Nonlocal turbulent cascades in nonlinear Schrödinger (Gross-Pitaevski) equation

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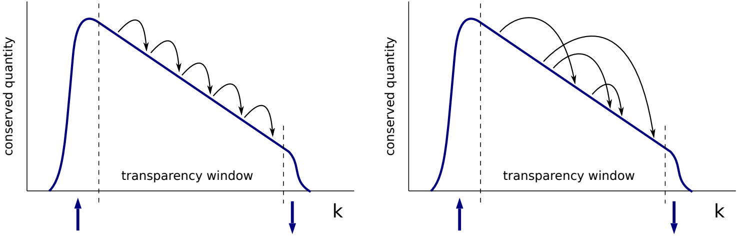
"Non-local cascades" — oxymoron?

Perhaps. But we can still talk about non-local turbulence.



local interactions

nonlocal interations

- The spectra (two-point correlation functions) can be non-local and non-universal (dependent on forcing and dissiption scales).
- There must be a high-order correlation function which is universal.

Kolmogorov turbulence:

$$\langle \left(\delta v_{\|}(\mathbf{r},\ell)\right)^{3} \rangle = -\frac{4}{5}\epsilon\ell$$

NLS turbulence:

???

Nonlinear Schrödinger (Gross-Pitaevski) equation as a model for wave turbulence

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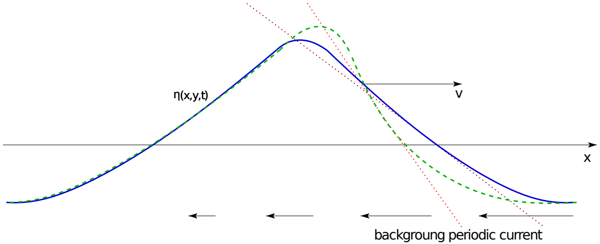
References cited in this talk are incomplete and subjective.

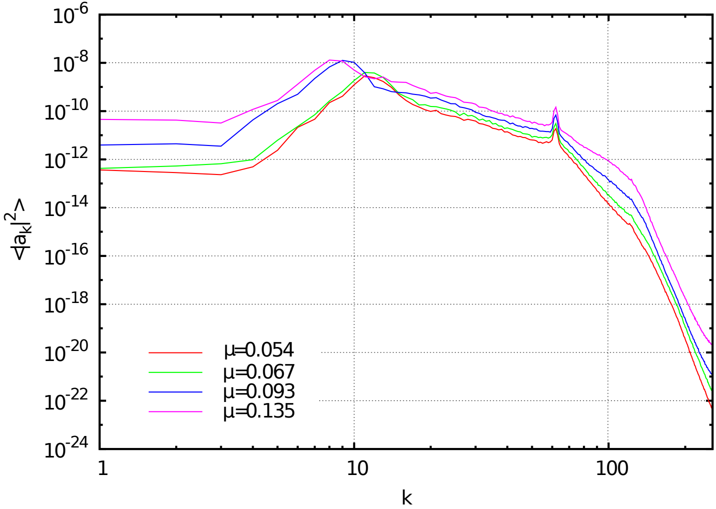
Gravity waves on water surface (A. Korotkevich)



- Formulation in terms of surface elevation $\eta(\mathbf{r}, t)$ and velocity potential on the surface, $\Phi = \phi(\mathbf{r}, \eta, t)$, where $\mathbf{v} = \nabla\phi$.
- Hamiltonian is expanded in powers of steepness, $\mu = \sqrt{|\nabla\eta|^2}$.
- Complex canonical (normal) variables a_k are introduced instead of real $\Phi(\mathbf{r}, t)$ and $\eta(\mathbf{r}, t)$.
- a_k is an elementary excitation (plane wave). Inverse cascase of $|a_k|^2$ is studied.

A. Korotkevich <alexkor@math.unm.edu>, private communications (2014)

Energy spectra of gravity waves (A. Korotkevich)



Mid-range forcing and small-scale damping result in establishing of direct and inverse cascades and accumulation of wave action at small **k**.

Nonlinear Schrödinger (Gross-Pitaevski) equation

$$i\psi_t + \nabla^2\psi \pm |\psi|^2\psi = 0$$

universal model
narrow wave packet
envelope of waves

Benney & Newell (1967) — general settings
Zakharov (1968) — deep water waves
Hasegawa & Tappert (1973) — optical fibers

Why universal?

Linear wave:

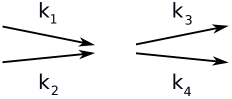
$$\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} = 0$$

$$\frac{\partial a_k}{\partial t} + i\omega a_k = 0$$

$$\frac{\partial a_k}{\partial t} = -i\frac{\partial H_2}{\partial a_k^*}$$

$$H_2 = \int \omega_k |a_k|^2 d\mathbf{k}$$

Nonlinearity:



$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{q}_k, \quad q_k \ll k_0$$

$$H_4 = \ldots$$

$$H = H_2 + H_4 = H_2 + \int T_{1234}\, a_1 a_2 a_3^* a_4^*\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$

Rewrite $\frac{\partial a_k}{\partial t} + i\omega a_k = -i\frac{\partial H_4}{\partial a_k^*}$ for the envelope, $a_k(t) = e^{-i\omega_0 t}\psi(q,t)$,

$$\frac{\partial \psi_q}{\partial t} - i\omega_0\psi_q + i\omega(q)\psi_q = -iT\int \psi_1^*\psi_2\psi_3\,\delta(\mathbf{q} + \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3$$

Why universal?

$$i\frac{\partial \psi_q}{\partial t} + \omega_0\psi_q - \omega(q)\psi_q = T\int \psi_1^*\psi_2\psi_3\,\delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3$$

Assume $\omega = \omega(k)$ and expand for small $\mathbf{q}$

$$\omega(q) = \omega_0 + q_i\left(\frac{\partial\omega}{\partial k_i}\right)_0 + \frac{1}{2}q_iq_j\left(\frac{\partial^2\omega}{\partial k_i\partial k_j}\right)_0 = \omega_0 + vq_\parallel + \frac{1}{2}\left(\omega'' q_\parallel^2 + \frac{v}{k_0}q_\perp^2\right)$$

