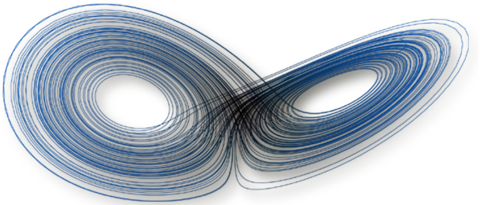

Chaotic convection and Lorenz-like dynamics

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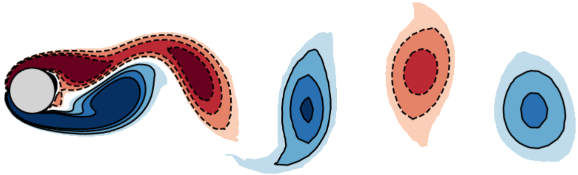
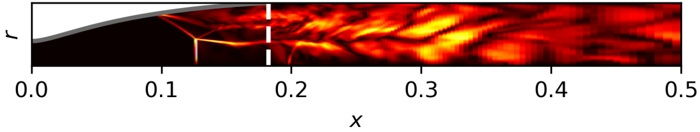
DynFluid Fluid Dynamics Laboratory

Arts et Métiers Institute of Technology

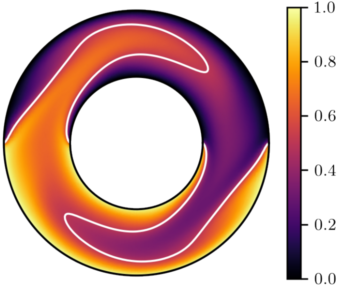
A gallery of fluid problems



Aerodynamics



Rocket Science (litterraly!)



Heat exchange

2D Thermosiphon

Our setup for today

- Two-dimensional flow governed by the incompressible Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \mathrm{Pr}\nabla^2 \boldsymbol{u} + \mathrm{Ra}\ \mathrm{Pr}\ \boldsymbol{\theta} \boldsymbol{e}_y$$
$$\frac{\partial \boldsymbol{\theta}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{\theta} = \nabla^2 \boldsymbol{\theta}.$$

- Ra and Pr are parameters defining our problem.
  - ↪ Ra is set to $Ra = 17\ 000$.
  - ↪ Pr is set to $Pr = 5$.

- Simulation is performed using a spectral element solver (`Nek5000`).
  - ↪ Approx. 50 000 grid points and $\Delta t = 2.5 \times 10^{-5}$.

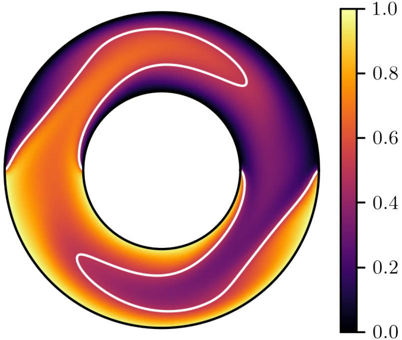
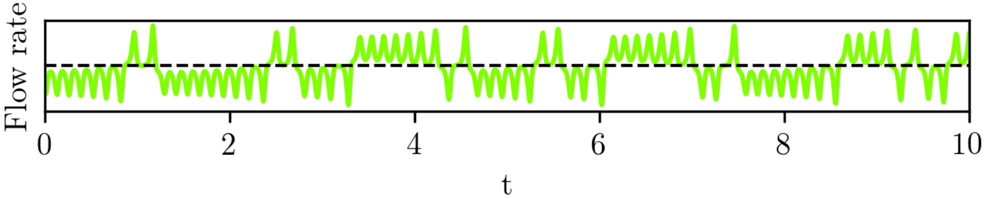


Fig: Chaotic thermosyphon temp. field.

