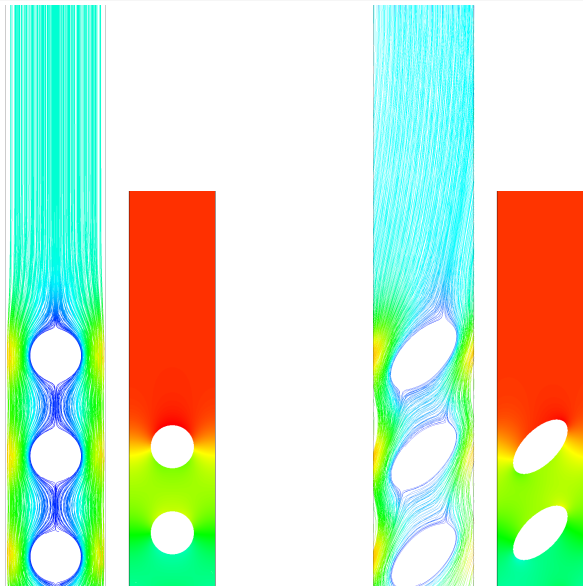
**[A]** The case of a forced infiltration. After T. Levy, E. Sanchez-Palencia, On boundary conditions for fluid flow in porous media, International Journal of Engineering Science, 13 (1975) 923-940, the interface conditions are (a) a **constant effective pressure** and (b) the **continuity of the normal velocities**. After T. Carraro, C. Goll, A. Marciniak-Czochra, A. Mikelić *Comput. Methods Appl. Mech. Engrg.*, 292 (2015) 195–220. the third interface condition is the **jump of the tangential velocity**.

# The tangential velocity close to the interface



Visualisation of  $v_1^\varepsilon$ : (a) Circular inclusions. (b) Elliptical inclusions.

# The microscopic velocity and pressure for (a) circular and (b) oval inclusions



# the interface between the fracture and the poroelastic medium

**[B]** The case of a tangential flow over a porous bed. After G.S. Beavers, D.D. Joseph, Boundary conditions at a naturally permeable wall, J. Fluid Mech., 30 (1967) 197-207, there is an effective slip law at the interface  $v_1 = C\sqrt{k}\frac{\partial v_1}{\partial x_2}$ . It was justified in

W. Jäger, A. Mikelić, On the interface boundary conditions by Beavers, Joseph and Saffman, SIAM J. Appl. Math., 60(2000) 1111-1127.

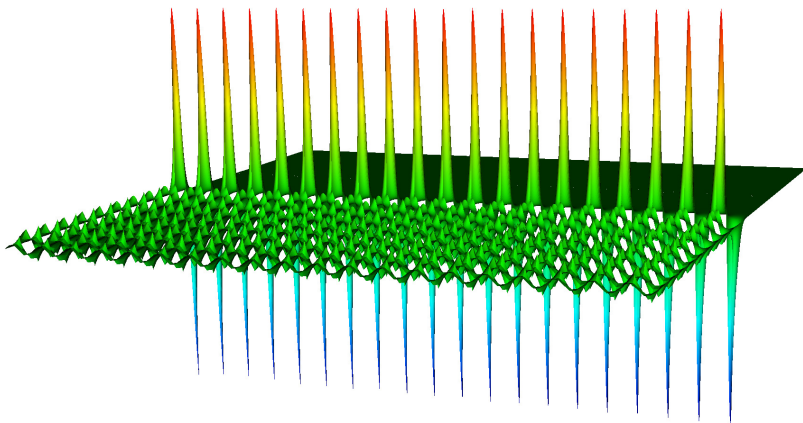
In addition, there is a pressure jump. See

T. Carraro, C. Goll, A. Marciniak-Czochra, A. Mikelić, Pressure jump interface law for the Stokes-Darcy coupling: Confirmation by direct numerical simulations, *J. Fluid Mech.*, Vol. 732 (2013), p. 510-536.

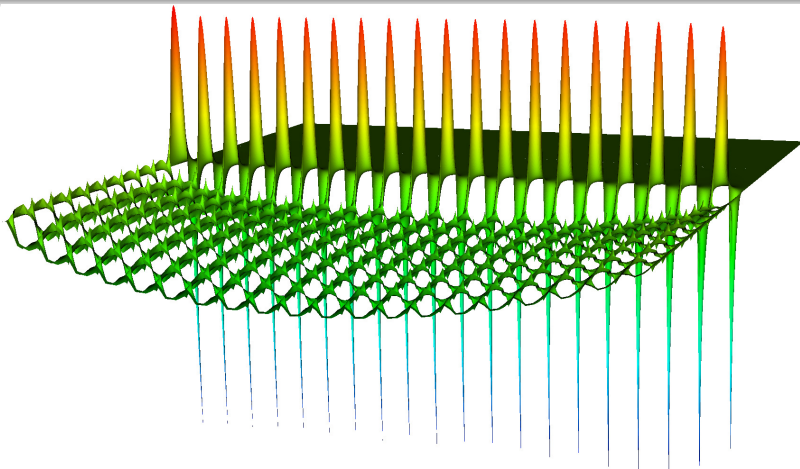
A. Marciniak-Czochra, A. Mikelić, Effective pressure interface law for transport phenomena between an unconfined fluid and a porous medium using homogenization, *SIAM Multiscale Model. Simul.*, Vol. 10, No. 2 (2012), p. 285-305.