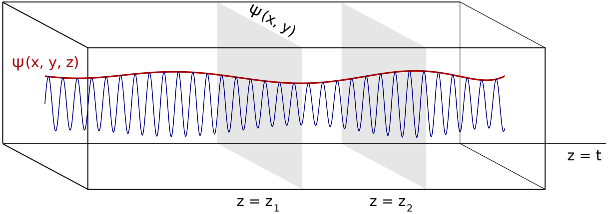
Back to r-space ($\mathbf{k}_0 \parallel \hat{\mathbf{z}}$):

$$\underbrace{i\left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial z}\right)}_{\frac{\partial\psi}{\partial t}\text{ in moving frame}} + \underbrace{\frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2}}_{\text{dispersion}} + \underbrace{\frac{v}{2k_0}\nabla_\perp^2\psi}_{\text{diffraction}} = \underbrace{T|\psi|^2\psi}_{\text{nonlinearity}}$$

Rescale ψ and spatial coordinates:

$$\boxed{i\psi_t + \nabla^2\psi \pm |\psi|^2\psi = 0}$$

Connection to nonlinear optics



$$\boxed{\frac{1}{c^2}\left(\epsilon E\right)_{tt} - \nabla^2 E = 0}$$

Stationary envelope: $E = \frac{1}{2}\psi(x,y,z)e^{ikz-i\omega t}$, with $\omega = \frac{kc}{\sqrt{\epsilon_0}}$.
Kerr nonlinearity: $\epsilon = \epsilon_0 + \epsilon_2|E|^2 = \epsilon_0 + \epsilon_2|\psi|^2$.

$$\frac{1}{c^2}(i\omega)^2(\epsilon_0 + \epsilon_2|\psi|^2)\psi - \left[\nabla^2\psi + 2ik\psi_z - k^2\psi\right] = 0$$

Neglecting $\frac{\partial^2\psi}{\partial z^2}$ and using $kx \to x$, $\frac{1}{2}kz \to z$, and $\psi|\frac{\epsilon_2}{k\epsilon_0}|^{\frac{1}{2}} \to \psi$,

$$\boxed{i\psi_z + \nabla_\perp^2\psi - T|\psi|^2\psi = 0,} \qquad \text{with} \quad T = \pm 1$$

Connection to hydrodynamics

$$i\psi_t + \nabla^2\psi - T|\psi|^2\psi = 0$$

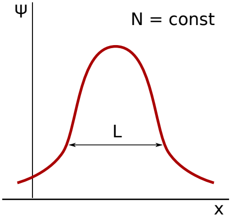
Change of variables: $\psi = Ae^{i\phi}$, $\rho = A^2$, $\mathbf{v} = 2\nabla\phi$.

$$\begin{aligned} \mathbf{v}_t + \nabla\frac{|\mathbf{v}|^2}{2} &= -\frac{1}{\rho}\nabla p \\ \rho_t + \nabla(\rho\mathbf{v}) &= 0 \end{aligned}$$

"Equation of state":

$$\frac{1}{\rho}\nabla p = \nabla\left[2T\rho - \frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right]$$

Collapses in focusing NSE



$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0$$

Integrals of motion

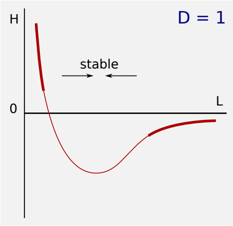
$$N = \int |\psi|^2 \, d^D r$$

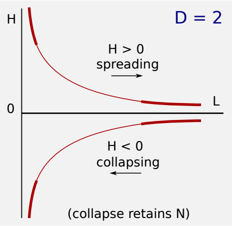
$$\mathcal{H} = \int \left(|\nabla\psi|^2 - \tfrac{1}{2}|\psi|^4 \right) \, d^D r$$

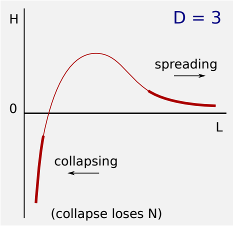
Within the packet

$$|\psi|^2 \sim N/L^D$$

$$\mathcal{H} \sim NL^{-2} - N^2 L^{-D}$$





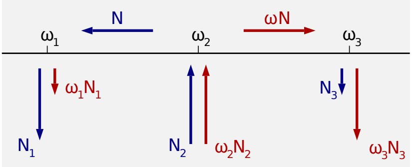


Zakharov & Kuznetsov (1986)

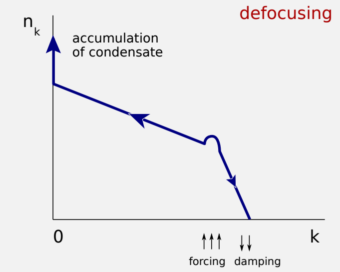
Cascades of turbulence

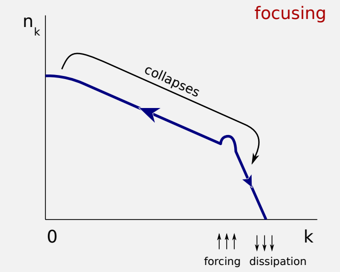
$$\mathcal{H} = \int \omega_k |a_k|^2 d\mathbf{k}$$
$$N = \int |a_k|^2 dk$$

$$N_1 + N_3 = N_2$$
$$\omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2$$



$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1} \approx N_2 \qquad \omega_1 N_1 \ll \omega_2 N_2$$
$$N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \ll N_2 \qquad \omega_3 N_3 \approx \omega_2 N_2$$





Dyachenko, Newell, Pushkarev, & Zakharov (1992)

Modulational instability

$$\boxed{i\psi_t = -\tfrac{1}{2}\omega''\nabla^2\psi + T|\psi|^2\psi}$$

Exact solution (condensate):

$$\Psi = \sqrt{N_0}e^{-iTN_0t}$$

For small perturbation $\psi := \Psi + \psi$,

$$i\psi_t = -\tfrac{1}{2}\omega''\nabla^2\psi + 2TN_0\psi + T\Psi^2\psi^* + O(|\psi|^2).$$

In k-space, using $(\psi^*)_k = \psi^*_{-k}$,

$$\begin{aligned} i\tfrac{d}{dt}\psi_k &= \left(\tfrac{1}{2}\omega''k^2 + 2TN_0\right)\psi_k + T\Psi^2\psi^*_{-k}, \\ -i\tfrac{d}{dt}\psi^*_{-k} &= \left(\tfrac{1}{2}\omega''k^2 + 2TN_0\right)\psi^*_{-k} + T\Psi^2\psi_k. \end{aligned}$$

Modulational instability

Looking for the solution in the form

$$\psi_k = \alpha e^{-i(TN_0+\Omega_k)t} \qquad \text{and} \qquad \psi^*_{-k} = \beta e^{i(TN_0-\Omega_k)t},$$

rewrite the system as

$$\begin{pmatrix} \frac{1}{2}\omega'' k^2 + TN_0 - \Omega_k & T\Psi^2 \\ T\Psi^{*2} & \frac{1}{2}\omega'' k^2 + TN_0 + \Omega_k \end{pmatrix} \begin{pmatrix} \alpha \, e^{-iTN_0 t} \\ \beta \, e^{iTN_0 t} \end{pmatrix} = 0$$

Bogoliubov dispersion relation:

$$\boxed{\Omega_k^2 = \omega'' TN_0 k^2 + \tfrac{1}{4}\omega''^{\,2} k^4}$$

Instability: $\omega'' T < 0$ (focusing nonlinearity).

