On the Power of Models of Quantum Computation

Manny

- Models of quantum computation.

  . . . many examples.
- Faithful simulation of QC?
- Counting models.
- Classical simulations of counting models?
- Lie algebraic models are weak.

  . . . such as true linear optics models.

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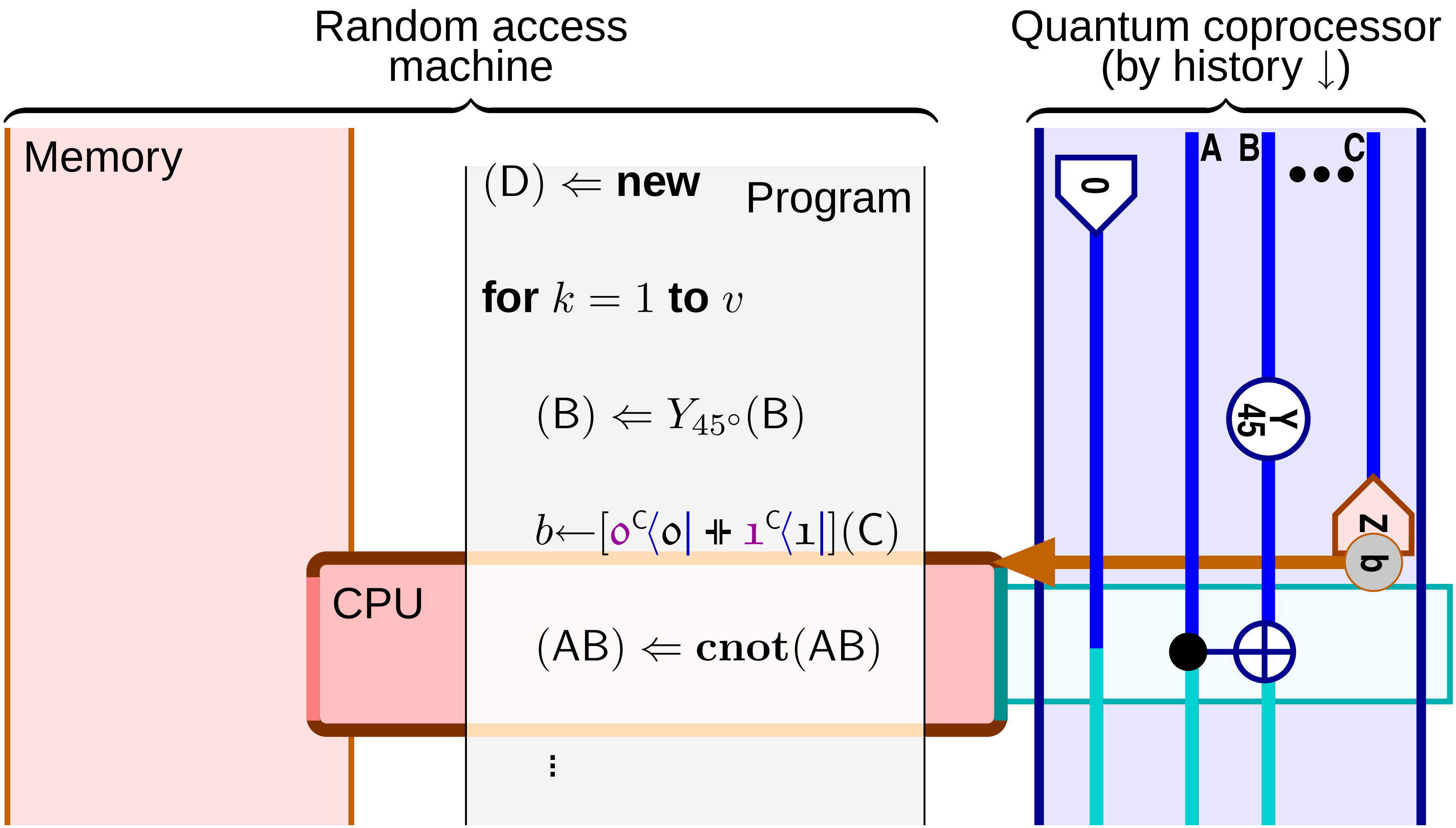
Computation with Coprocessors



```
$(\text{D}) \Leftarrow$ new
for $k = 1$ to $v$
    $(\text{B}) \Leftarrow Y_{45^\circ}(\text{B})$
    $b \leftarrow [0^{\text{C}}\langle 0| \Vdash 1^{\text{C}}\langle 1|](\text{C})$
    $(\text{AB}) \Leftarrow \mathbf{cnot}(\text{AB})$
    $\vdots$
```

- Quantum computation $=$
  deterministic computation $+$ quantum network coprocessor

Quantum Coprocessor Instances

1. Get resources.

- Ask for n qubits
- Cost: n.

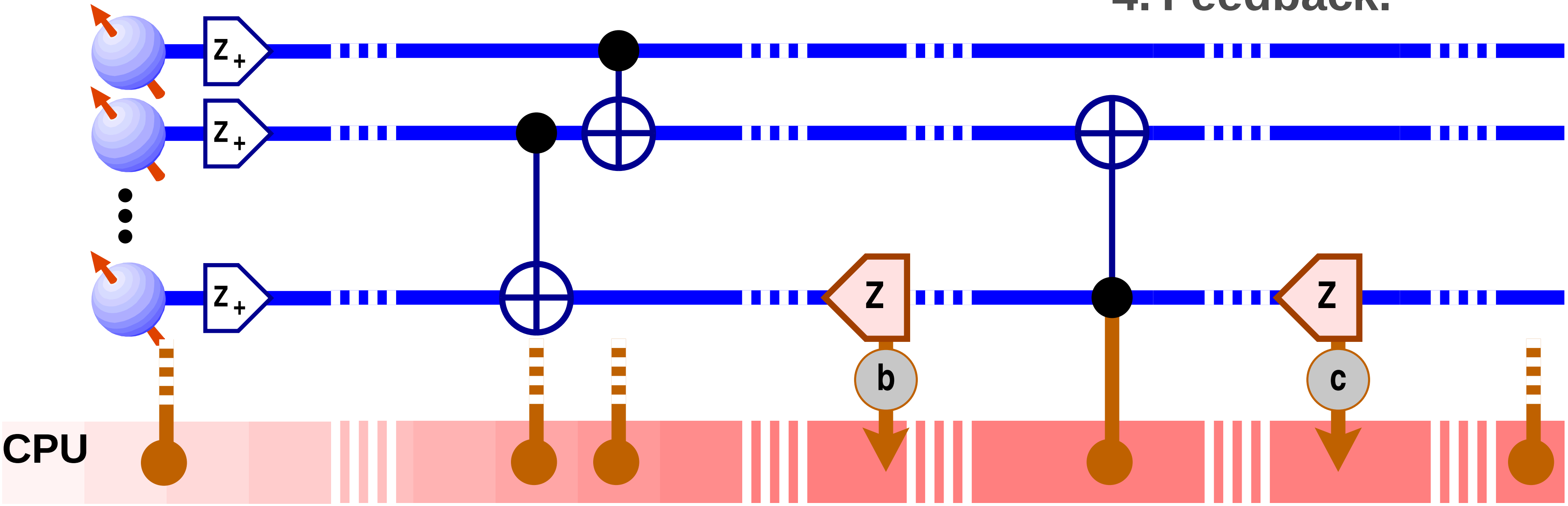
2. Apply operations.

- Unitary one and two qubit quantum gates.
- Cost: Constant per gate.

3. Read.

- One qubit measurement.
- Cost: Constant per measurement.

4. Feedback.



General Quantum Models

1. Get resources. A specified finite-cost quantum system $\mathcal{A}$.

- Initial state ρ.
- ◆ Cost: $C(\mathcal{A},\rho)$.