The pressure jump law at the interface reads

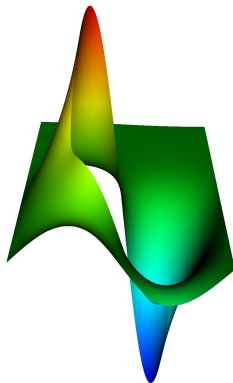
$$p^{fl} + C_\omega \frac{\partial v_1}{\partial x_2} = p^{porousmedium}.$$



**Figure:** Numerical simulation for  $p^\epsilon$ , corresponding to the conditions of the experiment by Beavers and Joseph: Circles as inclusions with  $\epsilon = 10^{-1}$ .



**Figure:** Numerical simulation for  $p^\epsilon$ , corresponding to the conditions of the experiment by Beavers and Joseph: Ellipses as inclusions with  $\epsilon = 10^{-1}$ .



**Figure:** Pressure close to the interface for ellipsoidal inclusions

# Direct modeling of the interface free fluid-poroelastic medium

Determination of the interface conditions between a poroelastic medium and an elastic medium, from the pore level model, was undertaken by A.M. and M.F. Wheeler in M3AS, Vol. 22 (2012). Determination of the interface conditions between a poroelastic medium and a free fluid flow using homogenization, from the pore level, is an open problem. After the pioneering "energy estimates compatible" conditions introduced by Showalter in 2005 and Murad in 2000, a more detailed physically based direct modeling is presented in articles by Badia, Quaini, Quarteroni, J. Comp. Phys. 228 (2009) and Bukač, Yotov, Zakerzadeh, Zunino CMAME 292 (2015).

For more details we refer to the talk of Ivan Yotov.

A good understanding of the interface is crucial for all approaches to the hydraulic fracturing.

# Summary of the existing approaches

Even in the most basic formulation, hydraulic fracturing is complicated to model since it involves the coupling of (i) mechanical deformation; (ii) the flow of fluids within the fracture and in the reservoir; (iii) fracture propagation.

Generally, rock deformation is modeled using the theory of linear elasticity which can be represented by an integral equation that determines a relationship between fracture width and the fluid pressure. Fluid flow in the fracture is modeled using lubrication theory that relates fluid flow velocity, fracture width and the gradient of pressure. Fluid flow in the reservoir is modeled as a Darcy flow and the respective fluids are coupled through a leakage term. The criterion for fracture propagation is usually given by the conventional energy-release rate approach of linear elastic fracture mechanics (LEFM) theory; that is the fracture propagates if the stress intensity factor at the tip matches the rock toughness.

# The goal of this research

The **goal of this talk** is to develop a complete and robust phase field model for a fluid-driven fracture propagation in a poroelastic medium with the solid structure displacement, pressure and the phase-field variable as unknowns.

The novelties in this coupled approach are:

- Solving a fluid equation inside the fracture interacting with the fluid in the bulk poroelastic medium;
- Constructing the appropriate Lyapunov functional for the coupled three-field (displacement-phase field-pressure) problem, which is linked to the phase-field modified free energy.
- Establishing mathematical stability of the three-field (displacement-phase field-pressure) equation, through constructing the corresponding Lyapunov functional and demonstrating effectiveness by providing several representative numerical tests.

# Equations in the fracture and the interface conditions I

Let  $\mathcal{C}$  denote any open set homeomorphic to an ellipsoid in  $\mathbb{R}^3$  (a crack set). Its boundary is a closed surface  $\partial\mathcal{C}$ . In most applications  $\mathcal{C}$  is a curved 3D domain, with one dimension significantly smaller than the dominant two. An example is a penny shape crack.

The crack set  $\mathcal{C}$  is surrounded by the poroelastic domain  $\Omega = B \setminus \bar{\mathcal{C}}$ , where  $B = (0, L)^3 \supset \supset \mathcal{C}$ .