2D Thermosiphon

Chaotic dynamics



- Time-evolution of the cross-sectional flow rate is indicative of Lorenz-like chaotic dynamics.
  - ↪ "Random" switching between clockwise and anti-clockwise rotation.
- **Hypothesis**: These dynamics can be captured by a low-order model.

2D Thermosiphon

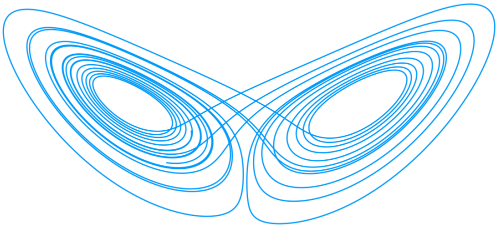
Objectives of today

- What physical properties should our reduced-order model have?
  - ↪ Analyze the physics prior to modeling.
- How to obtain a good low-dimensional embedding?
  - ↪ Dimensionality reduction.
- Can we identify the equations governing the dynamics in the embedded space?
  - ↪ System identification.
  - ↪ How to enforce the physical constraints?

Part I

Rayleigh-Bénard convection and the Lorenz system

- Rayleigh-Bénard convection:
  - ↪ Overview
  - ↪ Governing equations
  - ↪ Linear stability
- From Navier-Stokes to Lorenz:
  - ↪ Derivation of the low-order model
  - ↪ Properties of the Lorenz system



Rayleigh-Bénard convection

Overview

- Buoyancy-driven flow of a fluid heated from below and cooled from above.
- Applications in geophysics, astrophysics, meteorology, oceanography and engineering.
- Well-known model for nonlinear and chaotic dynamics, pattern formation and fully developed turbulence.

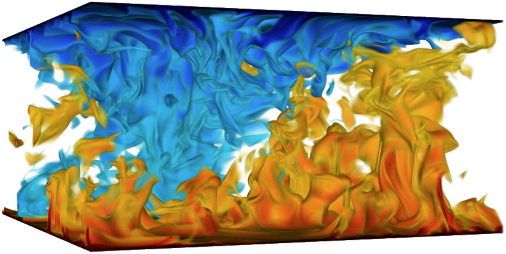


Illustration of turbulent Rayleigh-Bénard convection

Rayleigh-Bénard convection

Governing equations

- Under appropriate nondimensionalization, the governing equations read

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0 \\ \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} &= -\nabla p + \mathrm{Pr}\ \nabla^2 \boldsymbol{u} + \mathrm{Ra}\ \mathrm{Pr}\ \theta \boldsymbol{e}_y \\ \frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \theta &= \nabla^2 \theta, \end{aligned}$$

  where Pr is the Prandtl number and Ra the Rayleigh number.

- In the rest of this talk, we'll assume that the flow is two-dimensional, i.e. $\boldsymbol{u} = (v_x, v_y)^T$.

Rayleigh-Bénard convection

Fixed point: the conducting state

- The fixed point is solution to

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}y^2} = 0$$

with appropriate boundary conditions.

- The nondimensional temperature profile $\Theta(y)$ is given by

$$\Theta(y) = 1 - y.$$

- It corresponds to a pure conduction state (i.e. $\boldsymbol{u} = 0$).

Rayleigh-Bénard convection

Linear stability analysis

- Let us consider the linear stability of this conducting state towards two-dimensional perturbations.
- The linearized equations read

$$
\begin{aligned}
\frac{\partial}{\partial t} \nabla^2 \psi &= -\mathrm{Ra}\ \mathrm{Pr}\ \frac{\partial \theta}{\partial x} + \mathrm{Pr}\ \nabla^2 \psi \\
\frac{\partial \theta}{\partial t} &= -\frac{\partial \psi}{\partial x} + \nabla^2 \theta
\end{aligned}
$$

where ψ is the streamfunction of the perturbation.

Rayleigh-Bénard convection

Linear stability analysis

- Solutions are sought in the form of *normal modes*

$$\boldsymbol{q}(x, y, t) = \hat{\boldsymbol{q}}(y) e^{ikx + \lambda t} + c.c.,$$

where λ is the growth rate and k the perturbation's wavenumber.

