

Extreme Vortex States and the Hydrodynamic Blow-Up Problem

(Probing Fundamental Bounds in Hydrodynamics Using
Variational Optimization Methods)

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Agenda

Sharpness of Estimates as Optimization Problem

- Regularity Problem for Navier-Stokes Equation
- Research Program and Earlier Results
- Finite-Time Bounds in 1D Burgers Problem

Bounds for 2D Navier-Stokes Problem

- Bounds on Palinstrophy Growth
- Optimization Problems
- Computational Approach & Results

Bounds for 3D Navier-Stokes Problem

- Bounds on Enstrophy Growth & Optimization Problems
- Extreme Vortex States
- Discussion

- ▶ Navier-Stokes equation ($\Omega = [0, L]^d$, $d = 2, 3$)

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p - \nu \Delta \mathbf{v} = \mathbf{0}, & \text{in } \Omega \times (0, T] \\ \nabla \cdot \mathbf{v} = 0, & \text{in } \Omega \times (0, T] \\ \mathbf{v} = \mathbf{v}_0 & \text{in } \Omega \text{ at } t = 0 \\ \text{Boundary Condition} & \text{on } \Gamma \times (0, T] \end{cases}$$

- ▶ 2D Case

- ▶ Existence Theory Complete — smooth and unique solutions exist for arbitrary times and arbitrarily large data

- ▶ 3D Case

- ▶ Weak solutions (possibly nonsmooth) exist for arbitrary times
- ▶ Classical (smooth) solutions (possibly nonsmooth) exist for *finite* times only
- ▶ Possibility of “blow-up” (finite-time singularity formation)
- ▶ One of the Clay Institute “Millennium Problems” (\$ 1M!)
http://www.claymath.org/millennium/Navier-Stokes_Equations

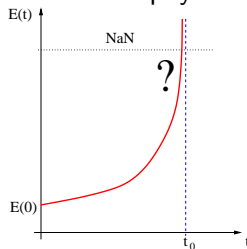
What is known? — Available Estimates

- ▶ A Key Quantity — Enstrophy

$$\mathcal{E}(t) \triangleq \int_{\Omega} |\nabla \times \mathbf{v}|^2 d\Omega \quad (= \|\nabla \mathbf{v}\|_2^2)$$

- ▶ Smoothness of Solutions \iff Bounded Enstrophy
(Foias & Temam, 1989)

$$\max_{t \in [0, T]} \mathcal{E}(t) < \infty \quad ???$$



- ▶ Can estimate $\frac{d\mathcal{E}(t)}{dt}$ using the momentum equation, Sobolev's embeddings, Young and Cauchy-Schwartz inequalities, ...
 - ▶ REMARK: incompressibility not used in these estimates

► 2D Case:

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{C^2}{\nu} \mathcal{E}(t)^2$$

- Gronwall's lemma and energy equation yield $\forall_t \mathcal{E}(t) < \infty$
- smooth solutions exist for all times

► 3D Case:

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3$$

- corresponding estimate not available
- upper bound on $\mathcal{E}(t)$ blows up in finite time

$$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$$

- singularity in finite time cannot be ruled out!

Problem of Lu & Doering (2008), I

- ▶ Can we actually find solutions which “saturate” a given estimate?
- ▶ Estimate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3$ at a *fixed* instant of time t

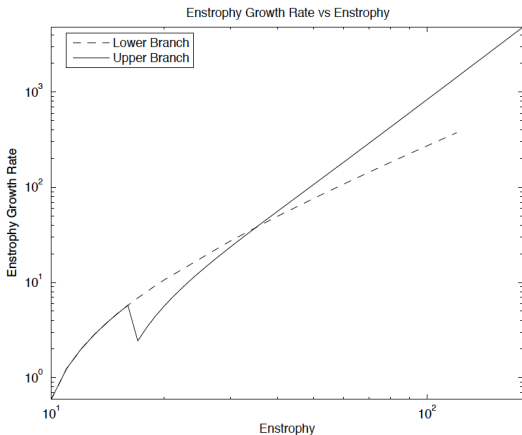
$$\max_{\mathbf{v} \in H^1(\Omega), \nabla \cdot \mathbf{v} = 0} \frac{d\mathcal{E}(t)}{dt}$$

subject to $\mathcal{E}(t) = \mathcal{E}_0$

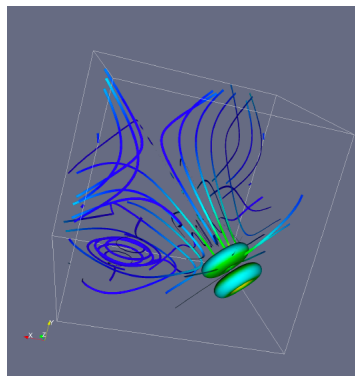
where

- ▶
$$\frac{d\mathcal{E}(t)}{dt} = -\nu \|\Delta \mathbf{v}\|_2^2 + \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} \, d\Omega$$
- ▶ \mathcal{E}_0 is a parameter
- ▶ Solution using a gradient-based descent method

Problem of Lu & Doering (2008), II



$$\left[\frac{d\mathcal{E}(t)}{dt} \right]_{max} = 8.97 \times 10^{-4} \mathcal{E}_0^{2.997}$$



vorticity field (top branch)

- ▶ How about solutions which saturate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3$ over a *finite* time window $[0, T]$?

$$\max_{\mathbf{v}_0 \in H^1(\Omega), \nabla \cdot \mathbf{v} = 0} \left[\max_{t \in [0, T]} \mathcal{E}(t) \right]$$

subject to $\mathcal{E}(t) = \mathcal{E}_0$

where

- ▶
$$\mathcal{E}(t) = \int_0^t \frac{d\mathcal{E}(\tau)}{d\tau} d\tau + \mathcal{E}_0$$
- ▶ \mathcal{E}_0 is a parameter
- ▶ $\max_{t \in [0, T]} \mathcal{E}(t)$ nondifferentiable w.r.t initial condition
 \implies non-smooth optimization problem
- ▶ In principle doable, but will try something simpler first ...