Due to the fracture size, the relative velocity satisfies Darcy's law

$$\mathbf{v}_F = \frac{\mathcal{K}_F}{\eta_F} (\mathbf{G} - \nabla p_F). \quad (5)$$

Note that for a penny shaped  $\mathcal{C}$  the dominant velocity part is two-dimensional. We have an explicit expression for the permeability  $\mathcal{K}_F$  as a function of the fracture shape. The term  $\mathbf{G}$  is a linear function of the gravity and leakage.

# Equations in the fracture and the interface conditions II

In the fracture  $\mathcal{C}$ , the pressure satisfies the mass conservation equation

$$c_F \partial_t p + \operatorname{div} \left( \frac{\mathcal{K}_F}{\eta_F} (\mathbf{G} - \nabla p) \right) = q. \quad (6)$$

Modeling the moving interface  $\partial\mathcal{C}$  is delicate. We impose the following interface conditions on the moving interface  $\partial\mathcal{C}$ :

$$p_R = p_F \quad \text{and} \quad \sigma^{por} \mathbf{n} = -p_F \mathbf{n}, \quad (7)$$

$$\mathbf{v}_F \cdot \mathbf{n} = \mathbf{v}_R \cdot \mathbf{n} = \frac{\mathcal{K}_R}{\eta_R} (\rho_R \mathbf{g} - \nabla p_R) \cdot \mathbf{n}, \quad (8)$$

where  $\mathbf{n}$  is the normal exterior to the fracture.

# Modeling fractures with a phase-field approach in solid mechanics

In fracture mechanics, Griffith's criterion states that crack propagation occurs when the elastic energy restitution rate reaches its critical value  $G_c$ . If  $\tau$  is the traction force applied at the part of the boundary  $\partial_N\Omega$ , then one can associate to the crack  $\mathcal{C}$  the following total energy

$$E(\mathbf{u}, \mathcal{C}) = \int_{\Omega} \frac{1}{2} \mathcal{G} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{u}) \, dx - \int_{\partial_N\Omega} \boldsymbol{\tau} \cdot \mathbf{u} \, dS - \int_{\Omega} F \operatorname{div} \mathbf{u} \, dx + G_c \mathcal{H}^2(\partial\mathcal{C}), \quad (9)$$

where  $F$  is a volume force. In fracture mechanics the crack is considered as a lower dimensional manifold and would correspond to  $\partial\mathcal{C}$ .

In their seminal work (J. Mech. Phys. Solids 46 (1998)), Francfort and Marigo introduced a phase field regularization of the term  $G_c \mathcal{H}^2(\partial\mathcal{C})$ .

# Specific challenges of phase-field fracture in poroelasticity

The situation in poroelasticity is more complicated than the classical one from solid mechanics.

First, we recall that our fracture is a (thin) 3D body and not a surface, with dimensions being small compared to  $B$ , but large compared to the characteristic pore size  $\ell$ . At the pore level we do have a complex interaction between fracture and the pore skeleton. Therefore we can not have the term  $G_c \mathcal{H}^2(\partial \mathcal{C})$ .

Modeling the pore scale fracture growth is presently out of reach.

Nevertheless, the experimental observation is that if the stresses (respectively the free energy) cross a threshold, the fracture will start growing and propagating.

# Specific challenges of phase-field fracture in poroelasticity II

Our strategy is to generalize the notion of the free energy for poroelasticity in order to include the phase-field. Essentially, we will follow the ideas of Francfort and Marigo and add to the free energy the phase field regularization of the term  $G_c \mathcal{H}^2(\partial C)$ .

Since the observed and simulated fractures correspond to smeared two-dimensional surfaces, our treatment is consistent with ideas from the literature (see e.g. C. Miehe et al, International Journal of Numerical Methods in Engineering, 83 (2010)).

# Biot's free energy density

Biot's free isothermal energy density of the fluid-solid system,  $W$  depends on the strain tensor  $e(\mathbf{u})$  and on the increment of the fluid content  $\zeta = p/M + \alpha \operatorname{div} \mathbf{u}$ , i.e.  $W = W(e, \zeta)$ .

$$p = \frac{\partial W}{\partial \zeta} \quad \text{and} \quad \sigma = \frac{\partial W}{\partial e},$$

$$W(e, \zeta) = \frac{1}{2} \mathcal{G} e(\mathbf{u}) : e(\mathbf{u}) + p^2 / (2M) = \frac{1}{2} \mathcal{G} e(\mathbf{u}) : e(\mathbf{u}) + \frac{M}{2} \left( \zeta - \alpha \operatorname{div} \mathbf{u} \right)^2.$$

Our idea is to generalize this energy and to use the "phase field" free energy. The crack phase field  $\varphi$  is intuitively a regularization of  $1 - \chi c$  and we impose its negative evolution  $\partial_t \varphi \leq 0$ .