- We obtain the following generalized eigenvalue problem

$$\lambda \begin{bmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \mathrm{Pr}(D^2 - k^2) & -ik\mathrm{RaPr} \\ -ik & D^2 - k^2 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\theta} \end{bmatrix}$$

where $D = {}^{\mathrm{d}}/_{\mathrm{d}y}$.

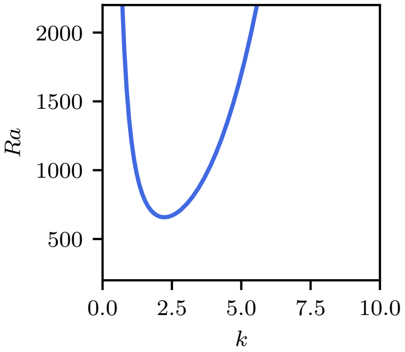
Rayleigh-Bénard convection

Linear stability analysis

- Problem solved analytically in 1916 with free-slip boundary conditions, i.e.

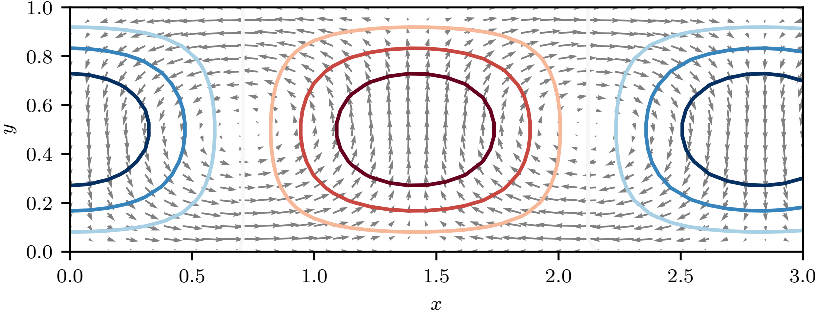
$$\psi(x, y, t) = \hat{\psi}(t) \sin(n\pi y) \sin(kx)$$
$$\theta(x, y, t) = \hat{\theta}(t) \sin(n\pi y) \cos(kx).$$

- The dispersion relation reduces to a quadratic equations.
- One then obtaines $Ra_c = {}^{27\pi^4}/_4$ and $k_c = {}^{\pi}/_{\sqrt{2}}$.



Rayleigh-Bénard convection

Linear stability analysis



Isocontours of temperature and velocity field of the instability mode.

From Navier-Stokes to Lorenz

Nonlinear equations

- The governing equations read

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2\psi - \mathrm{Pr}\nabla^2\psi + \mathrm{RaPr}\frac{\partial\theta}{\partial x} &= \mathcal{J}(\nabla^2\psi, \psi) \\ \frac{\partial\theta}{\partial t} - \nabla^2\theta + \frac{\partial\psi}{\partial x} &= \mathcal{J}(\theta, \psi),\end{aligned}$$

  where the nonlinear terms are expressed as a Jacobian operator $\mathcal{J}$ given by

$$\mathcal{J}(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

- We futhermore assume free-slip boundary conditions as before.

From Navier-Stokes to Lorenz

Truncated Galerkin expansion

- It has been shown by Saltzmann (1962) that the general solution can be expressed as an infinite Fourier series.

- Let us however consider a *truncated Galerkin expansion* such that

$$\psi(x,y,t) = a(t)\sin(\pi y)\sin(k\pi x) + \cdots$$
$$\theta(x,y,t) = b(t)\sin(\pi y)\cos(k\pi x) + c(t)\sin(2\pi y) + \cdots$$

- $a(t)$ and $b(t)$ correspond to the convection rolls with wavenumber k in the x-direction. The term $c(t)$ describes the modification of the mean temperature profile due to convection.

