Biot-Stokes modeling of flow in fractured poroelastic media

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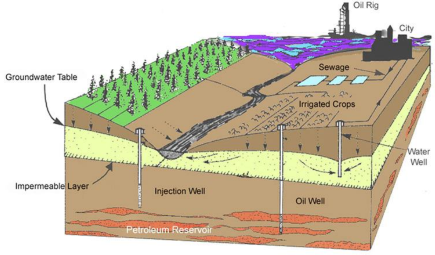
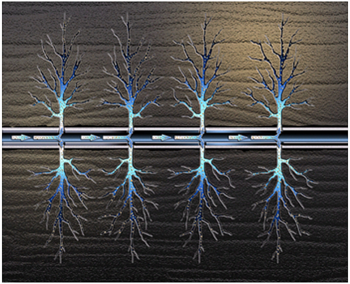
IPAM, April 3-7, 2017

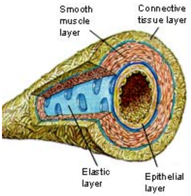
Applications



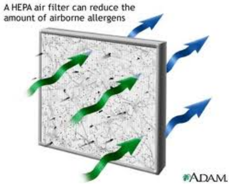
Surface-ground water systems



Hydraulic fracturing



Arterial flows



Industrial filters

Outline

1. Stokes-Biot model for flows in fractured poroelastic media
2. Interior penalty (Nitsche's coupling) formulation
3. Lagrange multiplier formulation
4. Dimensionally reduced fracture model
5. Noninear Stokes-Biot model

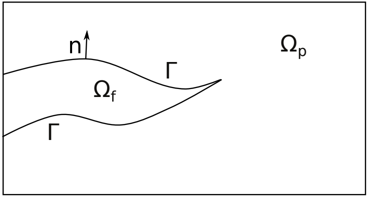
Models for flow in fractured porous media

- Darcy - Darcy, J. Jaffre, J. Roberts, et. al.
- Darcy - Forchheimer, J. Jaffre, J. Roberts, et. al.
- Darcy/Biot - Reynolds lubrication, M. F. Wheeler et. al.
- Darcy/Biot - Stokes/Brinkman/Navier Stokes, Quarteroni et. al.

Our work:

- Biot - Navier Stokes, pressure formulation for Darcy, operator splitting: Bukac, I.Y., Zunino, Num PDEs, 2015.
- Biot - Stokes, mixed formulation for Darcy, Nitsche's coupling: Bukac, I.Y., Zakerzadeh, Zunino, CMAME, 2015.
- Biot-Stokes, mortar formulation: Ambartsumyan, Khattatov, I.Y., Zunino, LNCS, 2015.
- Biot - Brinkman, reduced model: Bukac, I.Y., Zunino, M2AN, 2016.
- nonlinear Biot-Stokes: Ambartsumyan, Ervin, Nguyen, I.Y., preprint.

Biot-Stokes model for coupled flow with poroelastic structure



Biot system of poroelasticity in Ω_p:

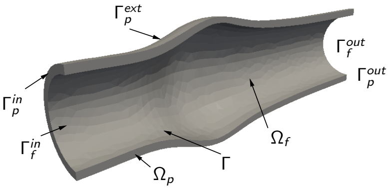
$$-\nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \mathbf{f}_p, \quad \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \lambda_p(\nabla \cdot \boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p \mathbf{I}$$

$$\frac{\partial}{\partial t}(s_0 p_p + \alpha \nabla \cdot \boldsymbol{\eta}_p) + \nabla \cdot \mathbf{u}_p = s, \quad \kappa^{-1}\mathbf{u}_p = -\nabla p_p$$

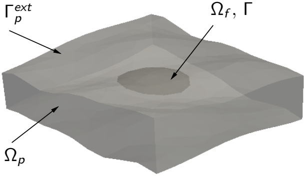
Stokes in Ω_f:

$$-\nabla \cdot \boldsymbol{\sigma}_f = \mathbf{f}_f, \quad \boldsymbol{\sigma}_f = -p_f \mathbf{I} + 2\mu_f \mathbf{D}(\mathbf{u}_f), \quad \nabla \cdot \mathbf{u}_f = g$$

Biot-Stokes applications



Arterial Flow (*i*)



Fracture flow (*ii*)

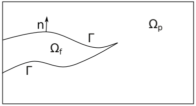
Biot-Stokes interface conditions on Γ

SHOWALTER [2000]

BADIA, QUAINI, QUARTERONI [2009]

LESINIGO, D'ANGELO, QUARTERONI [2011]

MIKELIC, WHEELER [2012]



Mass conservation:

$$\mathbf{u}_f \cdot \mathbf{n}_f + (\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p) \cdot \mathbf{n}_p = 0$$

No slip (i):

$$\mathbf{u}_f \cdot \boldsymbol{\tau}_f = \partial_t \boldsymbol{\eta}_p \cdot \boldsymbol{\tau}_f$$

or Beavers-Joseph-Saffman condition (slip with friction) (ii):

$$(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_f = -c_{BJS}(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_f$$

Balance of normal fluid stress:

$$(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -p_p$$

Conservation of momentum:

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0 \quad (p_f \mathbf{n}_f = \boldsymbol{\sigma}_p \mathbf{n}_p \text{ in Reynolds lubrication model})$$

Weak formulation (no flow, zero displacement BCs)

$$2\mu_f \int_{\Omega_f} \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f) - \int_{\Omega_f} p_f \nabla \cdot \mathbf{v}_f + \int_{\Omega_f} w_f \nabla \cdot \mathbf{u}_f$$

$$+ \int_{\Omega_p} \left(2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) : \mathbf{D}(\boldsymbol{\xi}_p) + \lambda_p \nabla \cdot \boldsymbol{\eta}_p \nabla \cdot \boldsymbol{\xi}_p\right) - \alpha \int_{\Omega_p} p_p \nabla \cdot \boldsymbol{\xi}_p$$

$$+ \int_{\Omega_p} \kappa^{-1} \mathbf{u}_p \cdot \mathbf{v}_p - \int_{\Omega_p} p_p \nabla \cdot \mathbf{v}_p$$

$$+ \int_{\Omega_p} s_0 \partial_t p_p \, w_p + \alpha \int_{\Omega_p} \nabla \cdot \partial_t \boldsymbol{\eta}_p w_p + \int_{\Omega_p} \nabla \cdot \mathbf{u}_p w_p$$

$$+ \int_{\Gamma} \left(-\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f - \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p + p_p \mathbf{v}_p \cdot \mathbf{n}_p\right) = F(\mathbf{v}_f, \boldsymbol{\xi}_p, \mathbf{v}_p)$$

Interface term

$$I_\Gamma = \int_\Gamma \left(-\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f - \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p + p_p \mathbf{v}_p \cdot \mathbf{n}_p \right)$$

Equlibrium: $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -p_p; \quad \boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0$

Slip with friction (BJS): $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_f = -c_{BJS} (\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_f$

$$I_\Gamma = a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \boldsymbol{\sigma}_f)$$

where

$$a_{BJS}(\mathbf{u}_f, \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) = \int_\Gamma c_{BJS} (\mathbf{u}_f - \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_f \, (\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_f,$$

$$b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \boldsymbol{\sigma}_f) = -\int_\Gamma \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f (\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p)$$

Interior penalty (Nitsche's) method [1]

$$b_\Gamma = -\int_\Gamma \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f (\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p)$$

Interior penalty for the essential continuity of normal flux condition:

$$\begin{aligned}\tilde{b}_\Gamma = &-\int_\Gamma \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f (\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p) \\ &+ \int_\Gamma \gamma_f \mu_f h^{-1} (\mathbf{u}_f \cdot \mathbf{n}_f + (\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p) \cdot \mathbf{n}_p)(\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p)\end{aligned}$$

[1] Bukac, I.Y., Zakerzadeh, Zunino, CMAME, 2015

Features of the Nitsche's method

+ Efficient non-iterative time-splitting: Elasticity, Darcy, Stokes
+ Non-matching grids across the interface
+ Optimal accuracy in space
- Penalty parameter
- Conditionally stable: $\tau \lesssim h$
- Sub-optimal accuracy in time: $O(\tau^{1/2})$
+ The split scheme is an optimal preconditioner for the monolithic scheme

Backward Euler - Monolithic scheme

$$\widetilde{\mathcal{A}}_h(\widetilde{\mathbf{y}}_h^n, \mathbf{z}_h) = \mathcal{F}^n(\mathbf{z}_h)$$

$$MON: \quad \begin{bmatrix} A_f & B_{fp}^T & \Gamma_{fq} & 0 & \Gamma_{fs} \\ B_{fp} & 0 & 0 & 0 & \Gamma_{ps} \\ \Gamma_{qf} & 0 & A_q & B_{qp}^T & \Gamma_{qs} \\ 0 & 0 & B_{qp} & A_p & B_{ps} \\ \Gamma_{sf} & \Gamma_{sp} & \Gamma_{sq} & B_{sp} & A_s \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_f \\ p_f \\ \mathbf{u}_p \\ p_p \\ \boldsymbol{\eta} \end{bmatrix} = F$$

f - flow; q - Darcy velocity; p - Darcy pressure; s - solid

Backward Euler - Monolithic scheme

Stability: *if γ_f is large enough, $\exists\, c, C$ $0 < c < 1 < C < \infty$ s.t.*

$$|||\mathbf{y}_h^N|||_{\heartsuit}^2 + c\tau \sum_{n=1}^{N} |||\mathbf{y}_h^n|||_{\nabla}^2 \le |||\mathbf{y}_h^0|||_{\heartsuit}^2 + \tau \sum_{n=1}^{N} \frac{C}{\mu_f} \|F(t^n)\|^2$$