$W$  has to take into the account the following effects:

# Generalization of Biot's free energy density

- The degradation of the elastic properties in the fracture, described by the multiplication of the Gassmann tensor  $\mathcal{G}$  by the function  $A$ , such that

$A(\varphi)$  is a smooth monotone increasing function satisfying  
 $A(0) = k > 0$ ,  $k \ll \varepsilon$ ,  $A'(0) = 0$  and  $A(1) = 1$ . (10)

A usual choice is  $A(\varphi) = (1 - k)\varphi^2 + k$ .

- The change of the porosity being different in the poroelastic part and in the fracture, and now described through a function  $c$  satisfying

$c(\varphi)$  is a smooth bounded function such that  $c(0) = c_F > 0$ ,  
 $\inf_z c(z) = c_{min} > 0$ ,  $c'(0) = 0$  and  $c(1) = 1/M$ . (11)

# Generalization of Biot's free energy density II

Note that the coefficients  $A$  and  $c$  are naturally defined on  $[0, 1]$ . We extend them for negative values of argument by the value at  $\varphi = 0$  and pose  $A(\varphi) = A(\varphi_+)$  and  $c(\varphi) = c(\varphi_+)$ . Due to constraint  $\partial_t \varphi \leq 0$ ,  $\varphi$  will always remain smaller or equal to 1. Our new 'phase-field' free energy density including fractures is now

$$W_\varepsilon(\mathbf{e}, \zeta, \varphi) = \frac{1}{2}A(\varphi_+)\mathcal{G}\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{u}) + \frac{c(\varphi_+)p^2}{2} + G_c \left( \frac{1}{2\varepsilon}(1-\varphi)^2 + \frac{\varepsilon}{2}|\nabla\varphi|^2 \right)$$

We note the presence of the term allowing the fracture to grow, once the threshold energy is crossed. Furthermore the phase field unknown  $\varphi$  should allow a smooth change from the fracture zone to the poroelastic zone.

# The elastic part energy functional

We suppose a given pressure, but bear in mind that this pressure is linked to the displacement and the phase-field unknown. Then we borrow the energy functional from fracture theory of solid mechanics and modify by taking into the account the porosity change. Using the phase-field functional for the fracture spreading, with given pressure, was initiated by B. Bourdin et al, A Variational Approach to the Numerical Simulation of Hydraulic Fracturing, SPE Journal, Conference Paper 159154-MS, 2013.

$$\begin{aligned}\tilde{\mathcal{E}}_\varepsilon(\mathbf{u}, \varphi) = & \int_B \frac{1}{2} A(\varphi) \mathcal{G} e(\mathbf{u}) : e(\mathbf{u}) \, dx - \int_{\partial_N B} \boldsymbol{\tau} \cdot \mathbf{u} \, dS - \int_{\partial C} \boldsymbol{\tau} \cdot \mathbf{u} \, dS \\ & - \int_B \alpha \varphi p \operatorname{div} \mathbf{u} \, dx + G_c \int_B \left( \frac{1}{2\varepsilon} (1 - \varphi)^2 + \frac{\varepsilon}{2} |\nabla \varphi|^2 \right) \, dx \\ & + \int_B \frac{1}{2} \left( \frac{1}{M} - c(\varphi_+) \right) p^2 \, dx.\end{aligned}\quad (12)$$

The last term is constant with respect to the displacement  $\mathbf{u}$  and does not contribute to the displacement equation.

# Interface law for the normal stresses on the fracture boundary

Now we observe that the interface term  $\int_{\partial\mathcal{C}} \boldsymbol{\tau} \cdot \mathbf{u} \, dS$  (12) has to be transformed into a volume term. Here, we model the fracture pressure using an interface law. After undertaking integrations by part,  $\tilde{\mathcal{E}}_\varepsilon$  transforms to

$$\begin{aligned} \hat{\mathcal{E}}_\varepsilon(\mathbf{u}, \varphi) &= \int_B \frac{1}{2} A(\varphi_+) \mathcal{G} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) \, dx - \int_{\partial_N B} (\boldsymbol{\tau} \cdot \mathbf{u} + p u_n) \, dS \\ &\quad + \int_\Omega ((1 - \alpha) p \operatorname{div} \mathbf{u} + \nabla p \cdot \mathbf{u}) \, dx \\ &+ G_c \int_B \left( \frac{1}{2\varepsilon} (1 - \varphi)^2 + \frac{\varepsilon}{2} |\nabla \varphi|^2 \right) \, dx + \int_B \left( \frac{1}{M} - \frac{c(\varphi_+)}{2} \right) p^2 \, dx. \end{aligned} \tag{13}$$

We suppose that the crack  $\mathcal{C}$  does not interact with  $\partial_N \Omega$ . Hence there is no need to insert  $\varphi$  in the boundary integral over  $\partial_N B$ .