From Navier-Stokes to Lorenz

Low-order model

- Finally, we obtain the folowing low-order model

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= -\mathrm{Pr}\ \pi^2(1-k^2)a - \frac{k\pi}{\pi^2(1+k^2)}\mathrm{Pr}\ \mathrm{Ra}\ b \\
\frac{\mathrm{d}b}{\mathrm{d}t} &= -k\pi a - \pi^2(1+k^2)b - k\pi^2 ac \\
\frac{\mathrm{d}c}{\mathrm{d}t} &= \frac{1}{2}k\pi^2 ab - 4\pi^2 c.
\end{aligned}
$$

- This low-dimensional model of thermal convection is a rescaled version of the one originally introduced by Lorenz in 1963.

Lorenz system

1963 model

- Lorenz-1963 model reads

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z,
\end{aligned}
$$

where $\sigma = \mathrm{Pr}$, $\rho = {}^{Ra}/_{Ra_c}$ and $\beta = {}^{2\pi^2}/_{\pi^2+k^2}$ is the aspect ratio of the convection cells.

- It is now well-known that this system exhibits chaotic dynamics.
  - ↪ Classical parameters: $\sigma = 10$, $\rho = 28$, $\beta = {}^8/_3$.

Lorenz system

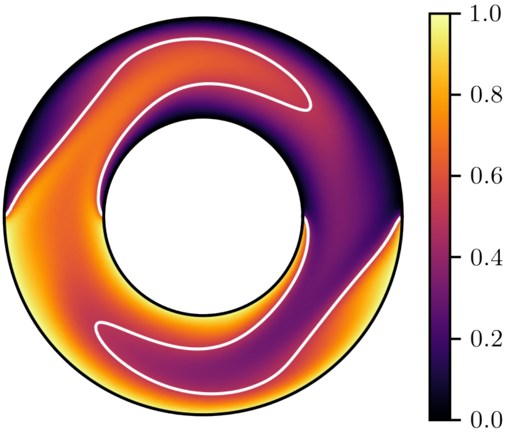
Properties

- **Symmetry**: Invariant wrt. the the transformation $(x, y, z) \to (-x, -y, z)$.
- **Invariant z-axis**: If $x(0) = y(0) = 0$, then $x(t) = y(t) = 0 \ \forall t$. Accordingly, $\dot{z} = -\beta z$ and hence $z(t) = e^{-\beta z}$. The z-axis is always part of the stable manifold for the equilibrium at the origin.
- **Dissipative**: We have $\nabla \cdot \boldsymbol{F}(\boldsymbol{x}) = -\sigma - 1 - \beta < 0$. Any given volume V of phase points eventually tends to 0 as $t \to \infty$.

Part II

Dimensionality reduction

- Problem formulation:
  - ↪ Original DMD formulation
  - ↪ Reduced Rank Regression
- Practical usage:
  - ↪ Extracting patterns from our flow
  - ↪ Low-dimensional embedding



Dimensionality reduction

Dynamic Mode Decomposition

- DMD is a system identification technique introduced by P. Schmid in 2010 in the field of fluid dynamics.
  - ↪ Initially presented as a generalization of eigenvalue analysis for nonlinear systems.
- DMD has been shown to provide an approximation of the so-called *Koopman operator* (Rowley *et al.*, *J. Fluid Mech.*, 2010).
  - ↪ Beyond the scope of the present work. For more details, ask Steve and Frank.
- Since its introduction, numerous variants have been proposed.
  - ↪ EDMD, DMD with control, ioDMD, kernel DMD, Forward-Backward DMD, sparsity-promoting DMD, optimized DMD, gEDMD, . . .