Bogoliubov (1947)

Why turbulence?

- Wide energy spectra; cascades
- Statistical description
- High probability of extreme events (intermittency)
- Coherent structures — condensate or collapses
- Steady (with damping/forcing) or decaying

Direct and inverse cascades in 2D NLS equation with defocusing nonlinearity before onset of the condensate

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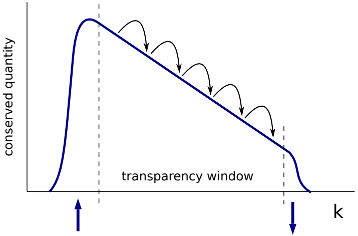
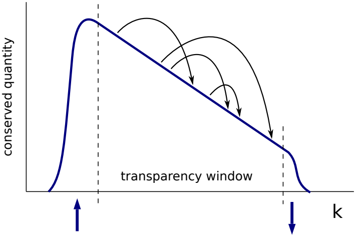


local interactions



nonlocal interations

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NLS turbulence:

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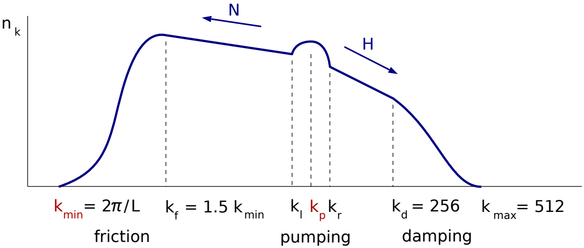
Numerical setup



Defocusing nonlinearity, forcing in **k**-space:

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}_k\psi + i\hat{g}_k.$$

Pumping: $g_k = |g_k|e^{i\phi_k}$, $|g_k| \propto \sqrt{(k^2 - k_l^2)(k_r^2 - k^2)}$, random ϕ_k, $k_l < k < k_r$. Deposition rate $\alpha = \dot{N} \equiv \overline{|\psi|^2}$.

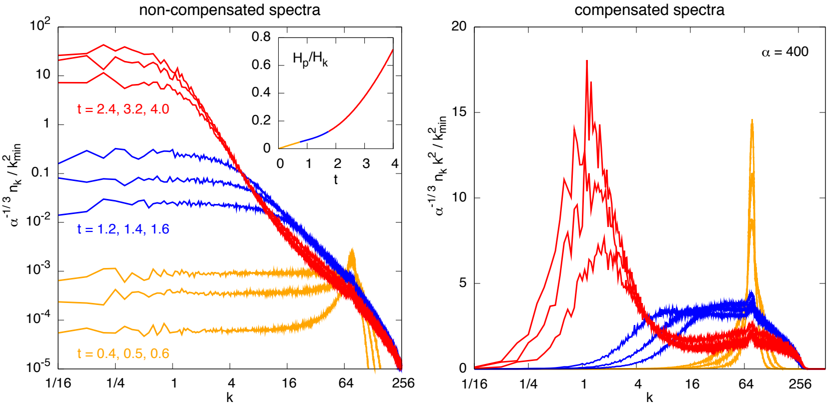
Small-scale damping: $f_k = -\beta(k/k_d)^4(k/k_d - 1)^2, \quad k > k_d.$

Large-scale friction: $f_k = -(1, 1, \frac{1}{\sqrt{2}})\,\gamma$ for $k = (0, 1, \sqrt{2})k_{\min}.$

Inverse Cascade

Evolution of spectra in simulations without friction
Spectra stabilized by friction
Comparison to nonlinear theory

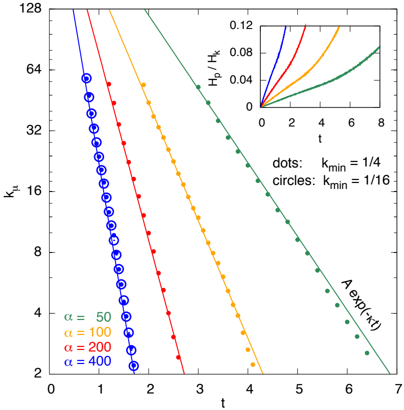
Inverse cascade: time evolution of non-stabilized spectra

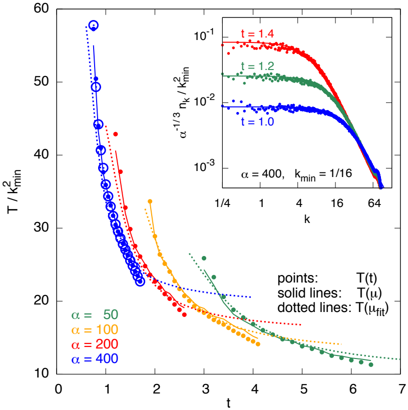

non-compensated spectra

compensated spectra

Early stage: Equipartitioned distribution of wave action.
Intermediate stage: Thermal quazi-equilibrium with chemical potential.
Late stage: Nonlinearity effects, moving pile-up at low k.

Thermal equilibrium with chemical potential $\mu = k_\mu^2$

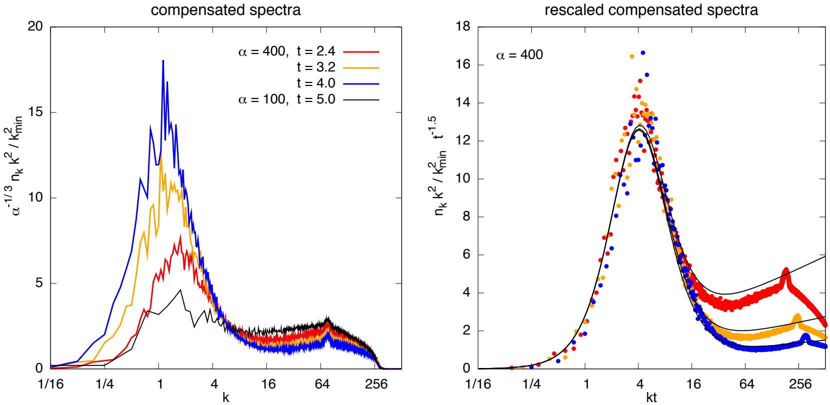




$$n_k = \frac{T(t)}{k_\mu^2(t) + k^2}$$

Assumption of $T(t) \to const$ leads to $k_\mu = Ae^{-\varkappa t}$.
From balance of wave action, $T(k_\mu) \propto (t-a)/(t-b)$.
Deviation is due to non-linear effects, not due to limited domain size.