# Phase-field formulation of the pressure cross terms

It remains to extend the pressure cross term

$$\int_{\Omega} ((1 - \alpha)p \operatorname{div} \mathbf{u} + \nabla p \cdot \mathbf{u}) \, dx$$

to a term with integration over  $B$  by adding  $\varphi$ . Due to the complexity of our model, it is not clear that we will be able to prove the nonnegativity of  $\varphi$ . Hence we multiply the integrand by  $\varphi_+^2$  yielding

$$\begin{aligned} \mathcal{E}_\varepsilon(\mathbf{u}, \varphi) &= \int_B \frac{1}{2} A(\varphi_+) \mathcal{G} e(\mathbf{u}) : e(\mathbf{u}) \, dx - \int_{\partial_N B} (\boldsymbol{\tau} \cdot \mathbf{u} + p u_n) \, dS \\ &\quad + \int_B \varphi_+^2 \left( (1 - \alpha)p \operatorname{div} \mathbf{u} + \nabla p \cdot \mathbf{u} \right) \, dx \\ &+ G_c \int_B \left( \frac{1}{2\varepsilon} (1 - \varphi)^2 + \frac{\varepsilon}{2} |\nabla \varphi|^2 \right) \, dx + \int_B \left( \frac{1}{M} - \frac{c(\varphi_+)}{2} \right) p^2 \, dx. \end{aligned} \tag{14}$$

# Fréchet derivative of $\mathcal{E}_\varepsilon$

As in the literature,  $\{\mathbf{u}, \varphi\}$  are critical points of the non-convex functional  $\mathcal{E}_\varepsilon$ . Taking the Fréchet derivatives yields

$$-\operatorname{div} \left( (A(\varphi_+) \mathcal{G}e(\mathbf{u})) \right) + \varphi_+^2 \nabla p + (\alpha - 1) \nabla(\varphi_+^2 p) = 0 \text{ in } B, \quad (15)$$

$$\begin{aligned} \mathcal{L}(\mathbf{u}, \varphi) = & -G_c \varepsilon \Delta \varphi - \frac{G_c}{\varepsilon} (1 - \varphi) + \frac{A'(\varphi_+)}{2} \mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) - \frac{c'(\varphi_+)}{2} p^2 \\ & + 2(1 - \alpha) \varphi_+ p \operatorname{div} \mathbf{u} + 2\varphi_+ \nabla p \cdot \mathbf{u} \leq 0 \text{ in } B, \end{aligned} \quad (16)$$

$$\partial_t \varphi \leq 0 \text{ in } B, \quad (17)$$

$$\partial_t \varphi \cdot \mathcal{L}(\mathbf{u}, \varphi) = 0 \text{ in } B. \quad (18)$$

Here, the last expression (18) is a complementarity condition (namely the Rice condition) linked to the irreversibility constraint (17).

# Adding reservoir and fracture pressure equations

Under the assumption of the weak compressibility of the fluid, the mass conservation equation reads (2) in the poroelastic part and (6) in the fracture. It can be written as the pressure diffusion problem

$$\theta \partial_t p - \operatorname{div} \left( K_{\text{eff}} (\nabla p - \tilde{\mathbf{G}}) \right) = \tilde{q} \quad \text{in } B, \quad (19)$$

where

$$\begin{aligned} \theta = \theta(x, t) &= \chi_{\Omega} \frac{1}{M} + \chi_C c_F, \quad \tilde{q} = \tilde{q}(x, t) := \chi_{\Omega} \frac{q_R}{\rho_R} - \alpha \chi_{\Omega} \partial_t \operatorname{div} \mathbf{u} \\ &+ \chi_C \frac{q_F}{\rho_F}; \quad \tilde{\mathbf{G}} = \chi_C \mathbf{G}, \quad K_{\text{eff}} = \chi_{\Omega} \frac{\mathcal{K}_R}{\eta_R} + \chi_C \frac{\mathcal{K}_F}{\eta_F}. \end{aligned}$$

# Adding reservoir and fracture pressure equations II

The formulation contains the interface between  $\Omega$  and  $\mathcal{C}$ , which changes in time, as the crack propagates. Our idea is to replace the diffraction problem (19) by a phase field equation for the pressure diffraction equation. Following the distinction between terms containing  $\alpha - 1$  as factor and others from equation (15), we have the following global problem for the pressure:

$$\begin{aligned} \partial_t(c(\varphi_+)p) - \operatorname{div} (K_{\text{eff}}(\varphi_+)(\nabla p - \tilde{\mathbf{G}}(\varphi_+))) + (\alpha - 1)\partial_t(\varphi_+^2 \operatorname{div} \mathbf{u}) \\ + \operatorname{div} (\partial_t(\varphi_+^2 \mathbf{u})) = \tilde{q} \text{ in } B. \end{aligned} \quad (20)$$