Dimensionality reduction

Dynamic Mode Decomposition (Original formulation)

- Given a discrete-time nonlinear system $\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k)$ with $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$, the aim of DMD is to find a (low-rank) linear operator $\boldsymbol{A}$ such that

$$\boldsymbol{x}_{k+1} \simeq \boldsymbol{A}\boldsymbol{x}_k.$$

- Eigenvalues and eigenvectors of $\boldsymbol{A}$ provide valuable information about the coherent structures and charateristics time scales of the dynamics of our system.

- Additionnally, if $\boldsymbol{A}$ is low-rank, we might be able to replace $\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k)$ by a lower-dimensional linear system

$$\boldsymbol{w}_{k+1} \simeq \boldsymbol{S}\boldsymbol{w}_k,$$

where $\boldsymbol{w} \in \mathbb{R}^s$ and $\boldsymbol{S} \in \mathbb{R}^{s \times s}$ the projection of $\boldsymbol{A}$ onto a s-dimensional subspace with $s \ll n$.

Dimensionality reduction

DMD as a Reduced Rank Regression problem

- Recast DMD as a *Reduced Rank Regression* (RRR) problem

$$\begin{aligned} \underset{\boldsymbol{A}}{\text{minimize}} \ & \|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}\|_F^2 \\ \text{subject to } & \text{rank}(\boldsymbol{A}) = r \end{aligned}$$

  with $\boldsymbol{Y} = \boldsymbol{X}_{k+1}$ and $\boldsymbol{X} = \boldsymbol{X}_k$.

- The closed-form solution to this minimization problem is actually known since the 1950's.

Dimensionality reduction

DMD as a Reduced Rank Regression problem

- In the unconstrained case, the least-square solution is

$$\begin{aligned}\boldsymbol{A}_{LS} &= \boldsymbol{Y}\boldsymbol{X}^H \left(\boldsymbol{X}\boldsymbol{X}^H\right)^{-1}, \\ &= \boldsymbol{C}_{\boldsymbol{yx}}\boldsymbol{C}_{\boldsymbol{xx}}^{-1}\end{aligned}$$

- Our RRR problem can now be rewritten as

$$\begin{aligned}&\underset{\boldsymbol{P}}{\text{minimize}} \ \|\boldsymbol{Y} - \boldsymbol{P}\boldsymbol{P}^H\boldsymbol{C}_{\boldsymbol{yx}}\boldsymbol{C}_{\boldsymbol{xx}}^{-1}\boldsymbol{X}\|_F^2 \\ &\text{subject to } \operatorname{rank}(\boldsymbol{P}) = r \\ &\text{and } \boldsymbol{P}^H\boldsymbol{P} = \boldsymbol{I}.\end{aligned}$$

Dimensionality reduction

DMD as a Reduced Rank Regression problem

- After some simple matrix manipulations, we finally arrive at

$$\begin{aligned} \underset{\boldsymbol{P}}{\text{maximize}} \ \operatorname{Tr}\left(\boldsymbol{P}^H \boldsymbol{C}_{\boldsymbol{yx}} \boldsymbol{C}_{\boldsymbol{xx}}^{-1} \boldsymbol{C}_{\boldsymbol{xy}} \boldsymbol{P}\right) \\ \text{subject to } \operatorname{rank}(\boldsymbol{P}) = r \\ \text{and } \boldsymbol{P}^H \boldsymbol{P} = \boldsymbol{I}. \end{aligned}$$

- The optimal solution is given by the first r eigenvectors of the spd matrix $\boldsymbol{C}_{\boldsymbol{yx}} \boldsymbol{C}_{\boldsymbol{xx}}^{-1} \boldsymbol{C}_{\boldsymbol{xy}}$.
  - ↪ If $\boldsymbol{X}$ is full rank, these correspond to the first r singular vectors of the output matrix $\boldsymbol{Y}$!
  - ↪ If not, these correspond to the first r singular vectors of the output matrix $\boldsymbol{Y} \boldsymbol{V}_{\boldsymbol{x}} \boldsymbol{V}_{\boldsymbol{x}}^H$.