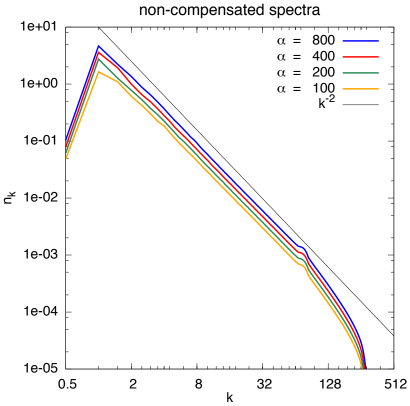
Non-linear effects in large boxes

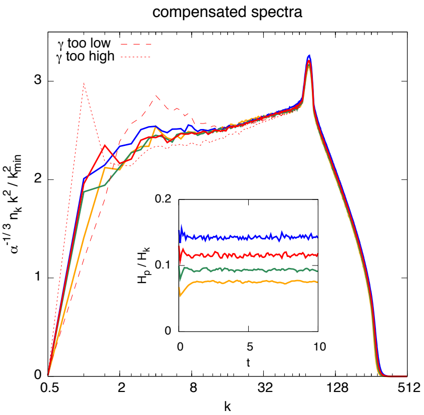


$$n_k = \frac{T(1 + c_1 c_2 k^2 \ln k)}{k_\mu^2 + k^2 + c_2 k^4},$$

Three intervals: equipartitioned, DNPZ-1992, $n_k \propto k^{-2} \ln k$.
Hump location moves as t^{-1}, amplitude grows as $t^{3/2}$ (botteneck?)
Pumping at lower rate α reduces piling-up and extends the spectrum.

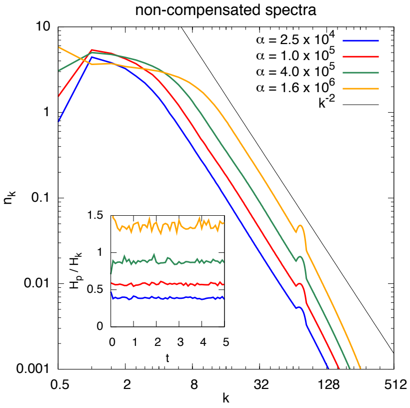
Stabilized spectra: effect of forcing and friction

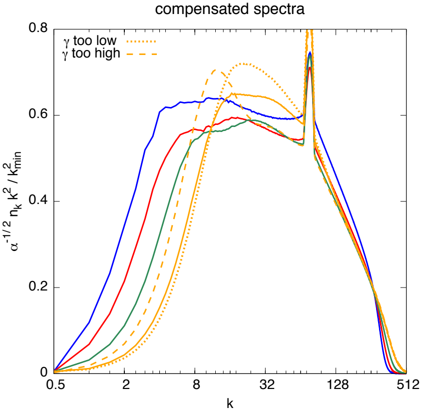




- Deviation from $n_k \sim k^{-2}$ is small.
- Weak turbulence, four-wave interactions are dominant, resulting in $n_k \sim \alpha^{1/3}$ scaling.
- Too high or too low γ leads to the distortion of spectrum at small k.

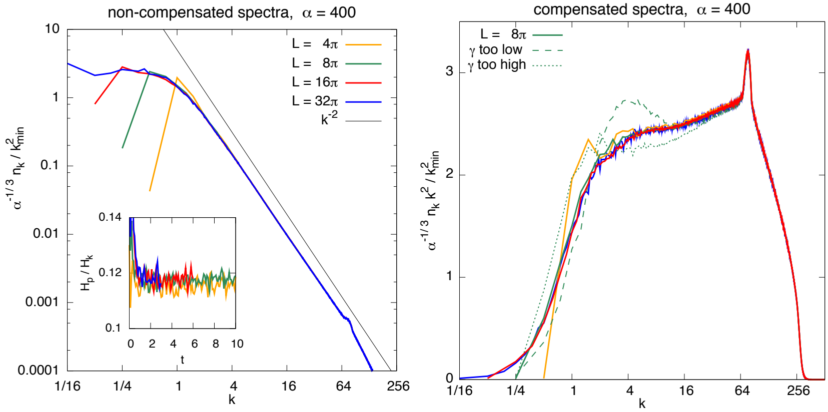
Stabilized spectra with high nonlinearities





- At large k, deviation from $n_k \sim k^{-2}$ is small; unlike at weak nonlinearity, compensated spectra have negative slopes.
- Strong turbulence, three-wave interactions are dominant, resulting in $n_k \sim \alpha^{1/2}$ scaling.
- Nonlinearity makes equipartitioned part of the spectrum wider.

Stabilized spectra: effect of domain size



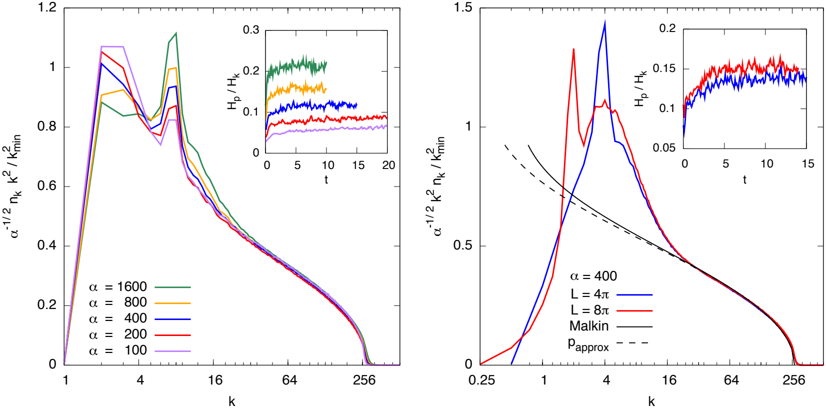
Can we extend the universal part of the spectrum by reducing $k_{\min}$?

- For given α, domain size does not affect k^{-2} part of the spectrum.
- Pushing $k_{\min} \to 0$ widens equipartitioned part, with $k_\mu = const$.
- Adjustment of friction does not extend universal part.
- Longer spectrum is expected for lower pumping rate α.