Relevant Estimates

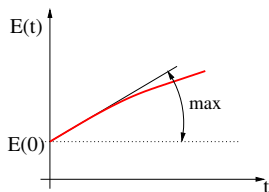
	BEST ESTIMATE	SHARP?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left(\frac{1}{\pi^2\nu}\right)^{1/3} \mathcal{E}(t)^{5/3}$	
1D Burgers finite-time	$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{t}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} \mathcal{E}_0 \right]^3$	
2D Navier-Stokes instantaneous	$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &\leq -\left(\frac{\nu}{\mathcal{E}}\right) \mathcal{P}^2 + C_1 \left(\frac{\mathcal{E}}{\nu}\right) \mathcal{P} \\ \frac{d\mathcal{P}(t)}{dt} &\leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2} \end{aligned}$	
2D Navier-Stokes finite-time	$\max_{t > 0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$	
3D Navier-Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3$	YES Lu & Doering (2008)
3D Navier-Stokes finite-time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4 \frac{C\mathcal{E}(0)^2}{\nu^3} t}}$	

► QUESTION #1 (“SMALL”)

Sharpness of *instantaneous* estimates
 (at some *fixed* t)

$$\max_{\mathbf{u}} \frac{d\mathcal{E}}{dt} \quad (1D, 3D)$$

$$\max_{\mathbf{u}} \frac{d\mathcal{P}}{dt} \quad (2D)$$

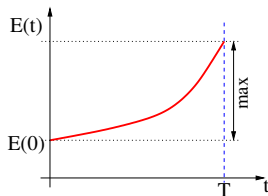


► QUESTION #2 (“BIG”)

Sharpness of *finite-time* estimates
 (at some time window $[0, T]$, $T > 0$)

$$\max_{\mathbf{u}_0} \left[\max_{t \in [0, T]} \mathcal{E}(t) \right] \quad (1D, 3D)$$

$$\max_{\mathbf{u}_0} \left[\max_{t \in [0, T]} \mathcal{P}(t) \right] \quad (2D)$$



PROBLEM I

INSTANTANEOUS AND FINITE-TIME BOUNDS FOR GROWTH OF ENSTROPY IN 1D BURGERS PROBLEM

- ▶ Burgers equation ($\Omega = [0, 1]$, $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega$$

$$u(x) = \phi(x) \quad \text{at } t = 0$$

Periodic B.C.

Enstrophy : $\mathcal{E}(t) = \frac{1}{2} \int_0^1 |u_x(x, t)|^2 dx$

- ▶ Solutions smooth for all times
- ▶ Questions of sharpness of enstrophy estimates still relevant

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left(\frac{1}{\pi^2 \nu} \right)^{1/3} \mathcal{E}(t)^{5/3}$$

- ▶ Best available finite-time estimate

$$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4} \right)^2 \left(\frac{1}{\pi^2 \nu} \right)^{4/3} \mathcal{E}_0 \right]^3 \xrightarrow{\mathcal{E}_0 \rightarrow \infty} C_2 \mathcal{E}_0^3$$

- ▶ Finite-time Estimates — a different approach without explicit time-integration of instantaneous estimates

Spectral Properties of Solutions of Burgers Equation with Small Dissipation

Andrei Birjuk

Functional Analysis and Its Applications. Vol. **35.**, no 1., 2001.

1 Introduction

This present paper concerns the initial value problem for the one dimensional ($\dim x = 1$) parabolic equation of Burgers type:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = \delta u_{xx}, \quad (1.1)$$

with the initial state

$$u(0, x) = u_0(x), \quad (1.2)$$

- ▶ Still unclear whether the resulting finite-time estimate is (much) sharper ...

“Small” Problem of Lu & Doering (2008), I

- ▶ Estimate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^{5/3}$ at a *fixed* instant of time t

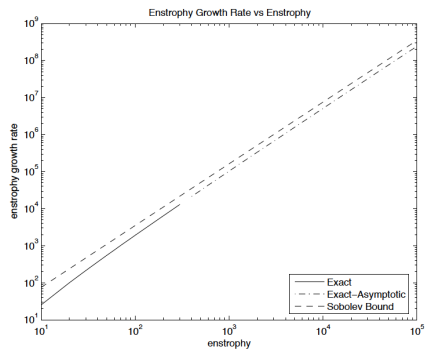
$$\max_{u \in H^1(\Omega)} \frac{d\mathcal{E}(t)}{dt}$$

subject to $\mathcal{E}(t) = \mathcal{E}_0$

where

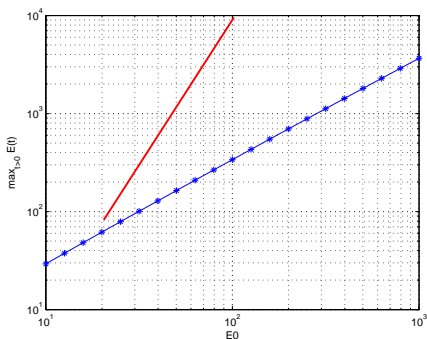
- ▶
$$\frac{d\mathcal{E}(t)}{dt} = -\nu \left\| \frac{\partial^2 u}{\partial x^2} \right\|_2^2 + \frac{1}{2} \int_0^1 \left(\frac{\partial u}{\partial x} \right)^3 d\Omega$$
- ▶ \mathcal{E}_0 is a parameter
- ▶ Solution (maximizing field) found analytically!
(in terms of elliptic integrals and Jacobi elliptic functions)

“Small” Problem of Lu & Doering (2008), II



$$\left[\frac{d\mathcal{E}(t)}{dt} \right]_{\max} = 0.2476 \frac{\mathcal{E}_0^{5/3}}{\nu^{1/3}}$$

instantaneous estimate is sharp



— $\max_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{1.048}$

— finite-time estimate
 (far from saturated)

Finite-Time Optimization Problem (I)