Dimensionality reduction

DMD as a Reduced Rank Regression problem

- The low-rank DMD operator is thus given by $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{Q}^H$ with

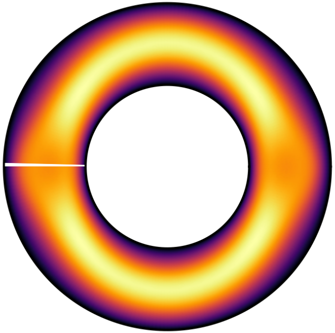
$$\boldsymbol{Q}^H = \boldsymbol{P}^H \boldsymbol{C}_{\boldsymbol{yx}} \boldsymbol{C}_{\boldsymbol{xx}}^{-1}.$$

- The right and left eigenvectors and the eigenvalues of $\boldsymbol{A}$ can easily be computed from its low-rank factorization.
  - ↪ It can be factorized as $\boldsymbol{A} = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^H$

Dimensionality reduction

Extracting coherent structures

- DMD analyis is performed independantly for each state variable on a sequence of 20 000 snapshots.
- Only one mode is retained for the azimuthal velocity.
  - ↪ It capture the large-scale motion of the convection cell.
- The first two modes are retained for the temperature.
  - ↪ They capture the left-right and up-down symmetries.



Mode 1 for velocity.

Dimensionality reduction

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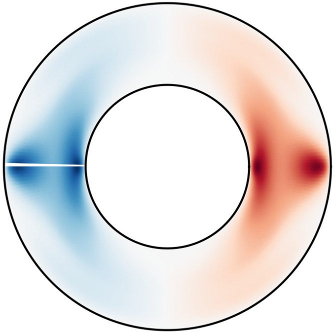

Mode 1 for temperature.

Dimensionality reduction

Extracting coherent structures

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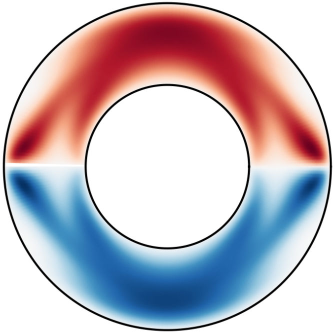
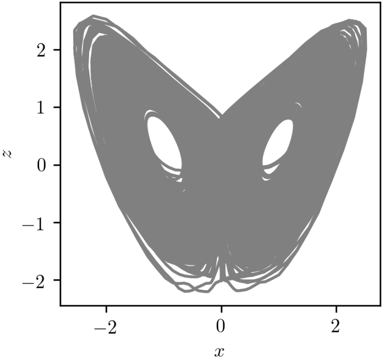


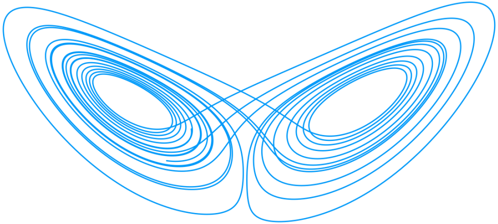
Mode 2 for temperature.

Dimensionality reduction

Low-dimensional embedding



- Projection onto the span of these DMD modes provides a relatively accurate embedding.
  - ↪ 90-95% of the fluctuation's kinetic and thermal energy is captured.
- Evolution of the system in this low-dimensional phase space clearly resembles Lorenz!