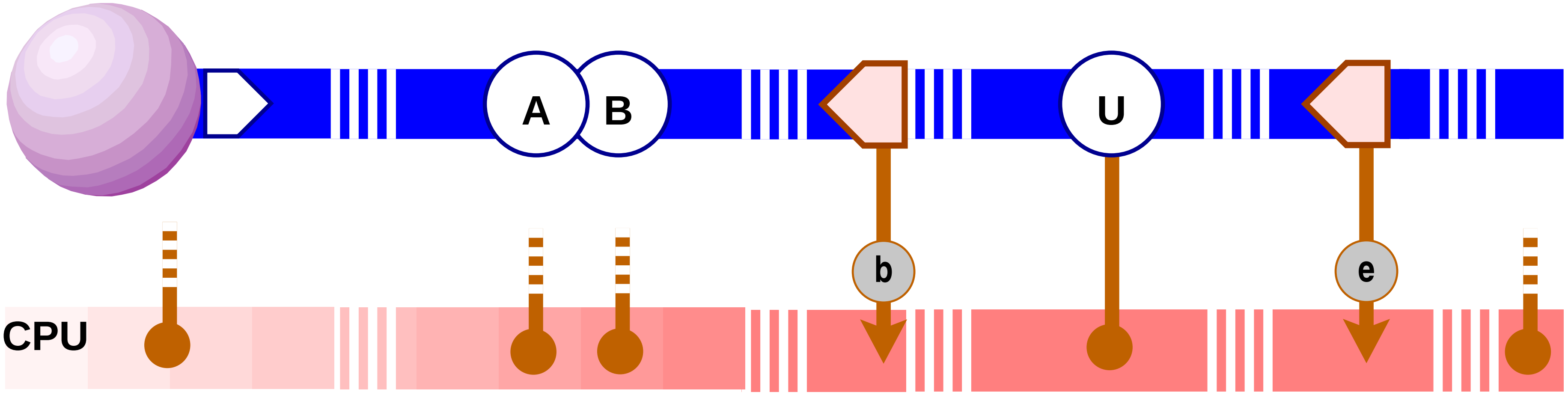
2. Apply operations.

- Unitary gate set.
- ◆ Cost: Constant per gate.

3. Read.

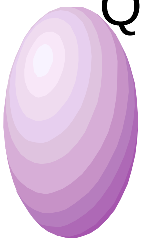
- Set of generalized measurements with backactions.
- ◆ Cost: Constant per measurement.

4. Feedback.



Some Examples

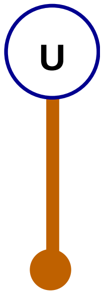
Coin flips ($\emptyset$).



Quantum system:

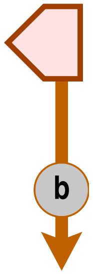
- One qubit. Hilbert space $\mathbb{C}^2$.
  Observables $\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$.
- Initial state: $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

◆ Physical cost: 1.



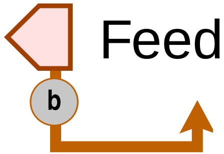
Operations:

- None.



Measurements:

- Von Neumann measurement of σ_z.
  $|+\rangle \rightarrow 0 \cdot |0\rangle \mathbin{+\!\!\!+} 1 \cdot |1\rangle$



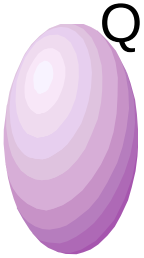
Feedback?

- Irrelevant.

Observation: $\emptyset \simeq$ Probabilistic computation.

Some Examples

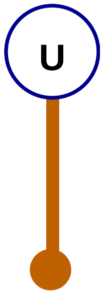
Standard quantum computation (QC).



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_i \sigma_u^{(i)}$.
- Initial state: $|00\ldots0\rangle$.

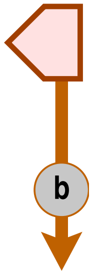
◆ Physical cost: 1 per qubit.



Operations:

- Universal set of one and two qubit gates.
- Example: $\exp\left(-i\sigma_u^{(i)}\pi/8\right),\ \exp\left(-i\sigma_z^{(k)}\sigma_z^{(l)}\pi/4\right)$.

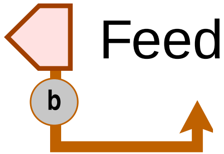
◆ Physical cost: 1 per gate.



Measurements:

- Measurement of qubit 1.
  $|\psi\rangle \quad \rightarrow \quad \underbrace{0}_{\text{probabilistic output}} \cdot |0\rangle^{1}_{1}\langle 0||\psi\rangle \ +\!\!\!\!+\ 1 \cdot \underbrace{|1\rangle^{1}_{1}\langle 1|}_{\text{projector on qubit 1}}|\psi\rangle$.

◆ Physical cost: 1 per measurement.

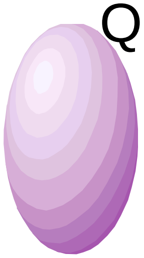


Feedback?

- Yes.

Some Examples

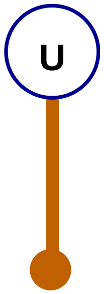
Pauli normalizer (Clifford).



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_{\mathrm{i}} \sigma_u{}^{(\mathrm{i})}$.
- Initial state: $|00\ldots0\rangle$.

◆ Physical cost: 1 per qubit.

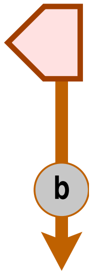


Operations:

- Set of one and two qubit gates.
- $\exp\left(-i\sigma_u{}^{(\mathrm{i})}\pi/4\right),\ \exp\left(-i\sigma_z{}^{(\mathrm{k})}\sigma_z{}^{(\mathrm{l})}\pi/4\right)$.

. . . generates the “normalizer” of the Pauli-Heisenberg group.

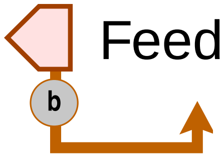
◆ Physical cost: 1 per gate.



Measurements:

- Measurement of qubit 1.
  $|\psi\rangle \quad \to \quad \underbrace{0}_{\text{probabilistic output}} \cdot |0\rangle^1_1\langle 0||\psi\rangle \mathbin{+\!\!\!+} 1 \cdot \underbrace{|1\rangle^1_1\langle 1|}_{\text{projector on qubit 1}}|\psi\rangle$.

◆ Physical cost: 1 per measurement.



Feedback?

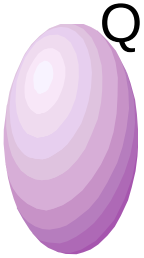
- Yes.

Relationship: $\emptyset \simeq_{\text{poly}}$ Clifford

. . . yields only stabilizer states. Gottesman’98[1]

Some Examples

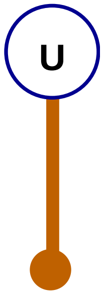
Standard quantum computation (QC).



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_i \sigma_u{}^{(i)}$.
- Initial state: $|00\ldots0\rangle$.

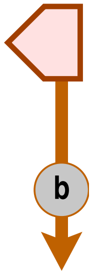
◆ Physical cost: 1 per qubit.



Operations:

- Universal set of one and two qubit gates.
- Example: $\exp\left(-i\sigma_u{}^{(i)}\pi/8\right),\ \exp\left(-i\sigma_z{}^{(k)}\sigma_z{}^{(l)}\pi/4\right)$.

◆ Physical cost: 1 per gate.

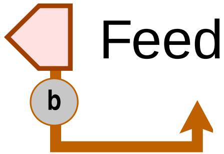


Measurements:

- Measurement of qubit 1.

$$|\psi\rangle \quad\to\quad \underbrace{0}_{\text{probabilistic output}} \cdot |0\rangle{}^{1}_{1}\langle 0||\psi\rangle \;\parallel\!\!\!+\; 1 \cdot \underbrace{|1\rangle{}^{1}_{1}\langle 1|}_{\text{projector on qubit 1}}|\psi\rangle.$$

◆ Physical cost: 1 per measurement.

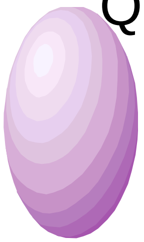


Feedback?

- Yes.

Some Examples

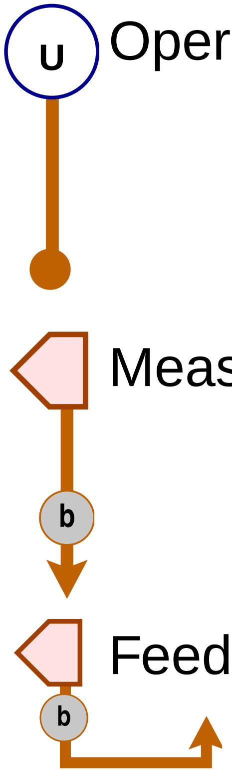
Noisy quantum computing ($\Gamma(\epsilon)$QC).



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_i \sigma_u^{(i)}$.
- Initial state: $|00\ldots0\rangle$.

◆ Physical cost: 1 per qubit.



Operations:

- Noisy universal set of one and two qubit gates.
- $\Gamma_{u,\pi/8}\exp\left(-i\sigma_u^{(i)}\pi/8\right),\ \Gamma_{zz,\pi/4}\exp\left(-i\sigma_z^{(k)}\sigma_z^{(l)}\pi/4\right)$.

