Approaching the Asymptotic Regime of Rapidly Rotating Convection: Boundary Layer vs Interior Dynamics

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Mathematics of Turbulence Long Program

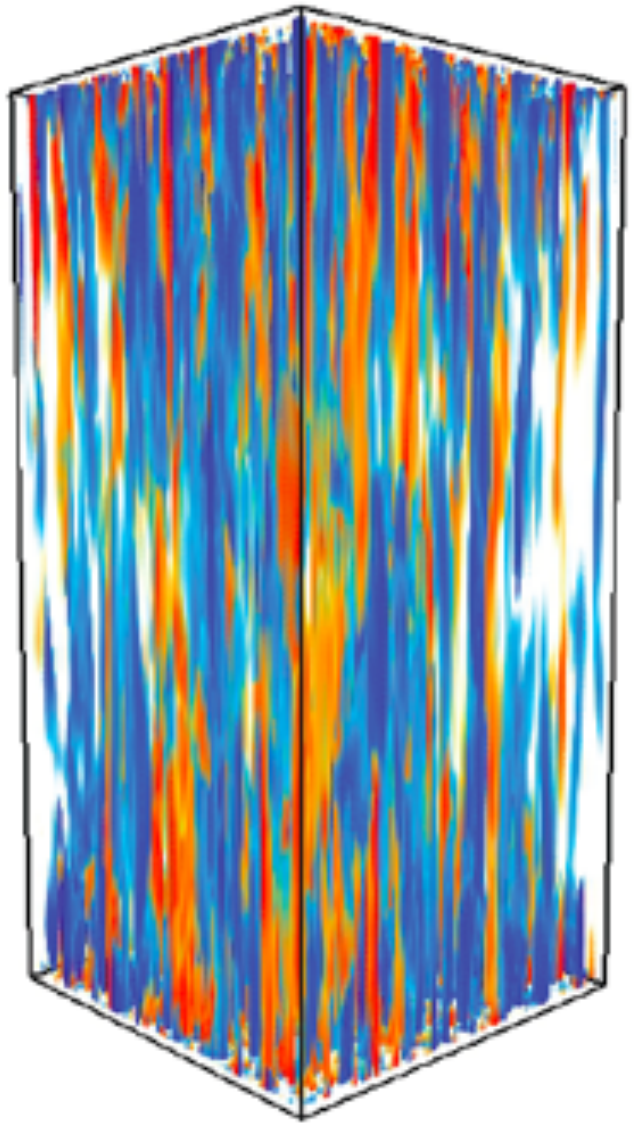
Geophysical and Astrophysical Turbulence

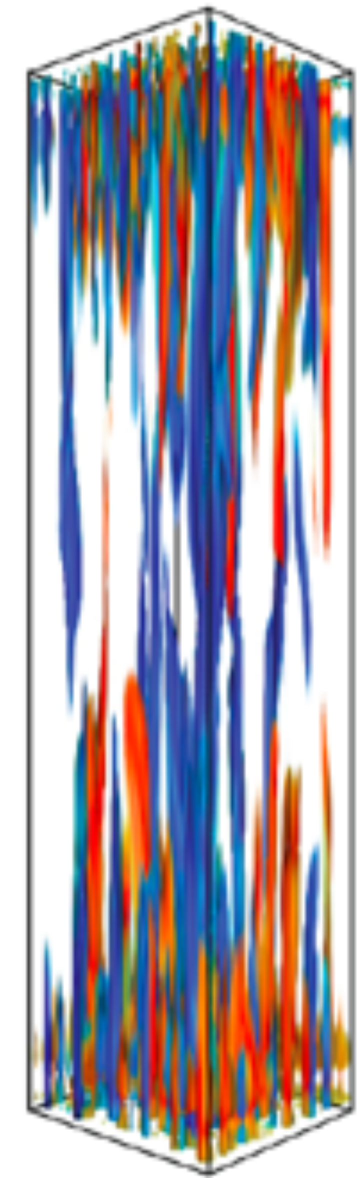
IPAM, UCLA

Lab Experiments

Jon Aurnou

Eric King, Jonathan Cheng





Reduced Modeling & Simulation (Colorado & Berkeley)

Edgar Knobloch, Jeff Weiss

Ian Grooms, Antonio Rubio,

Geoff Vasil [3],

Michael Calkins, Philippe Marti

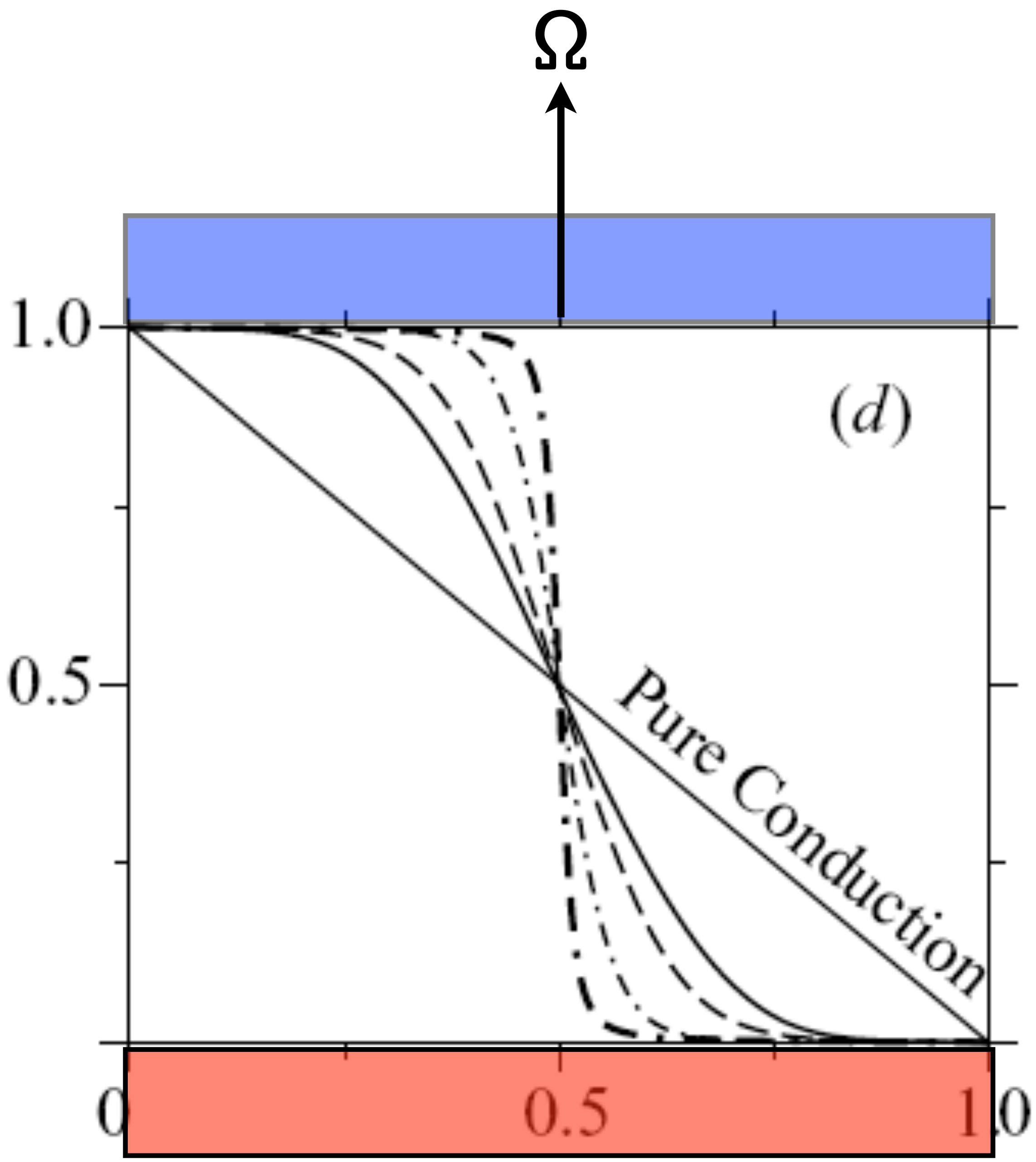
DNS (U. Muenster)

Stephan Stellmach

M. Lischper

Support: NSF FRG DMS-0855010
NSF EAR CSEDI-1067944

Rotating Rayleigh-Benard Convection



$Ra \gg 1$: conduction to turbulent motions

- Mixed interiors with thermal boundary layers
- Rich in flow morphologies

Rayleigh, Prandtl number

$$Ra = \frac{g\alpha\Delta T H^3}{\nu\kappa}, \quad \sigma = \frac{\nu}{\kappa}$$

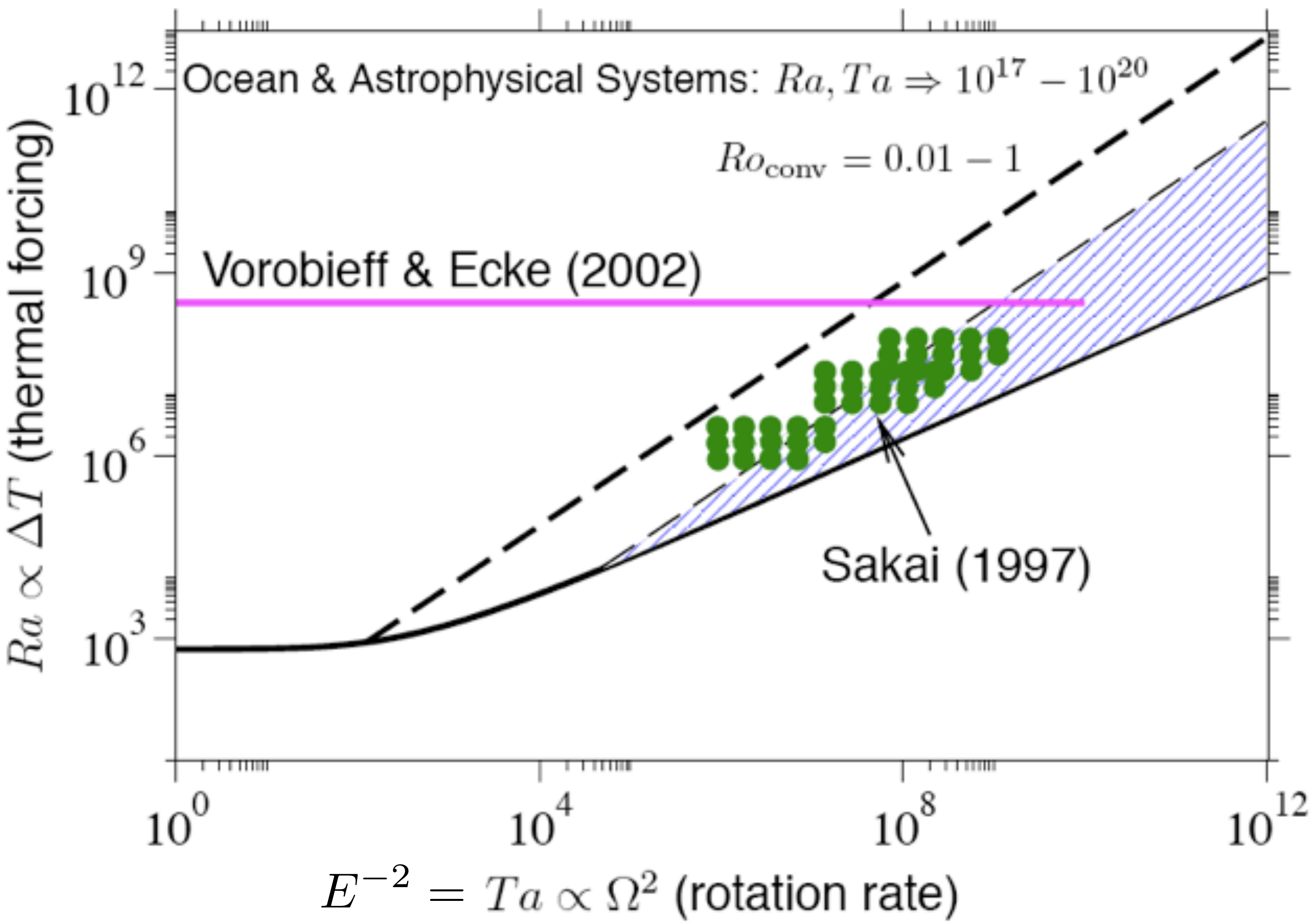
Ekman number

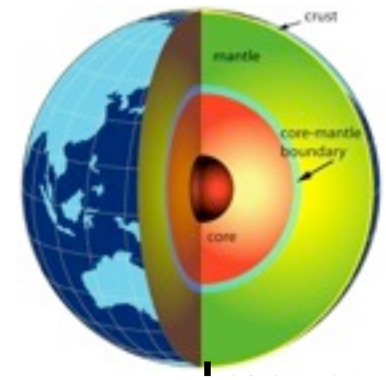
$$E = \frac{\nu}{2\Omega H^2}$$

Convective Rossby number

$$Ro = \sqrt{\frac{Ra}{\sigma}} E = \frac{\sqrt{g\alpha\Delta T/H}}{2\Omega}$$

Rotationally Constrained Parameter Space: Low *Ro*, High *Ra*





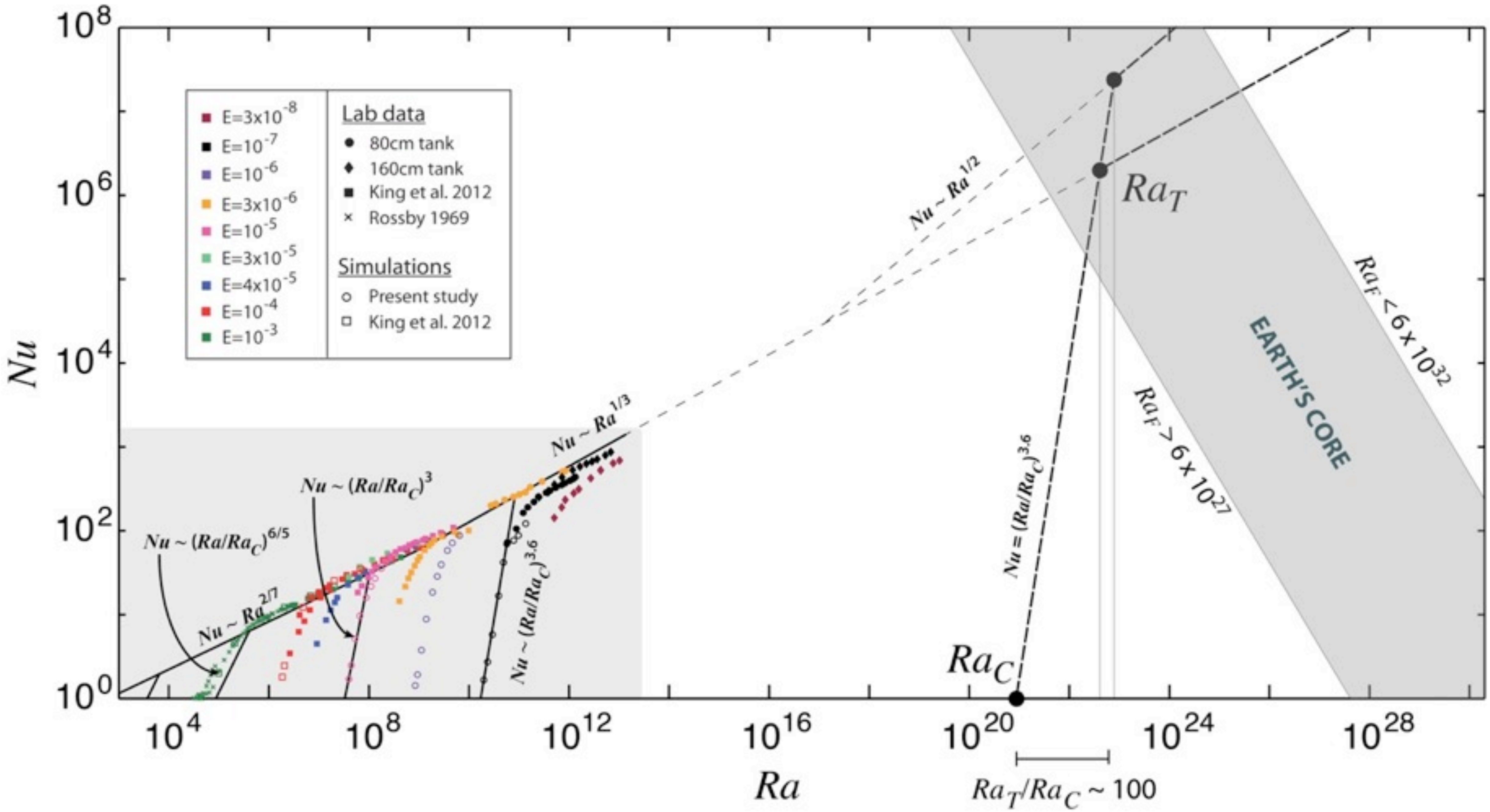
$Ro \sim 10^{-7}$
$Re \sim 10^{8}$
$Ek \sim 10^{-15}$