Norms:

$$|||\mathbf{y}_h^n|||_{\heartsuit}^2 := \frac{1}{2}\rho_f ||\mathbf{u}_f^n||_{L^2(\Omega_f)}^2 \ \textbf{(Fluid)}$$

$$\textbf{(Biot)}\ + \frac{1}{2}\Big(2\mu_p||\mathbf{D}(\boldsymbol{\eta}^n)||_{L^2(\Omega_p)}^2 + \lambda_p||\nabla\cdot\boldsymbol{\eta}^n||_{L^2(\Omega_p)}^2 + s_0||p_p^n||_{L^2(\Omega_p)}^2\Big)$$

$$|||\mathbf{y}_h^n|||_{\nabla}^2 := 2\mu_f\|\mathbf{D}(\mathbf{u}_f^n)\|_{\Omega_f}^2 + \kappa^{-1}\|\mathbf{u}_p^n\|_{\Omega_p}^2 \quad \textbf{(dissipation)}$$
$$+ \mu_f h^{-1}\|\big(\mathbf{u}_f^n - \mathbf{u}_p^n - d_\tau\boldsymbol{\eta}^n\big)\cdot\mathbf{n}\|_{\Gamma}^2$$
$$+ \frac{\tau}{2}\big(\rho_f\|d_\tau\mathbf{u}_f^n\|_{\Omega_f}^2 + s_0\|d_\tau p_p\|_{\Omega_p}^2 + \lambda_p\|d_\tau\nabla\cdot\boldsymbol{\eta}\|_{\Omega_p}^2\big)$$

Splitting strategy

1. given $\mathbf{u}_f^{n-1}|_\Gamma,\ p_f^{n-1}|_\Gamma$

   $\Rightarrow$**drained split for Biot**; Robin BCs

$$A_s\boldsymbol{\eta}^n = -\Gamma_{sf}\mathbf{u}_f^{n-1} - \Gamma_{sp}p_f^{n-1} - \Gamma_{sq}\mathbf{u}_p^{n-1} - B_{sp}p_p^{n-1}$$

$$\begin{bmatrix} A_q & B_{qp}^T \\ B_{qp} & A_p \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_p^n \\ p_p^n \end{bmatrix} = \begin{bmatrix} -\Gamma_{qf}\mathbf{u}_f^{n-1} - \Gamma_{qs}\boldsymbol{\eta}^n \\ -B_{ps}\boldsymbol{\eta}^n \end{bmatrix}$$

2. given $(\boldsymbol{\eta}^n, \mathbf{u}_p^n, p_p^n)|_\Gamma$

   $\Rightarrow$ Stokes solve; Dirichlet BCs

$$\begin{bmatrix} A_f & B_{fp}^T \\ B_{fp} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_f^n \\ p_f^n \end{bmatrix} = \begin{bmatrix} -\Gamma_{fq}\mathbf{u}_p^n - \Gamma_{fs}\boldsymbol{\eta}^n \\ -\Gamma_{ps}\boldsymbol{\eta}^n \end{bmatrix}$$

- Stabilization of the fully explicit scheme is required (Burman-Fernandez);
- Additional stabilization terms are required for splitting Biot

Splitting, stabilization terms & stability

Partitioned scheme (PART): $\widehat{\mathcal{A}}_h(\widehat{\mathbf{y}}_h^n, \mathbf{z}_h) = \widehat{\mathcal{F}}^n(\mathbf{z}_h)$

$$\widetilde{\mathcal{A}}_h(\mathbf{y}_h^n, \mathbf{z}_h) \quad \text{(MON)}$$

$$+\gamma_{stab}\frac{h\tau}{\gamma_f \mu_f}\int_\Gamma d_\tau p_f\, \psi_f \quad \text{(Burman-Fernandez)}$$

$$+\gamma'_{stab}\gamma_f \mu_f \frac{\tau}{h}\int_\Gamma d_\tau \mathbf{u}_p\cdot\mathbf{n}\mathbf{r}\cdot\mathbf{n} \quad \text{(New! to control Biot splitting ...}$$

$$+\gamma'_{stab}\gamma_f \mu_f \frac{\tau}{h}\int_\Gamma d_\tau \mathbf{u}_f\cdot\mathbf{n}\boldsymbol{\varphi}_f\cdot\mathbf{n} \quad \text{... but not needed in practice!)}$$

$$= \mathcal{F}^n(\mathbf{z}_h)$$

$$+\int_\Gamma \gamma_f \mu_f h^{-1}\big((\mathbf{u}_f^n - \mathbf{u}_f^{n-1}) - (\mathbf{u}_p^n - \mathbf{u}_p^{n-1})\big)\cdot\mathbf{n}\,(-\boldsymbol{\varphi}_p)\cdot\mathbf{n}$$

$$+2\mu_f\int_\Gamma \mathbf{n}\cdot\big(\mathbf{D}(\mathbf{u}_f^n) - \mathbf{D}(\mathbf{u}_f^{n-1})\big)\mathbf{n}\,(\boldsymbol{\varphi}_f - \mathbf{r} - \boldsymbol{\varphi}_p)\cdot\mathbf{n} \quad \text{(splitting residuals)}$$

$$-\int_\Gamma \mathbf{n}\cdot(p_f^n - p_f^{n-1})\mathbf{n}\,(\boldsymbol{\varphi}_f - \mathbf{r} - \alpha\boldsymbol{\varphi}_p)\cdot\mathbf{n} - \int_{\Omega_p}(p_p^n - p_p^{n-1})\nabla\cdot\boldsymbol{\varphi}_p$$

Splitting, stabilization terms & stability

Stability of PART: if γ_f is large enough and

$$\frac{\alpha^2}{\lambda_p s_0} < \frac{1}{2} \quad \text{(only due to Biot splitting)} \quad \exists\, c', C' \; 0 < c' < 1 < C' < \infty \; s.t.$$

$$|||\mathbf{y}_h^N|||_{\heartsuit}^2 + c'\tau \sum_{n=1}^{N} |||\mathbf{y}_h^n|||_{\nabla}^2 \quad \text{(as for MON)}$$

$$+\tau \frac{\gamma_{stab}}{2} \frac{h}{\gamma_f \mu_f} \|p_f^N\|_\Gamma^2 + \tau \frac{\gamma'_{stab}}{2} \frac{\gamma_f \mu_f}{h} \left(\|\mathbf{u}_f^N \cdot \mathbf{n}_f\|_\Gamma^2 + \|\mathbf{u}_p^N \cdot \mathbf{n}_p\|_\Gamma^2 \right)$$

$$\leq |||\mathbf{y}_h^0|||_{\heartsuit}^2 + C'\mu_f^{-1}\tau \sum_{n=1}^{N} \|F(t^n)\|^2$$

$$+ C'\mu_f \left(\frac{\tau}{h} \|\mathbf{u}_f^0 \cdot \mathbf{n}\|_\Gamma^2 + \frac{\tau}{h} \|\mathbf{u}_f^0 \cdot \boldsymbol{\tau}\|_\Gamma^2 + \frac{\tau}{h} \|\mathbf{u}_p^0 \cdot \mathbf{n}\|_\Gamma^2 + \|\mathbf{D}(\mathbf{u}_f^0)\|_{L^2(\Omega_f)}^2 + \mu_f^{-2} \|p_f^0\|_\Gamma^2 \right)$$

MON & PART share stability properties in equivalent norms!

Convergence analysis - time discretization error

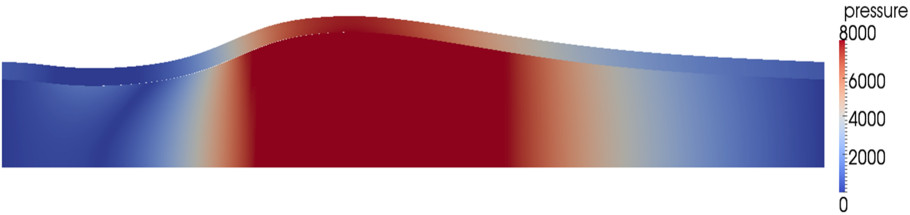
Under the assumption $\tau < ch^2$:

$$\sqrt{|||\widehat{\mathbf{e}}_h^N|||_{\heartsuit}^2 + \tau \sum_{n=1}^{N} |||\widehat{\mathbf{e}}_h^n|||_{\nabla}^2} = \mathcal{O}(\frac{\tau}{h} + \tau^2 + \tau) = \mathcal{O}(\tau^{\frac{1}{2}})$$

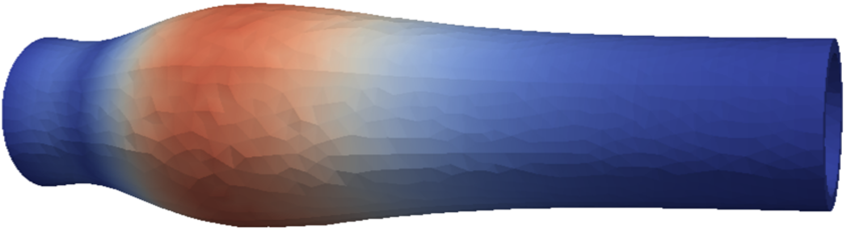
Numerical results

Benchmark problem: propagation of a single (sinusoidal) pressure wave;
5 cm poroelastic tube (artery); time scale 6 ms.