◆ Physical cost: 1 per ga

As specified: (No memory error)
or (Threshold = 0).
Aharonov&Ben-Or’96[2]

Measurements:

- Measurement of q

$$|\psi\rangle \rightarrow 0\cdot\left(|0\rangle^{1}_{1}\langle 0| + \Gamma_{m,0}\right)|\psi\rangle \mathbin{+\!\!\!+} 1\cdot\left(|1\rangle^{1}_{1}\langle 1| + \Gamma_{m,1}\right)|\psi\rangle$$

◆ Physical

Feedback?

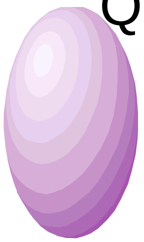
- Yes.

Error Thresholds:

Threshold(Γ) $= \sup\{\epsilon \mid \text{QC} \preccurlyeq_{\text{poly}} \Gamma(\epsilon)\text{QC}\}$

Some Examples

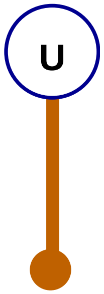
Standard quantum computation (QC).



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_{\mathsf{i}} \sigma_u^{(\mathsf{i})}$.
- Initial state: $|00\ldots0\rangle$.

◆ Physical cost: 1 per qubit.



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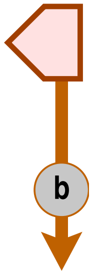
- Example: $\exp\left(-i\sigma_u^{(\mathsf{i})}\pi/8\right),\ \exp\left(-i\sigma_z^{(\mathsf{k})}\sigma_z^{(\mathsf{l})}\pi/4\right)$.

◆ Physical cost: 1 per gate.

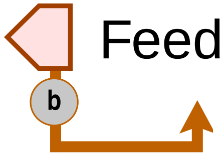


Measurements:

- Measurement of qubit 1.

$$|\psi\rangle \quad\rightarrow\quad \underbrace{0}_{\text{probabilistic output}} \cdot |0\rangle{}^{1}_{1}\langle 0||\psi\rangle \mathbin{+\!\!\!+} 1 \cdot \underbrace{|1\rangle{}^{1}_{1}\langle 1|}_{\text{projector on qubit 1}}|\psi\rangle.$$

◆ Physical cost: 1 per measurement.

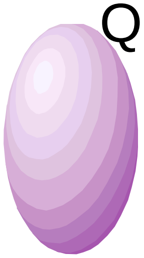


Feedback?

- Yes.

Some Examples

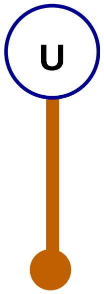
Destructive read quantum computation (QC$_d$).



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_{\mathsf{i}} \sigma_u^{(\mathsf{i})}$.
- Initial state: $|00\ldots0\rangle$.

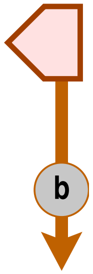
◆ Physical cost: 1 per qubit.



Operations:

- Universal set of one and two qubit gates.
- Example: $\exp\left(-i\sigma_u^{(\mathsf{i})}\pi/8\right),\ \exp\left(-i\sigma_z^{(\mathsf{k})}\sigma_z^{(\mathsf{l})}\pi/4\right)$.

◆ Physical cost: 1 per gate.

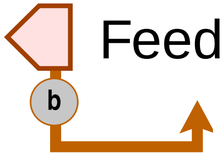


Measurements:

- Measurement of qubit 1.

$$|\psi\rangle \quad\rightarrow\quad \underbrace{0}_{\text{probabilistic output}} \cdot |0\rangle^{1}_{1}\langle 0||\psi\rangle \mathbin{+\!\!\!\!+} 1 \cdot \underbrace{|1\rangle^{1}_{1}\langle 1|}_{\text{projector on qubit 1}}|\psi\rangle.$$

◆ Physical cost: 1 per measurement.



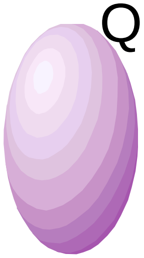
Feedback?

- No.

Some Examples

One bit QC (QC1).

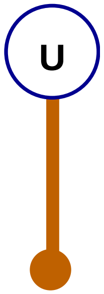
High-temperature NMR QC: Knill&Laflamme'98[3]



Quantum system:

- n qubits. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Observables $\prod_{\text{i}} \sigma_u^{(\text{i})}$.
- Initial state: $|0\rangle^1_1\langle 0|\; \mathbb{1}^{(2\ldots\text{n})}/2^{n-1}$

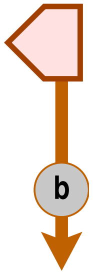
◆ Physical cost: 1 per qubit.



Operations:

- Universal set of one and two qubit gates.
- Example: $\exp\left(-i\sigma_u^{(\text{i})}\pi/8\right),\ \exp\left(-i\sigma_z^{(\text{k})}\sigma_z^{(\text{l})}\pi/4\right)$.

◆ Physical cost: 1 per gate.

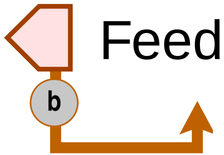


Measurements:

- Measurement of qubit 1.

$$|\psi\rangle \quad\rightarrow\quad \underbrace{0}_{\text{probabilistic output}} \cdot |0\rangle^1_1\langle 0||\psi\rangle \mathbin{+\!\!+} 1 \cdot \underbrace{|1\rangle^1_1\langle 1|}_{\text{projector on qubit 1}}|\psi\rangle .$$

◆ Physical cost: 1 per measurement.



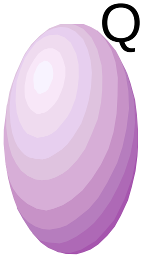
Feedback?

- No.

Relationships: $\emptyset \preccurlyeq_{\text{poly}}$ QC1 $\preccurlyeq_{\text{poly}}$ QC.
$\emptyset \overset{?}{\prec}_{\text{poly}}$ QC1 $\overset{?}{\prec}_{\text{poly}}$ QC ?

Some Examples

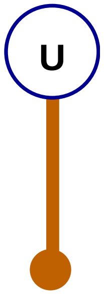
Fermionic linear optics quantum computation (FLOQC).



Quantum system:

- n fermionic modes. Hilbert space $(\mathbb{C}^2)^{\otimes n}$.
  Algebra generated by annihilation ops.:
  $$\mathbf{a}_1, \ldots \mathbf{a}_n\text{:}\ \ \{\mathbf{a}_i, \mathbf{a}^\dagger{}_j\} = \delta_{i,j},\ \ \{\mathbf{a}_i, \mathbf{a}_j\} = 0.$$
- Initial state: $\langle \mathbf{0}|\mathbf{a}^\dagger{}_i\mathbf{a}_i|\mathbf{0}\rangle = 0$ (vacuum).

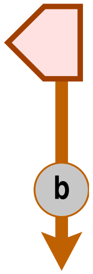
◆ Physical cost: 1 per mode.



Operations:

- Linear optics gates.
  Particle preserving: $\exp\left(-i\alpha\mathbf{a}_i\mathbf{a}^\dagger{}_j\right)$. $(|\alpha| \le 1)$
  Squeezing: $\exp\left(-i(\alpha\mathbf{a}_i\mathbf{a}_j + \bar{\alpha}\mathbf{a}^\dagger{}_i\mathbf{a}^\dagger{}_j)\right)$.
  Translation: $\exp\left(-i(\alpha\mathbf{a}_i + \bar{\alpha}\mathbf{a}^\dagger{}_i)\right)$.

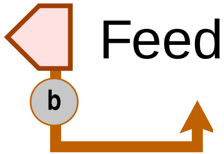
◆ Physical cost: 1 per gate.



Measurements:

- Particle detectors.
  $$|\psi\rangle \quad \rightarrow \quad \mathtt{0} \cdot |\mathbf{0}\rangle_k{}^k\langle\mathbf{0}||\psi\rangle \ ⫲\ \mathtt{1} \cdot (\mathbb{1} - |\mathbf{0}\rangle_k{}^k\langle\mathbf{0}|)|\psi\rangle.$$

◆ Physical cost: 1 per measurement.

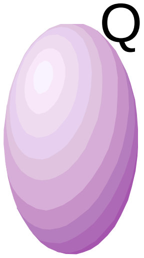


Feedback?

- Yes.