► Statement

$$\max_{\phi \in H^1(\Omega)} \mathcal{E}(T)$$

subject to $\mathcal{E}(t) = \mathcal{E}_0$

T, \mathcal{E}_0 — parameters

► Optimality Condition

$$\forall_{\phi' \in H^1} \quad \mathcal{J}'_{\lambda}(\phi; \phi') = - \int_0^1 \frac{\partial^2 u}{\partial x^2} \Big|_{t=T} u' \Big|_{t=T} dx - \lambda \int_0^1 \frac{\partial^2 \phi}{\partial x^2} \Big|_{t=0} u' \Big|_{t=0} dx$$

Finite-Time Optimization Problem (II)

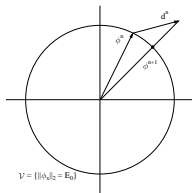
► Gradient Descent

$$\begin{aligned}\phi^{(n+1)} &= \phi^{(n)} - \tau^{(n)} \nabla \mathcal{J}(\phi^{(n)}), & n = 1, \dots, \\ \phi^{(0)} &= \phi_0,\end{aligned}$$

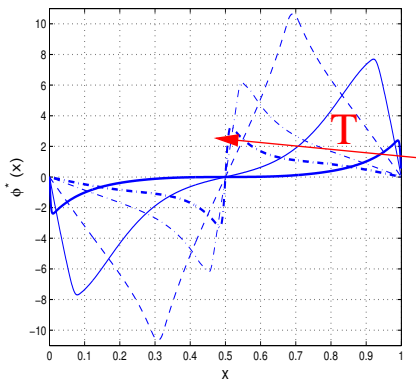
where $\nabla \mathcal{J}$ determined from *adjoint system* via H^1 Sobolev preconditioning

$$\begin{aligned}-\frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - \nu \frac{\partial^2 u^*}{\partial x^2} &= 0 & \text{in } \Omega \\ u^*(x) &= -\frac{\partial^2 u}{\partial x^2}(x) & \text{at } t = T \\ & \text{Periodic B.C.}\end{aligned}$$

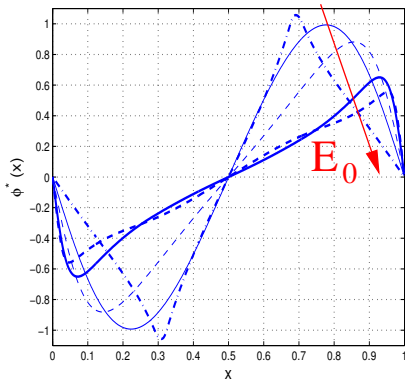
► Step size $\tau^{(n)}$ found via *arc minimization*



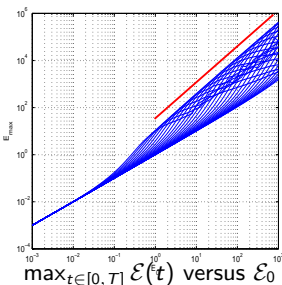
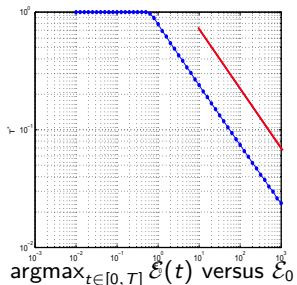
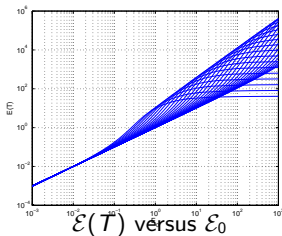
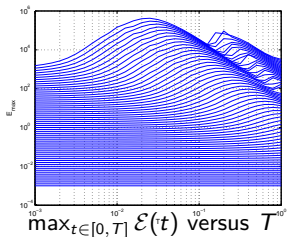
- ▶ Two parameters: T, \mathcal{E}_0 ($\nu = 10^{-3}$)
- ▶ Optimal initial conditions corresponding to initial guess with wavenumber $m = 1$ (local maximizers)



Fixed $\mathcal{E}_0 = 10^3$, different T



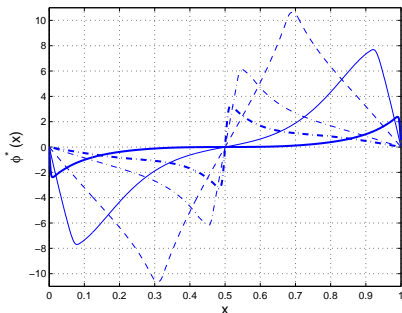
Fixed $T = 0.0316$, different \mathcal{E}_0



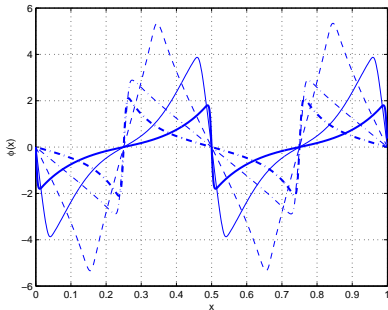
$$\operatorname{argmax}_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{-0.5}$$

$$\max_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{1.5}$$

- Sol'ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi mx)$, $m = 1, 2, \dots$



$$m = 1, \epsilon_0 = 10^3$$



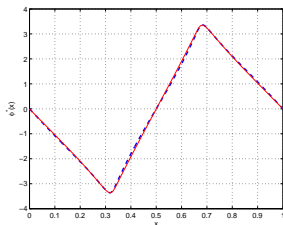
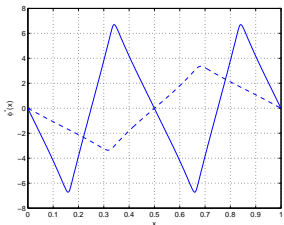
$$m = 2, \epsilon_0 = 10^3$$

- Change of variables leaving Burgers equation invariant ($L \in \mathbb{Z}^+$):

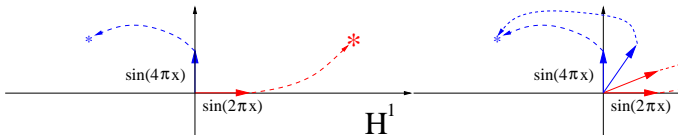
$$x = L\xi, \quad (x \in [0, 1], \xi \in [0, 1/L]), \quad \tau = t/L^2$$

$$v(\tau, \xi) = Lu(x(\xi), t(\tau)), \quad \mathcal{E}_v(\tau) = L^4 \mathcal{E}_u \left(\frac{t}{L^2} \right)$$