2D test case, 3.5 ms snapshot, pressure surface plot:



3D test case:



Convergence in time for the arterial flow problem

$$\mathcal{E}_{f,h}^N := \rho_f \|\mathbf{u}_f^N - \mathbf{u}_f^{N,ref}\|^2_{L^2(\Omega_f)},\ \mathcal{E}_{p,h}^N(a) := \rho_p \|\dot{\boldsymbol{\eta}}^N - \dot{\boldsymbol{\eta}}^{N,ref}\|^2_{L^2(\Omega_p)},\ \mathcal{E}_{p,h}^N(c) := s_0 \|p_p^N - p_p^{N,ref}\|^2_{L^2(\Omega_p)}$$

$$\mathcal{E}_{p,h}^N(b) := 2\mu_p \|\mathbf{D}(\boldsymbol{\eta}^N - \boldsymbol{\eta}^{N,ref})\|^2_{L^2(\Omega_p)} + \lambda_p \|\nabla\cdot(\boldsymbol{\eta}^N - \boldsymbol{\eta}^{N,ref})\|^2_{L^2(\Omega_p)},$$

monolithic	$\sqrt{\mathcal{E}_{f,h}^N}$	rate	$\sqrt{\mathcal{E}_{p,h}^N(a)}$	rate	$\sqrt{\mathcal{E}_{p,h}^N(b)}$	rate	$\sqrt{\mathcal{E}_{p,h}^N(c)}$	rate
$\tau_0 = 10^{-4}$	2.14E-01		1.48E-01		5.24E-01		1.32E-02	
$\tau_0/2$	1.05E-01	1.02	7.89E-02	0.91	2.82E-01	0.90	6.95E-03	0.92
$\tau_0/4$	5.13E-02	1.04	4.03E-02	0.97	1.44E-01	0.97	3.53E-03	0.98
$\tau_0/8$	2.45E-02	1.07	1.98E-02	1.03	7.07E-02	1.03	1.72E-03	1.03

partitioned	$\sqrt{\mathcal{E}_{f,h}^N}$	rate	$\sqrt{\mathcal{E}_{p,h}^N(a)}$	rate	$\sqrt{\mathcal{E}_{p,h}^N(b)}$	rate	$\sqrt{\mathcal{E}_{p,h}^N(c)}$	rate
$\tau_0 = 10^{-4}$	2.87E-01		1.84E-01		7.71E-01		1.96E-02	
$\tau_0/2$	1.49E-01	0.94	9.91E-02	0.89	4.15E-01	0.90	1.01E-02	0.95
$\tau_0/4$	7.58E-02	0.98	5.16E-02	0.94	2.13E-01	0.96	5.09E-03	0.99
$\tau_0/8$	3.75E-02	1.01	2.59E-02	0.99	1.06E-01	1.01	2.49E-03	1.03

PART as preconditioner for the monolithic scheme

Solve:

$$\mathbf{P}^{-1}\begin{bmatrix} A_f & B_{fp}^T & \Gamma_{fq} & 0 & \Gamma_{fs} \\ B_{fp} & 0 & 0 & 0 & \Gamma_{ps} \\ \Gamma_{qf} & 0 & A_q & B_{qp}^T & \Gamma_{qs} \\ 0 & 0 & B_{qp} & A_p & B_{ps} \\ \Gamma_{sf} & \Gamma_{sp} & \Gamma_{sq} & B_{sp} & A_s \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ p_f \\ \mathbf{q} \\ p_p \\ \boldsymbol{\eta} \end{bmatrix} = F$$

where $\mathbf{P}$ is

$$A_s\boldsymbol{\eta} = -\Gamma_{sf}\mathbf{v}^{n-1} - \Gamma_{sp}\mathbf{p}_f^{n-1} - \Gamma_{sq}\mathbf{q}^{n-1} - B_{sp}\mathbf{p}_p^{n-1}$$

$$\begin{bmatrix} A_q & B_{qp}^T \\ B_{qp} & A_p \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q} \\ p_p \end{bmatrix} = \begin{bmatrix} -\Gamma_{qf}\mathbf{v}^{n-1} - \Gamma_{qs}\boldsymbol{\eta}^n \\ -B_{ps}\boldsymbol{\eta}^n \end{bmatrix}$$

$$\begin{bmatrix} A_f & B_{fp}^T \\ B_{fp} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ p_f \end{bmatrix} = \begin{bmatrix} -\Gamma_{fq}\mathbf{q}^n - \Gamma_{fs}\boldsymbol{\eta}^n \\ -\Gamma_{ps}\boldsymbol{\eta}^n \end{bmatrix}$$

PART as preconditioner for the monolithic scheme

Solve:

$$\mathbf{P}^{-1}\begin{bmatrix} A_f & B_{fp}^T & \Gamma_{fq} & 0 & \Gamma_{fs} \\ B_{fp} & 0 & 0 & 0 & \Gamma_{ps} \\ \Gamma_{qf} & 0 & A_q & B_{qp}^T & \Gamma_{qs} \\ 0 & 0 & B_{qp} & A_p & B_{ps} \\ \Gamma_{sf} & \Gamma_{sp} & \Gamma_{sq} & B_{sp} & A_s \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ p_f \\ \mathbf{q} \\ p_p \\ \boldsymbol{\eta} \end{bmatrix} = F$$

if $\tau \lesssim h$, MON and PART satisfy stability properties in equivalent norms:

$$|||\mathbf{y}_h|||^2 := |||\mathbf{y}_h|||^2_{\heartsuit} + |||\mathbf{y}_h|||^2_{\nabla}$$

Generalized Rayleigh quotient, $\exists, c, C > 0$ independent of h, τ s.t.:

$$c\frac{\widetilde{\mathcal{A}}_h(\mathbf{y}_h, \mathbf{y}_h)}{|||\mathbf{y}_h|||^2} \le \frac{\widehat{\mathcal{A}}_h(\mathbf{y}_h, \mathbf{y}_h)}{|||\mathbf{y}_h|||^2} \le C\frac{\widetilde{\mathcal{A}}_h(\mathbf{y}_h, \mathbf{y}_h)}{|||\mathbf{y}_h|||^2}$$

 # GMRES iterations for $P^{-1}A$ is proportional to τ/h

(G.Starke, Field-of-values analysis of preconditioned iterative methods..., Num. Math. '97)

Preconditioner test

Artery test case (i)

$\tau = 10^{-4}$	$h = 0.05$	$h = 0.025$	$h = 0.0125$
# GMRES($\widetilde{\mathrm{A}}_h^s$)	211.4	446.4	1282.9
# GMRES($(\widehat{\mathrm{A}}_h)^{-1}\widetilde{\mathrm{A}}_h^s$)	10.9	12	13.9

Artery test case (i)

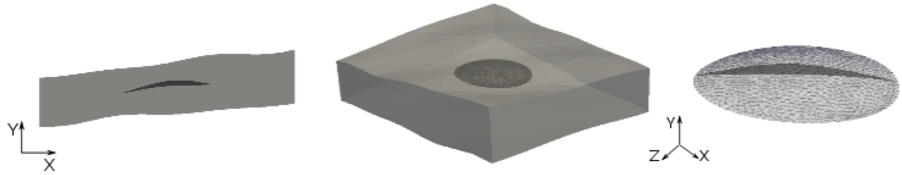
$\tau = 10^{-5}$	$h = 0.05$	$h = 0.025$	$h = 0.0125$
# GMRES($\widetilde{\mathrm{A}}_h^s$)	362.1	498.3	1194.4
# GMRES($(\widehat{\mathrm{A}}_h)^{-1}\widetilde{\mathrm{A}}_h^s$)	8	10	12.9

Fractured reservoir test case (ii)

$\tau = 10^{-3}$	$h = 1.06$	$h = 0.450$	$h = 0.212$
# GMRES($\widetilde{\mathrm{A}}_h^s$)	53.8	101.7	245.4
# GMRES($(\widehat{\mathrm{A}}_h)^{-1}\widetilde{\mathrm{A}}_h^s$)	2	2	2

Table : Average number of GMRES iterations (# GMRES) required to reduce the relative residual of a factor 10^{-6}.

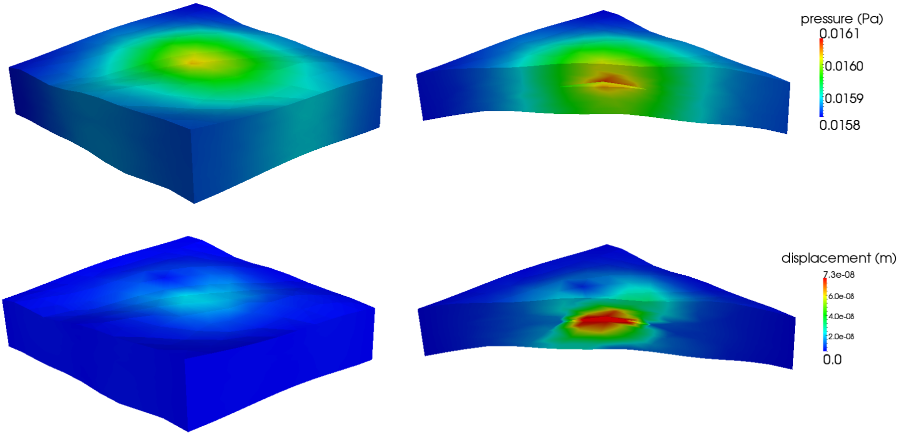
Flow through a fractured poroelastic reservoir



Parameter	Symbol	Units	Value
Poroelastic wall density	ρ_p	(kg/m^3)	897
Fluid density	ρ_f	(kg/m^3)	897
Dyn. viscosity	μ	(Pa s)	10^{-3}
Lamé coeff.	μ_p	(Pa)	2.92×10^{10}
Lamé coeff.	λ_p	(Pa)	1.94×10^{10}
Hydraulic conductivity (2D)	$\boldsymbol{\kappa}$	$(\text{m}^2/\text{Pa s})$	$diag(200, 50) \times 10^{-12}$
Hydraulic conductivity (3D)	$\boldsymbol{\kappa}$	$(\text{m}^2/\text{Pa s})$	$diag(200, 50, 200) \times 10^{-12}$
Mass storativity coeff.	s_0	(Pa^{-1})	6.9×10^{-5}
Biot-Willis constant	α		1
Beavers-Joseph-Saffman coefficient (2D)	β	$(\text{m}^2/\text{Pa s})$	$2.88 \cdot 10^{-4}$
Beavers-Joseph-Saffman coefficient (3D)	β	$(\text{m}^2/\text{Pa s})$	$3.88 \cdot 10^{-4}$
Total simulation time	T	(s)	18600

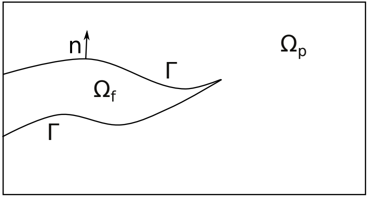
Parameters from Girault, Wheeler, Ganis, Mear 2013

Simulation results - flow in fractured reservoir



Pressure change (top) and displacement (bottom) in reservoir in 3D at final time T.