Some Examples

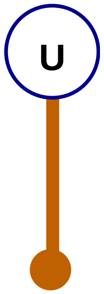
Bosonic linear optics quantum computation (eLOQC).



Quantum system:

- n bosonic modes. Hilbert space $l_2(\mathbb{N})^{\otimes n}$.
  Algebra generated by annihilation ops.:
  $$\mathbf{a}_1, \ldots \mathbf{a}_{\mathsf{n}}\colon\ [\mathbf{a}_{\mathsf{i}}, \mathbf{a}^\dagger{}_{\mathsf{j}}] = \delta_{\mathsf{i,j}},\ \ [\mathbf{a}_{\mathsf{i}}, \mathbf{a}_{\mathsf{j}}] = 0.$$
- Initial state: $\langle\mathbf{0}|\mathbf{a}^\dagger{}_{\mathsf{i}}\mathbf{a}_{\mathsf{i}}|\mathbf{0}\rangle = 0$ (vacuum).

◆ Physical cost: 1 per mode.

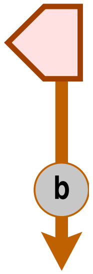


Operations:

- Linear optics gates.

Particle preserving:	$\exp\left(-i\alpha\mathbf{a}_{\mathsf{i}}\mathbf{a}^\dagger{}_{\mathsf{j}}\right)$.	$(\lvert\alpha\rvert \leq 1)$
Squeezing:	$\exp\left(-i(\alpha\mathbf{a}_{\mathsf{i}}\mathbf{a}_{\mathsf{j}} + \bar{\alpha}\mathbf{a}^\dagger{}_{\mathsf{i}}\mathbf{a}^\dagger{}_{\mathsf{j}})\right)$.	
Translation:	$\exp\left(-i(\alpha\mathbf{a}_{\mathsf{i}} + \bar{\alpha}\mathbf{a}^\dagger{}_{\mathsf{i}})\right)$.	

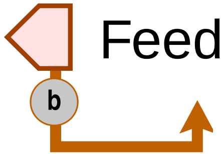
◆ Physical cost: 1 per gate.



Measurements:

- Particle detectors.
  $$|\psi\rangle \quad \rightarrow \quad \mathtt{0} \cdot |\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}||\psi\rangle \ \mathbin{+\!\!\!\Vert}\ \mathtt{1} \cdot (\mathbb{1} - |\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}|)|\psi\rangle.$$

◆ Physical cost: 1 per measurement.

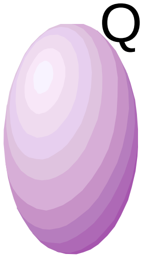


Feedback?

- Yes.

Some Examples

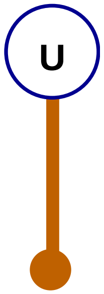
Bosonic linear optics QC with homodyne meas. (BLOQC).



Quantum system:

- n bosonic modes. Hilbert space $l_2(\mathbb{N})^{\otimes n}$.
  Algebra generated by annihilation ops.:
  $$\mathbf{a}_1,\ldots\mathbf{a}_\mathsf{n}\text{:}\quad [\mathbf{a}_\mathsf{i},\mathbf{a}^\dagger{}_\mathsf{j}]=\delta_{\mathsf{i},\mathsf{j}},\quad [\mathbf{a}_\mathsf{i},\mathbf{a}_\mathsf{j}]=0.$$
- Initial state: $\langle\mathbf{0}|\mathbf{a}^\dagger{}_\mathsf{i}\mathbf{a}_\mathsf{i}|\mathbf{0}\rangle=0$ (vacuum).

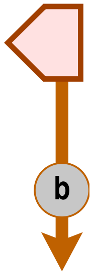
◆ Physical cost: 1 per mode.



Operations:

- Linear optics gates.
  Particle preserving: $\exp\left(-i\alpha\mathbf{a}_\mathsf{i}\mathbf{a}^\dagger{}_\mathsf{j}\right)$. $(|\alpha|\leq 1)$
  Squeezing: $\exp\left(-i(\alpha\mathbf{a}_\mathsf{i}\mathbf{a}_\mathsf{j}+\bar{\alpha}\mathbf{a}^\dagger{}_\mathsf{i}\mathbf{a}^\dagger{}_\mathsf{j})\right)$.
  Translation: $\exp\left(-i(\alpha\mathbf{a}_\mathsf{i}+\bar{\alpha}\mathbf{a}^\dagger{}_\mathsf{i})\right)$.

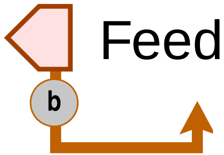
◆ Physical cost: 1 per gate.



Measurements:

- Destructive homodyne detectors.
  $$|\psi\rangle\quad\rightarrow\quad\int_\mathfrak{p}\mathfrak{p}\cdot\underbrace{|\mathbf{0}\rangle_\mathsf{j}\,{}_\mathsf{j}\langle p|}_{\text{measure } i(\mathbf{a}_\mathsf{j}-\mathbf{a}^\dagger{}_\mathsf{j})}|\psi\rangle.$$

◆ Physical



Feedback?

- Yes.

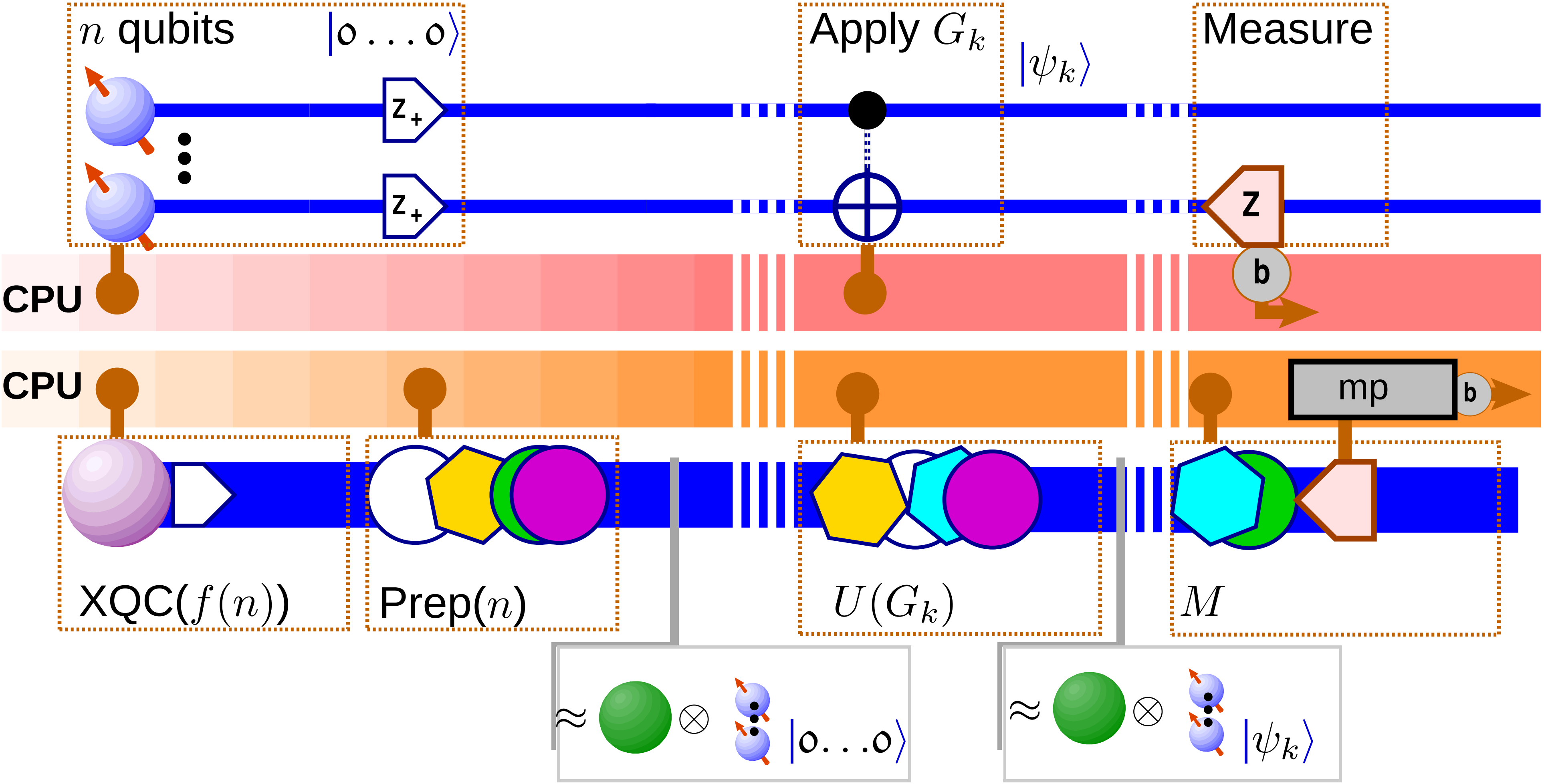
Relationship: $\emptyset\simeq_{\text{poly}}$ BLOQC.