$Ro \sim 10^{-2}$
$Re \sim 10^{16}$
$Ek \sim 10^{-18}$

Characterization - Heat Transport
Low *Ro* branch characterized by steep branch in *Nu-Ra* space

$$Ro_{crit} \sim E^{1/3} < Ro < 1, \quad Ra \sim E^{-4//3}$$

UCLA Spin-Lab: King et al Nature 2009
U. Muenster: Stellmach DNS simulation effort



$$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left(\frac{Ra}{Ra_c}\right)^{\beta}$$

$$\beta_{rot} > 1$$
$$\beta_{norot} = 2/7$$

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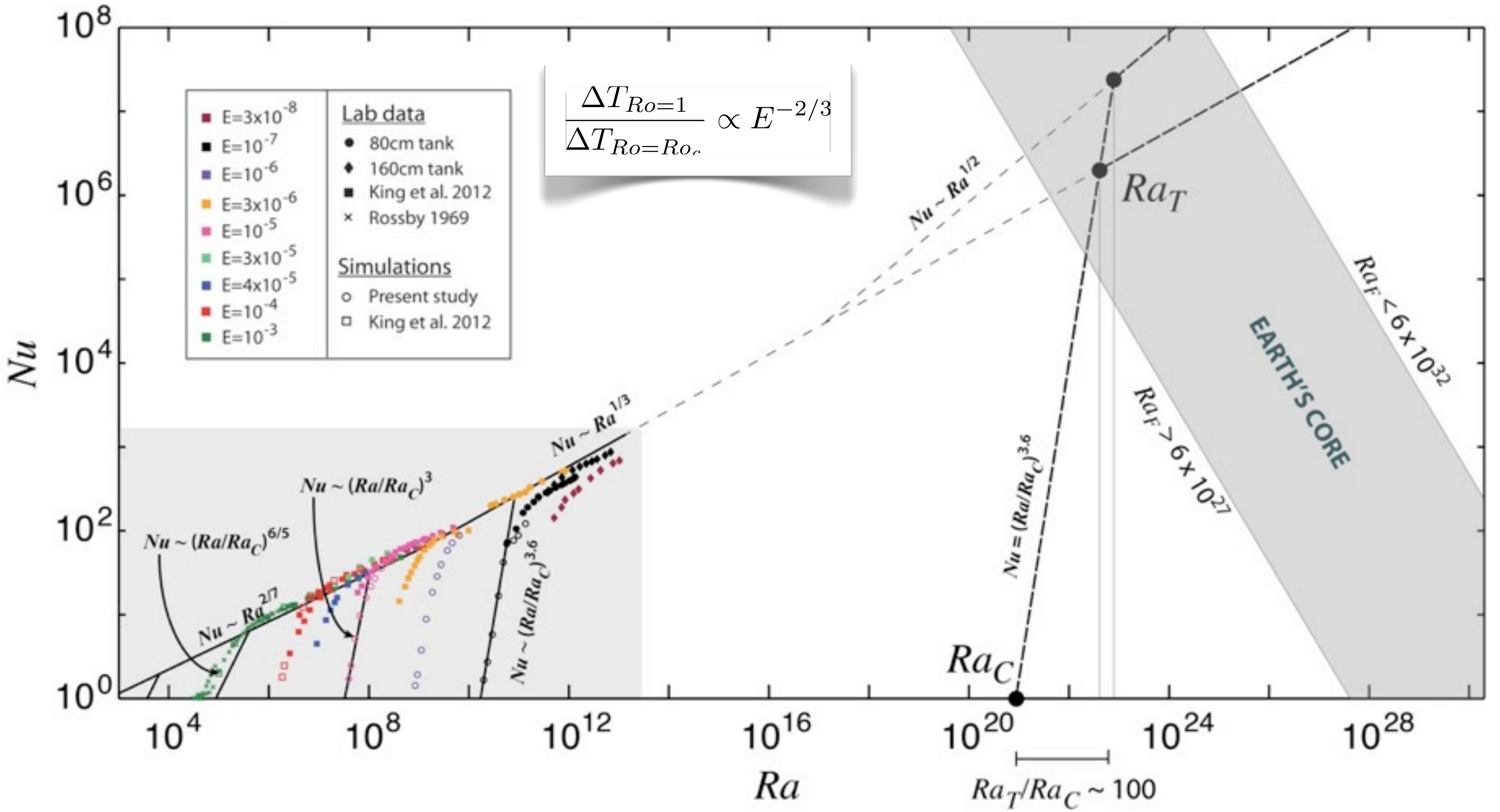
$$\frac{\Delta T_{Ro=1}}{\Delta T_{Ro=Ro_c}} \propto E^{-2/3}$$



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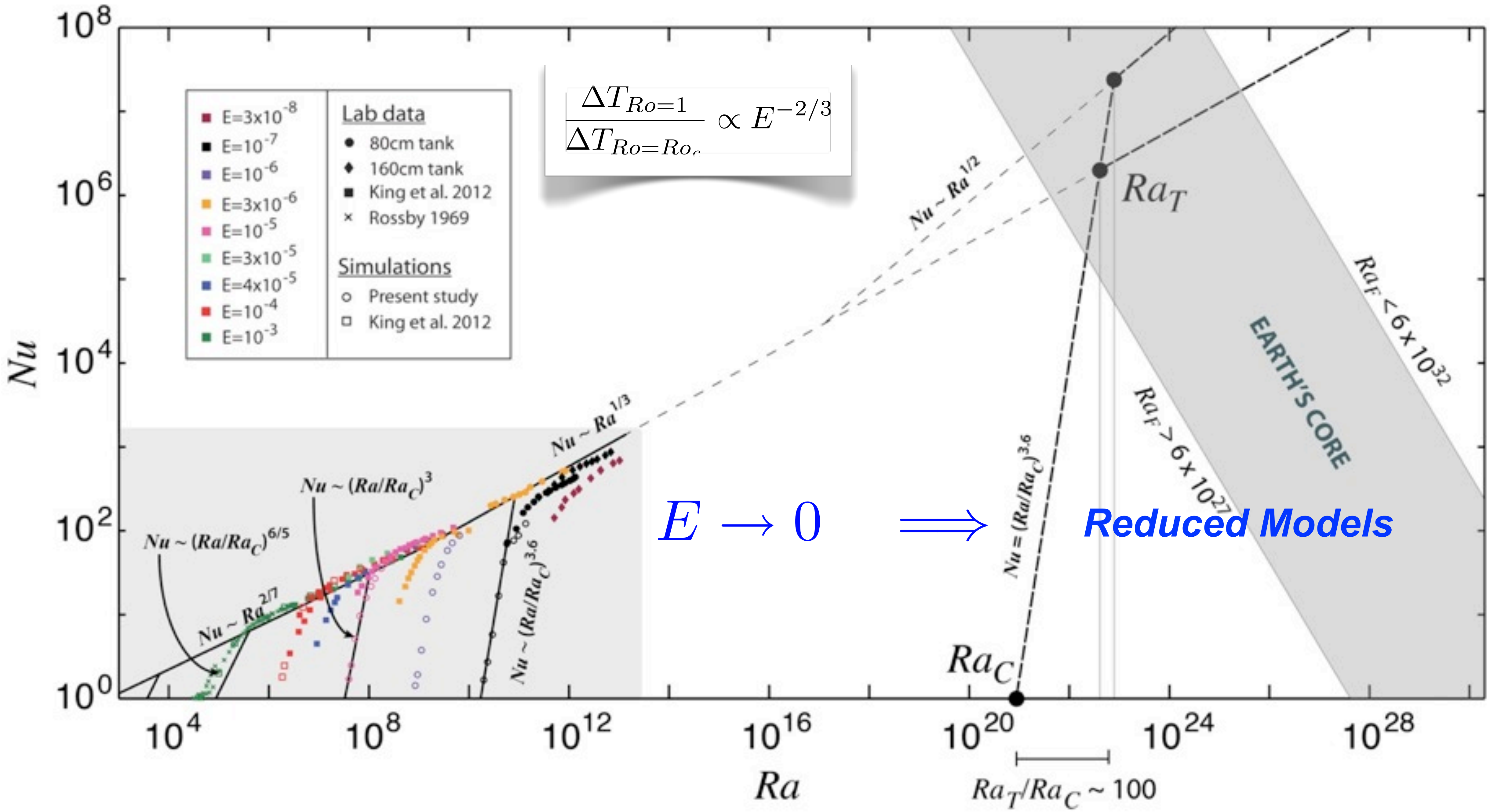

$$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left(\frac{Ra}{Ra_c}\right)^{\beta}$$

$$\beta_{rot} > 1$$

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Governing Equations in the Rotationally Constrained Limit

$$D_t^{\perp}\boldsymbol{u}_{\perp} + w\partial_Z \boldsymbol{u}_{\perp} + \frac{1}{E}\widehat{\boldsymbol{z}}\times\boldsymbol{u}_{\perp} = -\nabla_{\perp}p + \left(\nabla_{\perp}^2 + \partial_{ZZ}\right)\boldsymbol{u}_{\perp}$$

$$D_t^{\perp}w + w\partial_Z w = -\partial_Z p + \frac{Ra}{\sigma}T + \left(\nabla_{\perp}^2 + \partial_{ZZ}\right)w$$

$$D_t^{\perp}T + w\partial_z T = \frac{1}{\sigma}\left(\nabla_{\perp}^2 + \partial_{ZZ}\right)T$$

$$\nabla_{\perp}\cdot\boldsymbol{u}_{\perp} + \partial_Z w = 0$$

Incompressible Navier-Stokes

- Challenge: spanning the entire rotationally constrained branch
- Resolving BL's, fast waves at large Ra as *Ro,E* ⇒ *0*

$$D_t^{\perp}\zeta - \partial_Z w = \nabla_{\perp}^2\zeta$$

$$D_t^{\perp}w = -\,\partial_Z\psi + \frac{RaE^{4/3}}{\sigma}\theta + \nabla_{\perp}^2 w$$

$$D_t^{\perp}\theta + w\partial_Z\overline{T} = \frac{1}{\sigma}\nabla_{\perp}^2\theta$$

$$\partial_\tau\overline{T} + \partial_Z\left(\overline{w\theta}\right) = \frac{1}{\sigma}\partial_{ZZ}\overline{T}$$

$$p = \psi, \quad \boldsymbol{u}_{\perp} = (-\partial_x\psi, \partial_y\psi, 0), \quad \zeta = \nabla_{\perp}^2\psi$$

NonHydrostatic Quasi-Geostrophic Eqns

- Challenge: robustness of asymptotics as *Ro,E* increase away from *0*

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$$D_t^\perp \boldsymbol{u}_\perp + w\partial_Z \boldsymbol{u}_\perp + \frac{1}{E}\widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\nabla_\perp p + \left(\nabla_\perp^2 + \partial_{ZZ}\right)\boldsymbol{u}_\perp$$

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Incompressible Navier-Stokes

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Boundary conditions

- Fixed Temperature
- Impenetrable, Stress-Free or No-Slip

Boundary conditions

- Fixed Temperature
- Impenetrable ⟹ Stress-Free

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Incompressible Navier-

- Challenge: spanning rotationally constrain
- Resolving BL's, fast v large Ra as *Ro,E* ⇒ 0

Overlap in (Ra, Ro, E) permissible by Lab Exp's, DNS and Reduced Models

$$D_t^{\perp}\zeta - \partial_Z w = \nabla_{\perp}^2\zeta$$

$$D_t^{\perp}w = -\,\partial_Z\psi + \frac{RaE^{4/3}}{\sigma}\theta + \nabla_{\perp}^2 w$$

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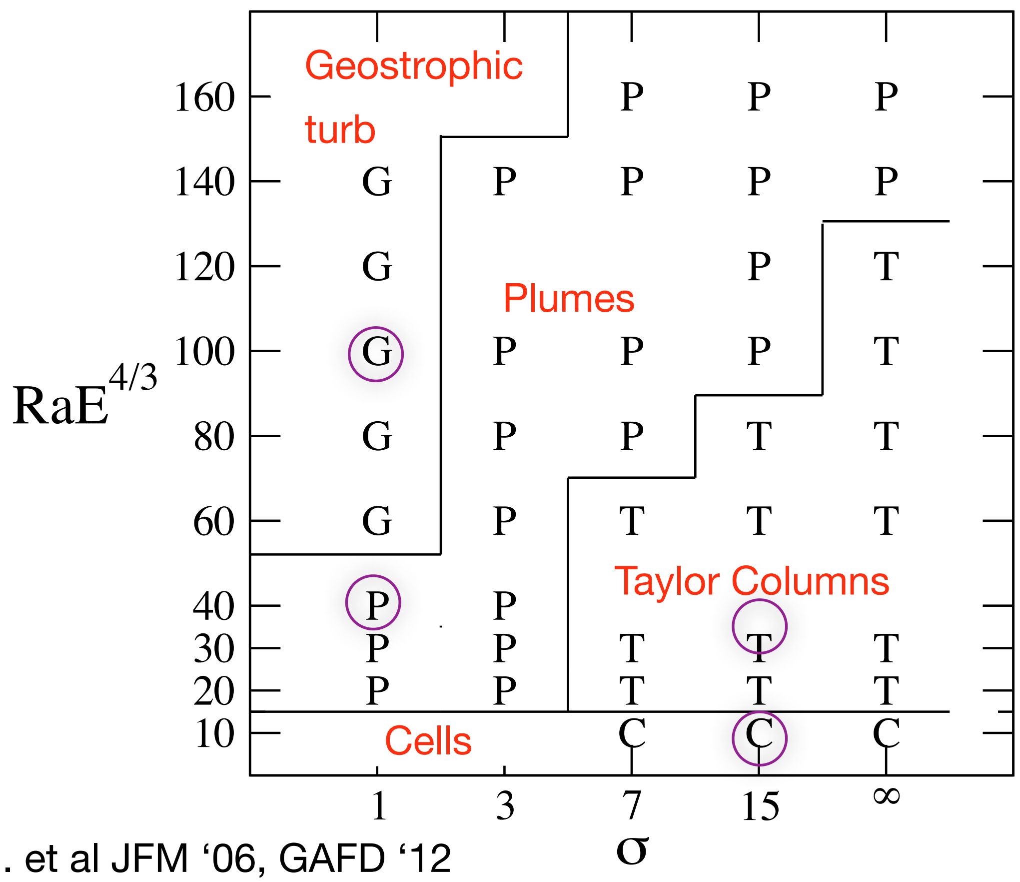
NonHydrostatic Quasi-Geostrophic Eqns

- Challenge: robustness of asymptotics as *Ro,E* increase away from *0*

Quasi-Geostrophic RBC Flow Regimes

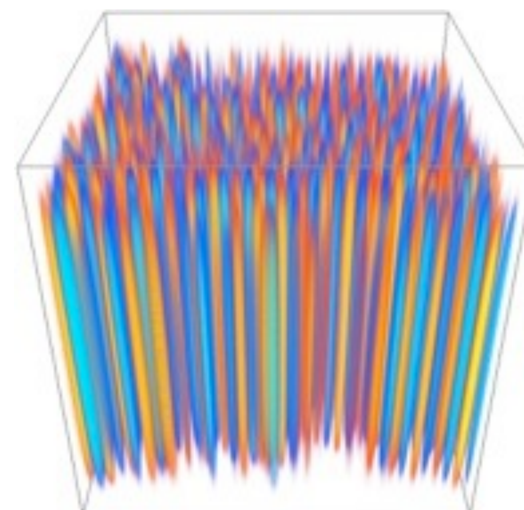
Impenetrable Stress-Free Boundaries $Ro \rightarrow 0$ limit



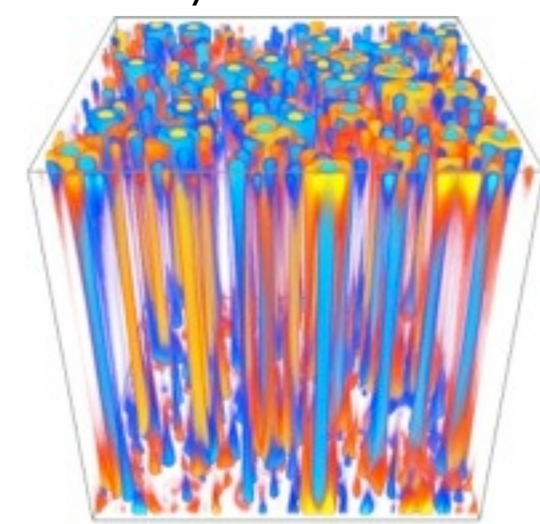
J. et al JFM '06, GAFD '12

Thermal anomaly

$RaE^{4/3} = 10, \sigma = 15$
cells



$RaE^{4/3} = 40, \sigma = 15$
Taylor Columns



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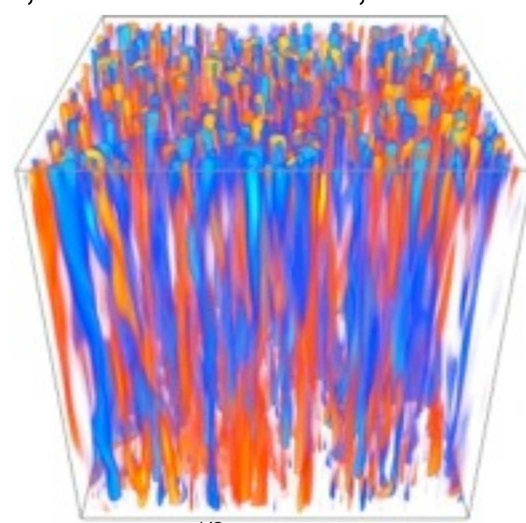
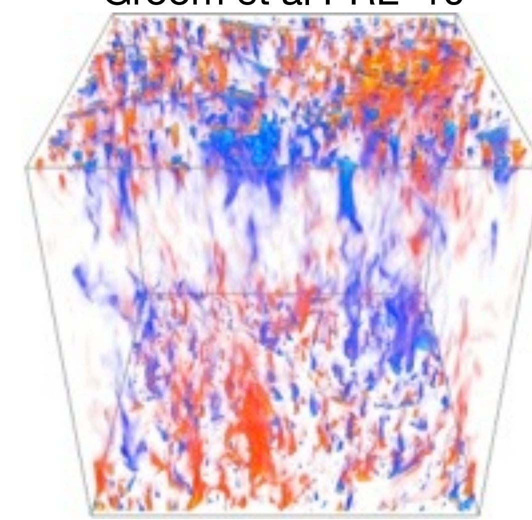
g