# The fully coupled system for $\{\mathbf{u}, \varphi, p\}$

Find  $\mathbf{u}, \varphi, p$  such that:

$$-\operatorname{div} \left( A(\varphi_+) \mathcal{G}e(\mathbf{u}) \right) + \varphi_+^2 \nabla p + (\alpha - 1) \nabla(\varphi_+^2 p) = 0 \text{ in } B, \quad (21)$$

$$\mathbf{u} = 0 \text{ and } -K_{\text{eff}}(\varphi_+) (\nabla p - \tilde{\mathbf{G}}(\varphi_+)) \cdot \mathbf{n} = v^{\text{inj}} \text{ on } \partial_D B, \quad (22)$$

$$-\mathcal{G}e(\mathbf{u}) \mathbf{n} = \tau + p^{\text{bdry}} \mathbf{n} \text{ on } \partial_N B, \quad (23)$$

---

$$\begin{aligned} -G_c \varepsilon \Delta \varphi - \frac{G_c}{\varepsilon} (1 - \varphi) + \frac{1}{2} A'(\varphi_+) \mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) - \frac{1}{2} c'(\varphi_+) p^2 \\ + 2\varphi_+ \left( ((1 - \alpha)p \operatorname{div} \mathbf{u} + \nabla p \cdot \mathbf{u}) \right) \leq 0 \text{ in } B, \end{aligned} \quad (24)$$

$$\partial_t \varphi \leq 0 \text{ on } B \quad (25)$$

$$\begin{aligned} \left( -G_c \varepsilon \Delta \varphi - \frac{G_c}{\varepsilon} (1 - \varphi) + \frac{1}{2} A'(\varphi_+) \mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) - \frac{1}{2} c'(\varphi_+) p^2 \right. \\ \left. + 2\varphi_+ \left( ((1 - \alpha)p \operatorname{div} \mathbf{u} + \nabla p \cdot \mathbf{u}) \right) \right) \partial_t \varphi = 0 \text{ in } B, \end{aligned} \quad (26)$$

# The fully coupled system for $\{\mathbf{u}, \varphi, p\}$

$$\varphi(x, 0) = \varphi^0(x) \text{ and } \frac{\partial \varphi}{\partial \mathbf{n}} = 0 \text{ on } \partial B; \quad (27)$$

---

$$\begin{aligned} \partial_t(c(\varphi_+)p) - \operatorname{div} \left( K_{\text{eff}}(\varphi_+)(\nabla p - \tilde{\mathbf{G}}(\varphi_+)) \right) + (\alpha - 1)\partial_t(\varphi_+^2 \operatorname{div} \mathbf{u}) \\ + \operatorname{div} \left( \partial_t(\varphi_+^2 \mathbf{u}) \right) = \tilde{q} \text{ in } B, \end{aligned} \quad (28)$$

$$p(x, 0) = p^0(x) \quad \text{on } B; \quad p = p^{\text{bdry}} \quad \text{on } \partial_N B, \quad (29)$$

where (22) and (29) contain the boundary conditions for the displacement and the pressure.

# Construction of the Lyapunov functional

The energy equality:

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left\{ \int_B \left( A(\varphi_+) \mathcal{G} e(\mathbf{u}) : e(\mathbf{u}) + G_c(\varepsilon |\nabla \varphi|^2 + \frac{1}{\varepsilon} (1 - \varphi)^2) + c(\varphi_+) p^2 - \right. \right. \\
 & \left. \left. 2c(\varphi_+) p p^{bdry} - 2(\alpha - 1) \varphi_+^2 \operatorname{div} \mathbf{u} p^{bdry} - 2 \operatorname{div} (\varphi_+^2 \mathbf{u}) p^{bdry} \right) dx \right. \\
 & \left. - 2 \int_{\partial_N B} (\boldsymbol{\tau} \cdot \mathbf{u} + (1 - \alpha) p^{bdry} u_n) dS \right\} + \int_B K_{eff}(\varphi_+) |\nabla p|^2 dx = \\
 & - \int_{\partial_N B} (\partial_t \boldsymbol{\tau} \cdot \mathbf{u} + \partial_t p^{bdry} u_n) dS - \int_B K_{eff}(\varphi_+) (\nabla p - \tilde{\mathbf{G}}(\varphi_+)) \cdot \nabla p^{bdry} dx \\
 & - \int_B c(\varphi_+) p \partial_t p^{bdry} dx - (\alpha - 1) \int_B \varphi_+^2 \operatorname{div} \mathbf{u} \partial_t p^{bdry} dx \\
 & - \int_B \operatorname{div} (\varphi_+^2 \mathbf{u}) \partial_t p^{bdry} dx + \int_B K_{eff}(\varphi_+) \tilde{\mathbf{G}}(\varphi_+) \cdot \nabla p^{bdry} dx \\
 & - \int_{\partial_D B} v^{inj} (p - p^{bdry}) dS + \int_B \tilde{q} (p - p^{bdry}) dx. \quad (30)
 \end{aligned}$$