Stabilized Spectra of Direct Cascade

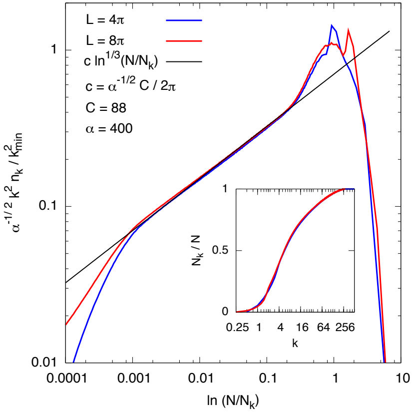
Comparison to Weakly-Nonlinear Theory

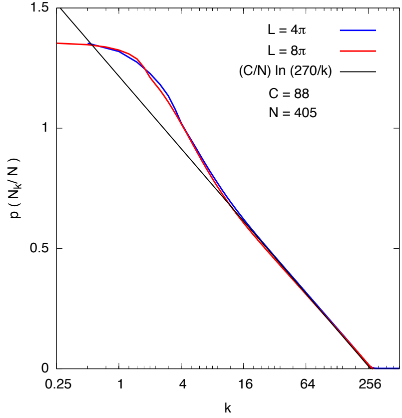
Direct cascade: compensated spectra



- Three-wave interactions are dominant, $n_k \sim \alpha^{1/2}$.
- Spectra at larger scales are distorted due to nonlinearity and sensitive to friction, γ.
- Spectra at small scales are universal and well-described by Malkin's theory (1996).

Comparison to weakly-nonlinear theory (Malkin, 1996)



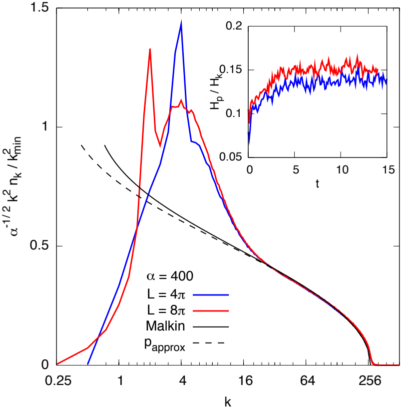


Implicit description in terms of the fraction of wave action contained within a sphere of radius k, N_k/N, and energy flux P,

$$\frac{n_k k^2}{k_{\min}^2} = \frac{C}{2\pi}\left[\ln\frac{N}{N_k}\right]^{\frac{1}{3}}, \qquad \frac{C}{N}\ln\frac{k_d}{k} = p\left(\frac{N_k}{N}\right).$$

Here, $p(m) = \int_m^1 \left[\ln y^{-1}\right]^{-\frac{1}{3}} dy$ and $C \propto P^{\frac{1}{3}}$. We show that $C \propto \alpha^{\frac{1}{2}}$.

Comparison to weakly-nonlinear theory (Malkin, 1996)



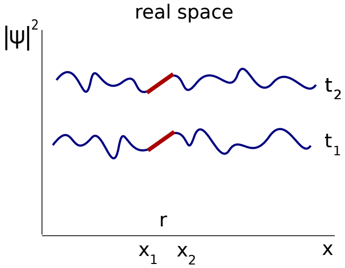
The parametric representation does not provide explicit expression for $n_k(k)$. Using approximation $p_{\text{approx}}(m) = \frac{3}{2}(1-m)^{\frac{2}{3}}$, we obtain

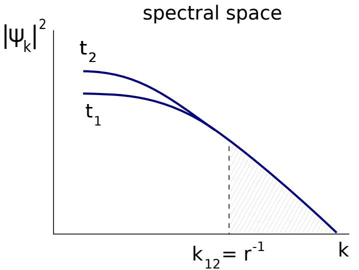
$$\frac{n_k k^2}{k_{\text{min}}^2} = \frac{C}{2\pi} \ln^{\frac{1}{3}} \left[1 - \left(\frac{2C}{3N} \ln \frac{k_d}{k} \right)^{\frac{3}{2}} \right].$$

Low pumping rates (smaller nonlinearity) might extent the range of appicability.

Fluxes of Wave Action and Energy

Flux of wave action





$N = \langle |\psi|^2 \rangle$ grows in time and long modes appear, but

$$\langle |\psi_1 - \psi_2|^2 \rangle = const$$

$$\langle |\psi_1 - \psi_2|^2 \rangle = \int |\psi_k|^2 (1 - \cos kr) dk$$

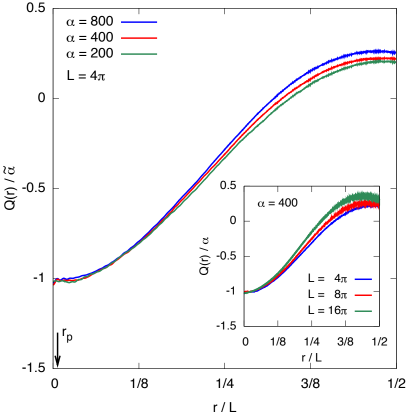
$$\langle |\psi_1 - \psi_2|^2 \rangle \sim \int_{1/r}^{\infty} |\psi_k|^2 dk = const$$

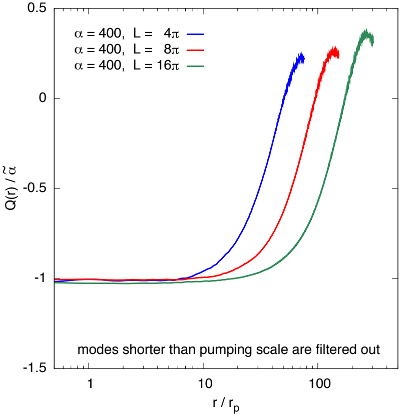
Take time derivative of $\langle |\psi_1 - \psi_2|^2 \rangle = 2N - \langle \psi_1 \psi_2^* + \psi_1^* \psi_2 \rangle$ to obtain,

$$Q(r) \equiv 2\,\mathrm{Im}\langle \psi_1^* |\psi_2|^2 \psi_2 \rangle = -\dot{N}$$

$Q(r)$ does not depend on distance between two points, r.
Analog of Kolmogorov's 4/5-law!