- Solutions for $m = 1$ and $m = 2$, after rescaling



- Using initial guess: $\phi^{(0)}(x) = \sin(2\pi m x)$, $m = 1$, or $m = 2$
 $\phi^{(0)}(x) = \epsilon \sin(2\pi m x) + (1 - \epsilon) \sin(2\pi n x)$, $m \neq n$, $\epsilon > 0$



- All local maximizers with $m = 2, 3, \dots$ are *rescaled copies* of the $m = 1$ maximizer

Relevant Estimates

	BEST ESTIMATE	SHARP?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left(\frac{1}{\pi^2\nu}\right)^{1/3} \mathcal{E}(t)^{5/3}$	YES Lu & Doering (2008)
1D Burgers finite-time	$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{t}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} \mathcal{E}_0 \right]^3$	NO Ayala & P. (2011)
2D Navier-Stokes instantaneous	$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &\leq -\left(\frac{\nu}{\mathcal{E}}\right) \mathcal{P}^2 + C_1 \left(\frac{\mathcal{E}}{\nu}\right) \mathcal{P} \\ \frac{d\mathcal{P}(t)}{dt} &\leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2} \end{aligned}$	
2D Navier-Stokes finite-time	$\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$	
3D Navier-Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3$	YES Lu & Doering (2008)
3D Navier-Stokes finite-time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4 \frac{C\mathcal{E}(0)^2}{\nu^3} t}}$	

PROBLEM II

INSTANTANEOUS BOUNDS FOR GROWTH OF PALINSTROPHY IN 2D NAVIER-STOKES PROBLEM

Relevant Estimates

	BEST ESTIMATE	SHARP?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left(\frac{1}{\pi^2\nu}\right)^{1/3} \mathcal{E}(t)^{5/3}$	YES Lu & Doering (2008)
1D Burgers finite-time	$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{t}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} \mathcal{E}_0 \right]^3$	NO Ayala & P. (2011)
2D Navier-Stokes instantaneous	$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &\leq -\left(\frac{\nu}{\mathcal{E}}\right) \mathcal{P}^2 + C_1 \left(\frac{\mathcal{E}}{\nu}\right) \mathcal{P} \\ \frac{d\mathcal{P}(t)}{dt} &\leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2} \end{aligned}$	
2D Navier-Stokes finite-time	$\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$	
3D Navier-Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3$	YES Lu & Doering (2008)
3D Navier-Stokes finite-time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4 \frac{C\mathcal{E}(0)^2}{\nu^3} t}}$	

- 2D VORTICITY EQUATION IN A PERIODIC BOX ($\omega = \mathbf{e}_z \cdot \boldsymbol{\omega}$)

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \Delta \omega \quad \text{where } J(f, g) = f_x g_y - f_y g_x$$

$$- \Delta \psi = \omega$$

- Enstrophy uninteresting in 2D flows (w/o boundaries)

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \omega^2 d\Omega = -\nu \int_{\Omega} (\nabla \omega)^2 d\Omega < 0$$

- Evolution equation for the vorticity gradient $\nabla \omega$

$$\frac{\partial \nabla \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \nabla \omega = \nu \Delta \nabla \omega + \underbrace{\nabla \omega \cdot \nabla \mathbf{u}}_{\text{"STRETCHING" TERM}}$$

► Palinstrophy

$$\mathcal{P}(t) \triangleq \int_{\Omega} (\nabla \omega(t, \mathbf{x}))^2 d\Omega = \int_{\Omega} (\nabla \Delta \psi(t, \mathbf{x}))^2 d\Omega$$

► Also of interest — Kinetic Energy

$$\mathcal{K}(t) \triangleq \int_{\Omega} \mathbf{u}(t, \mathbf{x})^2 d\Omega = \int_{\Omega} (\nabla \psi(t, \mathbf{x}))^2 d\Omega$$

► Poincaré's inequality

$$\mathcal{K} \leq (2\pi)^{-2} \mathcal{E} \leq (2\pi)^{-2} \mathcal{P}$$

- ▶ Estimates for the Rate of Growth of Palinstrophy

$$\frac{d\mathcal{P}(t)}{dt} = \int_{\Omega} J(\Delta\psi, \psi)\Delta^2\psi \, d\Omega - \nu \int_{\Omega} (\Delta^2\psi)^2 \, d\Omega \triangleq \mathcal{R}_{\mathcal{P}}(\psi)$$

$$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}}\right) \mathcal{P}^2 + C_1 \left(\frac{\mathcal{E}}{\nu}\right) \mathcal{P} \quad (\text{Doering \& Lunasin, 2011})$$

$$\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2} \quad (\text{Ayala, 2012})$$

- ▶ Using Poincaré's inequality (may not be sharp)

$$\frac{d\mathcal{P}(t)}{dt} \leq \frac{C}{\nu} \mathcal{P}^2,$$

- ▶ Bound on growth in finite time

$$\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2 \quad (\text{Ayala, 2012})$$

- ▶ Maximum Growth of $\frac{d\mathcal{P}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0, \mathcal{P}_0 > (2\pi)^2 \mathcal{E}_0$

$$\max_{\psi \in \mathcal{S}_{\mathcal{P}_0, \mathcal{E}_0}} \mathcal{R}_{\mathcal{P}_0}(\psi) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{P}_0, \mathcal{E}_0} = \left\{ \psi \in H^4(\Omega) : \begin{array}{l} \frac{1}{2} \int_{\Omega} (\nabla \Delta \psi)^2 d\Omega = \mathcal{P}_0 \\ \frac{1}{2} \int_{\Omega} (\Delta \psi)^2 d\Omega = \mathcal{E}_0 \end{array} \right\}$$

- ▶ Maximum Growth of $\frac{d\mathcal{P}(t)}{dt}$ for fixed $\mathcal{K}_0 > 0, \mathcal{P}_0 > (2\pi)^4 \mathcal{K}_0$

$$\max_{\psi \in \mathcal{S}_{\mathcal{P}_0, \mathcal{K}_0}} \mathcal{R}_{\mathcal{P}_0}(\psi) \quad \text{where}$$