Lagrange multiplier (mortar) method [2]



Biot system of poroelasticity in Ω_p:

$$-\nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \mathbf{f}_p, \quad \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \lambda_p(\nabla \cdot \boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p \mathbf{I}$$

$$\frac{\partial}{\partial t}(s_0 p_p + \alpha \nabla \cdot \boldsymbol{\eta}_p) + \nabla \cdot \mathbf{u}_p = s, \quad \kappa^{-1}\mathbf{u}_p = -\nabla p_p$$

Stokes in Ω_f:

$$-\nabla \cdot \boldsymbol{\sigma}_f = \mathbf{f}_f, \quad \boldsymbol{\sigma}_f = -p_f \mathbf{I} + 2\mu_f \mathbf{D}(\mathbf{u}_f), \quad \nabla \cdot \mathbf{u}_f = g$$

[2] Ambartsumyan, Khattatov, I.Y., Zunino, LNCS, 2015

Weak formulation (no flow, zero displacement BCs)

$$2\mu_f \int_{\Omega_f} \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f) - \int_{\Omega_f} p_f \nabla \cdot \mathbf{v}_f + \int_{\Omega_f} w_f \nabla \cdot \mathbf{u}_f$$

$$+ \int_{\Omega_p} \left(2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) : \mathbf{D}(\boldsymbol{\xi}_p) + \lambda_p \nabla \cdot \boldsymbol{\eta}_p \nabla \cdot \boldsymbol{\xi}_p\right) - \alpha \int_{\Omega_p} p_p \nabla \cdot \boldsymbol{\xi}_p$$

$$+ \int_{\Omega_p} \kappa^{-1} \mathbf{u}_p \cdot \mathbf{v}_p - \int_{\Omega_p} p_p \nabla \cdot \mathbf{v}_p$$

$$+ \int_{\Omega_p} s_0 \partial_t p_p w_p + \alpha \int_{\Omega_p} \nabla \cdot \partial_t \boldsymbol{\eta}_p w_p + \int_{\Omega_p} \nabla \cdot \mathbf{u}_p w_p$$

$$+ \int_\Gamma \left(-\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f - \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p + p_p \mathbf{v}_p \cdot \mathbf{n}_p\right) = F(\mathbf{v}_f, \boldsymbol{\xi}_p, \mathbf{v}_p)$$

Interface term

$$I_\Gamma = \int_\Gamma \left(-\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f - \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p + p_p \mathbf{v}_p \cdot \mathbf{n}_p\right)$$

Equlibrium: $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -p_p; \quad \boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0$

Slip with friction (BJS): $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_f = -c_{BJS}(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_f$

Lagrange multiplier:

$$\lambda = -(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p$$

$$I_\Gamma = a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda)$$

where

$$a_{BJS}(\mathbf{u}_f, \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) = \int_\Gamma c_{BJS}(\mathbf{u}_f - \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_f \, (\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_f,$$

$$b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \mu) = \int_\Gamma \left(\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p\right) \mu$$

Spaces and bilinear forms

$$\begin{aligned} \text{Stokes:} \quad & V_f = H^1(\Omega_f)^d,\ W_f = L^2(\Omega_f) \\ \text{Darcy:} \quad & V_p = H(\text{div};\Omega_p),\ W_p = L^2(\Omega_p) \\ \text{Elasticity:} \quad & X_p = H^1(\Omega_p)^d \\ \text{Lagrange multiplier:} \quad & \Lambda = H^{1/2}(\Gamma_{fp}) \end{aligned}$$

Bilinear forms related to Stokes, Darcy and elasticity operators:

$$\begin{aligned} a_f(\mathbf{u}_f, \mathbf{v}_f) &= (2\mu_f \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f))_{\Omega_f} \\ a_p^d(\mathbf{u}_p, \mathbf{v}_p) &= \left(\kappa^{-1}\mathbf{u}_p, \mathbf{v}_p\right)_{\Omega_p} \\ a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) &= (2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) : \mathbf{D}(\boldsymbol{\xi}_p))_{\Omega_p} + (\lambda_p \nabla\cdot\boldsymbol{\eta}_p, \nabla\cdot\boldsymbol{\xi}_p)_{\Omega_p} \\ b_\star(\mathbf{v}_\star, w_\star) &= -(\nabla\cdot\mathbf{v}_\star, w_\star)_{\Omega_\star},\ \star = f, p \end{aligned}$$

Weak formulation

Find $\mathbf{u}_f(t) \in \mathbf{V}_f$, $p_f(t) \in W_f$,
$\mathbf{u}_p(t) \in \mathbf{V}_p$, $p_p(t) \in W_p$,
$\boldsymbol{\eta}_p(t) \in \mathbf{X}_p, \lambda(t) \in \Lambda$,

$$a_f(\mathbf{u}_f, \mathbf{v}_f) + a_p^d(\mathbf{u}_p, \mathbf{v}_p) + a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p)$$

$$+ b_f(\mathbf{v}_f, p_f) + b_p(\mathbf{v}_p, p_p) + \alpha b_p(\boldsymbol{\xi}_p, p_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda) = f(\mathbf{v}_f, \boldsymbol{\xi}_p),$$

$$(\partial_t s_0 p_p, w_p)_{\Omega_p} - \alpha b_p(\partial_t \boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f) = q(w_f, w_p),$$

$$b_\Gamma(\mathbf{u}_f, \mathbf{u}_p, \partial_t \boldsymbol{\eta}_p; \mu) = 0$$

$p_p(0) = p_{p,0}$, $\boldsymbol{\eta}_p(0) = \boldsymbol{\eta}_{p,0}$

Space discretization

- Affine finite element partitions of Ω_f and Ω_p
- Allow for non-matching grids across Γ
- Stokes: stable spaces $\mathbf{V}_{f,h} \times W_{f,h} \subset H^1(\Omega_f)^d \times L^2(\Omega_f)$
- Darcy: stable spaces $\mathbf{V}_{p,h} \times W_{p,h} \subset H(\mathrm{div};\Omega_p) \times L^2(\Omega_p)$
- Elasticity: conforming space $\mathbf{X}_{p,h} \subset H^1(\Omega_p)^d$
- Mortar: $\Lambda_h = \mathbf{V}_{p,h} \cdot \mathbf{n}_p \not\subset H^{1/2}(\Gamma)$ (non-conforming)
- Λ_h-norm: $\|\mu_h\|^2_{\Lambda_h} = \|\mu_h\|^2_{L^2(\Gamma)} + |\mu_h|^2_{\Lambda_h}$, $|\mu_h|^2_{\Lambda_h} = a_p^d(\mathbf{u}^*_{p,h}(\mu_h), \mathbf{u}^*_{p,h}(\mu_h))$

Lemma (Galvis-Sarkis, ETNA 2007)

There exists a constant $\beta > 0$ independent of h such that

$$\inf_{(w_h,\mu_h)\in W_h\times\Lambda_h} \sup_{(\mathbf{v}_h,\boldsymbol{\xi}_{p,h})\in \mathbf{V}_h\times\mathbf{X}_{p,h}} \frac{b(\mathbf{v}_h,\boldsymbol{\xi}_{p,h}; w_h) + b_\Gamma(\mathbf{v}_h,\boldsymbol{\xi}_{p,h};\mu_h)}{\|(\mathbf{v}_h,\boldsymbol{\xi}_{p,h})\|_{\mathbf{V}\times\mathbf{X}_p}\|(w_h,\mu_h)\|_{W\times\Lambda_h}} \geq \beta.$$

Semidiscrete formulation: system of DAEs

$$\mathbf{E}\,\partial_t X(t) + \mathbf{H}\,X(t) = L(t),$$

$$X(t) = \begin{pmatrix} \overline{\mathbf{u}}_f(t) \\ \overline{\mathbf{u}}_p(t) \\ \overline{\boldsymbol{\eta}}_p(t) \\ \overline{p}_f(t) \\ \overline{p}_p(t) \\ \overline{\lambda}(t) \end{pmatrix}, \quad L(t) = \begin{pmatrix} \mathcal{F}_{\mathbf{u}_f} \\ 0 \\ \mathcal{F}_{\boldsymbol{\eta}_p} \\ \mathcal{F}_{p_f} \\ \mathcal{F}_{p_p} \\ 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 0 & A_{fe}^{BJS} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{ee}^{BJS} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha B_{ep} & 0 & s_0 M_p & 0 \\ 0 & 0 & -B_{e,\Gamma} & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} A_f + A_{ff}^{BJS} & 0 & 0 & -B_{ff}^T & 0 & B_{f,\Gamma}^T \\ 0 & A_p & 0 & 0 & -B_{pp}^T & B_{p,\Gamma}^T \\ A_{fe}^{BJS,T} & 0 & A_e & 0 & -\alpha B_{ep}^T & B_{e,\Gamma}^T \\ B_{ff} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{pp} & 0 & 0 & 0 & 0 \\ -B_{f,\Gamma} & -B_{p,\Gamma} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Existence of a solution

DAE theory:

$$\mathbf{E}\,\partial_t X(t) + \mathbf{H}\,X(t) = L(t)$$

has a solution if $s\mathbf{E} + \mathbf{H}$ is non-singular for some $s \neq 0$.

$$\mathbf{E} + \mathbf{H} = \begin{pmatrix} A_f + A_{ff}^{BJS} & 0 & A_{fe}^{BJS} & -B_{ff}^T & 0 & B_{f,\Gamma}^T \\ 0 & A_p & 0 & 0 & -B_{pp}^T & B_{p,\Gamma}^T \\ A_{fe}^{BJS,T} & 0 & A_e + A_{ee}^{BJS} & 0 & -\alpha B_{ep}^T & B_{e,\Gamma}^T \\ B_{ff} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{pp} & \alpha B_{ep} & 0 & s_0 M_p & 0 \\ -B_{f,\Gamma} & -B_{p,\Gamma} & -B_{e,\Gamma} & 0 & 0 & 0 \end{pmatrix}$$

Existence and uniqueness of a solution

$$\mathbf{E} + \mathbf{H} = \begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ -\mathbf{B} & \mathbf{C} \end{pmatrix},$$

Lemma

If $\mathbf{A}$ *and* $\mathbf{C}$ *are positive semidefinite and* $ker(\mathbf{A}) \cap ker(\mathbf{B}) = ker(\mathbf{C}) \cap ker(\mathbf{B}^T) = \{0\}$, *then* $\mathbf{E} + \mathbf{H}$ *is invertible.*

Uniqueness follows from the energy equality (for zero data)

$$\frac{1}{2}\left(s_0 \|p_{p,h}(t)\|^2_{L^2(\Omega_p)} + a_p^e(\boldsymbol{\eta}_{p,h}(t), \boldsymbol{\eta}_{p,h}(t)) \right) + \int_0^t \left[\left| \mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h} \right|^2_{a_{BJS}} + a_f(\mathbf{u}_{f,h}, \mathbf{u}_{f,h}) + a_p(\mathbf{u}_{p,h}, \mathbf{u}_{p,h}) \right] ds = 0$$

and the inf-sup conditions for $p_{p,h}$, $p_{f,h}$, and λ_h.