Flux of wave action in inverse cascade, $r \gg r_p$





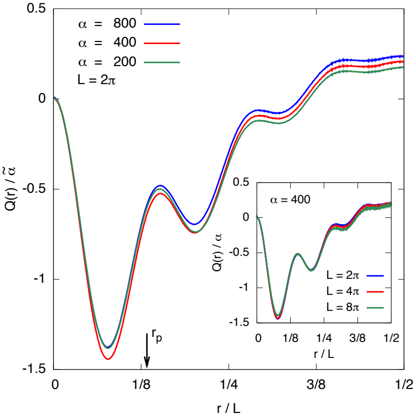
$$Q(r) \equiv 2\,\mathrm{Im}\langle \psi_1^* |\psi_2|^2 \psi_2 \rangle = -\dot{N}$$

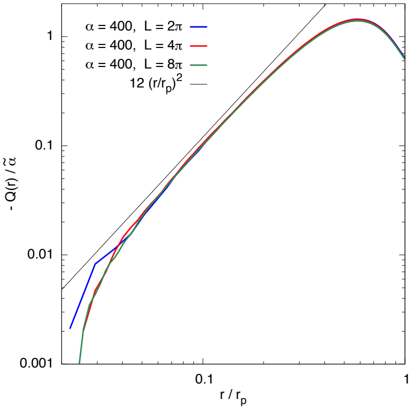
Simulations confirm:

$-Q(r) \propto \dot{N} = \tilde{\alpha} \approx 0.9\alpha$ for all scales.
$-Q(r) = \dot{N}$ for $r_p \lesssim r \lesssim L/16$.

$Q(r)$ is constant across the scales in inverse cascade.

Flux of energy in direct cascade, $r \ll r_p$





Simulations show:

$$-Q(r) \propto \dot{N} = \tilde{\alpha} \approx 0.9\alpha \quad \text{for all scales.}$$

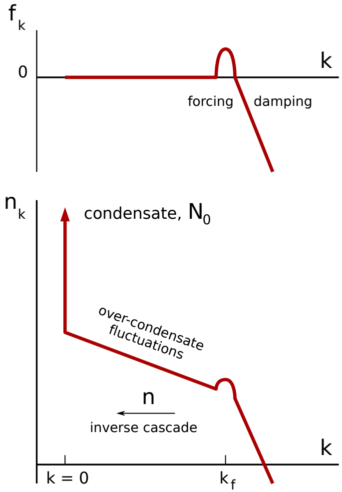
$$-Q''(r) = const, \quad \text{therefore} \;\; P \sim Qr^{-2} = const \;\; \text{for} \quad r \ll r_p.$$

$P(r)$ is constant across the scales in direct cascade.

NLS turbulence
after onset of the condensate

Defocusing nonlinear Schrödinger equation

$$\boxed{i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}\psi}$$



Condensate

$$\Psi = \sqrt{N_0}\exp(-iN_0 t)$$

Notation:

$$N = \overline{|\psi|^2}$$
$$N_0 = |\overline{\psi}|^2$$
$$n = N - N_0 = \int |\psi_k|^2 d^2k$$

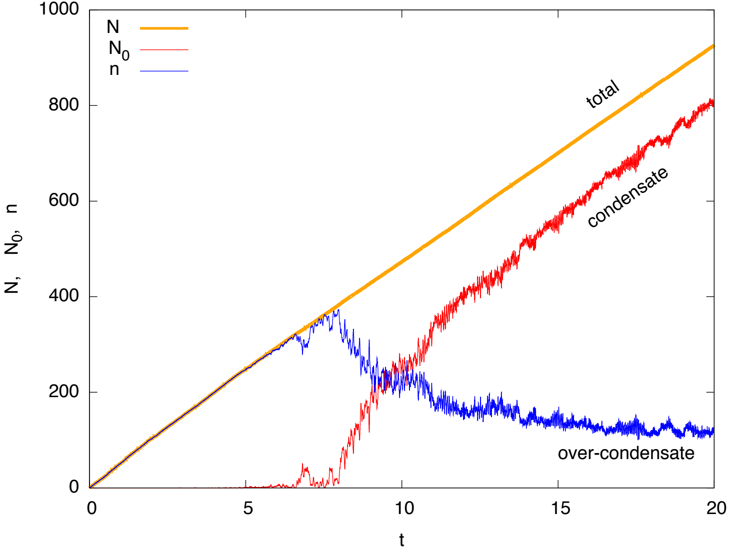
We consider large condensate

$$N_0 \gg n$$

Statistically quasi-steady

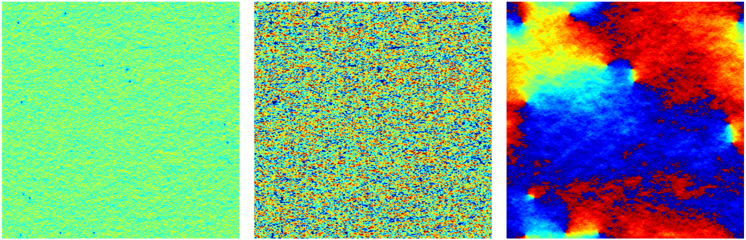
$$t \sim 10^4 \;\gg\; \frac{1}{\omega} \sim 10^{-3}$$