J., Knobloch PoF 99, JMP '07

Groom et al PRL '10



$RaE^{4/3} = 40, \sigma = 1$
Plumes



$RaE^{4/3} = 100, \sigma = 1$
Geostrophic Turbulence

Rubio et al PRL '14

Favier et al PoF 2014

Guervilly et al JFM 2014

Stellmach et al PRL 2014

- Four Flow Regimes as Ra ↑ : laminar to turbulent

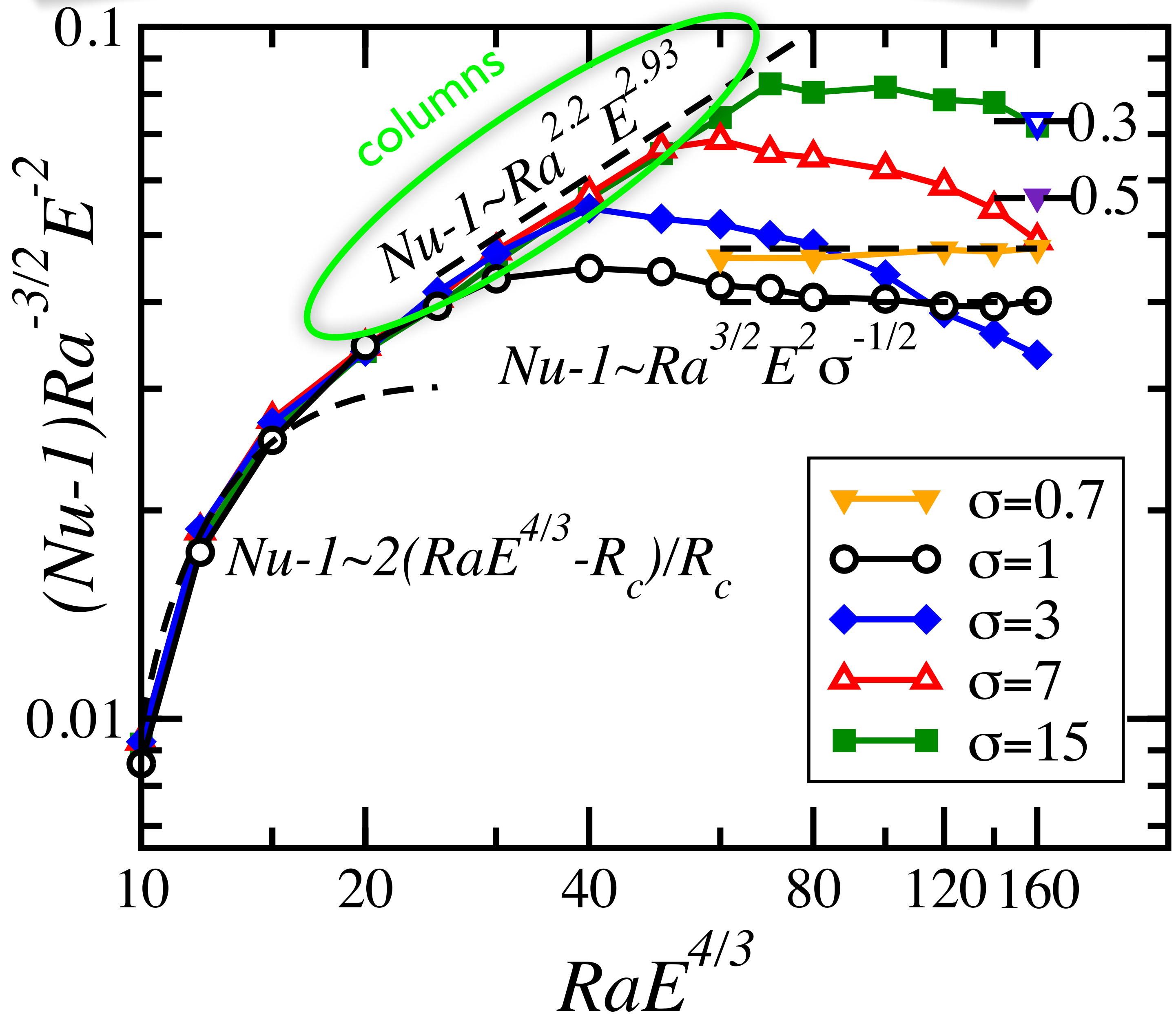
  Cells → CTC's via TBL instability & synchronization of TBL's

  CTC's: Shielded vortical columns with zero circulation

  Geostrophic turbulence regime occurs when TBL are unable to synchronize

NH-QGE
Geostrophic turbulence is the greatest throttle on heat transport

$$Nu \propto \sigma^{\alpha}(Ra/Ra_c)^{\beta}, \;\; Ra_c = E^{-4/3}$$



- marginally stable tbl's (Malkus, '54):

$$\beta = 3$$

- depth independence (Priestley, '59):

$$\beta = 3$$

- ultimate (dissipation-free) turbulent law (Kraichnan '63, Howard '63):

$$\beta = 3/2 < 3, \alpha = -1/2$$

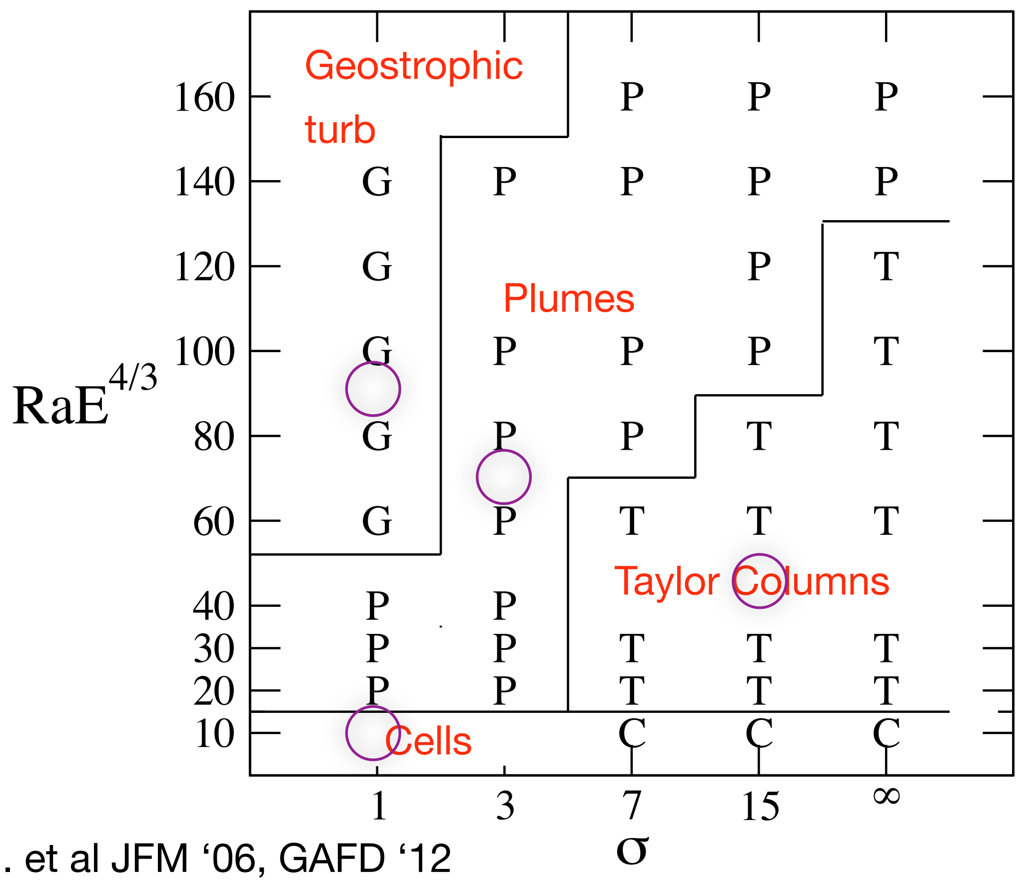
$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g\alpha \Delta T H^3}{\nu\kappa}, \quad E = \frac{\nu}{2\Omega H^2}$$

$$\sigma = \frac{\nu}{\kappa}$$

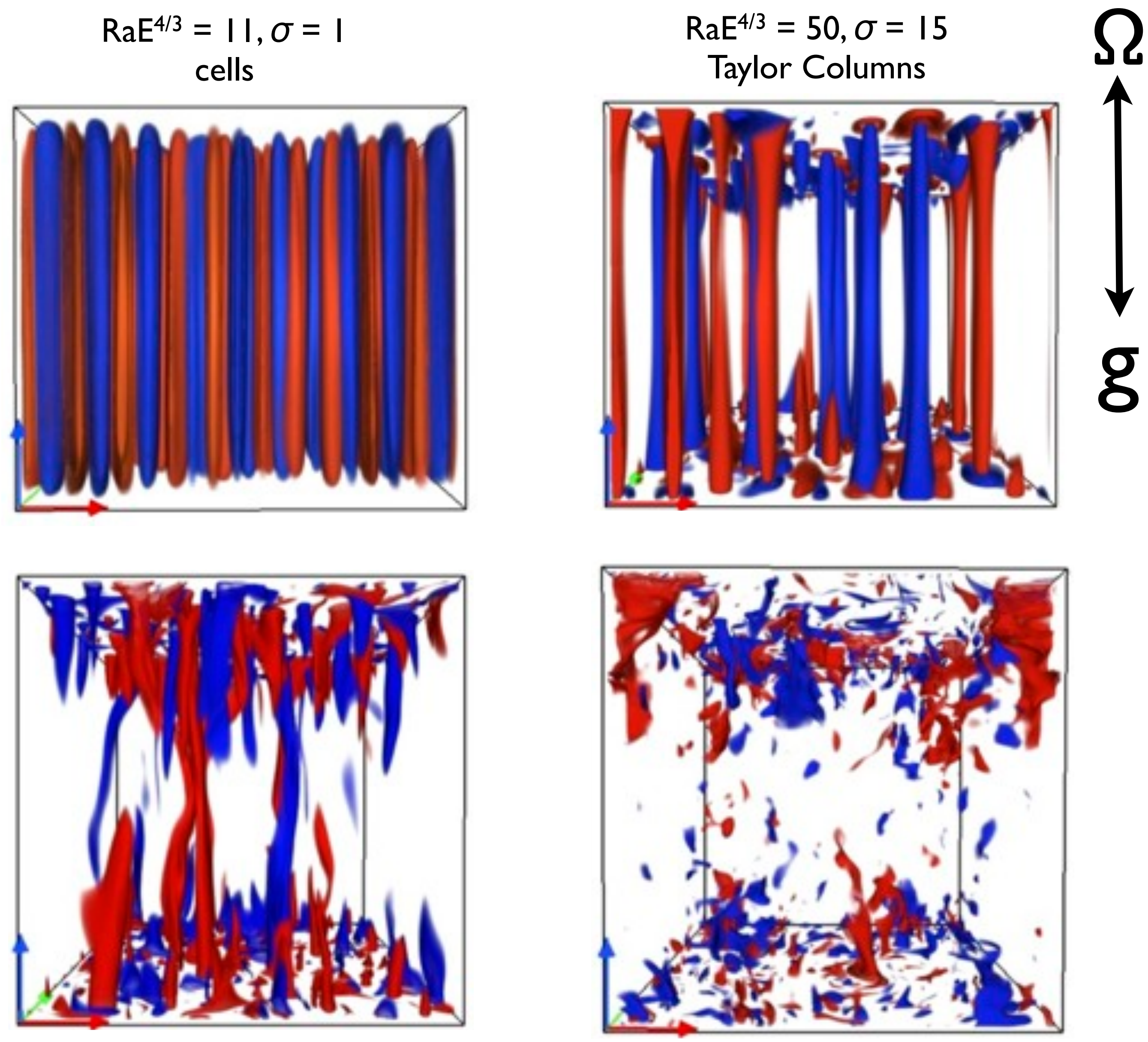
DNS RBC Flow Regimes

Impenetrable Stress-Free Boundaries $Ro \to 0$ limit

Stellmach ..,J,.., Aurnou, PRLsub arxiv' 14



J. et al JFM '06, GAFD '12



$RaE^{4/3} = 11, \sigma = 1$
cells

$RaE^{4/3} = 50, \sigma = 15$
Taylor Columns

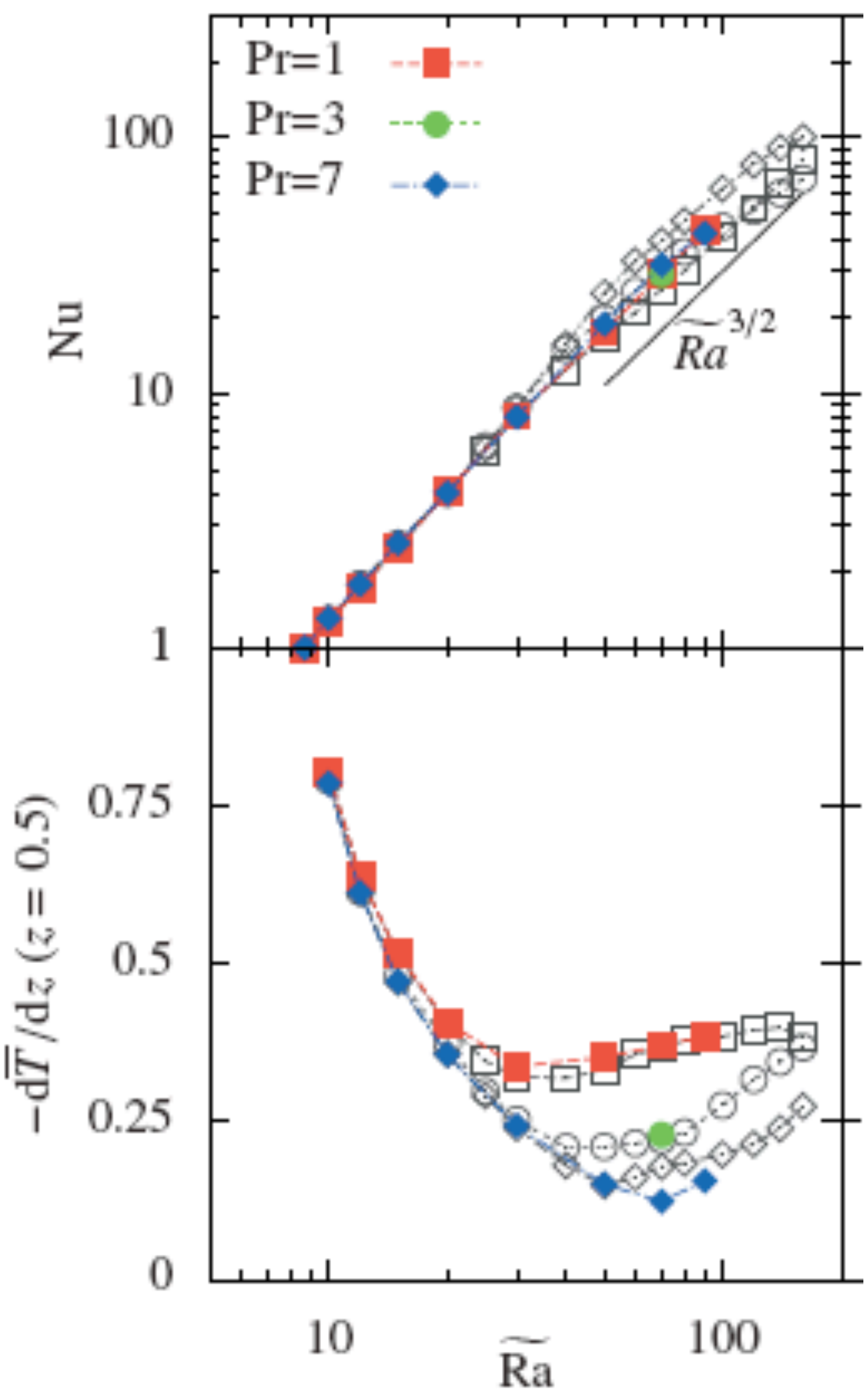
$RaE^{4/3} = 70, \sigma = 3$
Plumes

$RaE^{4/3} = 90, \sigma = 1$
Geostrophic Turbulence

- Four flow regimes captured by DNS
  $Ek = 10^{-7}$

DNS RBC vs NH-QGE
Impenetrable Stress-Free Boundaries

Stellmach ..,J,.., Aurnou, PRLsub arxiv' 14



Good quantitative agreement

- *NH-QGE (Open Symbols)*
- *DNS (Closed Symbols)* $Ek = 10^{-7}$

Experiments, GAFD vs NH-QGE?