Stability estimate

Theorem

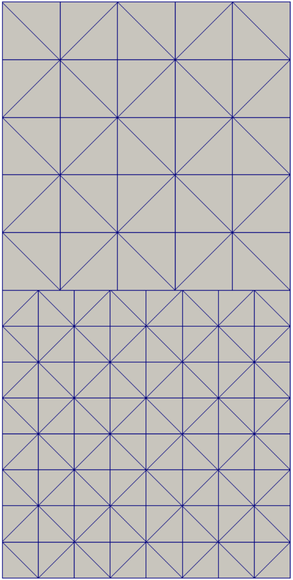
$$
\begin{aligned}
& s_0 \|p_{p,h}\|_{L^\infty(L^2(\Omega_p))} + \|\boldsymbol{\eta}_{p,h}\|_{L^\infty(H^1(\Omega_p))} + \|\mathbf{u}_f\|_{L^2(H^1(\Omega_f))} + \|\mathbf{u}_p\|_{L^2(L^2(\Omega_p))} \\
& \quad + \|p_{f,h}\|_{L^2(L^2(\Omega_f))} + \|p_{p,h}\|_{L^2(L^2(\Omega_p))} + \|\lambda_h\|_{L^2(\Lambda_h)} + \left|\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}\right|_{L^2(a_{BJS})} \\
& \lesssim \Big(\|\mathbf{f}_{p,h}\|_{L^\infty(L^2(\Omega_p))} + \|\mathbf{f}_{p,h}\|_{L^2(L^2(\Omega_p))} \|\mathbf{f}_{f,h}\|_{L^2(L^2(\Omega_f))} \\
& \quad + \|\partial_t \mathbf{f}_{p,h}\|_{L^2(L^2(\Omega_p))} + \|q_{f,h}\|_{L^2(L^2(\Omega_f))} + \|q_{p,h}\|_{L^2(L^2(\Omega_p))} \Big)
\end{aligned}
$$

Error estimate

Theorem

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(H^1(\Omega_f))} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|_{L^2(L^2(\Omega_p))} + \|p_f - p_{f,h}\|_{L^2(L^2(\Omega_f))} \\
&\quad + \|p_p - p_{p,h}\|_{L^2(L^2(\Omega_p))} + \|\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p\|_{L^\infty(H^1(\Omega_p))} + s_0\|p_p - p_{p,h}\|_{L^\infty(L^2(\Omega_p))} \\
&\quad + \|\lambda - \lambda_h\|_{L^2(\Lambda_h)} + \left|(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) - (\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h})\right|_{L^2(a_{BJS})} \\
&\quad \lesssim h^{k_f} + h^{k_p+1} + h^{k_s}
\end{aligned}
$$

Numerical results: convergence test



$$\mathbf{u}_f = \pi\cos(\pi t)\begin{pmatrix}-3x+\cos(y)\\ y+1\end{pmatrix}$$

$$p_f = \mathbf{e}^t\sin(\pi x)\cos(\frac{\pi y}{2}) + 2\pi\cos(\pi t)$$

$$\mathbf{u}_p = \pi\mathbf{e}^t\begin{pmatrix}\cos(\pi x)\cos(\frac{\pi y}{2})\\ \frac{1}{2}\sin(\pi x)\sin(\frac{\pi y}{2})\end{pmatrix}$$

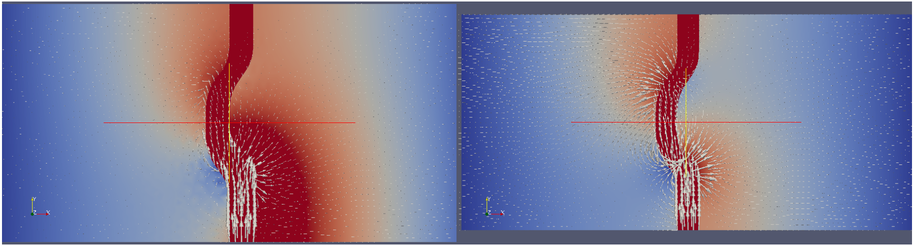
$$p_p = \mathbf{e}^t\sin(\pi x)\cos(\frac{\pi y}{2})$$

$$\boldsymbol{\eta}_p = \sin(\pi t)\begin{pmatrix}-3x+\cos(y)\\ y+1\end{pmatrix}$$

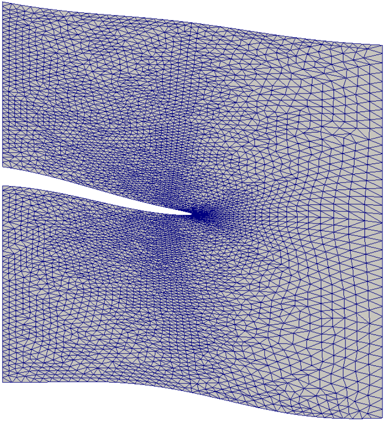
Numerical results: convergence test

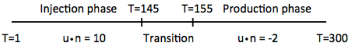
$\mathcal{P}_1^b - \mathcal{P}_1$, $\mathcal{RT}_0 - \mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_0$										
	$\Vert \mathbf{e}_f \Vert_{l^2(H^1(\Omega_f))}$		$\Vert e_{fp} \Vert_{l^2(L^2(\Omega_f))}$		$\Vert \mathbf{e}_p \Vert_{l^2(L^2(\Omega_p))}$		$\Vert e_{pp} \Vert_{l^\infty(L^2(\Omega_p))}$		$\Vert \mathbf{e}_s \Vert_{l^\infty(H_1(\Omega_p))}$	
h_{Biot}	error	rate	error	rate	error	rate	error	rate	error	rate
1/8	1.43E-02	–	6.06E-03	–	1.05E-01	–	1.03E-01	–	5.09E-02	–
1/16	7.16E-03	1.0	1.79E-03	1.8	5.23E-02	1.0	5.17E-02	1.0	1.34E-02	1.9
1/32	3.58E-03	1.0	5.81E-04	1.6	2.61E-02	1.0	2.59E-02	1.0	3.94E-03	1.8
1/64	1.79E-03	1.0	1.95E-04	1.6	1.31E-02	1.0	1.29E-02	1.0	1.43E-03	1.5
1/128	8.94E-04	1.0	6.77E-05	1.5	6.53E-03	1.0	6.47E-03	1.0	6.32E-04	1.2

Comparison of Nitsche (left) and Lagrange multiplier (right) on a curved fracture



Injection-production example (Lagrange multiplier)

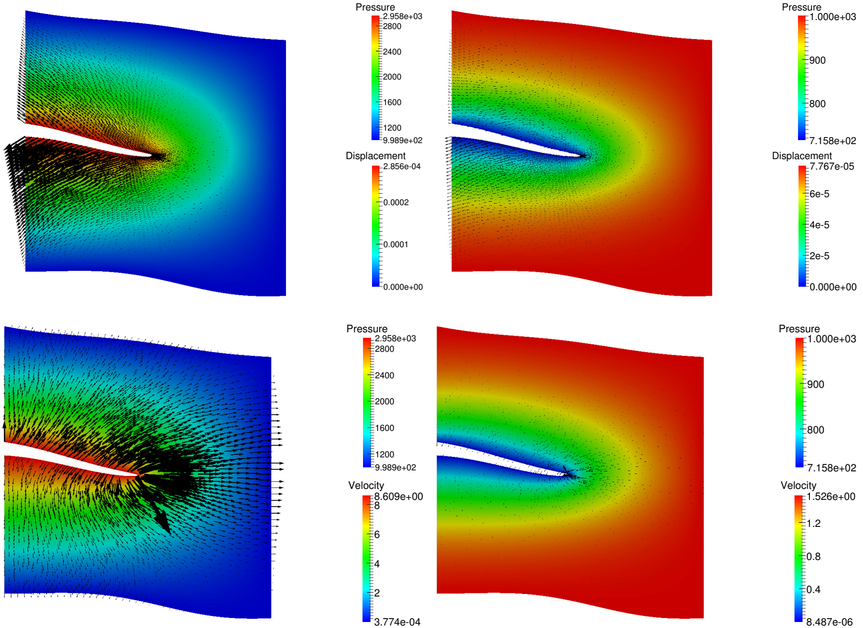




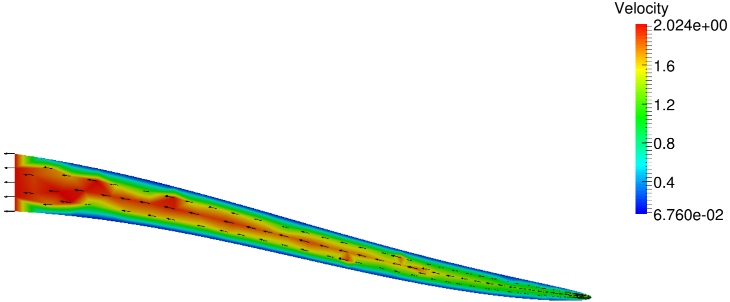
Parameter	Units	Value
Fluid density	kg/m^3	897
Dyn. viscosity	$KPa \cdot s$	10^{-6}
Hydraulic conductivity	$m^2/KPa \cdot s$	5×10^{-5}
Mass storativity	KPa^{-1}	6.89×10^{-2}
Biot-Willis constant		1
Total simulation time	s	300
Time step	s	1
BJS constant		1

$$
\begin{aligned}
u_f \cdot n &= 10, \text{ on } \Gamma_{inflow} \\
u_f \cdot \tau &= 0, \text{ on } \Gamma_{inflow} \\
u_p \cdot n &= 0, \text{ on } \Gamma_{left} \\
p_p &= 1000, \text{ on } \Gamma_{top} \cup \Gamma_{right} \cup \Gamma_{bottom} \\
\eta \cdot n &= 0, \text{ on } \Gamma_{top} \cup \Gamma_{right} \cup \Gamma_{bottom} \\
\eta \cdot \tau &= 0, \text{ on } \Gamma_{top} \cup \Gamma_{right} \cup \Gamma_{bottom}
\end{aligned}
$$