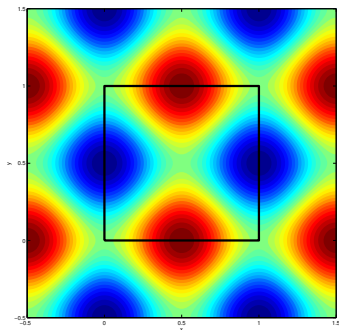
$$\mathcal{S}_{\mathcal{P}_0, \mathcal{K}_0} = \left\{ \psi \in H^4(\Omega) : \begin{array}{l} \frac{1}{2} \int_{\Omega} (\nabla \Delta \psi)^2 d\Omega = \mathcal{P}_0 \\ \frac{1}{2} \int_{\Omega} (\nabla \psi)^2 d\Omega = \mathcal{K}_0 \end{array} \right\}$$

- Small Palinstrophy Limit: $\mathcal{P}_0 \rightarrow (2\pi)^2 \mathcal{E}_0$

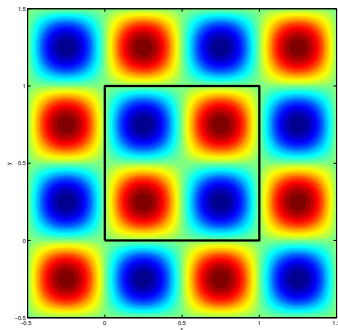
$$\tilde{\varphi}_0 = \arg \max_{\varphi \in \mathcal{S}_0} \mathcal{R}_0(\varphi), \quad \mathcal{R}_0(\varphi) = -\nu \int_{\Omega} (\Delta^2 \varphi)^2 d\Omega,$$

$$\mathcal{S}_0 = \left\{ \varphi \in H^4(\Omega) : \frac{1}{2} \int_{\Omega} (\nabla \Delta \psi)^2 d\Omega = \frac{(2\pi)^2}{2} \int_{\Omega} (\Delta \psi)^2 d\Omega \right\}$$

- Optimizers: **Eigenfunctions of the Laplacian** ($\tilde{\varphi}_0 \in \text{Ker}(\Delta)$)



$$\varphi(x, y) = \sin(\pi(y-x)) \sin(\pi(y+x))$$



$$\varphi(x, y) = \sin(2\pi x) \sin(2\pi y)$$

Numerical Solution of Maximization Problem

- ▶ Discretization of Gradient Flow

$$\begin{aligned} \frac{d\psi}{d\tau} &= -\nabla^{H^4} \mathcal{R}_\nu(\psi), & \psi(0) &= \psi_0 \\ \psi^{(n+1)} &= \psi^{(n)} - \Delta\tau^{(n)} \nabla^{H^4} \mathcal{R}_\nu(\psi^{(n)}), & \psi^{(0)} &= \psi_0 \end{aligned}$$

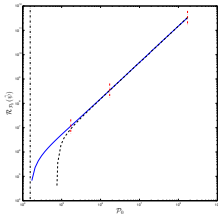
- ▶ Gradient in $H^4(\Omega)$ (via variational techniques)

$$\begin{aligned} [\text{Id} - L^8 \Delta^4] \nabla^{H^4} \mathcal{R}_\nu &= \nabla^{L^2} \mathcal{R}_\nu && \text{(Periodic BCs)} \\ \nabla^{L^2} \mathcal{R}_\nu(\psi) &= \Delta^2 J(\Delta\psi, \psi) + \Delta J(\psi, \Delta^2\psi) + J(\Delta^2\psi, \Delta\psi) - 2\nu\Delta^4\psi \end{aligned}$$

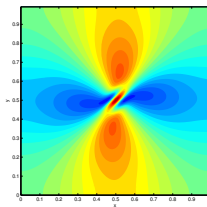
- ▶ Constraint satisfaction via arc minimization

Maximizers with Fixed $(\mathcal{K}_0, \mathcal{P}_0)$

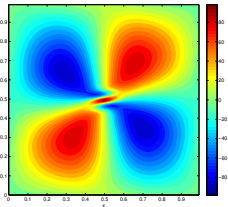
Estimate: $\frac{d\mathcal{P}(t)}{dt} \leq \frac{c_2}{\nu} \mathcal{K}_0^{1/2} \mathcal{P}_0^{3/2}$



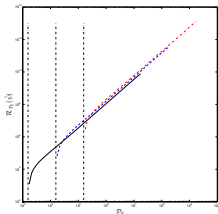
$\max \frac{d\mathcal{P}}{dt}$ vs. \mathcal{P}_0 , $\mathcal{K}_0 = 10$



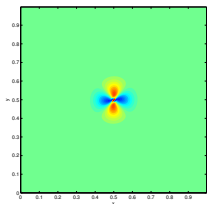
(a) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



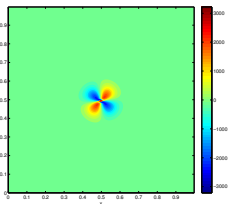
(b) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



$\max \frac{d\mathcal{P}}{dt} \sim \mathcal{P}_0^{3/2}$ as $\mathcal{P}_0 \rightarrow \infty$



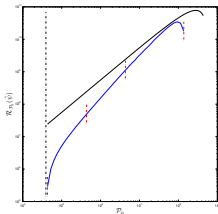
(c) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$



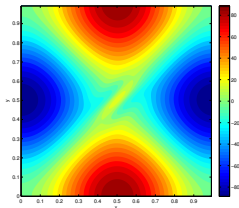
(d) $\mathcal{P}_0 \approx 10^4\mathcal{P}_c$

Maximizers with Fixed $(\mathcal{E}_0, \mathcal{P}_0)$

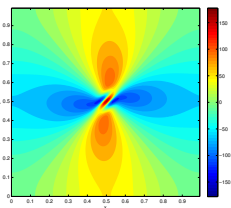
Estimate:
$$\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}_0}\right) \mathcal{P}_0^2 + C_1 \left(\frac{\mathcal{E}_0}{\nu}\right) \mathcal{P}_0$$