Bartlett&Sanders&Braunstein&Nemoto'01[10]

Faithful Simulation of QC

- Given quantum model XQC, is QC $\preccurlyeq$ XQC ?
  Sufficient: XQC *efficiently faithfully simulates* QC.

Simple faithful simulation:

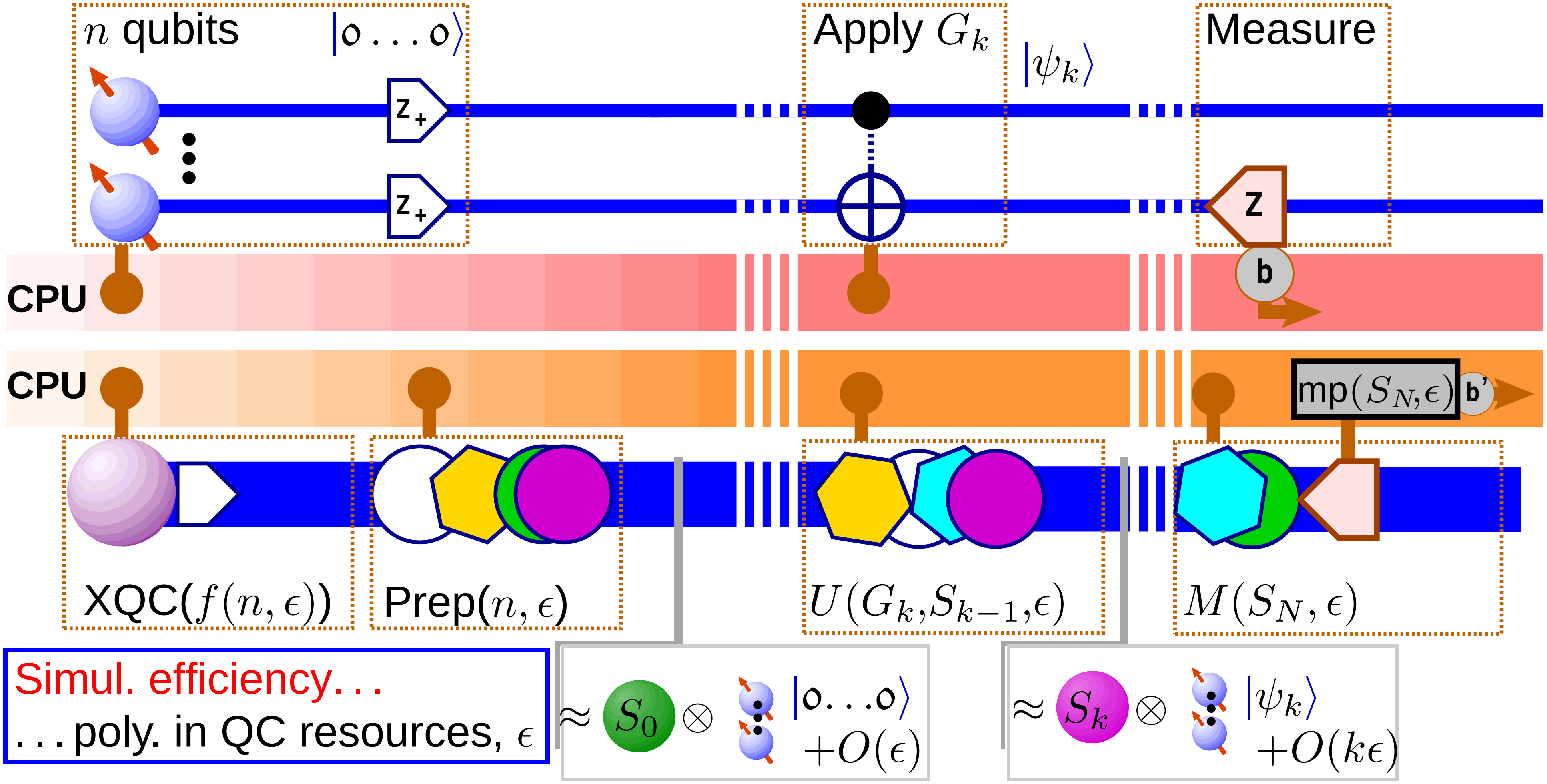


Faithful Simulation of QC

- Given quantum model XQC, is QC $\preccurlyeq$ XQC ?

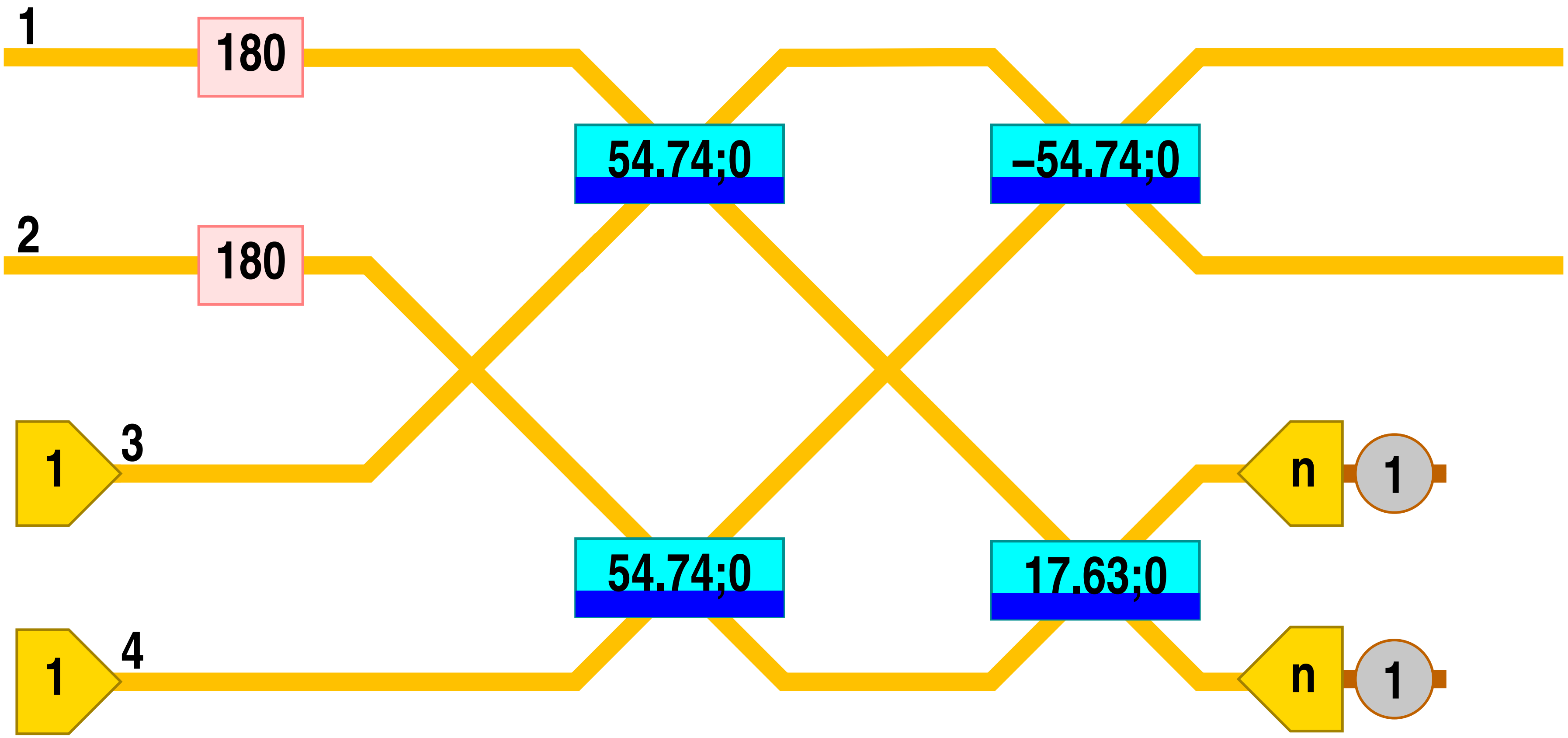
  Sufficient: XQC *efficiently faithfully simulates* QC.