- Laboratory Experiments & GAFD Flows

  - Physical boundaries ⇒ impenetrable no-slip boundaries

- DNS

  - Impenetrable no-slip in vertical, periodic in horizontal.

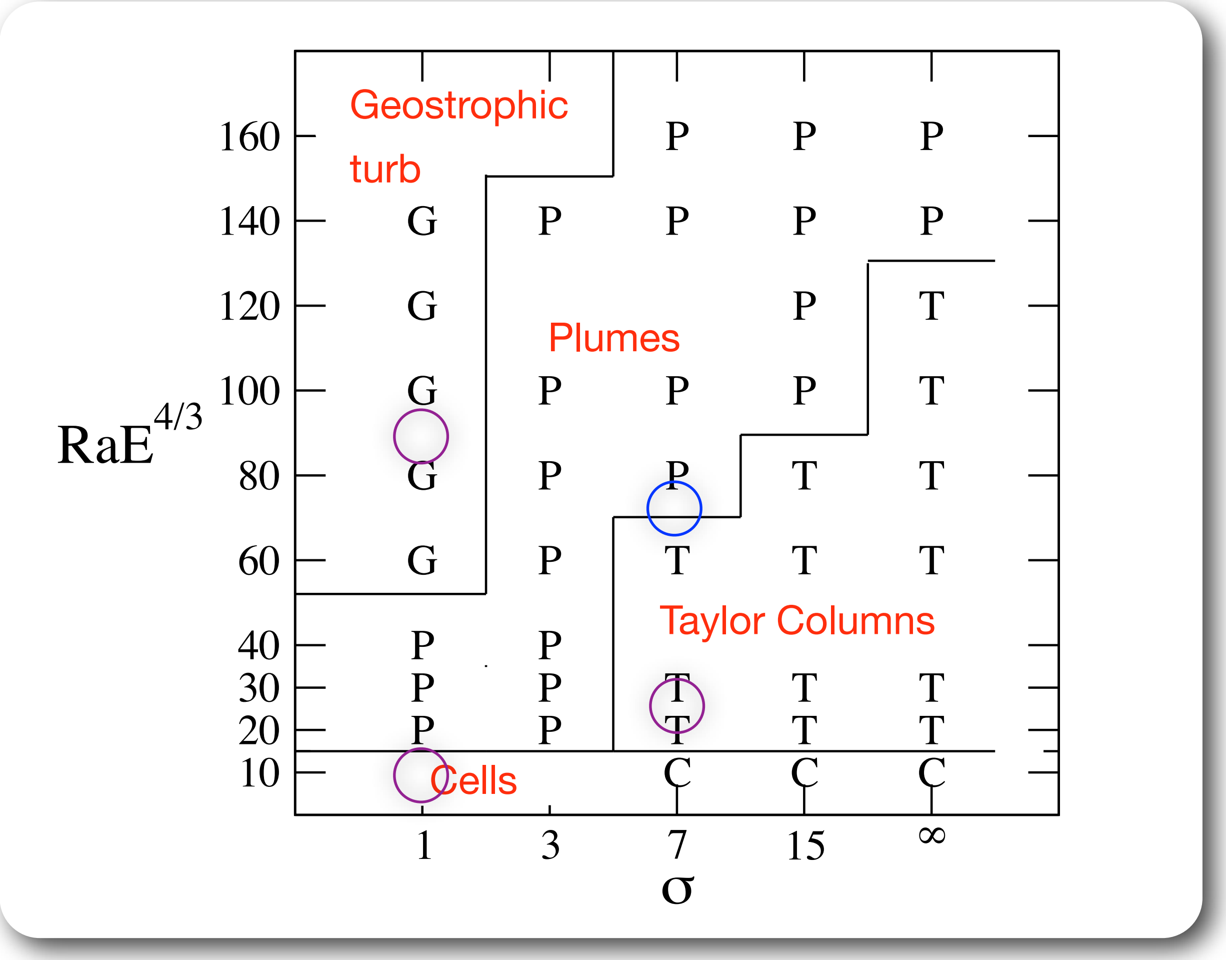
- Asymptotic Reduced Model

  - Impenetrable Stress-free vertical, periodic in horizontal
  - Linear Theory (Chandrasekhar, 1961; Niiler & Bisshopp, JFM 1965)

    differences between bc's are asymptotically small, confined to Ekman BL's
  - Strongly NonLinear Theory (Julien & Knobloch JFM 1998)

    EBL's passive, Ekman pumping asymptotically weak, again no difference

DNS RBC Flow Morphology
Impenetrable No-Slip Boundaries



J. et al JFM '06, GAFD '12

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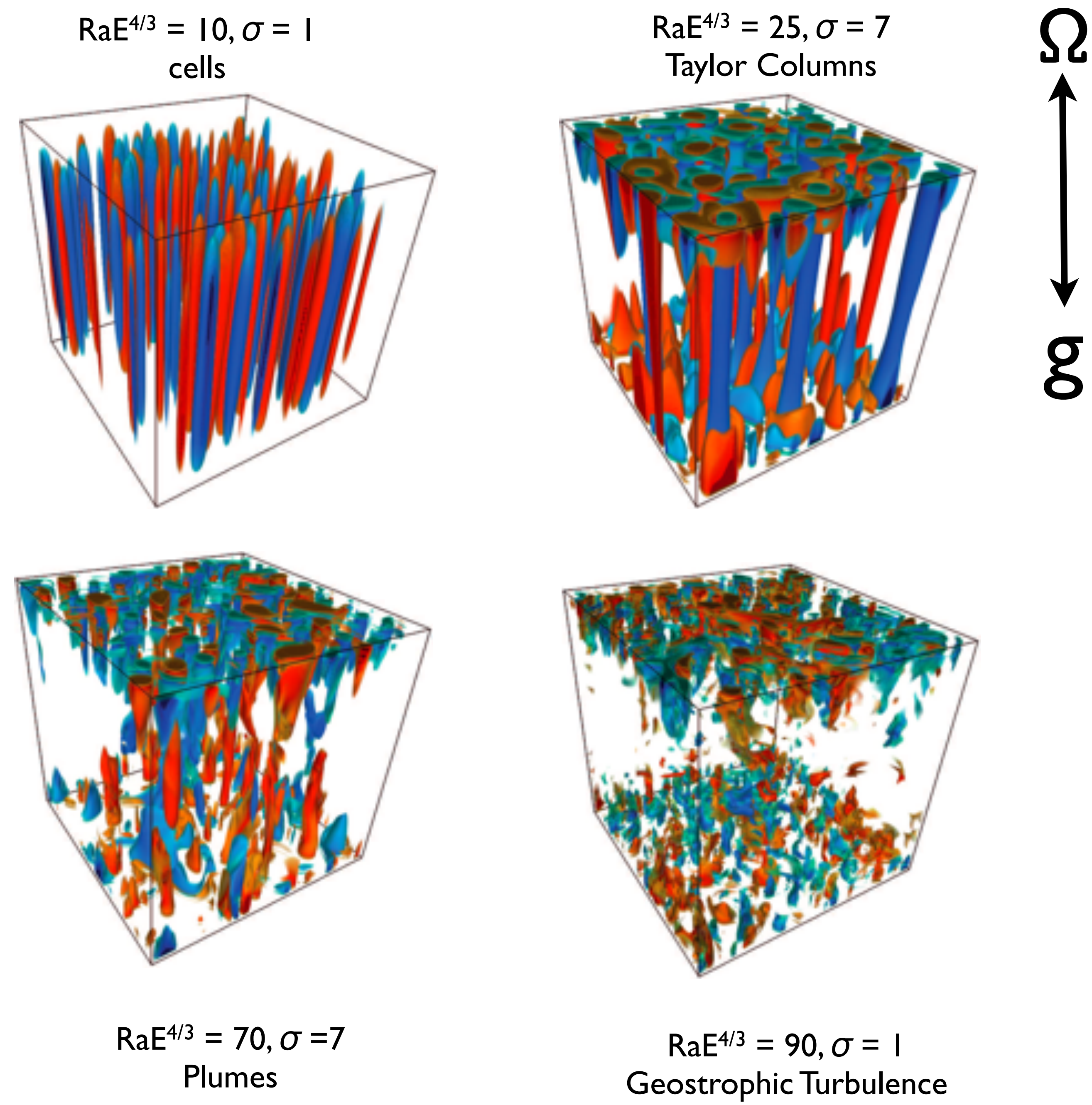
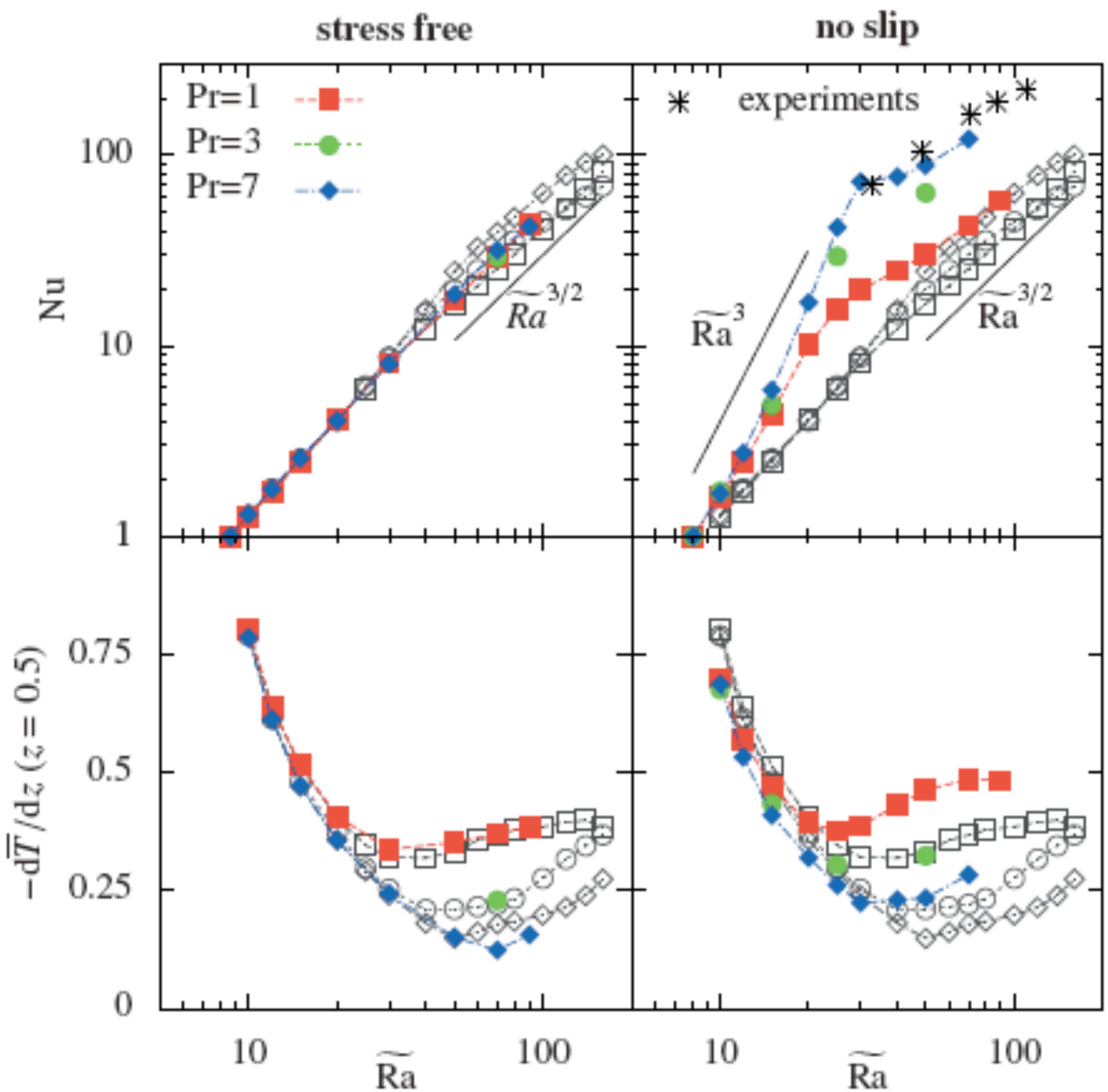


- Four flow regimes again captured by DNS
- Geostrophic vortex (LSV) suppressed

  $Ek = 10^{-7}$

DNS RBC vs NH-QGE
Impenetrable No-Slip Boundaries

Stellmach ..,J,.., Aurnou, PRLsub arxiv' 14



Significant departures

- *NH-QGE (Open Symbols)*
- *DNS (Closed Symbols)*
- *Exp (asterisks)*

$Ek = 10^{-7}$

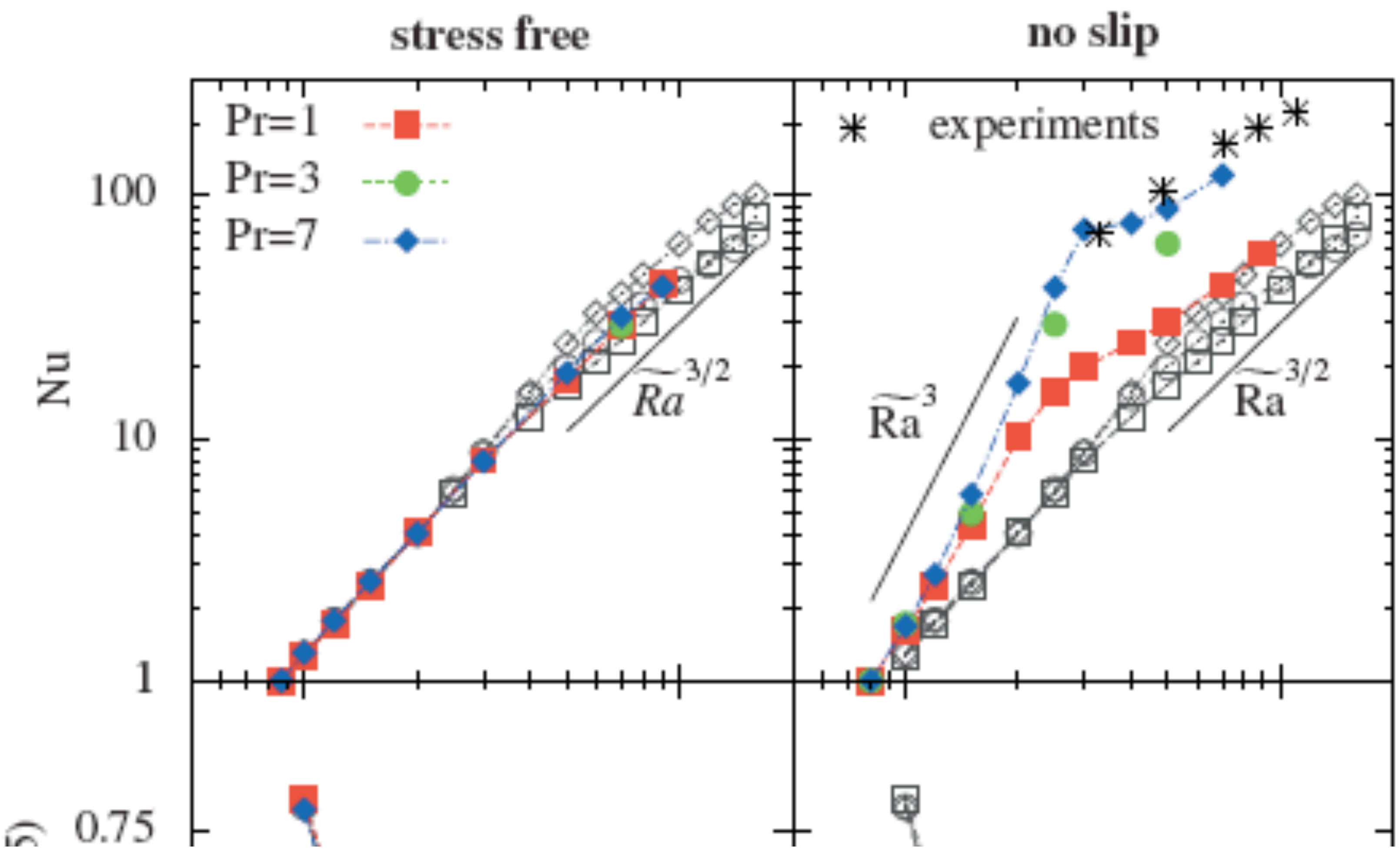
Some evidence for convergenc

- *GT regime*
- *Interior throttles heat flux*

Ekman layers are the only real difference between BC's!