Part III

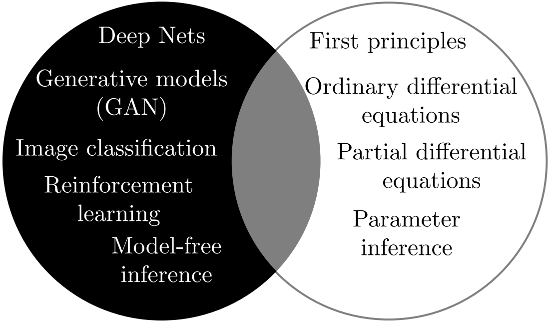
System Identification

- General overview:
  - ↪ Wiener's classification of models
  - ↪ (Nonlinear) state space models
- SINDy:
  - ↪ Problem formulation
  - ↪ Greedy algorithms
  - ↪ Practical tips
- Application to the chaotic thermosyphon:
  - ↪ A Lorenz-like system

System Identification

A hierarchy of models

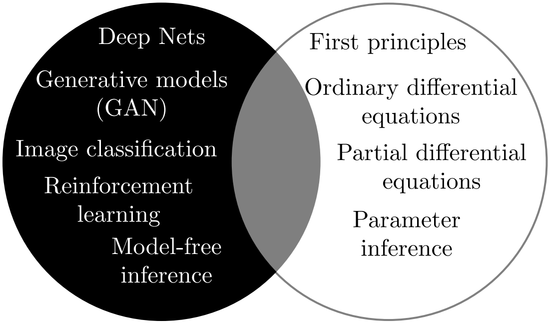
- Wiener classified models in three flavors:
  - ↪ **Black box:** Only input-output data.
  - ↪ **White box:** Input-output data + known model.
  - ↪ **Gray box:** Input-output data + partial knowledge of the model / inductive bias.



System Identification

A gray zone

- Often applied to physical systems for which we have some form of prior knowledge.
  - ↪ Symmetries, invariant quantities, etc.
- Exact form of the governing equations may be unknown.
  - ↪ dynamics modeled by a PDE but concentrate on a low-dim. manifold.



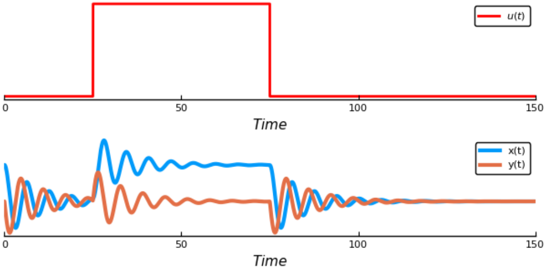
System Identification

Linear Time Invariant MIMO Systems

- Well established theory.
- Applications in a plethora of different scientific and industrial fields.
  - ↪ Optimal control, signal processing, . . .
- Extremely large body of literature and a real zoo of methodologies.
  - ↪ EigenRealization, N4SID, ARMAX, . . .
- Most of them rely on input-output data.
  - ↪ Experimentally accessible data.

State space model

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + w_k \\
\boldsymbol{y}_k &= \boldsymbol{C}\boldsymbol{x}_k + s_k
\end{aligned}
$$



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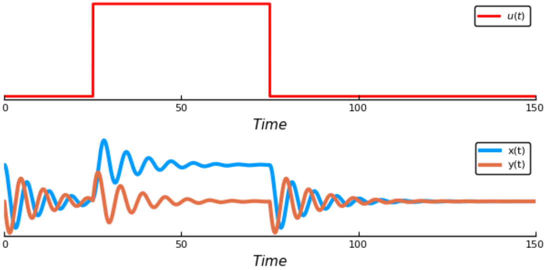
IIR / FIR filter

$$y_k = \sum_{i=0}^{N} h_i \cdot u_{k-i} + \sum_{i=0}^{N} h_i' \cdot w_i + s_k$$

$$\hat{y}(\omega) = \hat{H}(\omega)\hat{u}(\omega) + \hat{H}'(\omega)\hat{w}(\omega) + \hat{s}(\omega)$$