General faithful simulation:



Simul. efficiency…
…poly. in QC resources, ϵ

Zooming in on eLOQC



Gate model implemented: $\text{eLOQC}_{\text{postsel}(.074)}$

…to be simulated by eLOQC.

When Not to Expect Faithful Simulation

- Given model XQC.
- Let $\mathcal{S}$ be the state space of an XQC instance of cost n.
  Let $\mathcal{R} \subseteq \mathcal{S}$ be the set of *realizable states*
  (obtainable conditional on meas. record).

- Case 1: $|\mathcal{R}| = N = 2^{\textbf{poly}(n)}$. Examples: Clifford, ...
- **Theorem 1:** Efficient faithful simulation requires that N be doubly exponential in n.

  Proof: Try to simulate s qubits.

  A. To cover $\mathbb{C}^{2^s}$ with maximum distance $1/2$
  requires $2^{2^{\Omega(s)}}$ points. Knill'95[11]

  B.1 Can efficiently represent $2^{\textbf{poly}(s)}$ logical subsystems.

  B.2 Number of logical points: $N2^{\textbf{poly}(s)}$

  Requires: $2^{2^{\Omega(s)}} \leq N2^{\textbf{poly}(s)}$ ■

When Not to Expect Faithful Simulation

- Given model XQC.
- Let $\mathcal{S}$ be the state space of an XQC instance of cost n.
  Let $\mathcal{R} \subseteq \mathcal{S}$ be the set of *realizable states*
  (obtainable conditional on meas. record).

- Case 2: $\mathcal{R}$ has an N-dim. parametrization, $N = \textbf{poly}(n)$.
  Examples: BLOQC, FLOQC, fLAM, . . .
- . . . don't expect faithful simulatability.

When Not to Expect Faithful Simulation

- Given model XQC.
- Let $\mathcal{S}$ be the state space of an XQC instance of cost n.
  Let $\mathcal{R} \subseteq \mathcal{S}$ be the set of *realizable states*
  (obtainable conditional on meas. record).

- Case 3: Unsuitable state space.
  Example: QC1.
- No feedback: Weaken “faithful simulatability” by allowing any mapping of states and stat. inference of measurements.
- Case 4: Only **poly**$(n, 1/\epsilon)$ ϵ-*distinguishable* states.
  Example: QC1.
- **Theorem 2:** QC1 cannot “faithfully simulate” QC$_d$.

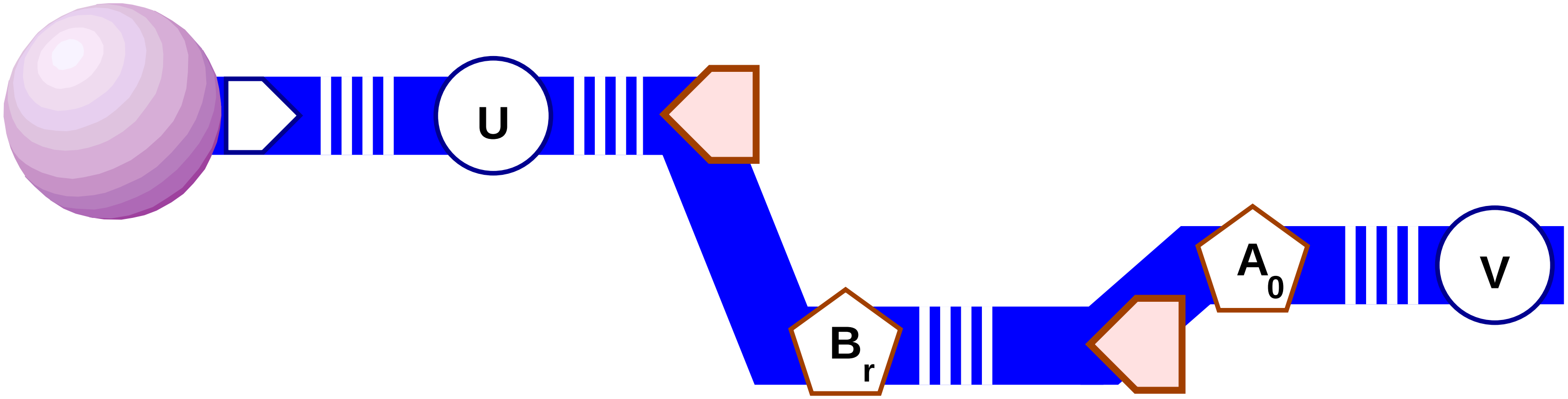
Ambainis&Schulman&Vazirani’00[12]

Classical Simulation

- Given quantum model XQC, is $\emptyset \simeq$ XQC?
- The output of a instance of XQC is the measurement record.
- $\emptyset \simeq$ XQC means...
  There is a probabilistic algorithm $\mathcal{A}$ such that:
  - Input: A instance of XQC, approx. parameter ϵ.
    (System spec., seq. of operations and measurements.)
  - Output: An instance of a measurement record.

  - The output distribution is within ϵ of the XQC instance's.
  - The resources used are poly(XQC instance resources).

Nondeterministic and Counting Models



Nondeterministic Models

1. Get resources. A specified finite-cost quantum system $\mathcal{A}$.

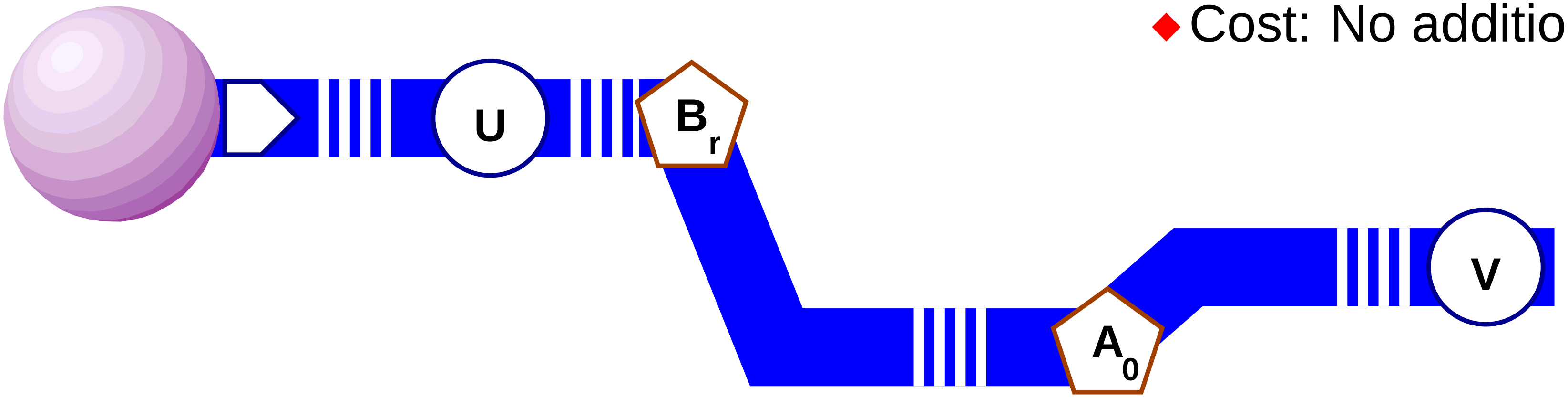
- Initial state ρ.
- Cost: $C(\mathcal{A}, \rho)$.

2. Apply operations.

- Linear operator gate set.
- Cost: Constant per gate.

3. Output.

- Answer to: $\rho_k = 0$?
- Cost: No additional cost.