# Construction of the Lyapunov functional II

**Corollary** The free energy for the problem is

$$\{\mathbf{u}, \varphi, p\} \rightarrow \frac{1}{2} \left( A(\varphi_+) \mathcal{G}e(\mathbf{u}) : e(\mathbf{u}) + G_c(\varepsilon |\nabla \varphi|^2 + \frac{1}{\varepsilon} (1-\varphi)^2) + c(\varphi_+) p^2 \right). \quad (31)$$

# An incremental formulation of the fully-coupled system

First, we derive an incremental form, i.e., we replace the time derivative with a discretized version; more precisely

$$\partial_t \varphi \rightarrow \partial_{\Delta t} \varphi = (\varphi - \Phi) / (\Delta t),$$

where  $\Delta t > 0$  is the time step and  $\Phi$  is the phase field from the previous time step. After time discretization, our quasistatic constrained minimization problem becomes a stationary problem, called the incremental problem.

The entropy condition is imposed in its discretized form and we introduce a convex set  $K$ :

$$K = \{\psi \in H^1(B) \mid \psi \leq \Phi \leq 1 \text{ a.e. on } B\}. \quad (32)$$

Note that the value of the phase field unknown  $\varphi$  from the previous time step enters only the convex set  $K$ , as the obstacle  $\Phi$ .  $U$  and  $P$  will denote the values of the displacement and the pressure from the previous time step.

# An incremental formulation of the fully-coupled system: Equations 1

$$- \operatorname{div} \left( A(\varphi_+) \mathcal{G}e(\mathbf{u}) \right) + \varphi_+^2 \nabla p + (\alpha - 1) \nabla(\varphi_+^2 p) = 0 \text{ in } B, \quad (33)$$

$$\mathbf{u} = 0 \text{ and } -K_{\text{eff}}(1)(\nabla p - \tilde{\mathbf{G}}(\Phi_+)) \cdot \mathbf{n} = v^{\text{inj}} \text{ on } \partial_D B, \quad (34)$$

$$\mathcal{G}e(\mathbf{u})\mathbf{n} = \tau + p^{\text{bdry}} \mathbf{n} \text{ on } \partial_N B, \quad (35)$$

---

$$-G_c \varepsilon \Delta \varphi - \frac{G_c}{\varepsilon} (1 - \varphi) + \frac{1}{2} \mathcal{A}(\varphi) \mathcal{G}e(\mathbf{U}) : e(\mathbf{U}) - \frac{1}{2} \mathcal{J}(\varphi) P(2p - P) \\ + \mathcal{D}(\varphi) \left( (1 - \alpha) p \operatorname{div} \mathbf{U} + \nabla p \cdot \mathbf{U} \right) \leq 0 \text{ in } B, \quad (36)$$

$$\partial_{\Delta t} \varphi \leq 0 \text{ on } B \quad (37)$$

# An incremental formulation of the fully-coupled system: Equations 2

$$\left( -G_c \varepsilon \Delta \varphi - \frac{G_c}{\varepsilon} (1 - \varphi) + \frac{1}{2} \mathcal{A}(\varphi) \mathcal{G} e(\mathbf{U}) : e(\mathbf{U}) + \frac{1}{2} \mathcal{J}(\varphi) P(2p - P) + \mathcal{D}(\varphi) \left( ((1 - \alpha)p \operatorname{div} \mathbf{U} + \nabla p \cdot \mathbf{U}) \right) \right) \partial_{\Delta t} \varphi = 0 \text{ in } B, \quad (38)$$

$$\varphi(x, 0) = \varphi^0(x) \text{ and } \frac{\partial \varphi}{\partial \mathbf{n}} = 0 \text{ on } \partial B; \quad (39)$$