Onset of condensate

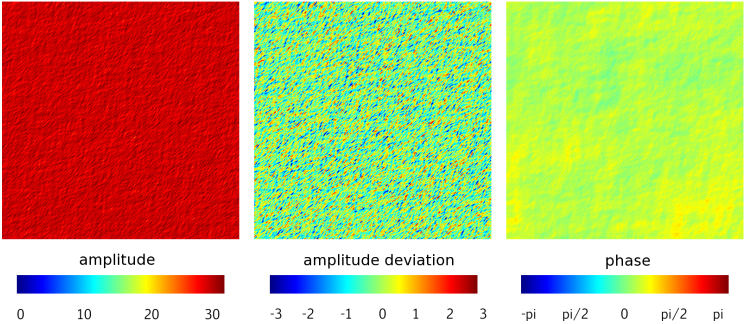


Onset of condensate

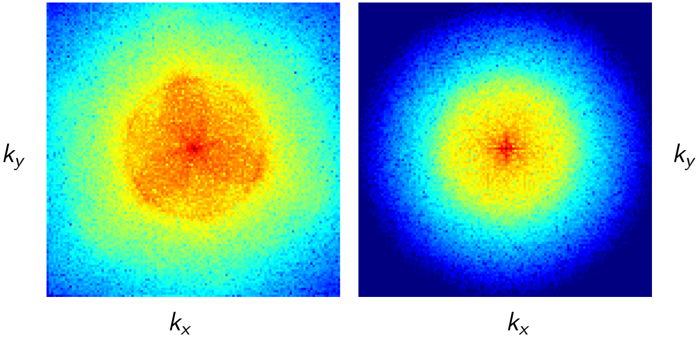
$t = 100:\quad N_0 = 58,\quad n = 160$



$t = 1500:\quad N_0 = 751,\quad n = 20$



Effect of forcing

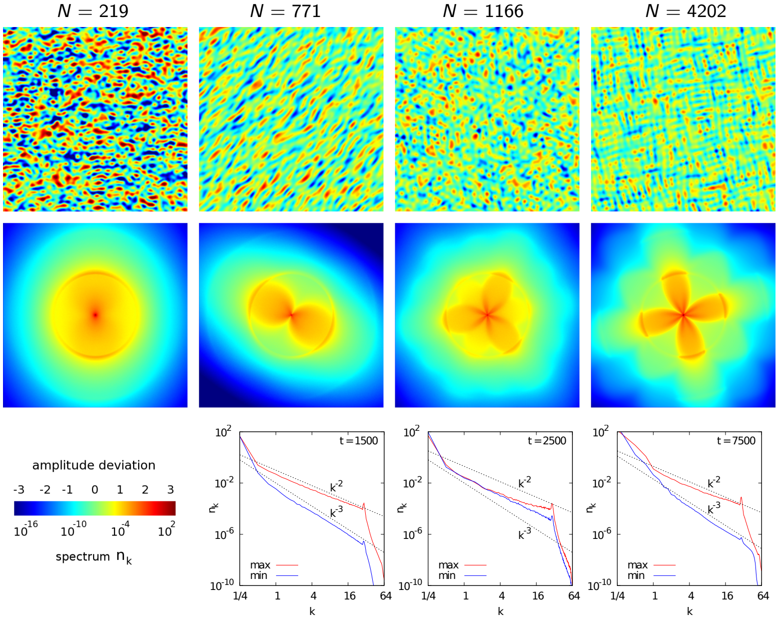


Instability-driven force

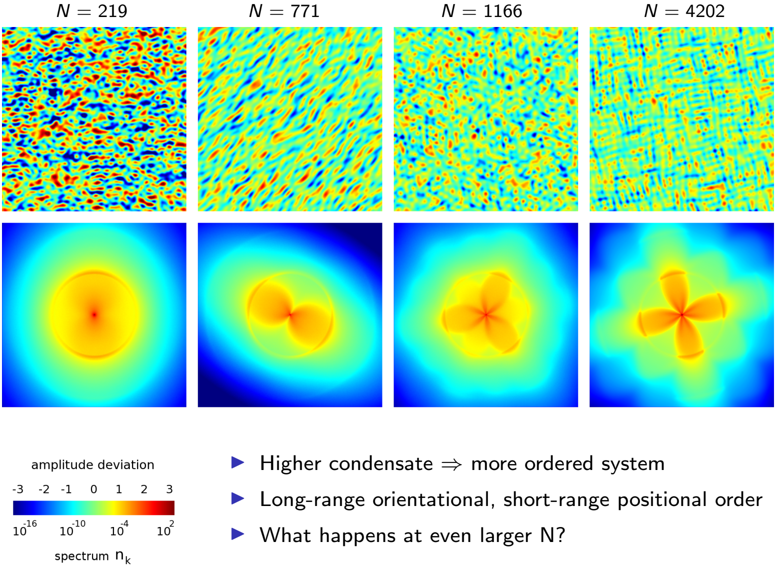
Random force

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}\psi \qquad i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{F}$$

Phase transitions: breakdown of symmetries



Phase transitions: breakdown of symmetries



- Higher condensate ⇒ more ordered system
- Long-range orientational, short-range positional order
- What happens at even larger N?

Vladimirova, Derevyanko, & Falkovich (2012)

Small perturbations

Compare quadratic and cubic terms in Hamiltonian

$$
\begin{aligned}
\langle \mathcal{H}_2 \rangle &= \Omega_k n = N_0^{1/2} k n \\
\langle \mathcal{H}_3 \rangle &= \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} V_{123} \langle \psi_{k_1} \psi_{k_2} \psi_{k_3}^* \rangle \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \\
&\simeq \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} |V_{123}|^2 n_1 n_2 \, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \delta(\Omega_1 + \Omega_2 - \Omega_3) \\
&\simeq \frac{|V|^2 n^2 c}{k^3} \frac{k}{c} \simeq \frac{n^2 k}{N_0^{1/2}}
\end{aligned}
$$

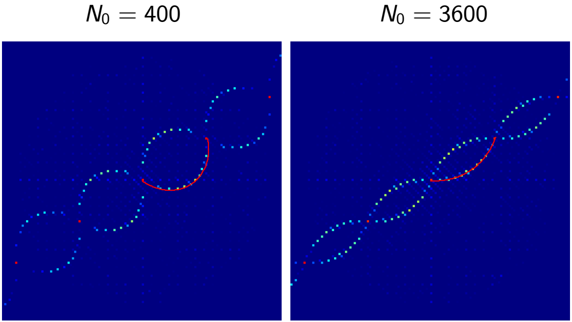
Effective nonlinearity parameter is small,

$$
\frac{\mathcal{H}_3}{\mathcal{H}_2} \simeq \frac{n}{N_0}.
$$

But: weak turbulence assumes random phases.

Angle of interaction: $k/c \sim k/\sqrt{N_0}$, where $c = \sqrt{2N_0}$.

Angle of interaction



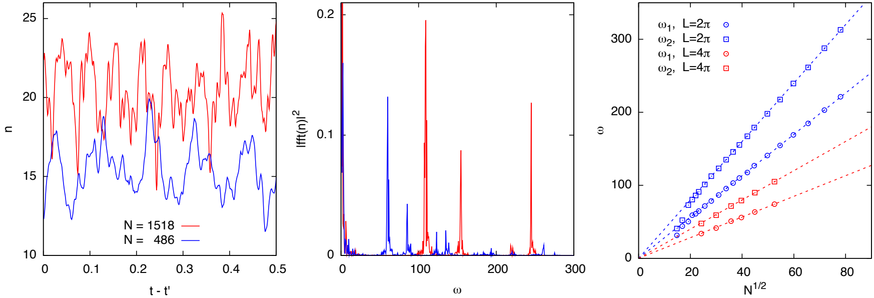
Arch grows in k-space from the condensate to a preset mode, $\mathbf{k}_0$.
Arch equation:

$$\begin{aligned} \omega(k_0) &= \omega(k) + \omega(|\mathbf{k}_0 - \mathbf{k}|) \\ \omega^2(k) &= 2N_0 k^2 + k^4 \end{aligned}$$

Angle of interaction: $\phi_{max} \approx \dfrac{k}{\sqrt{3N_0/2}} \sim \dfrac{k}{c}$

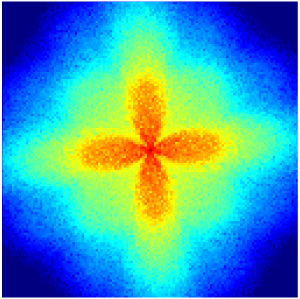
Condensate-turbulence oscillations

- The system periodicly oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- Predator-prey model?