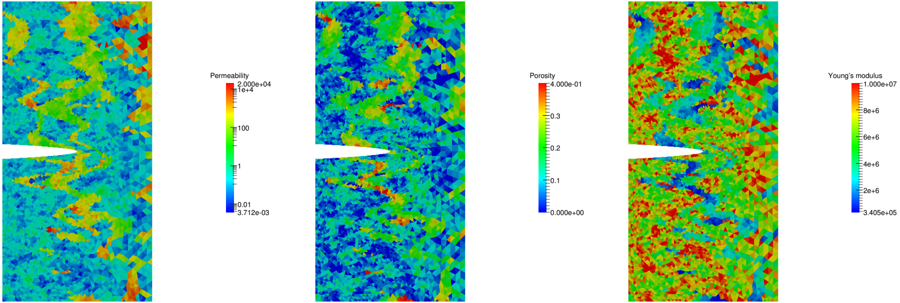
Injection-production example (Lagrange multiplier)



Injection-production example (Lagrange multiplier)



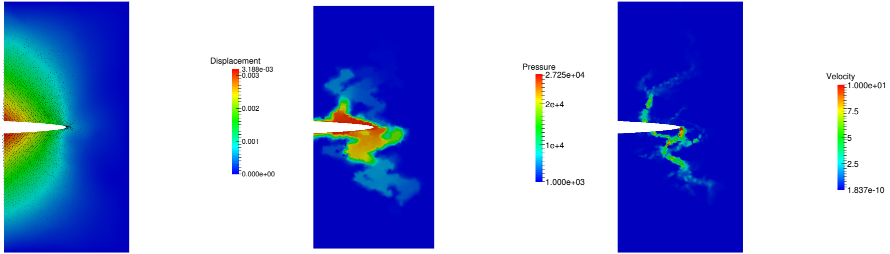
Heterogeneous example (Lagrange multiplier)



Young's modulus:

$$E = E_0(1 - \frac{\phi}{0.5})^{2.1}$$

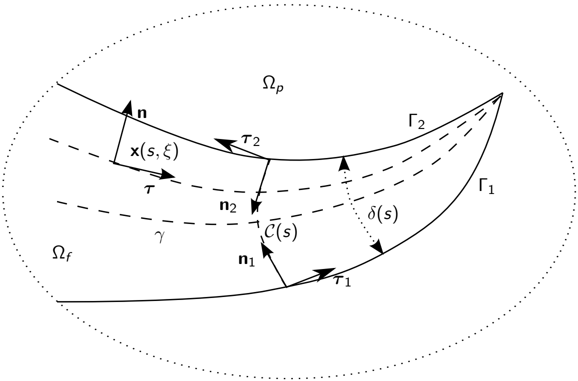
Heterogeneous example (Lagrange multiplier)



Reduced fracture model for the Biot-Stokes system [3]

Martin, Jaffre, Roberts [2005]

Lesinigo, D'Angelo, Quarteroni [2011]



$$\gamma = \{(s,0)\},\ \Gamma_1 = \{(s, \tfrac{-\delta(s)}{2})\},\ \Gamma_2 = \{(s, \tfrac{+\delta(s)}{2})\},\ \mathcal{C}(s) = \{(s,\xi) : \xi \in \frac{\delta(s)}{2}[-1,1]\ \},\quad s \in [0,L].$$

[3] Bukac, I.Y., Zunino, M2AN 2016

Reduced fracture model

Averaged variables:

$$U_n = \frac{1}{\delta}\int_{\mathcal{C}} \mathbf{u}_f \cdot \mathbf{n}\, d\mathbf{n}, \quad U_\tau = \frac{1}{\delta}\int_{\mathcal{C}} \mathbf{u}_f \cdot \boldsymbol{\tau}\, d\mathbf{n}, \quad P = \frac{1}{\delta}\int_{\mathcal{C}} p_f\, d\mathbf{n}$$

Reduced fracture flow equations:

$$-\mu_f \frac{\partial^2 U_n}{\partial \boldsymbol{\tau}^2} - F_n^f = \frac{1}{\delta}\left(p_{p,1}\big|_{\Gamma_1} - p_{p,2}\big|_{\Gamma_2} \right)$$

$$-\mu_f \frac{\partial^2 U_\tau}{\partial \boldsymbol{\tau}^2} + \frac{\partial P}{\partial \boldsymbol{\tau}} - F_\tau^f = -\frac{1}{\delta} c_{BJS}(\mathbf{u}_f - \frac{\partial \boldsymbol{\eta}_p}{\partial t}) \cdot \boldsymbol{\tau}\Big|_{\Gamma_1}^{\Gamma_2}$$

$$\frac{\partial U_\tau}{\partial \boldsymbol{\tau}} - H = \frac{1}{\delta}\left((\frac{\partial \boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p) \cdot \mathbf{n}_1\big|_{\Gamma_1} + (\frac{\partial \boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p) \cdot \mathbf{n}_2\big|_{\Gamma_2} \right)$$

Interface conditions for Biot on Γ_i

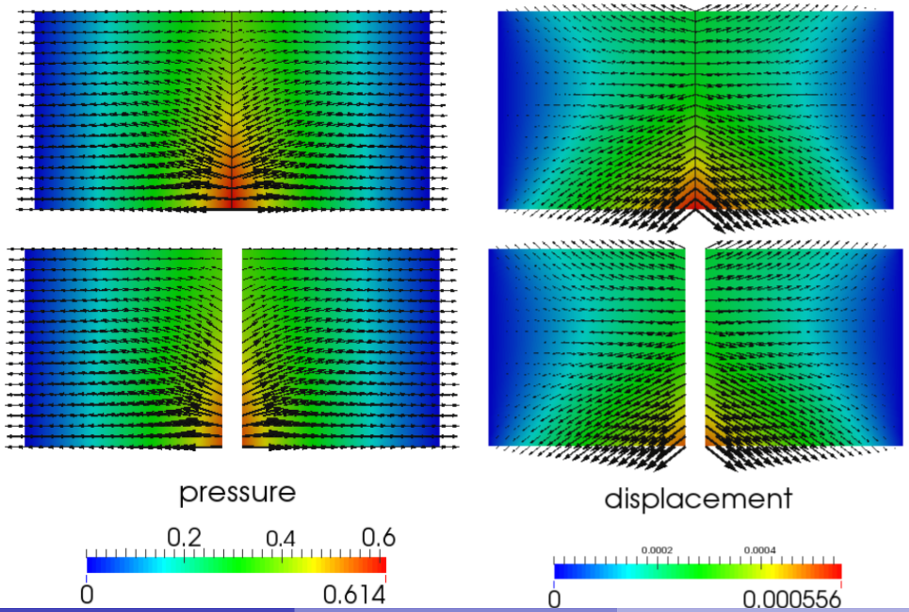
Closure: assume polynomial pressure and velocity profiles along the fracture width

$$(p_p - \beta(\frac{\partial \boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p) \cdot \mathbf{n}_i)|_{\Gamma_i} = F_i(P, U_n, p_p|_{\Gamma_j})$$

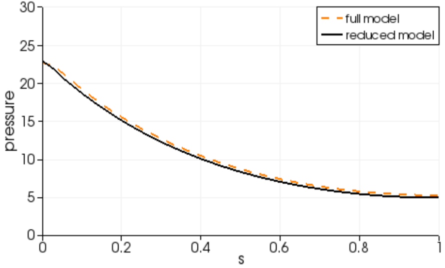
$$(\boldsymbol{\sigma}_p \mathbf{n}_i) \cdot \mathbf{n}_i|_{\Gamma_i} = -p_p|_{\Gamma_i}$$

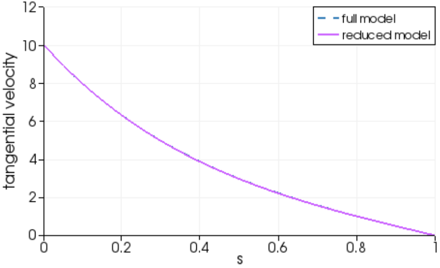
$$((\boldsymbol{\sigma}_p \mathbf{n}_i) \cdot \boldsymbol{\tau}_i - \beta_i \frac{\partial \boldsymbol{\eta}_p}{\partial t})|_{\Gamma_i} = G_i(U_\tau, \frac{\partial \boldsymbol{\eta}_p}{\partial t}|_{\Gamma_j})$$

Comparison with full (Nitsche) fracture model



Comparison with full (Nitsche) fracture model





Left: Average pressure P along the midline γ. Right: Average tangential velocity U_τ along the fracture midline γ.