Counting Models

1. Get resources. A specified finite-cost quantum system $\mathcal{A}$.

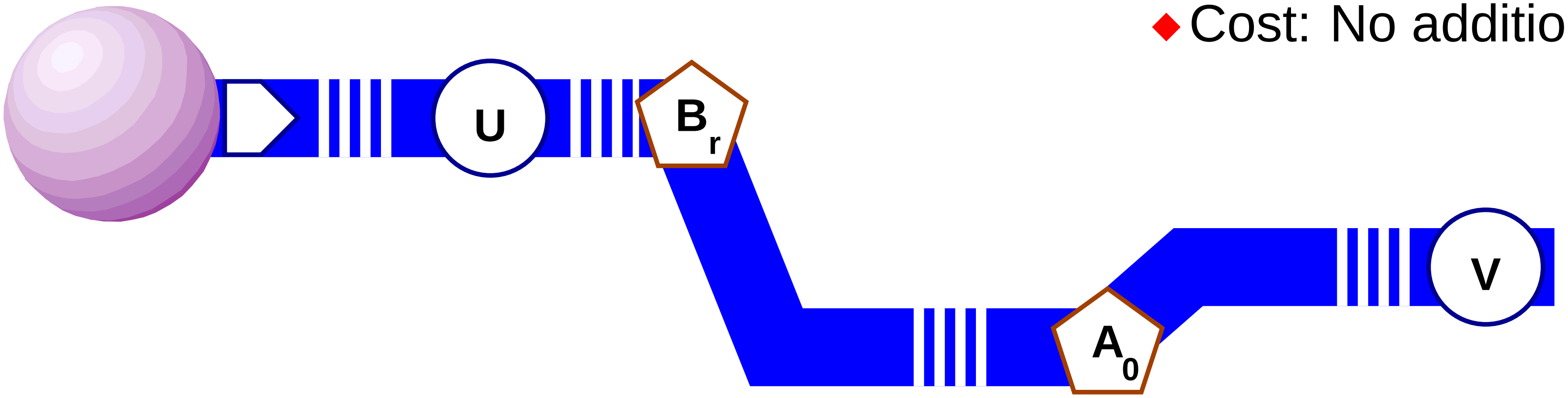
- Initial state ρ.
- Cost: $C(\mathcal{A}, \rho)$.

2. Apply operations.

- Linear operator gate set.
- Cost: Constant per gate.

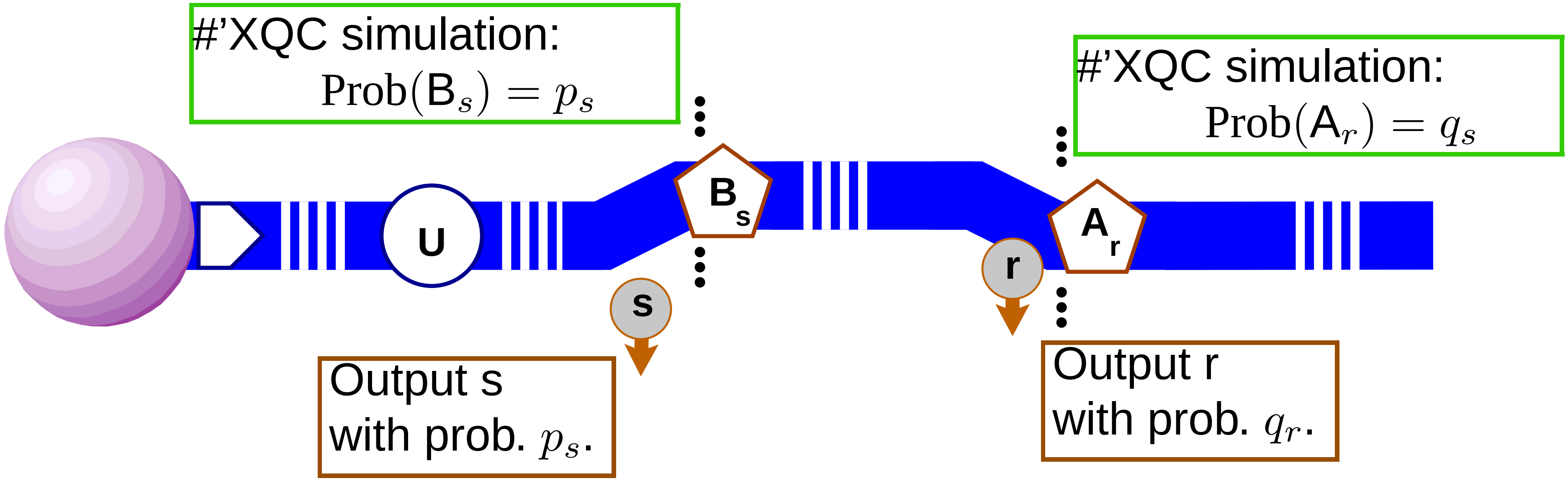
3. Output.

- $\mathrm{Prob}(\text{events})$: $\mathrm{tr}(\rho_k)$
- Cost: No additional cost.



(#'XQC $\simeq \emptyset$) $\Rightarrow$ (XQC $\simeq \emptyset$)?

- Assume that #'XQC $\simeq \emptyset$.
  - Attempt to construct an efficient classical simulation of XQC.



#'XQC simulation:
$\mathrm{Prob}(\mathsf{B}_s) = p_s$

#'XQC simulation:
$\mathrm{Prob}(\mathsf{A}_r) = q_s$

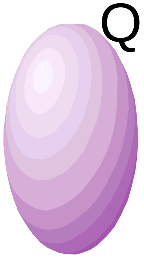
Output s
with prob. p_s.

Output r
with prob. q_r.

- When is this simulation efficient?
  - Must be able to sample efficiently from $\{p_0{:}0, \ldots, p_s{:}s, \ldots\}$.
  - Sufficient: Number of measurement outcomes are $O(\text{poly})$.

From FLOQC to Lie Algebraic Models

#’FLOQC.

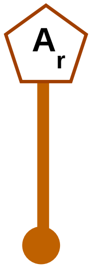


Quantum system:

- Lie algebra: $\mathrm{span}_{\mathbb{C}}(\mathbb{1}, \mathbf{a}_\mathrm{i}, \mathbf{a}^\dagger{}_\mathrm{i}, \mathbf{a}_\mathrm{i}\mathbf{a}^\dagger{}_\mathrm{j}, \mathbf{a}^\dagger{}_\mathrm{j}\mathbf{a}_\mathrm{i}, \mathbf{a}_\mathrm{i}\mathbf{a}_\mathrm{j}, \mathbf{a}^\dagger{}_\mathrm{i}\mathbf{a}^\dagger{}_\mathrm{j})_{1\leq i,j\leq n}$
  Commutation rules identify this as $\mathfrak{so}_{2n+1}\mathbb{C} \oplus \langle\mathbb{1}\rangle$
- Initial state and representation: $|\psi_0\rangle \in \Pi(\lambda, H)$.
  Unique ground state $|\psi_0\rangle$
  of Hermitian $H \in \mathfrak{so}_{2n+1}\mathbb{C}$ with $\lambda = \langle\psi_0|H|\psi_0\rangle$

See Barnum&Knill&Ortiz&Viola’02[13]

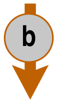
◆ Cost: $\dim(\mathfrak{so}_{2n+1}\mathbb{C}) + \log(|\lambda|/|H|_{\text{Killing}})$.



- For $A \in \mathfrak{so}_{2n+1}\mathbb{C}$, $t \in \mathbb{C}$: $\exp(A + t\mathbb{1})$.

◆ Cost: $|A|_{\text{Killing}} + \log(t)$.

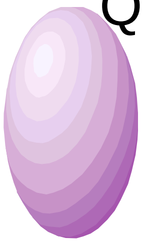
Output:



- $\mathrm{tr}($current state$)$.

Lie Algebraic Models

Finite-dim. semisimple Lie algebras (#'fLAM)



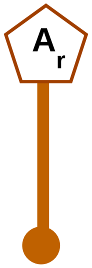
Quantum system:

- Finite-dimensional semisimple Lie algebra $\mathfrak{h}$ with "Hermitian transpose" $a \to a^\dagger$.

  ...given by basis and structure constants.
- "Generic" $H \in \mathfrak{h}$ with $H^\dagger = H$.
- Irreducible representation π with initial state $|\psi_0\rangle$ characterized by λ with:
  - $\lambda = \langle\psi_0|\pi(H)|\psi_0\rangle$.
  - $|\psi_0\rangle$ is the unique ground state of H in π.

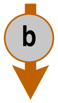
...restricts λ.

◆ Cost: $\dim(\mathfrak{h}) + \log(|\lambda|/|H|_{\text{Killing}})$.



- For $A \in \mathfrak{h}$, $t \in \mathbb{C}$: $\exp(\pi(A) + t\mathbb{1})$.

◆ Cost: $|A|_{\text{Killing}} + \log(t)$.



Output:

- $\mathrm{tr}($current state$)$.

#'fLAM $\approxeq \emptyset$

- **Theorem 3:** #'fLAM $\approxeq \emptyset$. …real output approximated to within $2^{-O(m)}$.

- **Proof:** Exhibit an efficient algorithm with

  Input: $\mathfrak{h}$, Hermitian $H \in \mathfrak{h}$, ground state energy λ.

  …defines representation π, ground state $|\psi_0\rangle$.