DNS RBC vs NH-QGE
LS: Diminishing influence of Ekman layers!

Stellmach ..,J,.., Aurnou, PRLsub arxiv' 14



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J. Fluid Mech. (1965), *vol.* 22, *part* 4, *pp.* 753–761
Printed in Great Britain

$$\left(RaE^{4/3}\right)_{NS} \approx 8.7_{SF} - \mathcal{O}\left(E^{1/6}\right)$$

On the influence of Coriolis force on onset of thermal convection

By P. P. NIILER* AND F. E. BISSHOPP
Division of Applied Mathematics, Brown University, Providence, Rhode Island

(Received 1 September 1964 and in revised form 29 January 1965)

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PELLEW, A. & SOUTHWELL, R. V. 1940 *Proc. Roy. Soc.* A, 176, 312.
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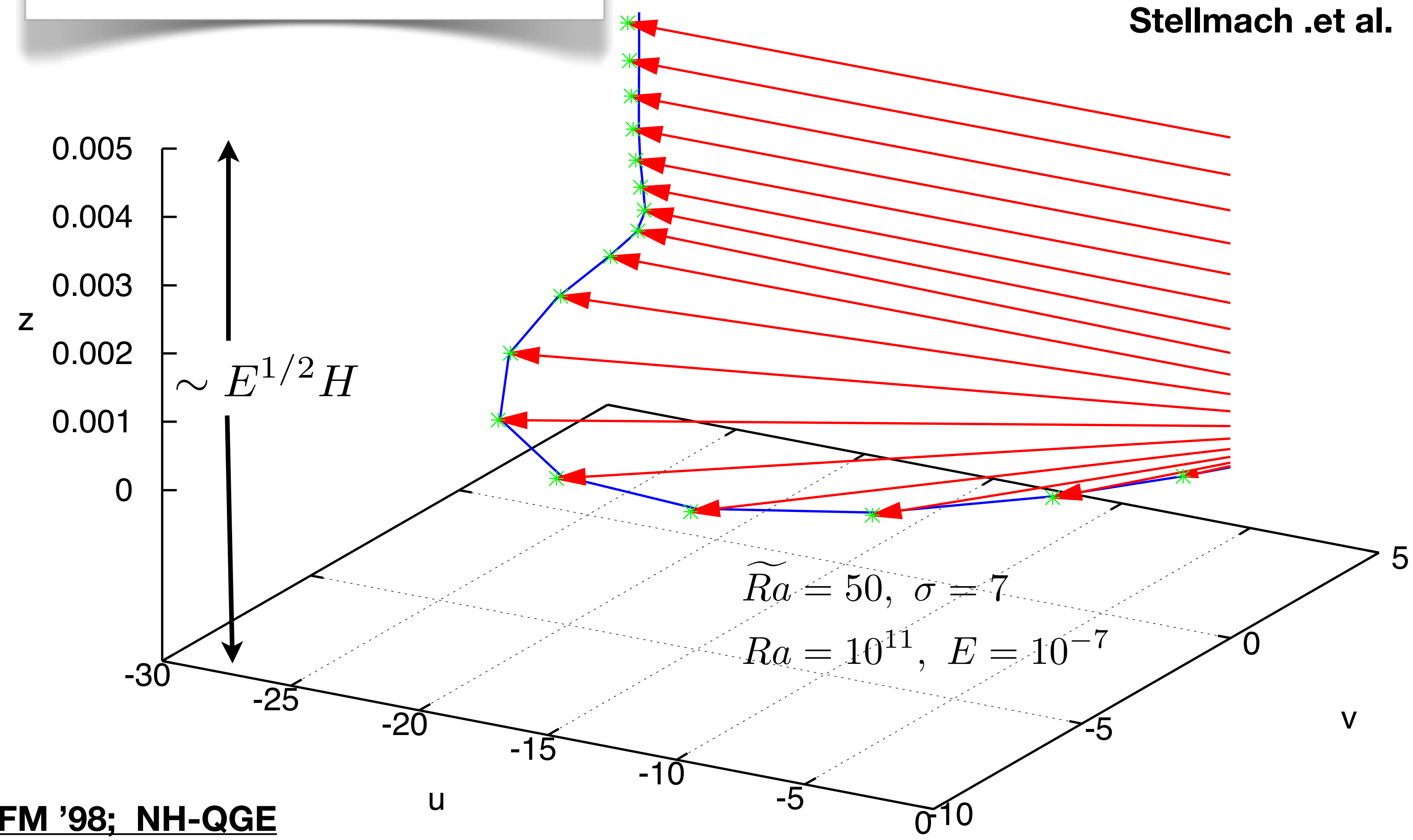
VERONIS, G. 1959 *J. Fluid Mech.* 5, 401.

NH-QG Theory for No-Slip Boundaries states Ekman layers are passive

Are Ekman layers passive?

$$\widehat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp}^{(e)} = \partial_{\delta\delta} \boldsymbol{u}_{\perp}^{(e)}$$



J. & Knobloch, JFM '98; NH-QGE

$$u_0^{(e)}(x,y,\delta,t) = U_0^{(g)}(x,y,0,t)\left(1 - e^{-\frac{\delta}{\sqrt{2}}}\cos\frac{\delta}{\sqrt{2}}\right) - V_0^{(g)}(x,y,0,t)e^{-\frac{\delta}{\sqrt{2}}}\sin\frac{\delta}{\sqrt{2}}$$

$$v_0^{(e)}(x,y,\delta,t) = V_0^{(g)}(x,y,0,t)\left(1 - e^{-\frac{\delta}{\sqrt{2}}}\cos\frac{\delta}{\sqrt{2}}\right) + U_0^{(g)}(x,y,0,t)e^{-\frac{\delta}{\sqrt{2}}}\sin\frac{\delta}{\sqrt{2}}$$

$$w_{1/2}^{(e)}(x,y,\delta,t) = \frac{1}{\sqrt{2}}\zeta_0^{(g)}(x,y,0,t)\left(1 - e^{-\frac{\delta}{\sqrt{2}}}\left[\cos\frac{\delta}{\sqrt{2}} + \sin\frac{\delta}{\sqrt{2}}\right]\right)$$

$$\theta_1^{(e)}(x,y,\delta,t) = \Theta_1^{(g)}(x,y,0,t) \equiv 0$$

$$p_1^{(e)}(x,y,\delta,t) = \Psi_0^{(g)}(x,y,0,t)$$

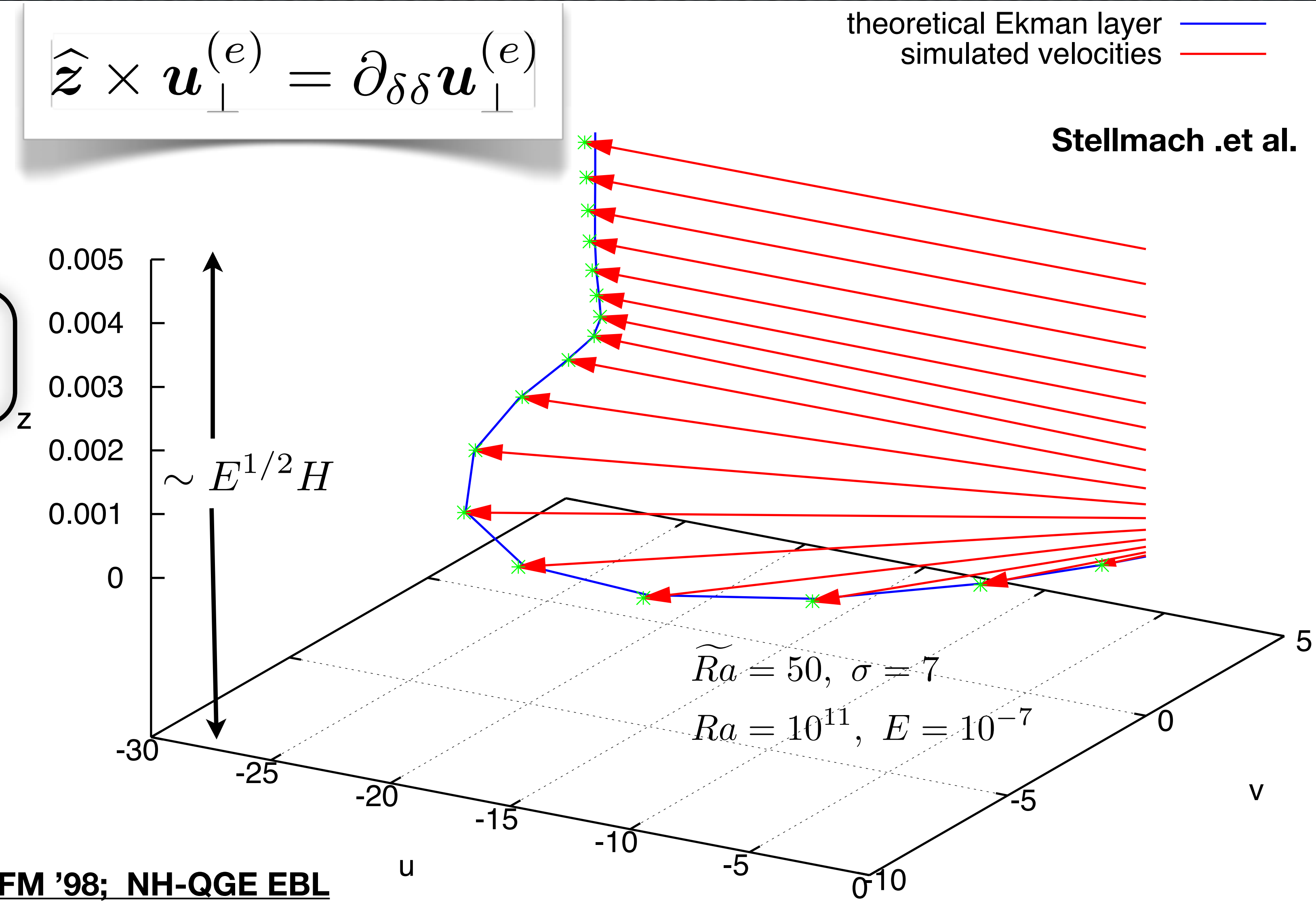
NH-QG Theory for No-Slip Boundaries Predicts Ekman layers are passive

Are Ekman layers passive?

As in atmos. and ocean applications filter Ekman layers by pumping/suction BC

$$w^{(e)}_{1/2}(x,y,Z_\pm,t) = \pm\frac{1}{\sqrt{2}}\zeta^{(g)}_0(x,y,Z_\pm,t)$$

$$\widehat{\boldsymbol{z}} \times \boldsymbol{u}^{(e)}_\perp = \partial_{\delta\delta}\boldsymbol{u}^{(e)}_\perp$$



J. & Knobloch, JFM '98; NH-QGE EBL

$$u^{(e)}_0(x,y,\delta,t) = U^{(g)}_0(x,y,0,t)\left(1 - e^{-\frac{\delta}{\sqrt{2}}}\cos\frac{\delta}{\sqrt{2}}\right) - V^{(g)}_0(x,y,0,t)e^{-\frac{\delta}{\sqrt{2}}}\sin\frac{\delta}{\sqrt{2}}$$

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$$w^{(e)}_{1/2}(x,y,\delta,t) = \frac{1}{\sqrt{2}}\zeta^{(g)}_0(x,y,0,t)\left(1 - e^{-\frac{\delta}{\sqrt{2}}}\left[\cos\frac{\delta}{\sqrt{2}} + \sin\frac{\delta}{\sqrt{2}}\right]\right)$$

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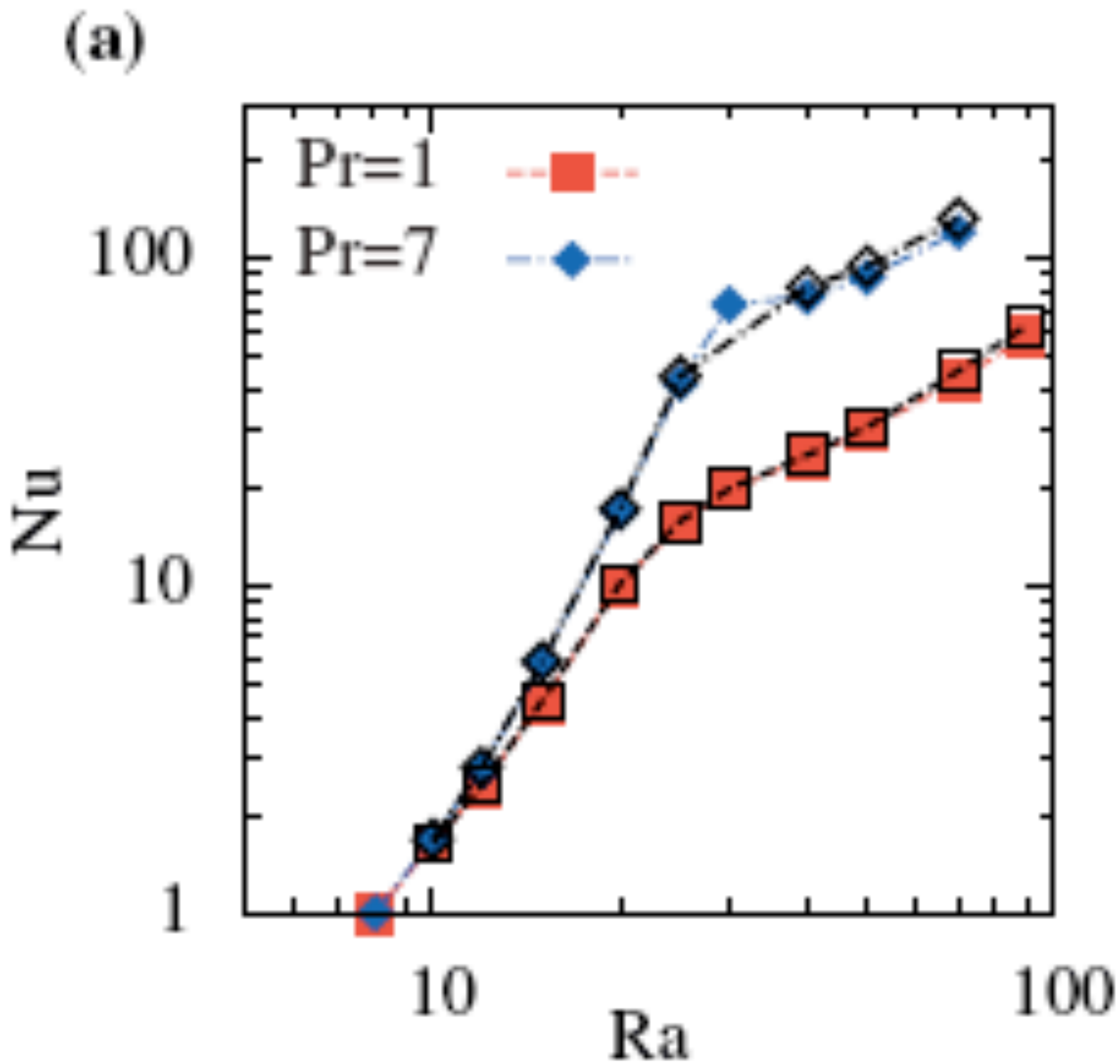
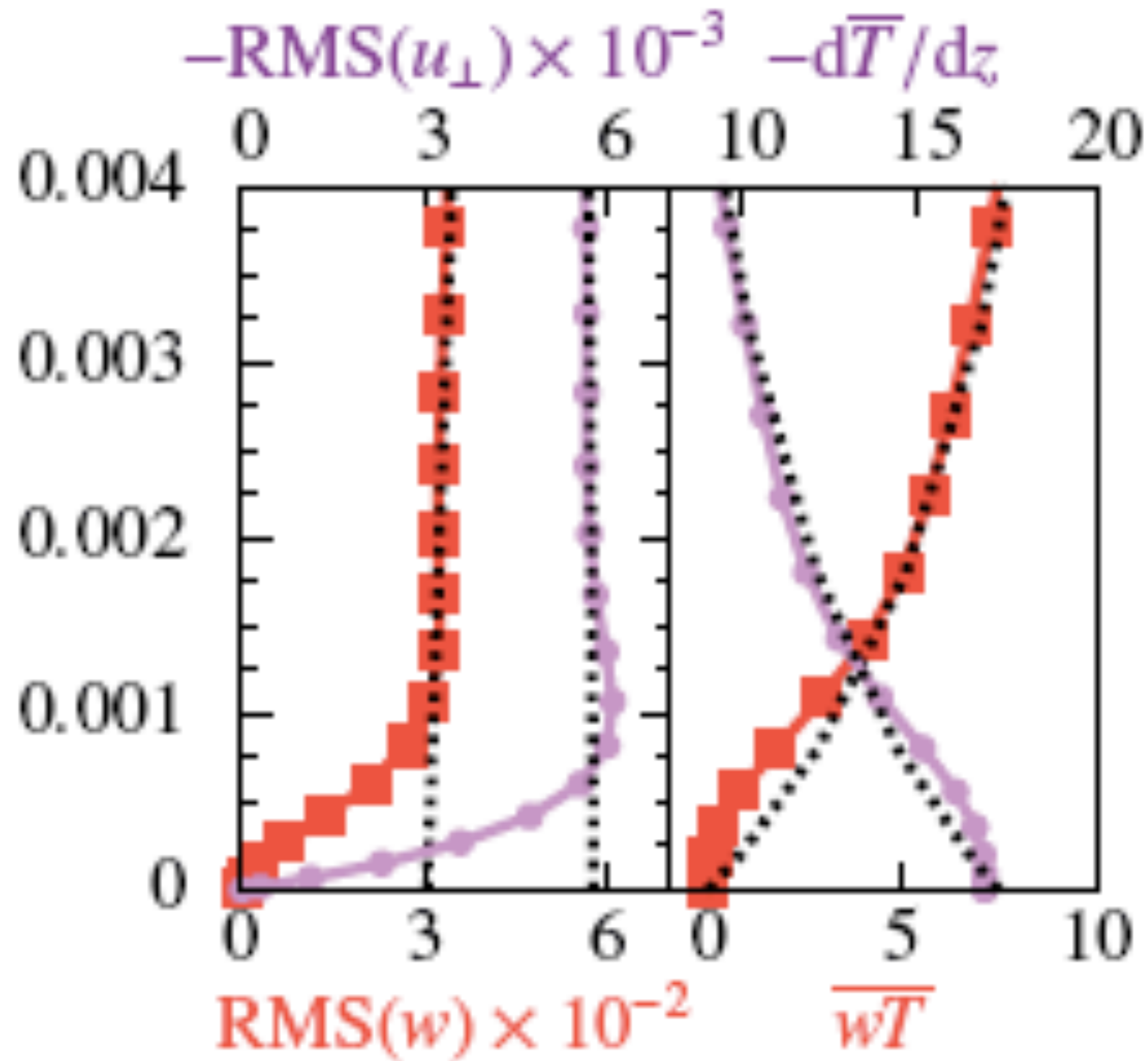
$$p^{(e)}_1(x,y,\delta,t) = \Psi^{(g)}_0(x,y,0,t)$$