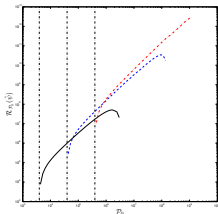
$\max \frac{d\mathcal{P}}{dt}$ vs. \mathcal{P}_0 , $\mathcal{E}_0 = 10^3$



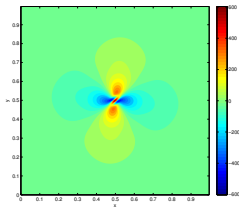
(a) $\mathcal{P}_0 \approx \mathcal{P}_c$



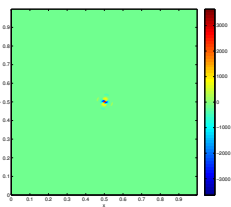
(b) $\mathcal{P}_0 \approx 10\mathcal{P}_c$



$\max \frac{d\mathcal{P}}{dt}$ vs. \mathcal{P}_0



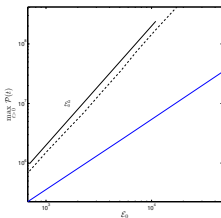
(c) $\mathcal{P}_0 \approx 10^2\mathcal{P}_c$



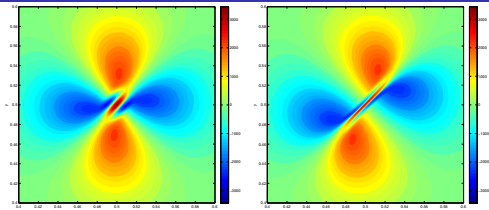
(d) $\mathcal{P}_0 \approx 10^{7/2}\mathcal{P}_c$

Maximizers with Fixed $(\mathcal{K}_0, \mathcal{P}_0)$

Finite-Time Estimate: $\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$

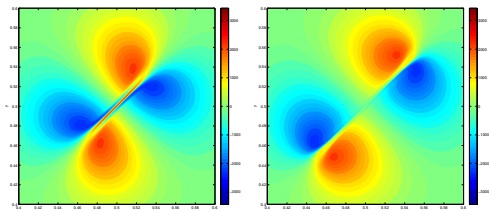


— \mathcal{P}_0 -constraint
 - - - $\{\mathcal{K}_0, \mathcal{P}_0\}$ -constraint



(a) $t = 0.000213$

(b) $t = 0.000458$



(c) $t = 0.000633$

(d) $t = 0.001265$

Relevant Estimates

	BEST ESTIMATE	SHARP?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left(\frac{1}{\pi^2\nu}\right)^{1/3} \mathcal{E}(t)^{5/3}$	YES Lu & Doering (2008)
1D Burgers finite-time	$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} \mathcal{E}_0 \right]^3$	No Ayala & P. (2011)
2D Navier-Stokes instantaneous	$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &\leq -\left(\frac{\nu}{\mathcal{E}}\right) \mathcal{P}^2 + C_1 \left(\frac{\mathcal{E}}{\nu}\right) \mathcal{P} \\ \frac{d\mathcal{P}(t)}{dt} &\leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2} \end{aligned}$	[YES] Ayala & P. (2013)
2D Navier-Stokes finite-time	$\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$	[YES] Ayala & P. (2013)
3D Navier-Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3$	YES Lu & Doering (2008)
3D Navier-Stokes finite-time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$???

PROBLEM III

INSTANTANEOUS BOUNDS FOR GROWTH OF ENSTROPY IN 3D NAVIER-STOKES PROBLEM (PRELIMINARY RESULTS)

Relevant Estimates

	BEST ESTIMATE	SHARP?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left(\frac{1}{\pi^2\nu}\right)^{1/3} \mathcal{E}(t)^{5/3}$	YES Lu & Doering (2008)
1D Burgers finite-time	$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2 \left(\frac{1}{\pi^2\nu}\right)^{4/3} \mathcal{E}_0 \right]^3$	No Ayala & P. (2011)
2D Navier-Stokes instantaneous	$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &\leq -\left(\frac{\nu}{\mathcal{E}}\right) \mathcal{P}^2 + C_1 \left(\frac{\mathcal{E}}{\nu}\right) \mathcal{P} \\ \frac{d\mathcal{P}(t)}{dt} &\leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2} \end{aligned}$	[YES] Ayala & P. (2013)
2D Navier-Stokes finite-time	$\max_{t>0} \mathcal{P}(t) \leq \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$	[YES] Ayala & P. (2013)
3D Navier-Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3$	YES Lu & Doering (2008)
3D Navier-Stokes finite-time	$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$???

- ▶ Rate of Growth of Enstrophy

$$\frac{d\mathcal{E}}{dt} = -\nu \int_{\Omega} |\Delta \mathbf{u}|^2 dx + \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} dx \triangleq \mathcal{R}_{\mathcal{E}_0}(\mathbf{u})$$

- ▶ Best available instantaneous upper bound

$$\frac{d\mathcal{E}}{dt} \leq \frac{C}{\nu^3} \mathcal{E}^3$$

- ▶ Finite-time estimates

$$\max_{t \geq 0} \mathcal{E}(t) \leq \frac{\mathcal{E}_0}{\sqrt{1 - \frac{4C_3}{\nu^3} \mathcal{E}_0^2 t}}$$

$$\frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq \frac{27}{(2\pi\nu)^4} [\mathcal{K}(0) - \mathcal{K}(t)]$$

- ▶ Single Constraint: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$

$$\max_{\mathbf{u} \in \mathcal{S}_{\mathcal{E}_0}} \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{E}_0} = \{\mathbf{u} \in H^2(\Omega) : \nabla \cdot \mathbf{u} = 0, \mathcal{E}(\mathbf{u}) = \mathcal{E}_0\}$$

- ▶ Two Constraints: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$ and $\mathcal{K}_0 < (2\pi)^{-2}\mathcal{E}_0$