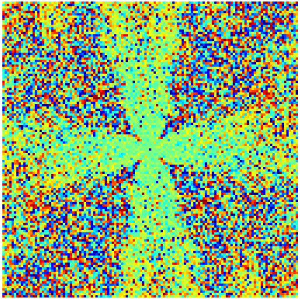


Phase coherence

n_k



$\theta_k = 2\phi_0 - \phi_k - \phi_{-k}$



$$\boxed{2\phi_0 - \phi_k - \phi_{-k} = \pi}$$

Three-wave model

Consider condensate interacting with two waves

$$\psi_{\pm k} = \sqrt{n}\exp(\pm ikx + iN_0 t + i\phi_{\pm k})$$

with $\theta = 2\phi_0 - \phi_k - \phi_{-k}$.

Hamiltonian:

$$H = 2k^2 n + \frac{1}{2}N^2 + 2n(N-2n)(1+cos\theta) + n^2$$

Equations of motion:

$$\begin{aligned} \dot{n} &= 2n(N-2n)\sin\theta \\ \dot{\theta} &= 2k^2 + 2(N-3n) + 2(N-4n)\cos\theta \end{aligned}$$

Stability points:

$\theta = \pi, \quad n = -\frac{1}{2}k^2 \quad \Rightarrow \quad$ unphysical

$\theta = 0, \quad n = (4N + k^2)/14 \quad \Rightarrow \quad$ too high n

---

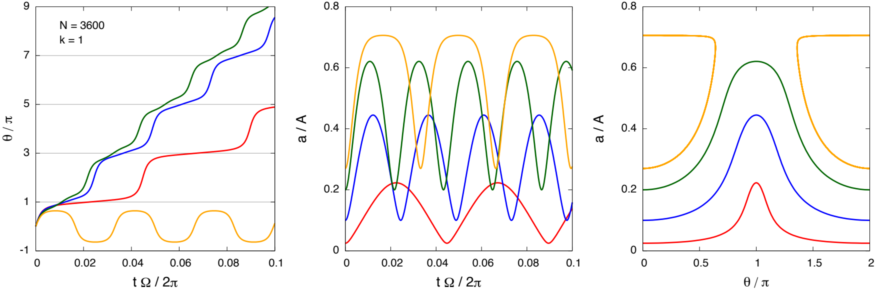
Falkovich (2011), Miller, Vladimirova & Falkovich (2013)

Predictions of three-wave model

$$\begin{aligned}\dot{n} &= 2n(N-2n)\sin\theta \\ \dot{\theta} &= 2k^2 + 2(N-3n) + 2(N-4n)\cos\theta\end{aligned}$$

For $n \ll N$:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations $2\Omega \approx 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a \equiv \sqrt{n(t)}$ exhibits complicated cusped shape



Individual modes in turbulence

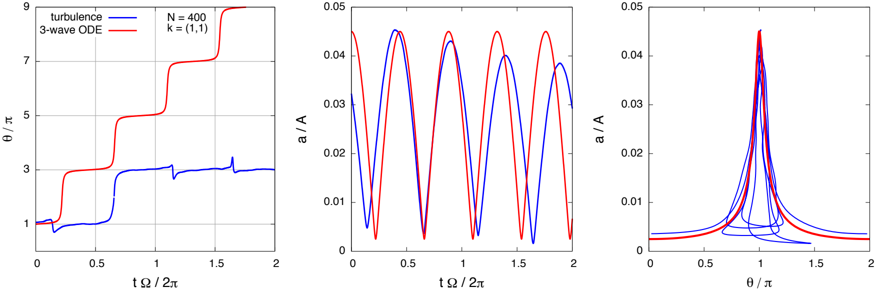
In turbulence, $n \ll N$ condition is well satisfied.

As predicted:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations approaches $2\Omega = 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a \equiv \sqrt{n(t)}$ exhibits complicated cusped shape

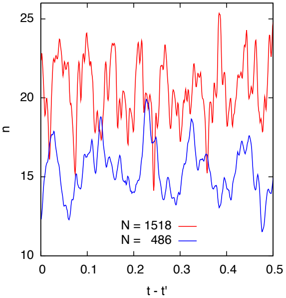
However:

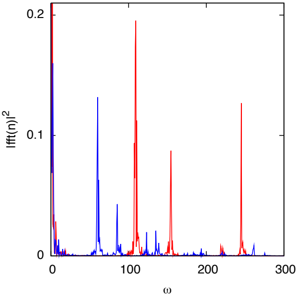
The 3-wave model cannot grasp closed trajectories with $\theta \approx \pi$.

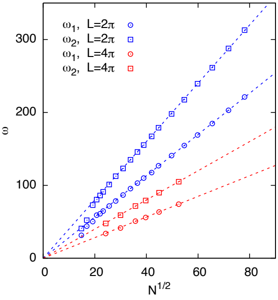


Collective oscillations

- The system periodically oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- The condensate imposes the phase coherence between the pairs of counter-propagating waves (anomalous correlation).
- Collective oscillations are not of a predator-prey type; they are due to phase coherence and anomalous correlations.







Conclusions - I

- When the driving term corresponds to an instability (but not a random force) high levels of condensate lead to a phase transitions — spontaneous breakdown of symmetries of small-scale over-condensate fluctuations: from the 2-fold to 3-fold to 4-fold.
- Collective oscillations are not of a predator-prey type; they are due to phase coherence, imposed by condensate, and anomalous correlations.

N. Vladimirova, S. Derevyanko S, G. Falkovich, Phys. Rev. E (2012)
P. Miller, N. Vladimirova, G. Falkovich, Phys. Rev. E (2013)

Conclusions - II

- Wave spectra (second-order moments) are close to slightly (logarithmically) distorted thermal equilibrium in both cascades.
- Correction by Dyachenko, Newell, Pushkarev, Zakharov (1992) for inverse cascade spectra works for intermediate k.
- Correction by Malkin (1996) for direct cascade spectra works.
- Analog of Kolmogorov's 4/5 law:

$$Q(r) \equiv 2\,\mathrm{Im}\langle \psi_1^* |\psi_2|^2 \psi_2 \rangle = -\dot{N} \qquad \text{for} \quad r > r_p;$$

the flux of wave action is independent of scale in inverse cascade, while the flux of energy is independent of scale in direct cascade.