DNS RBC vs DNS with Parameterized Pumping

$$w^{(e)}_{1/2}(x,y,Z_\pm,t) = \pm\frac{1}{\sqrt{2}}\zeta^{(g)}_0(x,y,Z_\pm,t)$$

$Ek = 10^{-7}$

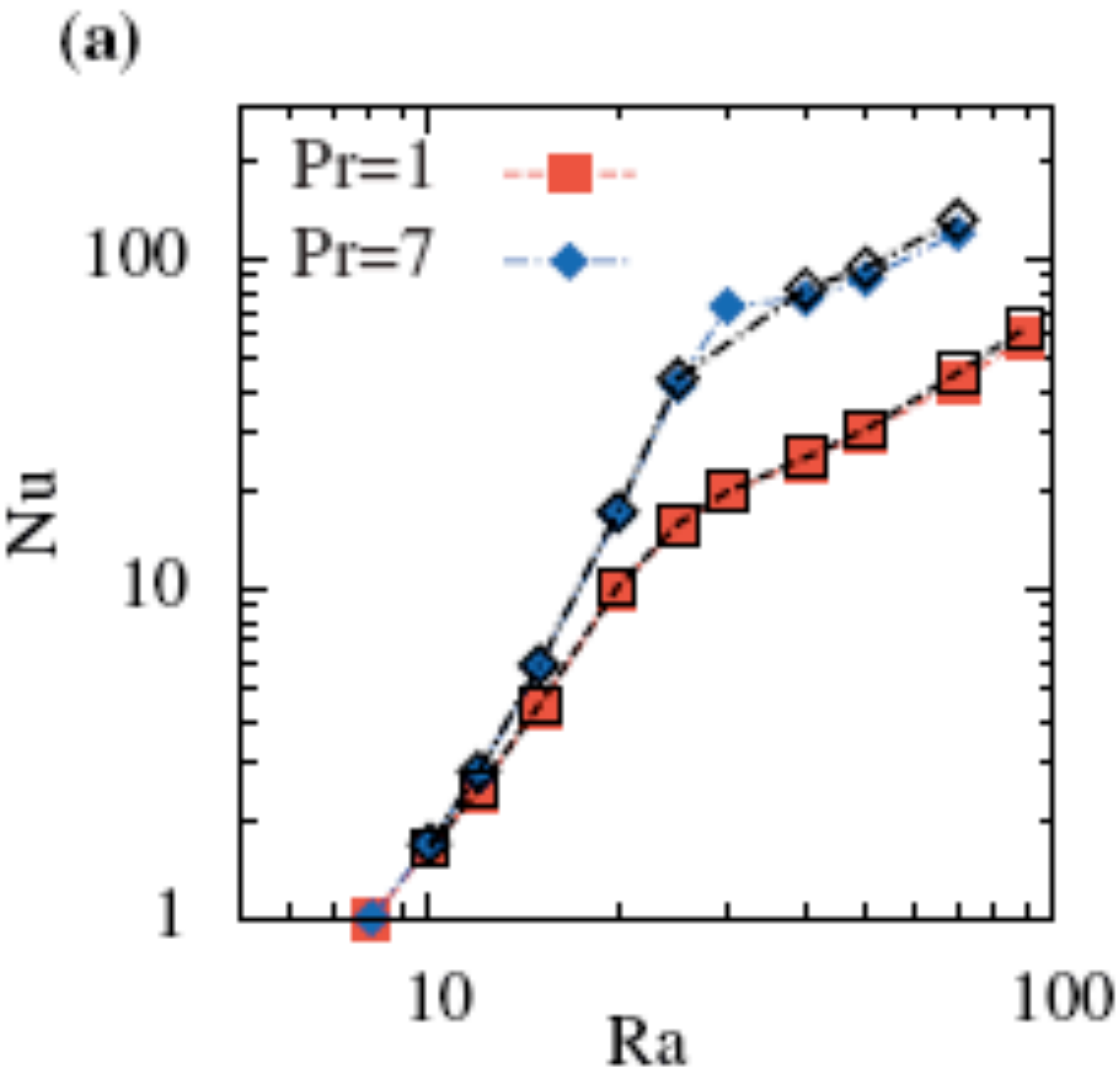




- *Filled symbols (DNS with No-Slip BC's)*
- Open symbols (DNS w/ Pumping BC's)
- *Convergence outside Ekman layers*
- Ekman layers can be filtered

DNS RBC vs DNS with Parameterized Pumping

$$w^{(e)}_{1/2}(x,y,Z_\pm,t) = \pm\frac{1}{\sqrt{2}}\zeta_0^{(g)}(x,y,Z_\pm,t)$$



Conclusion:

- *Ekman pumping responsible for enhanced HT*
- Either *E* to large

  as *E* ⇒ *0, DNS* ⇒ *NHQGE*

OR

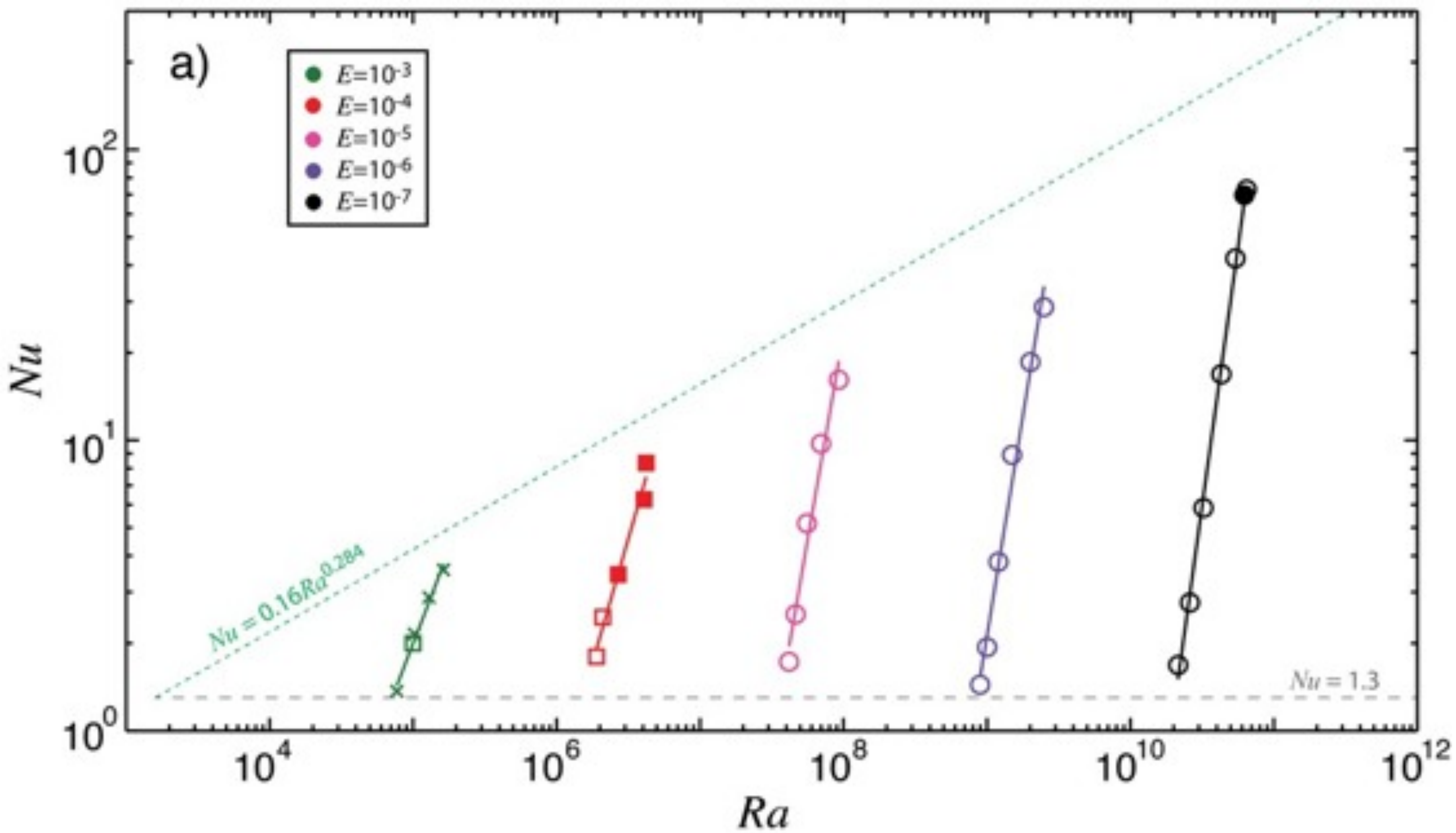
- Pumping remains important as *E* ⇒ *0*

  as *E* ⇒ *0, non-convergence*

- *Filled symbols (DNS with No-Slip BC's)*
- Open symbols (DNS w/ Pumping BC's)

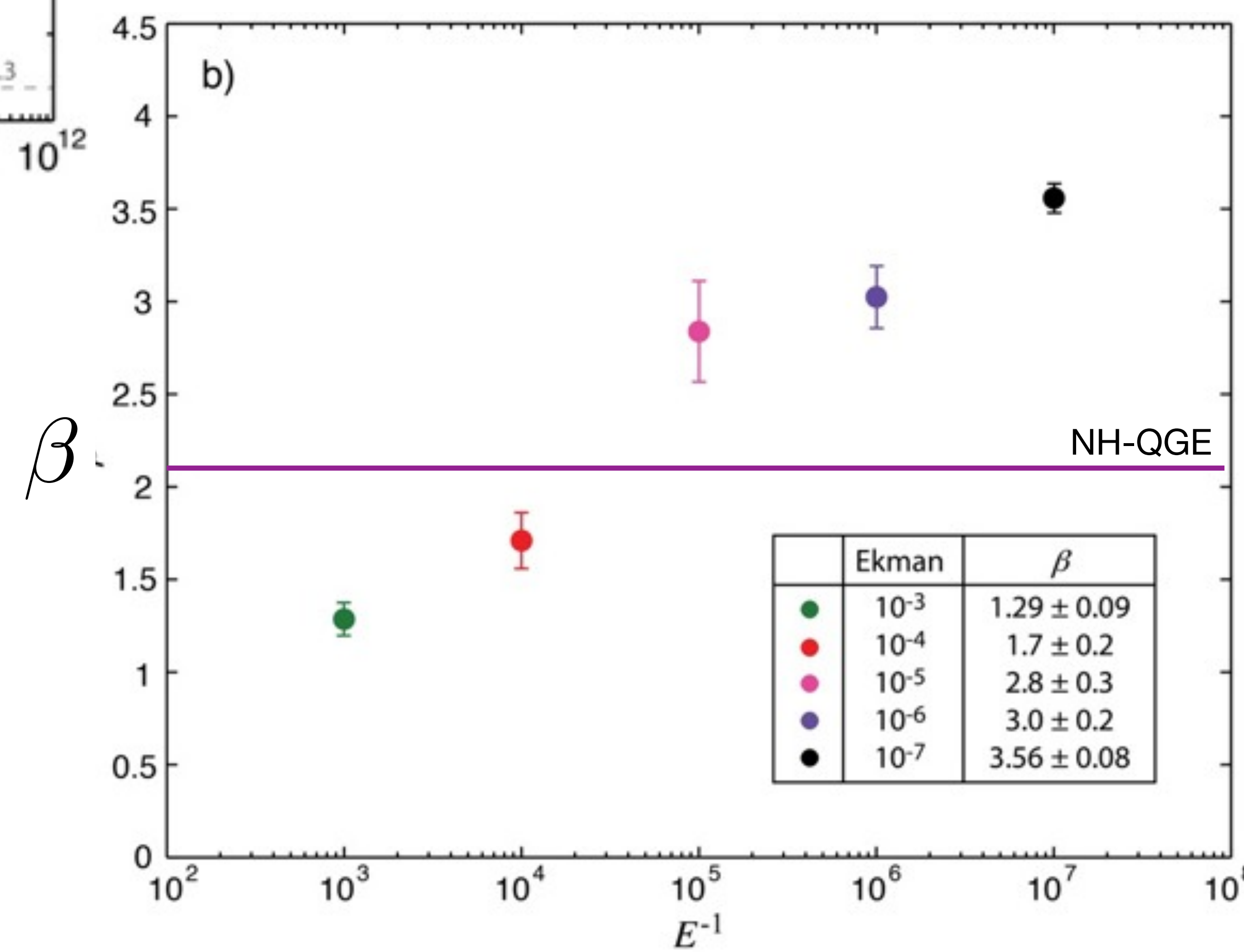
Heat Transport: Exponent vs Ekman number

$Nu \propto (Ra/Ra_c)^{\beta}$ nonconvergence: exponent appears to increase w/ decreasing *E* !



Conclusion:

- $\mathcal{O}(E^{1/6}H)$ EBL's having leading order affect on heat transport
- Not captured in asymptotic reduced model



Results from UCLA SpinLab

Courtesy Jon Aurnou, Jon Cheng

Asymptotic Development
Requires TBL Analysis of NH-QGE

NH-QGE

$$D_t^{\perp}\zeta_0 - \partial_Z w_0 = \nabla_{\perp}^2 \zeta_0$$

$$D_t^{\perp} w_0 = -\,\partial_Z \psi_0 + \frac{RaE^{4/3}}{\sigma}\theta_1 + \nabla_{\perp}^2 w_0$$

$$D_t^{\perp}\theta_1 + w_0 \partial_Z \overline{T}_0 = \frac{1}{\sigma}\nabla_{\perp}^2 \theta_1$$

$$\partial_\tau \overline{T}_0 + \partial_Z \left(\overline{w_0\theta_1}\right) = \frac{1}{\sigma}\partial_{ZZ}\overline{T}_0$$

Magnitudes:

$\Rightarrow \quad \theta_1 \sim w_0 \partial_Z \overline{T}_0$ thermal fluct.