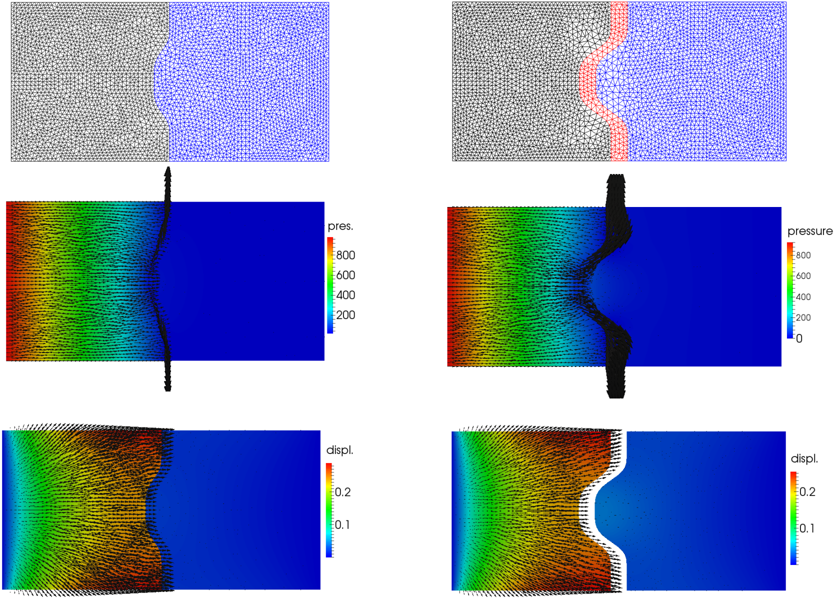
Comparison with full (Nitsche) fracture model

$$\mathcal{R}_{\Gamma_i} := \int_{\Gamma_i} \left(\mathbf{u}_f \cdot \mathbf{n}_i - (\frac{\partial \boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p) \cdot \mathbf{n}_i \right)$$

$$\mathcal{R}_{\gamma,i} := \int_{\gamma} \left(\frac{\delta}{4\mu_f} p_p|_{\Gamma_i} + \frac{\delta}{4\mu_f} p_p|_{\Gamma_j} - \frac{\delta}{2\mu_f} P - \left(\frac{\partial \boldsymbol{\eta}_p}{\partial t} \cdot \mathbf{n}_i + \mathbf{u}_p \cdot \mathbf{n}_i - \mathbf{U} \cdot \mathbf{n}_i \right) \Big|_{\Gamma} \right.$$

	Full model			Reduced model		
h	$\mathcal{R}_{\Gamma_1}$	$\mathcal{R}_{\Gamma_2}$	rate	$\mathcal{R}_{\gamma,1}$	$\mathcal{R}_{\gamma,2}$	rate
$1/20$	$4.56\mathrm{e}-3$	$4.56\mathrm{e}-3$	–	$1.4\mathrm{e}-2$	$1.4\mathrm{e}-2$	–
$1/40$	$1.32\mathrm{e}-3$	$1.32\mathrm{e}-3$	1.79	$5.1\mathrm{e}-3$	$5.1\mathrm{e}-3$	1.46
$1/80$	$3.44\mathrm{e}-4$	$3.44\mathrm{e}-4$	1.94	$2.8\mathrm{e}-3$	$2.8\mathrm{e}-3$	0.85
$1/160$	$8.76\mathrm{e}-5$	$8.76\mathrm{e}-5$	1.97	$1.4\mathrm{e}-3$	$1.4\mathrm{e}-3$	1.00

Comparison with full fracture model on a curved fracture



A nonlinear Biot-Stokes model for the interaction of a non-Newtonian fluid with poroelastic media [4]

References:

- Ervin, V.J., Jenkins, E.W., and Sun, S., *"Coupled Generalized Non-linear Stokes Flow with flow through a Porous Media"*, SINUM 2009
- Showalter, R. E., *"Nonlinear Degenerate Evolution Equations in Mixed Formulation"*, SIAMMA 2010
- X. Lopez, X., Valvatne, P.H. and Blunt, M.J. *"Predictive network modeling of single-phase non-Newtonian flow in a porous media"*, J. Colloid Int. Sci. 2003

[4] I. Ambartsumyan, V.J.Ervin, T. Nguen, I. Yotov (preprint)

Non-Newtonian fluid

Fluid region: the viscosity is a function of the magnitude of $\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f)$. Examples:

$$\text{Carreau model: } \nu(\mathbf{D}(\mathbf{u}_f)) = \nu_\infty + \frac{\nu_0 - \nu_\infty}{(1 + K|\mathbf{D}(\mathbf{u}_f)|^2)^{(2-r)/2}}$$

$$\text{Cross model: } \nu(\mathbf{D}(\mathbf{u}_f)) = \nu_\infty + \frac{\nu_0 - \nu_\infty}{1 + K|\mathbf{D}(\mathbf{u}_f)|^{2-r}}$$

$$\text{Power law model: } \nu(\mathbf{D}(\mathbf{u}_f)) = K\left(\frac{1}{|\mathbf{D}(\mathbf{u}_f)|}\right)^{2-r}$$

Porous media region: the viscosity is a function of the magnitude of $\mathbf{u}_p$. Examples:

$$\text{Cross model: } \nu_{eff}(\mathbf{u}_p) = \nu_\infty + \frac{\nu_0 - \nu_\infty}{1 + K|\mathbf{u}_p|^{2-r}}$$

$$\text{Power law model: } \nu_{eff}(\mathbf{u}_p) = K\left(\frac{\sqrt{\kappa} m_c}{|\mathbf{u}_p|}\right)^{2-r}$$

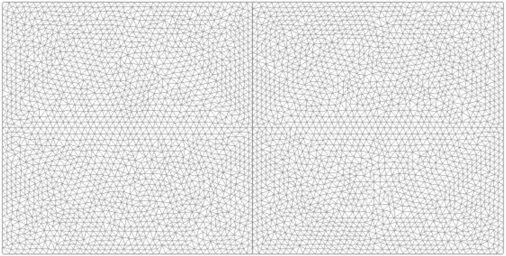
Here $r \in [1, 2)$ gives a shear thinning property and $r = 2$ corresponds to Newtonian fluid.

Semidiscrete formulation: error estimate

Theorem

$$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{L^2(0,T;\mathbf{V}_f)} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|^2_{L^2(0,T;L^r(\Omega_p))} + \|\mathcal{G}(\mathbf{u},\mathbf{u}_h)\|_{L^1(0,T)}$$
$$+ \|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|^2_{L^\infty(0,T;\mathbf{X}_p)} + \|\mathbf{u}_f - \mathbf{u}_{f,h} - \partial_s(\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\|^2_{L^2(0,T;L^r(\Gamma_{fp}))}$$
$$+ \|p_f - p_{f,h}\|^{r'}_{L^r(0,T;W_f)} + \|p_p - p_{p,h}\|^{r'}_{L^r(0,T;W_p)} + \|\lambda - \lambda_h\|^{r'}_{L^r(0,T;\Lambda)}$$
$$\lesssim h^{k_f} + h^{k_p+1} + h^{k_s}$$

Numerical results: filter example



Viscosity functions:

$$\nu_f = \nu_{f,\infty} + \frac{\nu_{f,0} - \nu_{f,\infty}}{1 + K_f|\mathbf{D}(\mathbf{u}_f)|^{2-r_f}}$$

$$\nu_p = \nu_{p,\infty} + \frac{\nu_{p,0} - \nu_{p,\infty}}{1 + K_p|\mathbf{u}_p|^{2-r_p}}$$

$$K_f = K_p = 1,\ r_f = r_p = 1.5$$

$$\nu_{f,\infty} = \nu_{p,\infty} = 1,\ \nu_{f,0} = \nu_{p,0} = 10$$

Boundary conditions:

$$\text{fluid:} \quad \begin{cases} p_f = 1, \text{ on } \Gamma_{left} \\ u_f \cdot \tau = 0, \text{ on } \Gamma_{left} \\ \text{no slip on } \Gamma_{top} \cup \Gamma_{bottom} \end{cases}$$

$$\text{structure:} \begin{cases} p_p = 0, \text{ on } \Gamma_{right} \\ \text{no flow on } \Gamma_{top} \cup \Gamma_{bottom} \\ \eta = 0, \text{ on } \Gamma_{top} \cup \Gamma_{right} \cup \Gamma_{bottom} \end{cases}$$

Assumption: $\alpha_{BJS} = 1$, independent of viscosity.

Computed solution for the filter example

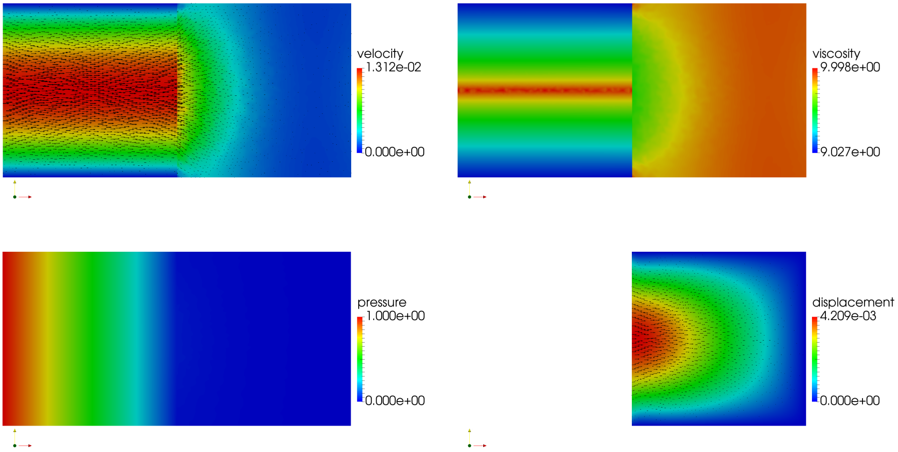
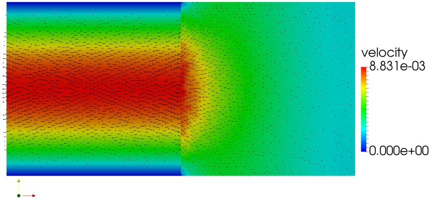
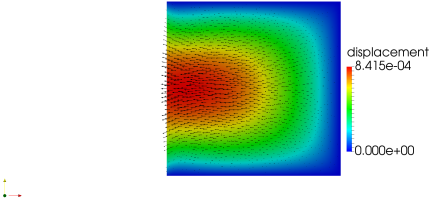


Figure : Solution at $t = 1$s: velocity (top left), viscosity (top right), pressure (bottom left), displacement (bottom right).