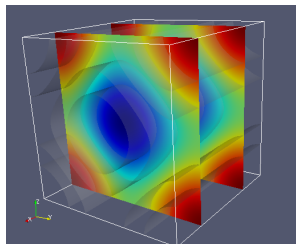
$$\max_{\mathbf{u} \in \mathcal{S}_{\mathcal{K}_0, \mathcal{E}_0}} \mathcal{R}_{\mathcal{E}_0}(\mathbf{u}) \quad \text{where}$$

$$\mathcal{S}_{\mathcal{K}_0, \mathcal{E}_0} = \{\mathbf{u} \in H^2(\Omega) : \nabla \cdot \mathbf{u} = 0, \mathcal{K}(\mathbf{u}) = \mathcal{K}_0, \mathcal{E}(\mathbf{u}) = \mathcal{E}_0\}$$

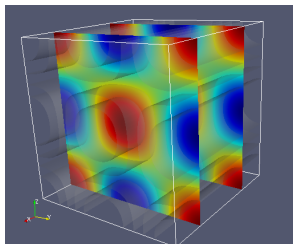
- ▶ Numerical solution via discretized gradient flow
 (required resolutions up to 512^3)

Extreme Vortex States for $\mathcal{E}_0 \rightarrow 0$ (single constraint)

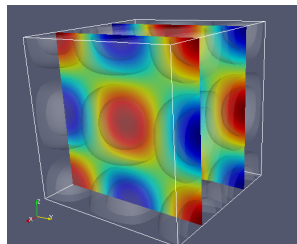
- ▶ In the limit $\mathcal{E}_0 \rightarrow 0$ optimal states found analytically
 \implies div-free eigenfunctions of vector Laplacian (3 branches)



(a) $|\mathbf{k}|^2 = 1$



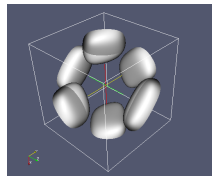
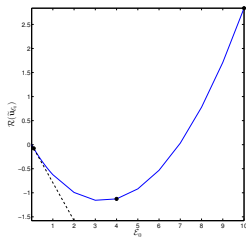
(b) $|\mathbf{k}|^2 = 2$



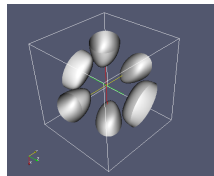
(c) $|\mathbf{k}|^2 = 3$

- ▶ Case (a): Largest value of $d\mathcal{E}/dt$
- ▶ Case (c): Taylor-Green vortex (Taylor & Green 1937)

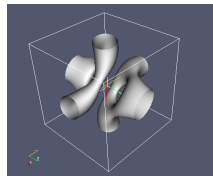
Single-constraint maximizers $\tilde{u}_{\varepsilon_0}$ (Lu & Doering 2008)



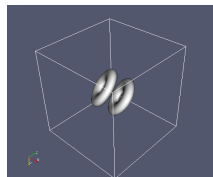
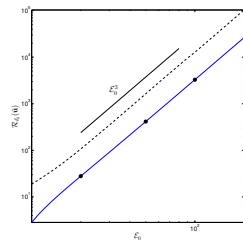
(a) $\varepsilon_0 = 1 \times 10^{-2}$



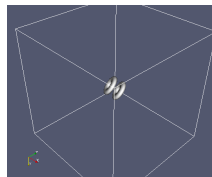
(b) $\varepsilon_0 = 4$



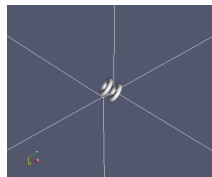
(c) $\varepsilon_0 = 10$



(d) $\varepsilon_0 = 20$

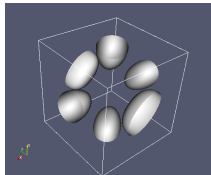
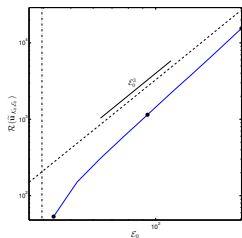


(e) $\varepsilon_0 = 50$

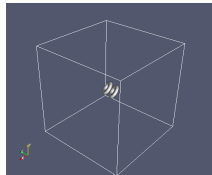


(f) $\varepsilon_0 = 100$

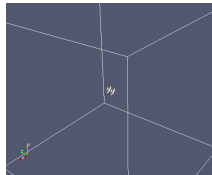
Two-constraint maximizers $\tilde{\mathbf{u}}_{\mathcal{K}_0, \mathcal{E}_0}$ ($\mathcal{K}_0 = 1$)



(a) $\mathcal{E}_0 \approx 40$

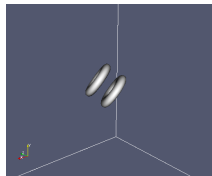
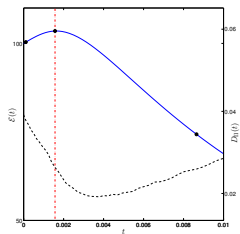
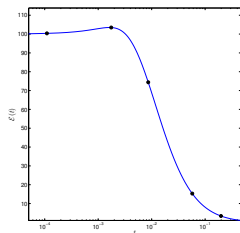


(b) $\mathcal{E}_0 \approx 90$

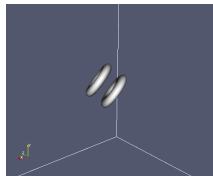


(c) $\mathcal{E}_0 \approx 200$

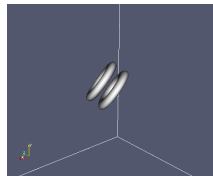
Time evolution of $\tilde{u}_{\mathcal{E}_0}$ (single constraint: $\mathcal{E}_0 = 100$)