$\Rightarrow \quad w_0\theta_1 \sim \partial_Z \overline{T}_0$ heat flux

Asymptotic Development Requires TBL Analysis of NH-QGE

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Boundary layer scaling for columnar regime;

$$Nu \sim \partial_Z \overline{T}_0 \sim \theta_1 \sim \widetilde{Ra}^{\beta}, \; \partial_Z \sim \widetilde{Ra}^{(1+\beta)/2}$$
$$w_0 \sim \widetilde{Ra}^{0}, \; \zeta_0 \sim \psi_0 \sim \widetilde{Ra}^{(1+\beta)/2}$$

Empirical result

$$\beta \approx 2$$

J. et al GAFD 2012

Asymptotic Development
Ekman pumping drives bulk corrections

Pumping bc's $\widehat{w}^{(p)} = \pm E^{1/6}\zeta_0$ suggest the inclusion of higher order correction to NH-QGE

Pose; with $\epsilon = E^{1/3}$

$$\widehat{w} = w_0 + \epsilon^{1/2} w^{(p)}_{1/2},\ \widehat{\zeta} = \zeta_0 + \epsilon^{1/2}\zeta^{(p)}_{1/2},\ \widehat{\psi} = \psi_0 + \epsilon^{1/2}\psi^{(p)}_{1/2},$$

$$\partial_Z \widehat{\overline{T}} = \partial_Z \overline{T}_0 + \epsilon^{1/2}\partial_Z \overline{T}_{1/2},\ \widehat{\theta} = \epsilon\theta_1 + \epsilon^{3/2}\theta^{(p)}_{3/2}$$

Asymptotic Development
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NH-QGEcorr
$$D_t^{\perp}\zeta_{1/2} + J[\psi_{1/2},\zeta_0] - \partial_Z w_{1/2} = \nabla_{\perp}^2\zeta_{1/2}$$

$$D_t^{\perp} w_{1/2} + J[\psi_{1/2},w_0] = -\,\partial_Z\psi_{1/2} + \frac{RaE^{4/3}}{\sigma}\theta_{3/2} + \nabla_{\perp}^2 w_{1/2}$$

$$D_t^{\perp}\theta_{3/2} + J[\psi_{1/2},\theta_1] + w_{1/2}\partial_Z\overline{T}_0 + w_0\partial_Z\overline{T}_{1/2} = \frac{1}{\sigma}\nabla_{\perp}^2\theta_{3/2}$$

$$\partial_\tau\overline{T}_{1/2} + \partial_Z\left(\overline{w_0\theta_{3/2} + w_{1/2}\theta_1}\right) = \frac{1}{\sigma}\partial_{ZZ}\overline{T}_{1/2}$$

Pose, with $\epsilon = E^{1/3}$

$$\widehat{w} = w_0 + \epsilon^{1/2} w_{1/2}^{(p)},\ \widehat{\zeta} = \zeta_0 + \epsilon^{1/2}\zeta_{1/2}^{(p)},\ \widehat{\psi} = \psi_0 + \epsilon^{1/2}\psi_{1/2}^{(p)},$$
$$\partial_Z\widehat{\overline{T}} = \partial_Z\overline{T}_0 + \epsilon^{1/2}\partial_Z\overline{T}_{1/2},\ \widehat{\theta} = \epsilon\theta_1 + \epsilon^{3/2}\theta_{3/2}^{(p)}$$

Asymptotic Development
Heat Transport by Ekman pumping is significant

Pumping bc's $w^{(p)} = \pm\zeta_0$ suggest the inclusion of higher order correction to the NH-QGE

NH-QGEcorr $$D_t^{\perp}\zeta_{1/2} + J[\psi_{1/2},\zeta_0] - \partial_Z w_{1/2} = \nabla_\perp^2\zeta_{1/2}$$

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Magnitudes:

pumping induced thermal fluct.

$$\Rightarrow \quad \theta_{3/2} \sim \epsilon^{1/2} w_{1/2}\partial_Z\overline{T}_0 \sim \epsilon^{1/2}\zeta_0\partial_Z\overline{T}_0$$

heat flux

$$\Rightarrow \quad w_0\theta_{3/2} + w_{1/2}\theta_1 \sim \epsilon^{1/2}\partial_Z\overline{T}_0$$

Asymptotic Development
Heat Transport by Ekman pumping is significant

Pumping bc's $w^{(p)} = \pm\zeta_0$ suggest the inclusion of higher order correction to the NH-QGE

NH-QGEcorr
$$D_t^{\perp}\zeta_{1/2} + J[\psi_{1/2}, \zeta_0] - \partial_Z w_{1/2} = \nabla_{\perp}^2 \zeta_{1/2}$$

$$D_t^{\perp} w_{1/2} + J[\psi_{1/2}, w_0] = -\,\partial_Z \psi_{1/2} + \frac{RaE^{4/3}}{\sigma}\theta_{3/2} + \nabla_{\perp}^2 w_{1/2}$$

$$D_t^{\perp}\theta_{3/2} + J[\psi_{1/2}, \theta_1] + w_{1/2}\partial_Z \overline{T}_0 + w_0 \partial_Z \overline{T}_{1/2} = \frac{1}{\sigma}\nabla_{\perp}^2 \theta_{3/2}$$

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Magnitudes:

$\Rightarrow$ pumping induced thermal fluct.
$$\theta_{3/2} \sim \epsilon^{1/2} w_{1/2}\partial_Z \overline{T}_0 \sim \epsilon^{1/2}\zeta_0 \partial_Z \overline{T}_0$$

$\Rightarrow$ heat flux
$$w_0\theta_{3/2} + w_{1/2}\theta_1 \sim \epsilon^{1/2}\partial_Z \overline{T}_0$$

Dynamics of higher system subdominant iff heat transport is ordered according to

$$w_0\theta_1 \gg \epsilon^{1/2}\left(w_0\theta_{3/2} + w_{1/2}\theta_1\right) \gg \epsilon w_{1/2}\theta_{3/2}$$
$$1 \gg \epsilon^{1/2}\zeta_0 \gg \epsilon\zeta_0$$

Always violated!

$$\zeta_0(\pm) = \mathcal{O}(\epsilon^{-1/2}) = o(\epsilon^{-1})$$
$$\widetilde{Ra}_{thres} = E^{-1/3(1+\beta)}$$

validity limit of NH-QGE

Asymptotic Development
Estimates of transitional values

$$\zeta_0(\pm) = \mathcal{O}(\epsilon^{-1/2}) = o(\epsilon^{-1}), \quad \epsilon = E^{1/3}$$

$$\widetilde{Ra}_{thres} = E^{-1/3(1+\beta)} = E^{-1/9}, \quad \beta \approx 2$$

$$\widetilde{Ra}_c = 8.7$$

DNS, Lab ⟶

$$E = 10^{-7},\ \epsilon^{1/2} = 0.068,\ \widetilde{Ra}_t \approx 6.0$$

$$E = 10^{-8},\ \epsilon^{1/2} = 0.046,\ \widetilde{Ra}_t \approx 7.7$$

$$E = 10^{-10},\ \epsilon^{1/2} = 0.022,\ \widetilde{Ra}_t \approx 12.9$$

$$E = 10^{-12},\ \epsilon^{1/2} = 0.010,\ \widetilde{Ra}_t \approx 21.5$$

$$E = 10^{-14},\ \epsilon^{1/2} = 0.005,\ \widetilde{Ra}_t \approx 35.9$$

Earth's core ⟶

$$E = 10^{-15},\ \epsilon^{1/2} = 0.003,\ \widetilde{Ra}_t \approx 46.4$$

Asymptotic Development
Composite Reduced Model

CNH-QGE

$$\widehat{D}_t^{\perp}\widehat{\zeta} - \partial_Z \widehat{w} = \nabla_{\perp}^2 \widehat{\zeta}$$

$$\widehat{D}_t^{\perp}\widehat{w} = -\,\partial_Z \widehat{\psi} + \frac{RaE^{4/3}}{\sigma}\widehat{\theta} + \nabla_{\perp}^2 \widehat{w}$$

$$\widehat{D}_t^{\perp}\widehat{\theta} + \widehat{w}\partial_Z \overline{\widehat{T}} = \frac{1}{\sigma}\left(\nabla_{\perp}^2 + \epsilon^2 \partial_{ZZ}\right)\widehat{\theta}$$

$$\partial_\tau \overline{\widehat{T}} + \partial_Z \left(\overline{\widehat{w}\widehat{\theta}}\right) = \frac{1}{\sigma}\partial_{ZZ}\overline{\widehat{T}}$$

$$\widehat{w} = \pm E^{1/6}\zeta_0$$

$$\overline{T}(0) = 1, \overline{T}(1) = 0$$

Heat flux correction due to Ekman pumping captured

Asymptotic Development
Composite Reduced Model

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$$\widehat{D}_t^{\perp}\widehat{\zeta} - \partial_Z \widehat{w} = \nabla_{\perp}^2 \widehat{\zeta}$$

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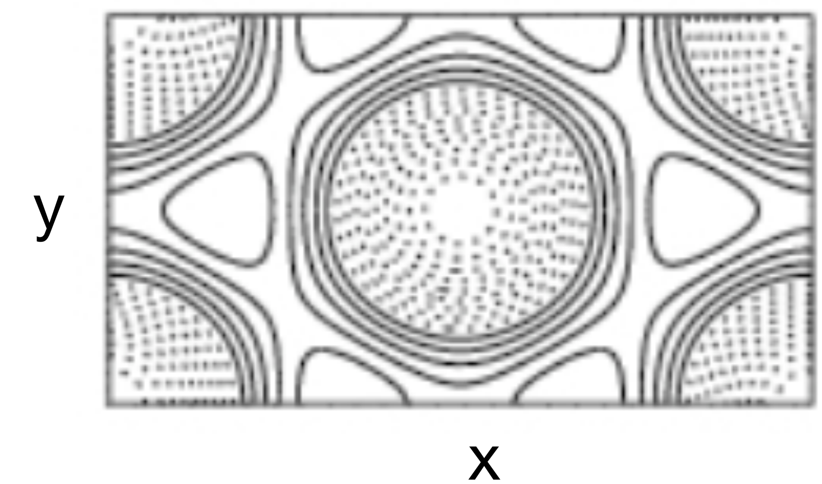
$$\overline{T}(0) = 1, \overline{T}(1) = 0$$

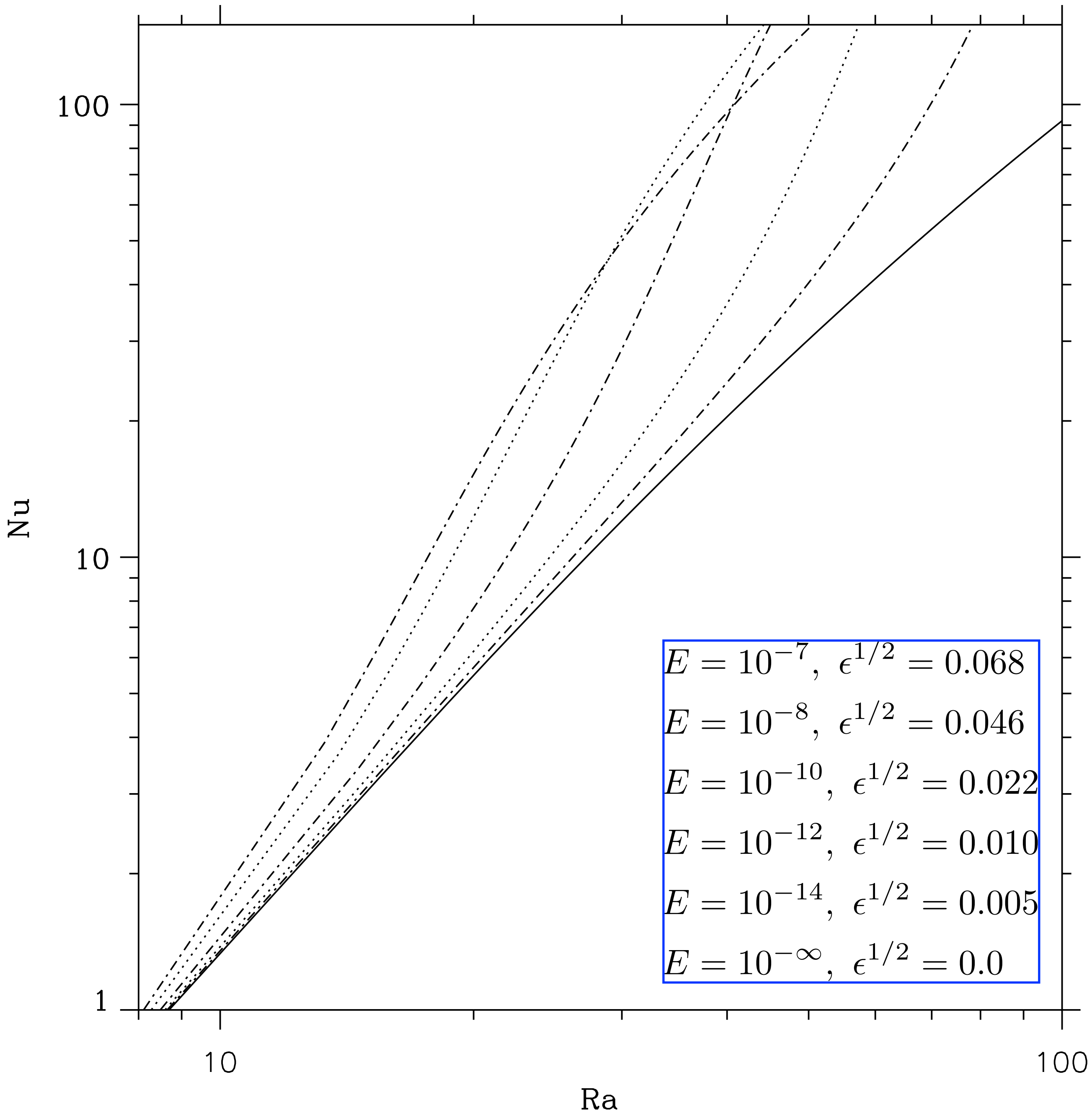
Heat flux correction due to Ekman pumping captured

$$\boldsymbol{u}_{\perp} \cdot \nabla_{\perp} = J[\widehat{\psi}, \widehat{f}] \equiv 0$$

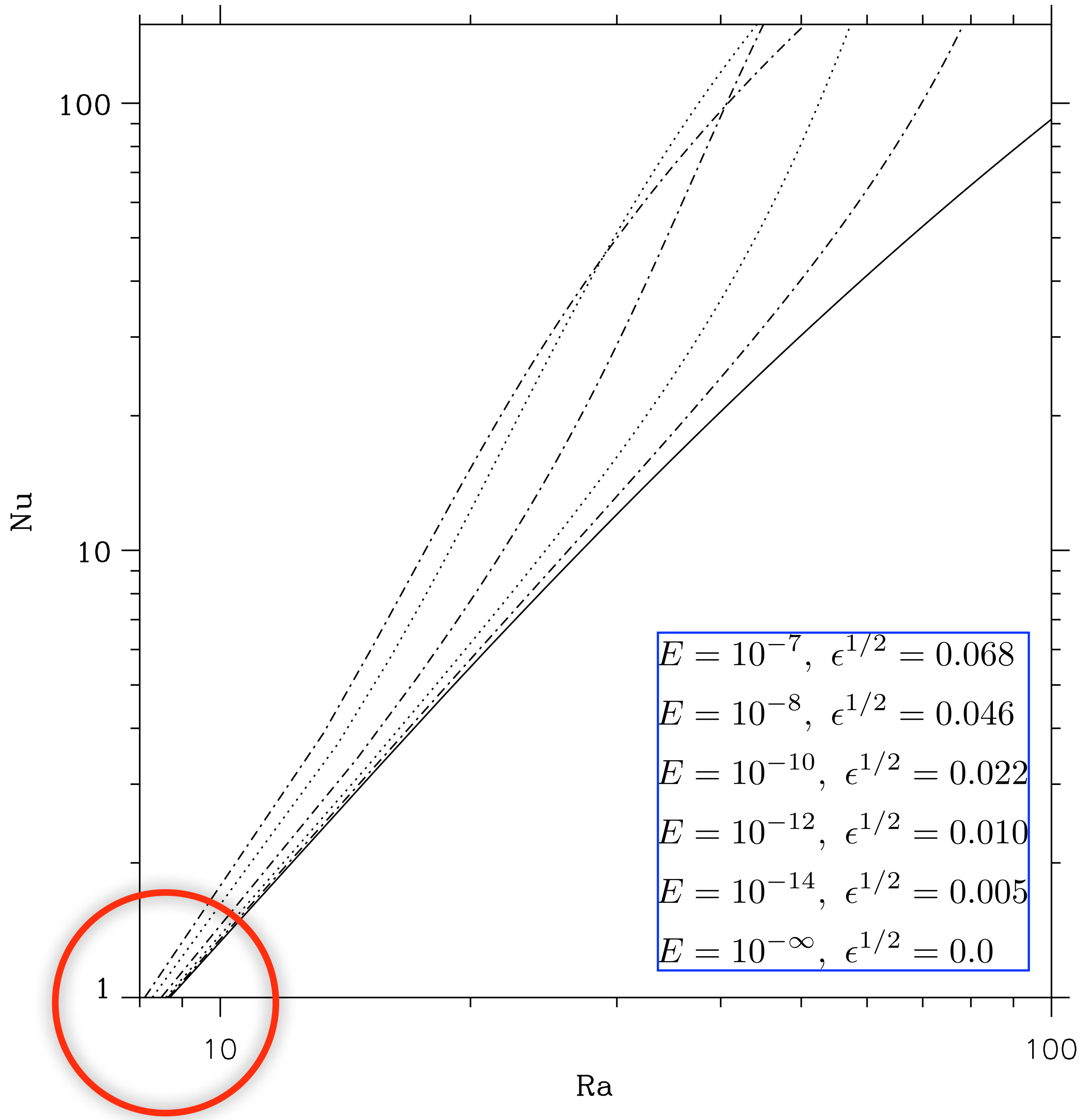


Preliminary investigation: single mode theory gives further reduction to a 1-D vertical model.