---

$$\partial_{\Delta t} (c(\varphi_+) p) - \operatorname{div} \left( K_{\text{eff}}(\Phi_+) (\nabla p - \tilde{\mathbf{G}}(\Phi_+)) \right) + (\alpha - 1) \partial_{\Delta t} (\varphi_+^2 \operatorname{div} \mathbf{u}) + \operatorname{div} \left( \partial_{\Delta t} (\varphi_+^2 \mathbf{u}) \right) = \tilde{q} \text{ in } B, \quad (40)$$

$$\varphi(x, 0) = \varphi^0(x) \text{ and } p(x, 0) = p^0(x) \text{ on } B; \quad p = p^{\text{bdry}} \text{ on } \partial_N B, \quad (41)$$

# An incremental formulation of the fully-coupled system: Equations 3

where (38) is the strong form of Rice' condition. The functions  $\mathcal{A}$ ,  $\mathcal{J}$  and  $\mathcal{D}$  correspond to discrete derivatives of  $A$ ,  $c$  and  $\varphi^2$  and are given by

$$\mathcal{A}(\varphi) = \frac{A(\varphi_+) - A(\Phi_+)}{\varphi - \Phi};$$
$$\mathcal{J}(\varphi) = \frac{c(\varphi_+) - c(\Phi_+)}{\varphi - \Phi} \quad \text{and} \quad \mathcal{D}(\varphi) = \frac{\varphi_+^2 - \Phi_+^2}{\varphi - \Phi}.$$

As functional spaces of admissible displacements and pressures, we choose

$$V_U = \{ \mathbf{z} \in H^1(B)^3 \mid \mathbf{z} = 0 \text{ on } \partial_D B \}$$
$$V_P = \{ \pi \in H^1(B) \mid \pi = 0 \text{ on } \partial_N B \}.$$

# Well-posedness of the incremental model

**Theorem 1** Assume regularity of the data. Then there exists at least one variational solution  $\{\mathbf{u}, \varphi, p\} \in V_U \times H^1(B) \cap K \times V_P$  for problem (33)-(41).

We introduce the incremental Lyapunov functional (or the incremental free energy functional) by

$$J^N = \frac{1}{2} \int_B \left( A(\tilde{\varphi}_N) \mathcal{G}e(\mathbf{u}_N) : e(\mathbf{u}_N) + G_c(\varepsilon |\nabla \varphi_N|^2 + \frac{1}{\varepsilon} (1 - \varphi_N)^2) + c(\tilde{\varphi}_N) p_N^2 \right) dx - \int_{\partial_N B} \tau(t_N) \cdot \mathbf{u}_N dS. \quad (42)$$

Then we have the following result

# Lyapunov functional for the incremental model

**Theorem 2** Let us suppose regularity of data and, in addition,  $p^{bdry} = 0$ . Then the following estimate holds:

$$\begin{aligned} J^N + \Delta t \sum_{n=0}^{N-1} \int_B K_{eff}(\tilde{\varphi}_n) \nabla p_{n+1} \cdot \nabla p_{n+1} \, dx &\leq J^0 + \\ \Delta t \sum_{n=0}^{N-1} \int_B K_{eff}(\tilde{\varphi}_n) \tilde{\mathbf{G}}(\tilde{\varphi}_n) \cdot \nabla p_{n+1} \, dx - \Delta t \sum_{n=0}^{N-1} \int_{\partial_N B} \partial_{\Delta t} \tau(t_{n+1}) \cdot \mathbf{u}_{n+1} \, dS \\ + \Delta t \sum_{n=0}^{N-1} \int_{\partial_D B} v^{inj}(t_{n+1}) p_{n+1} \, dS + \Delta t \sum_{n=0}^{N-1} \int_B \tilde{q}(t_{n+1}) p_{n+1} \, dx, \end{aligned} \quad (43)$$

where  $J^0$  is calculated using the initial values of the unknowns.

# Conclusion

In this talk, we formulated a fully-coupled phase-field model describing propagating cracks in a poroelastic medium. The model consists of coupled nonlinear PDEs for the displacement, the pressure and the phase field unknowns. The phase-field algorithm is based on an incremental formulation and existence of a weak solution is established. Furthermore, the corresponding Lyapunov functional, based on the free energy, is constructed. Numerical tests demonstrate the correctness of the model. Specifically, this approach can treat crack growth in heterogeneous porous media, pressurized crack evolutions and multiple interacting cracks allowing them to join and branch. Moreover, the numerical simulations confirm findings from literature that low (negative) pressures arise at fracture tips.

- Incorporate the proppant flow in the fracture (addressed in a CMME paper with S. Lee from 2016).
- Incorporate more effects concentrated at the interface, like the Beavers-Joseph slip condition by including interface laws into the Biot free energy.
- Extension to the hydraulic fracturing involving multifluids.
- Extension to the chemo-poroelasticity.
- Extension to the thermo-poroelasticity (work in progress with C.J. van Duijn).