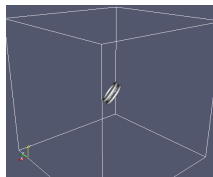
(a) $t = 0.0$



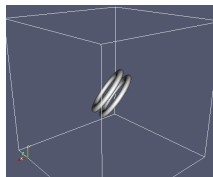
(b) $t = 1.1 \times 10^{-4}$



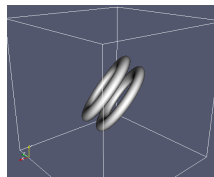
(c) $t = 1.75 \times 10^{-3}$



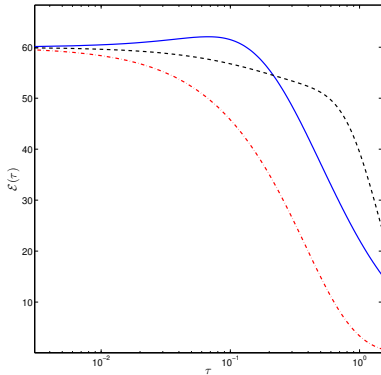
(d) $t = 8.63 \times 10^{-3}$



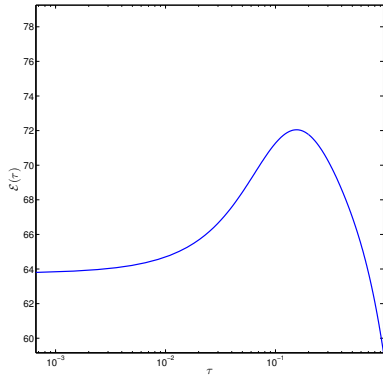
(e) $t = 5.74 \times 10^{-2}$



(f) $t = 0.198$



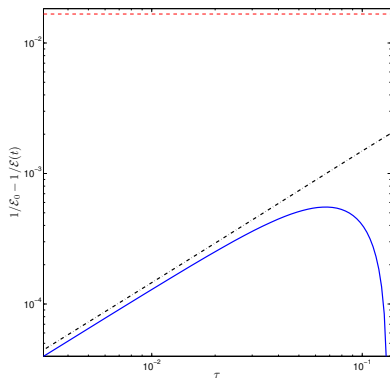
(a) single constraint ($\mathcal{E}_0 = 60$)



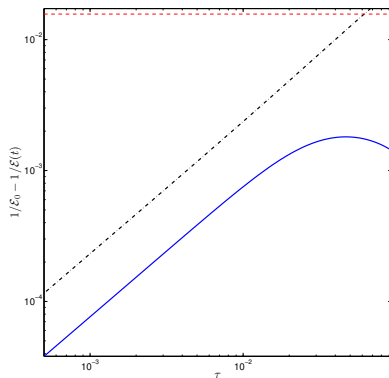
(b) two constraints ($\mathcal{K}_0 = 1$, $\mathcal{E}_0 = 64$)

- extreme (instantaneously optimal) states $\tilde{\mathbf{u}}_{\mathcal{E}_0}$, $\tilde{\mathbf{u}}_{\mathcal{K}_0, \mathcal{E}_0}$
- - - Taylor-Green vortex
- . - Kida-Pelz vortex

$$\frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq C [\mathcal{K}(0) - \mathcal{K}(t)]$$



(a) single constraint ($\mathcal{E}_0 = 60$)

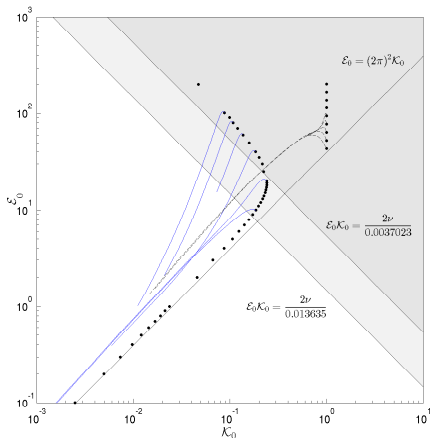


(b) two constraints ($\mathcal{K}_0 = 1$, $\mathcal{E}_0 = 64$)

— LHS of the estimate, - - - $1/\mathcal{E}(0)$
 - - - RHS of the estimate

$$\frac{d\mathcal{E}}{dt} \leq C' \mathcal{E}^3 \implies \frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq C [\mathcal{K}(0) - \mathcal{K}(t)]$$

$$\implies \max_{t \geq 0} \mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{1 - C \mathcal{K}(0) \mathcal{E}(0)}, \quad C, C' - \text{numerical fit}$$



Conclusions

- ▶ Found extreme vortex states in 2D and 3D saturating the worst-case mathematical bounds (although, in contrast to some recent studies in 3D, e.g., Bustamante & Brachet 2012, Kerr et al. 2013, 2014, the Reynolds numbers are small).
- ▶ Identified regions in the initial data phase-space $\{\mathcal{K}_0, \mathcal{E}_0\}$ for which global regularity is guaranteed.
- ▶ So far, no evidence of blow-up in 3D, although due to small Reynolds numbers, the results are not conclusive.
- ▶ Need to solve the optimization problem on *finite* time windows, i.e.,

$$\max_{\mathbf{u}_0} \mathcal{E}(T)$$

Open Questions and Open Problems

- ▶ Why do two-constraint optimizers exhibit a larger finite-time growth in 2D, but not in 3D? (finite Re effect?)
- ▶ Regularity problem for 3D Euler equation.
- ▶ Singularity formation in “active scalar” equations (fractional Burgers equation, surface quasi-geostrophic equation, etc.).
- ▶ Extreme behavior in the presence of noise.

References

- ▶ L. Lu and C. R. Doering, “Limits on Enstrophy Growth for Solutions of the Three-dimensional Navier-Stokes Equations” *Indiana University Mathematics Journal* **57**, 2693–2727, 2008.
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- ▶ D. Ayala and B. Protas, “On Maximum Enstrophy Growth in a Hydrodynamic System”, *Physica D* **240**, 1553–1563, 2011.
 - ▶ D. Ayala and B. Protas, “Maximum Palinstrophy Growth in 2D Incompressible Flows: Instantaneous Case”, *Journal of Fluid Mechanics* **742** 340–367, 2014.
 - ▶ D. Ayala and B. Protas, “Vortices, Maximum Growth and the Problem of Finite-Time Singularity Formation”, *Fluid Dynamics Research (Special Issue for IUTAM Symposium on Vortex Dynamics)*, **46**, 031404, 2014.
 - ▶ D. Ayala and B. Protas, “Extreme Vortex States and the Growth of Enstrophy in 3D Incompressible Flows”, (in preparation), 2014.