  $A_1, \dots A_N \in \mathfrak{h} \oplus \langle \mathbb{1} \rangle$.

  Output: $\left| e^{\pi(A_N)} \dots e^{\pi(A_1)} |\psi_0\rangle \right|^2$.

  Outline:

  1. Compute $U = e^{\pi(A_N)} \dots e^{\pi(A_1)}$ in the adjoint representation.
  2. Choose t large enough so that $P_t = e^{(\lambda - \pi(H))t} \approx |\psi_0\rangle\langle\psi_0|$.
  3. Compute $Q_t = U e^{(\lambda - H)t} U^\dagger$.
  4. Obtain $X \in \mathfrak{h}$, ν such that $e^{X+\nu} = Q_t$ in the adjoint representation.
     …$e^{\pi(X)+\nu}$ is close to a projector onto a minimum weight state $|\psi_1\rangle$.
  5. Use Cartan subalgebra and root system analysis
     to determine α such that $(\pi(X) + \nu)|\psi_1\rangle = \alpha|\psi_1\rangle$.
  6. Return e^α.

- …the m'th digit of the output can be obtained efficiently.

Consequences of #’XQC $\approxeq \emptyset$

- Suppose that #’XQC $\approxeq \emptyset$.
  - Is it possible that QC $\preccurlyeq$ XQC?

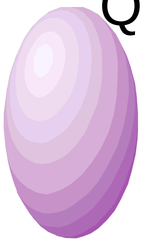
    . . . for bounded measurement outcomes,
    this implies $QC \simeq \emptyset$.
  - Is it possible that XQC can faithfully simulate QC?

    . . . if the simulation is exponentially accurate,
    this implies that **Poly**$^{\textbf{\#'P}}$ $\subseteq$ **Poly**.
    i.e. there would be an efficient classical algorithm
    for counting the number of satisfying assignments.

Bosonic LAMs

eLOQC versus BLOQC

- Lie algebraic, but not semisimple.
- π is infinite dim.



Quantum system:

- Lie alg. $\mathfrak{h}$: $\mathrm{span}_{\mathbb{C}}(\mathbb{1}, \mathbf{a}_{\mathsf{i}}, \mathbf{a}^\dagger{}_{\mathsf{i}}, \mathbf{a}_{\mathsf{i}}\mathbf{a}^\dagger{}_{\mathsf{j}}, \mathbf{a}^\dagger{}_{\mathsf{j}}\mathbf{a}_{\mathsf{i}}, \mathbf{a}_{\mathsf{i}}\mathbf{a}_{\mathsf{j}}, \mathbf{a}^\dagger{}_{\mathsf{i}}\mathbf{a}^\dagger{}_{\mathsf{j}})_{1\leq i,j\leq n}$
  $[\mathbf{a}^\dagger{}_{\mathsf{i}}, \mathbf{a}_{\mathsf{j}}] = \delta_{i,j}, [\mathbf{a}_{\mathsf{i}}, \mathbf{a}_{\mathsf{j}}] = 0, [\mathbf{a}^\dagger{}_{\mathsf{i}}, \mathbf{a}^\dagger{}_{\mathsf{j}}] = 0, \ldots$
- Representation π and initial state $|\mathbf{0}\rangle$:
  - $|\mathbf{0}\rangle$ is the unique ground state of $\mathbf{n}$ in π.
  - $\mathbf{n}|\mathbf{0}\rangle = 0.$ $(\mathbf{n} = \sum_k \mathbf{a}^\dagger{}_{\mathsf{k}}\mathbf{a}_{\mathsf{k}})$

Operations:

- For H Hermitian in $\mathfrak{h}$: $\exp -iH$.

◆ Physical cost: $|H|_{\text{generators}}$.

Measurements:

eLOQC

- Particle detectors:
  $\mathfrak{o} \cdot |\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}|$
  $+\!\!\!\!\| \;\; \mathfrak{1} \cdot (\mathbb{1} - |\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}|)$.

BLOQC

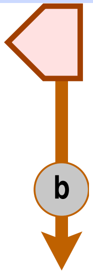
- Electric field measurement:
  $\int_{\mathfrak{p}} \mathfrak{p} \cdot |p\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle p|$.

Feedback?

- Yes.

Bosonic LAMs

eLOQC versus BLOQC



eLOQC	BLOQC
Particle detectors: $\mathfrak{o} \cdot \|\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}\| \mathbin{+\!\!\!+} \mathfrak{1} \cdot (\mathbb{1} - \|\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}\|)$.	Electric field measurement: $\int_{\mathfrak{p}} \mathfrak{p} \cdot \|p\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle p\|$.

- What explains BLOQC $\simeq \emptyset$ but eLOQC $\simeq$ QC?

- BLOQC measurement projectors are Lie algebraic:
  From $\left(|\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}| = \lim_{s\to\infty} e^{-\mathbf{n}_{\mathsf{k}} s}\right)$ to $|p\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle p|$
  ...by squeezing and shifting:
  $$|p\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle p| \quad = \quad \lim_{s,t\to\infty} e^{-ip\mathbf{p}} e^{-(\mathbf{a}_{\mathsf{k}}\mathbf{a}_{\mathsf{k}} - \mathbf{a}^{\dagger}{}_{\mathsf{k}}\mathbf{a}^{\dagger}{}_{\mathsf{k}})t} e^{-\mathbf{n}_{\mathsf{k}} s} e^{(\mathbf{a}_{\mathsf{k}}\mathbf{a}_{\mathsf{k}} - \mathbf{a}^{\dagger}{}_{\mathsf{k}}\mathbf{a}^{\dagger}{}_{\mathsf{k}})t} e^{ip\mathbf{p}}$$
  $\in$ closure of $\exp(\mathfrak{h})$

- eLOQC:
  $\left(\mathbb{1} - |\mathbf{0}\rangle_{\mathsf{k}}{}^{\mathsf{k}}\langle\mathbf{0}|\right)$ is not in the closure of $\exp(\mathfrak{h})$.

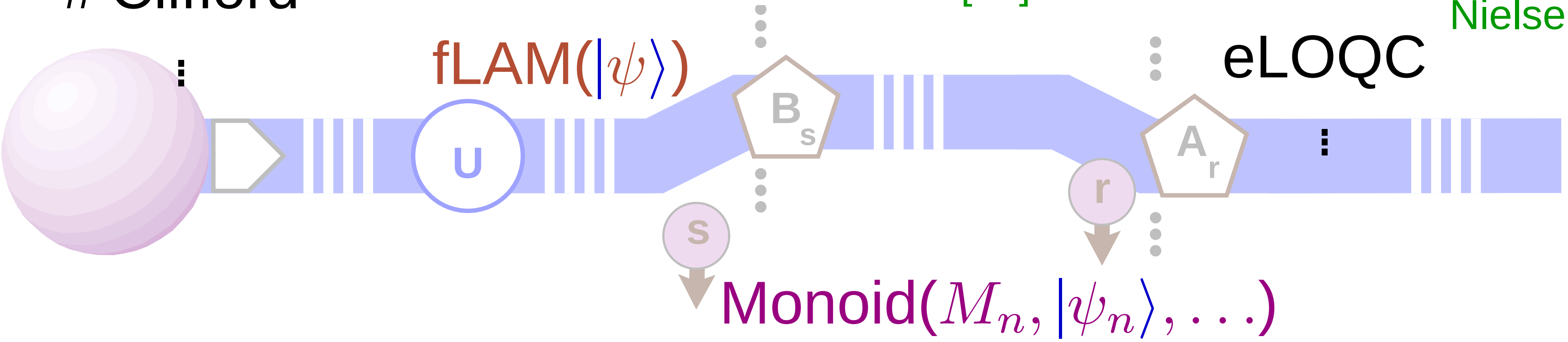
Summary

$\underline{\emptyset} \preccurlyeq$ QC1 $\preccurlyeq \underline{\text{QC}}$

BLOQC, #'BLOQC
#'FLOQC
#'fLAM
#'Clifford

$\Gamma(\epsilon > \epsilon_{\text{trsh}})$QC
BLOQC($|\psi\rangle$)
Bartlett&Sanders'02[16]
fLAM($|\psi\rangle$)

$\Gamma(\epsilon < \epsilon_{\text{thrsh}})$QC
QC$_d$
QC$_{noop}$
Raussendorf&Briegel'00[14]
Nielsen'01[15]
eLOQC



Monoid($M_n, |\psi_n\rangle, \ldots$)
Group($G_n \rhd N_n, \ldots$)

- Lots of room for exploration ...

Have fun!

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