Difference between the nonlinear and the linear model



Difference in velocity



Difference in displacement

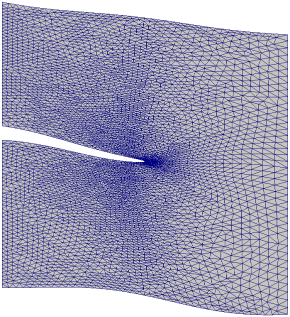
Convergence study

h	$\frac{\lVert \mathbf{u}_{f,h}^{ref} - \mathbf{u}_{f,h} \rVert_{l^2(0,T;H^1(\Omega_f))}}{\lVert \mathbf{u}_{f,h}^{ref} \rVert_{l^2(0,T;H^1(\Omega_f))}}$ error	order	$\frac{\lVert \mathbf{u}_{p,h}^{ref} - \mathbf{u}_{p,h} \rVert_{l^2(0,T;L^2(\Omega_p))}}{\lVert \mathbf{u}_{p,h}^{ref} \rVert_{l^2(0,T;L^2(\Omega_p))}}$ error	order	$\frac{\lVert p_{f,h}^{ref} - p_{f,h} \rVert_{l^2(0,T;L^2(\Omega_f))}}{\lVert p_{f,h}^{ref} \rVert_{l^2(0,T;L^2(\Omega_f))}}$ error	order
1/20	4.83E-03	–	1.55E-01	–	2.75E-02	–
1/40	2.31E-03	1.06	8.63E-02	0.85	1.03E-02	1.41
1/80	1.04E-03	1.16	4.08E-02	1.08	4.62E-03	1.16
1/160	3.94E-04	1.40	2.07E-02	0.98	2.14E-04	1.11

h	$\frac{\lVert p_{p,h}^{ref} - p_{p,h} \rVert_{l^2(0,T;L^2(\Omega_p))}}{\lVert p_{p,h}^{ref} \rVert_{l^2(0,T;L^2(\Omega_p))}}$ error	order	$\frac{\lVert p_{p,h}^{ref} - p_{p,h} \rVert_{l^\infty(0,T;L^2(\Omega_p))}}{\lVert p_{p,h}^{ref} \rVert_{l^\infty(0,T;L^2(\Omega_p))}}$ error	order	$\lVert \frac{\boldsymbol{\eta}_{p,h}^{ref} - \boldsymbol{\eta}_{p,h} \rVert_{l^\infty(0,T;H^1(\Omega_p))}}{\boldsymbol{\eta}_{p,h}^{ref} \rVert_{l^\infty(0,T;H^1(\Omega_p))}}$ error	order
1/20	4.10E-02	–	1.15E-01	–	4.98E-02	–
1/40	1.92E-02	1.10	5.28E-02	1.12	2.88E-02	0.79
1/80	8.24E-03	1.22	2.25E-02	1.23	1.61E-02	0.84
1/160	2.75E-03	1.58	7.48E-03	1.59	6.59E-03	1.29

Table : Convergence for $\mathcal{P}_1 b \times \mathcal{P}_1 \times \mathcal{RT}_0 \times \mathcal{P}_0 \times \mathcal{P}_1 \times \mathcal{P}_0$ elements.

Hydraulic fracturing example



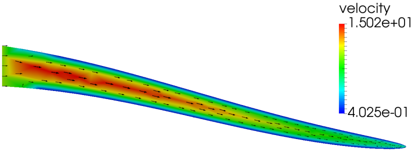
Viscosity functions:

$$\nu_f = \nu_{f,\infty} + \frac{\nu_{f,0} - \nu_{f,\infty}}{1 + K_f |\mathbf{D}(\mathbf{u}_f)|^{2-r_f}}$$

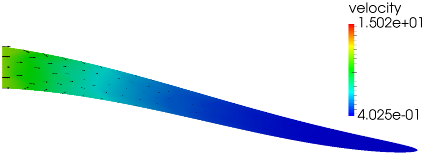
$$\nu_p = \nu_{p,\infty} + \frac{\nu_{p,0} - \nu_{p,\infty}}{1 + K_p |\mathbf{u}_p|^{2-r_p}}$$

Parameter		Units	Values
Young's modulus	E	(KPa)	10^7
Poisson's ratio	σ		0.2
Fluid viscosity	$\nu_{f,\infty}$	(KPa s)	$1.98 \cdot 10^{-8}$
	$\nu_{f,0}$	(KPa s)	$1.98 \cdot 10^{-6}$
	K_f		1
	r_f		1.35
Effective viscosity	$\nu_{p,\infty}$	(KPa s)	$1.98 \cdot 10^{-8}$
	$\nu_{p,0}$	(KPa s)	$1.98 \cdot 10^{-6}$
	K_p		1
	r_p		1.35
Lame coefficient	μ_p	(KPa)	$5/12 \cdot 10^7$
Lame coefficient	λ_p	(KPa)	$5/18 \cdot 10^7$
Permeability	K	(m^2)	$(200, 50) \times 10^{-12}$
Mass storativity	s_0	(KPa^{-1})	6.89×10^{-2}
Biot-Willis const.	α		1.0
BJS coeff.	α_{BJS}		1.0
Total time	T	(s)	300

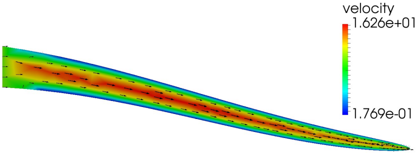
Stokes velocity



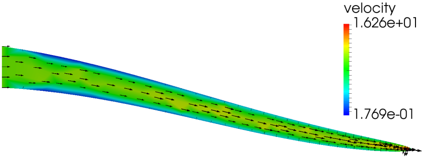
linear, $t = 1$



nonlinear, $t = 1$

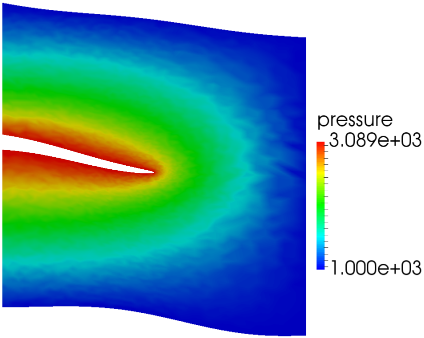


linear, $t = 300$

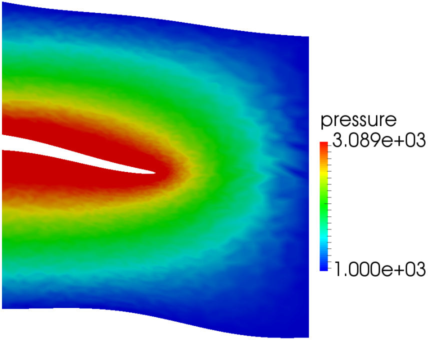


nonlinear, $t = 300$

Darcy pressure

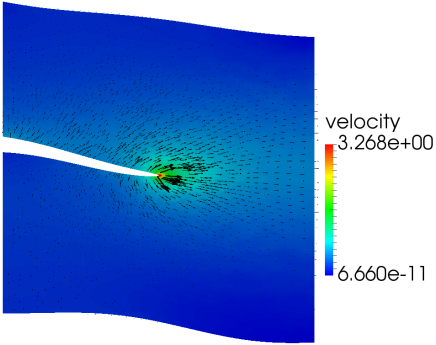


linear, $t = 300$

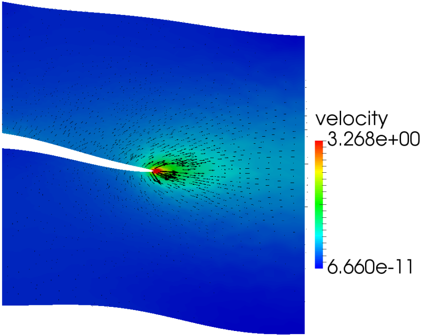


nonlinear, $t = 300$

Darcy velocity

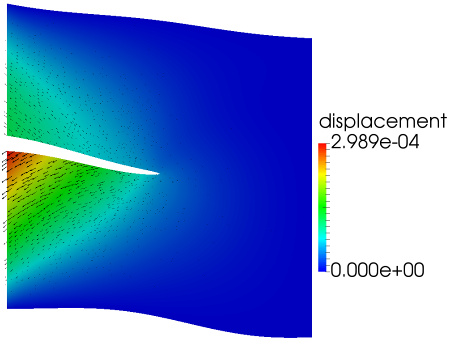


linear, $t = 300$

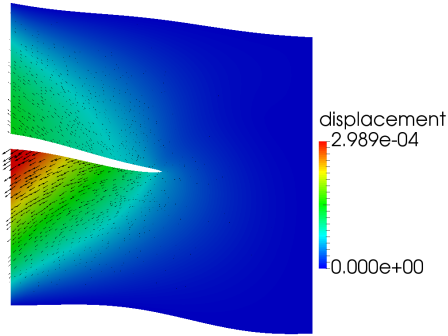


nonlinear, $t = 300$

Displacement



linear, $t = 300$



nonlinear, $t = 300$

Summary

- Couplings of Stokes and Biot equations: accurate modeling of flows in deformable fractured porous media
- Interior penalty for the normal flux: efficient non-iterative time-splitting
- Mortar formulation: accurate multiscale discretization; suitable for parallel domain decomposition
- Dimensionally reduced fracture model: reduced computational cost with comparable accuracy to the full fracture model
- Noninear Stokes-Biot model: polymer flow in hydraulic fracturing

Current and Future Work

- Non-overlapping domain decomposition for mortar formulations
- Multirate time-stepping schemes
- Proppant transport