Results with Ekman pumping
Nu vs Ra



Results with Ekman pumping
Nu vs Ra

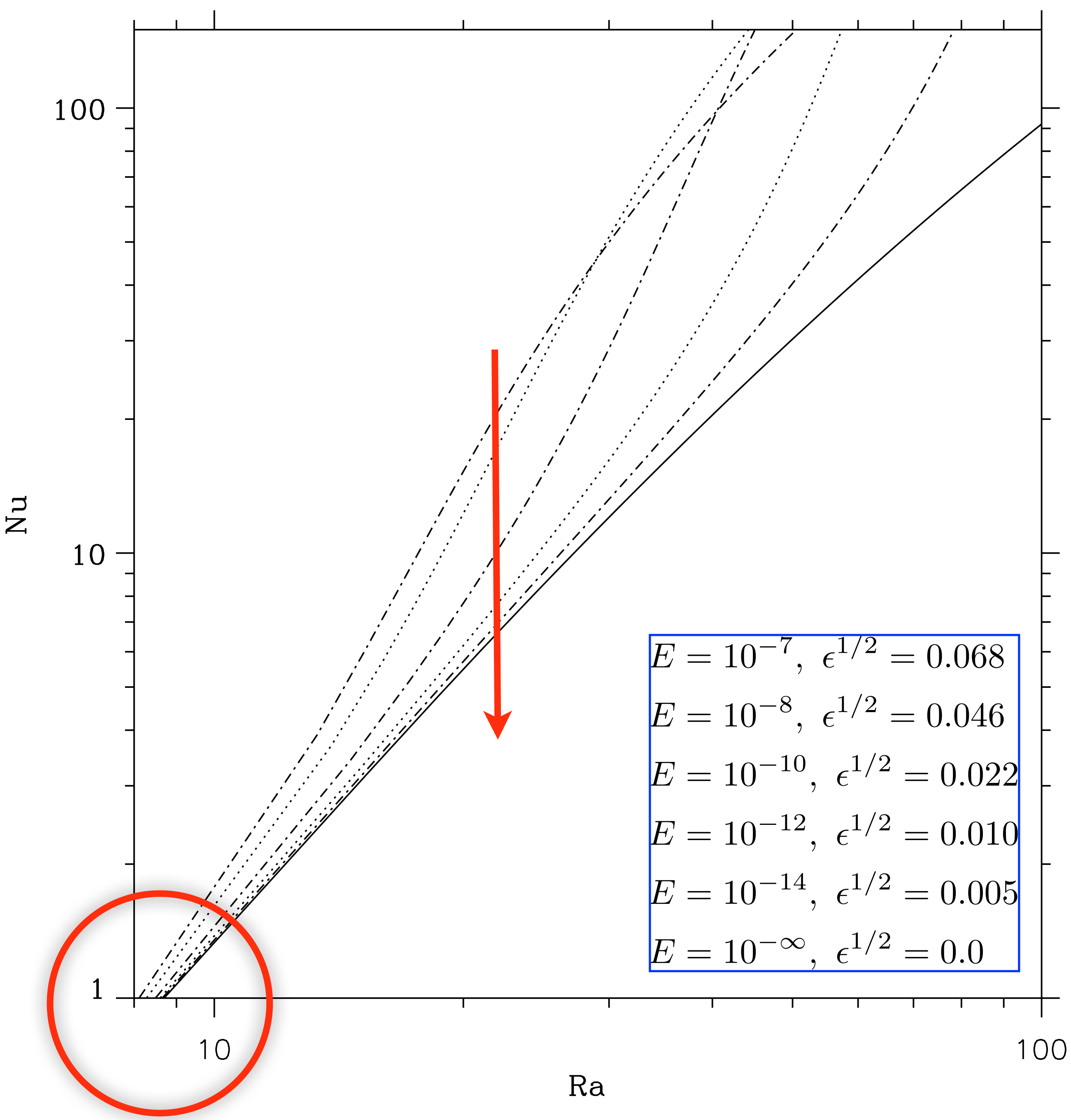


- Linear theory recovered,

  no slip easier to destabilize

  $$\left(RaE^{4/3}\right)_{NS} \approx 8.7_{SF} - \mathcal{O}\left(E^{1/6}\right)$$

Results with Ekman pumping
Nu vs Ra



- NH-QGE recaptures as *E ⇒ 0* at fixed *Ra.*

- Linear theory recovered, no slip easier to destabilize

$$\left(RaE^{4/3}\right)_{NS} \approx 8.7_{SF} - \mathcal{O}\left(E^{1/6}\right)$$

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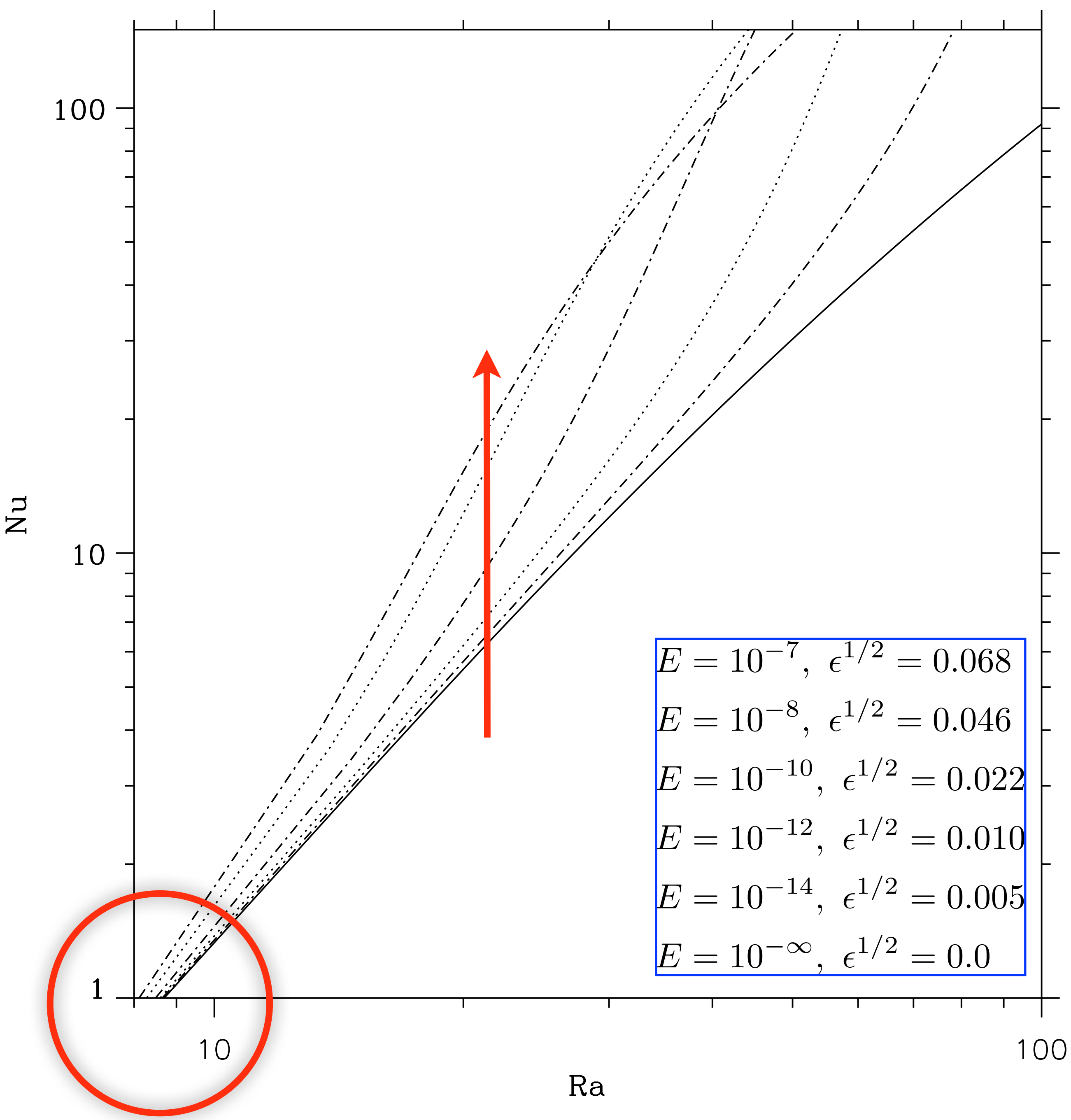
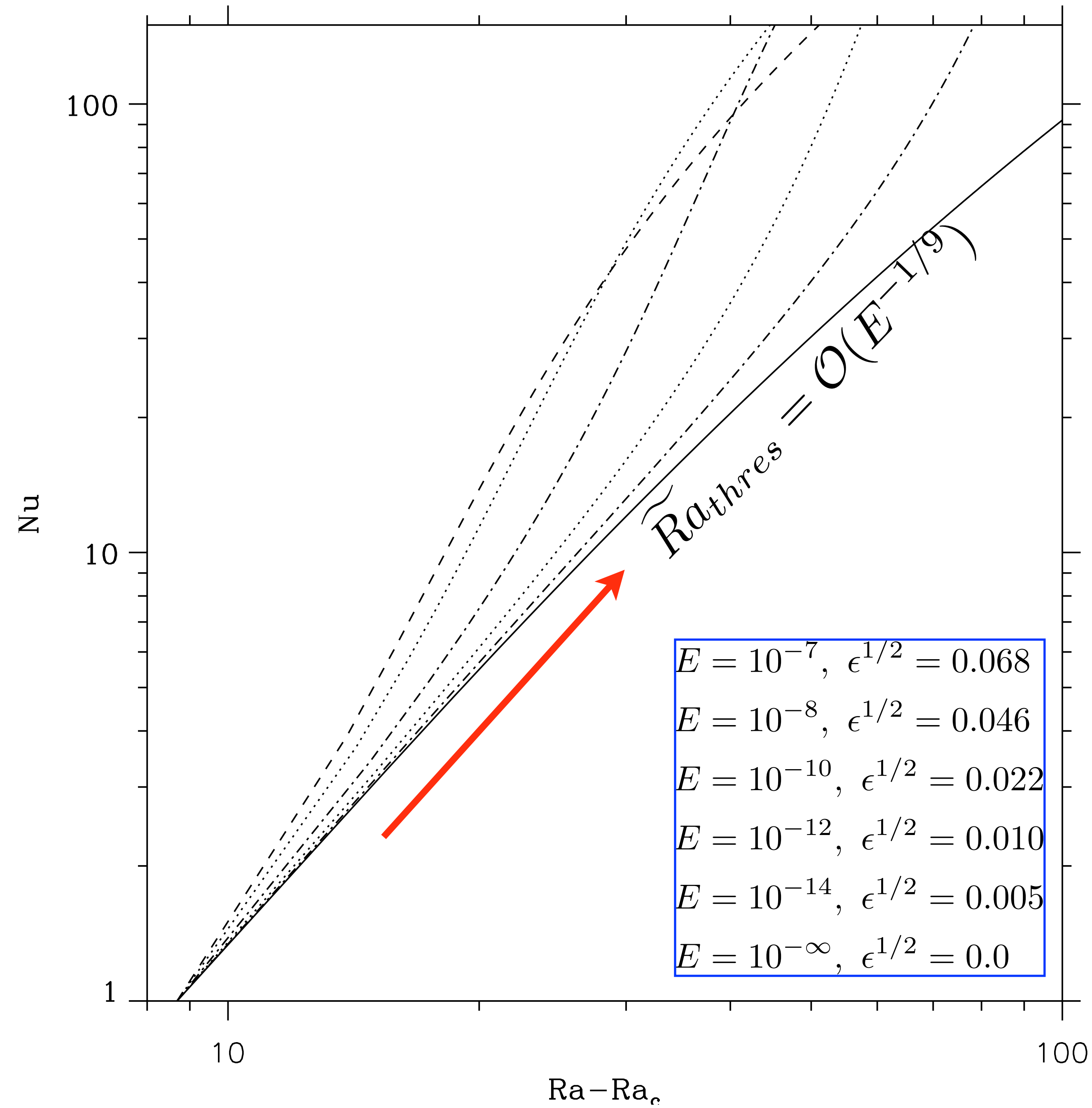
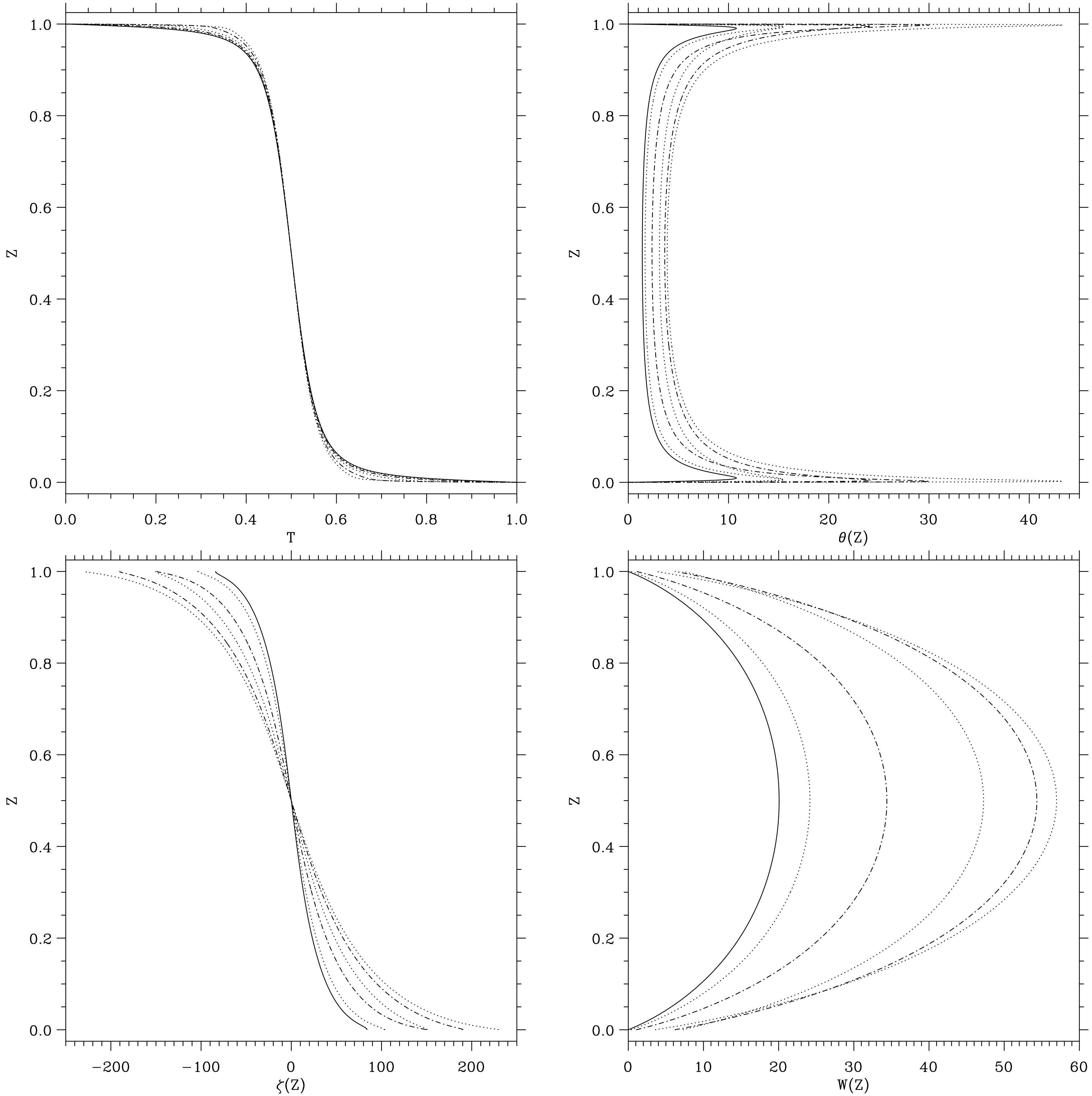




- Limit *E* ⇒ *0* pumping always becomes important, GAFD implications

Results with Ekman pumping
Nu vs Ra



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$E = 10^{-\infty},\ \epsilon^{1/2} = 0.0$

Summary

- Ekman pumping appears to have a non-diminishing effect
  - as $E \Rightarrow 0$ contribution to heat transport remains $O(1)$
  - has implication for GAFD flows that exist on low Ro branch
- EBL Effect can be captured by parameterized pumping BC's
  - DNS can be performed at lower E barring CFL constraints
  - NH-QGE can be extended to include pumping
- Effort requires synergy between Lab exp's, DNS, and reduced modeling

Some Open Questions for NH-QGE

- Analysis (MTWK1):
  - Global Existence, Regularity, ... *No, Nonlinear Vortex Stretching, Reynolds Closure prob.*
  - *Bounds on HT : (Stress-Free done. Grooms & Whitehead '14)*
- Mixing and Transport (MTWK2):
  - Mixing efficiency? RRBC has semi-analytic flows (Cells, CTCs)
- Engineering Applications (MTWK4)
  - Need greater understanding of EBL's and TBL's (Data analysis, Visualizations)