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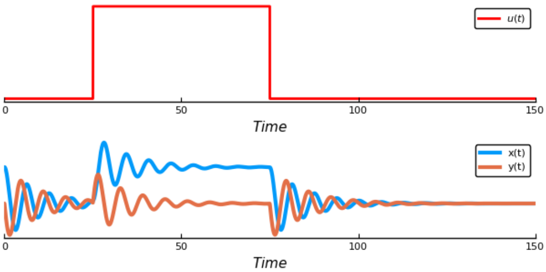
ARMAX process

$$y_{k+1} = \sum_{i=0}^{N} \alpha_i y_{k-i} + \sum_{i=0}^{M} \beta_i u_{k-i} + \sum_{i=0}^{L} \theta_i \epsilon_{k-i}$$

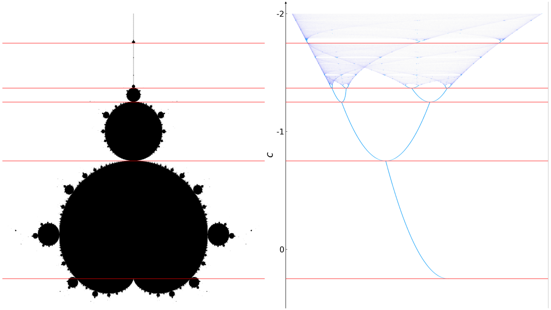


System Identification

Discrete-time nonlinear systems

Logistic equation

$$x_{k+1} = \mu x_k(1 - x_k) + u_k$$

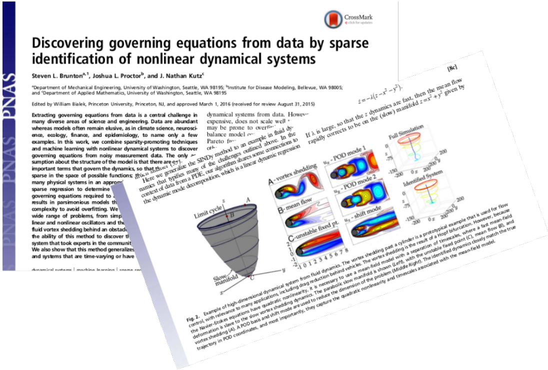


- First practical theory dates back to 1887 by Volterra.
  - ↪ So-called *Volterra series*.
- Since the late 1980's, **NARMAX** has become a key player.
  - ↪ Generalization of ARMAX to nonlinear systems.
  - ↪ Extremely versatile methodology.
  - ↪ Can account for latent variables.
- Limited to a discrete-time framework.
  - ↪ Recasting the identified model into an ODE is far from being straightforward!

SINDy

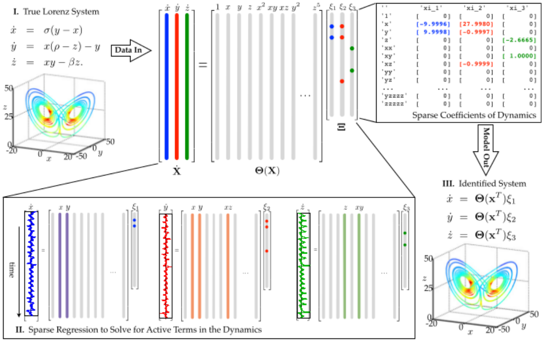
Identification of continuous-time nonlinear systems

- Identification of continuous-time nonlinear systems is a hard task.
- SINDy seems like a quite promising and versatile approach to do so.
  - ↪ Introduced by Steve in 2015.
- Relies on a dictionnary of functions and sparsity-promoting regression.



SINDy

A combinatorial problem



- Given $\boldsymbol{\Theta}(\boldsymbol{x})$, one aims to solve

$$\underset{\boldsymbol{\xi}}{\text{minimize}} \ \|\boldsymbol{\xi}\|_0$$
$$\text{subject to } \|\boldsymbol{\Theta}(\boldsymbol{x})\boldsymbol{\xi} - \dot{\boldsymbol{x}}\|_2^2 \leq \sigma.$$

- This is a combinatorial problem which becomes rapidly intractable.
- Convex relaxation and/or greedy algorithms needed.

SINDy

Convex relaxation and greedy algorithms

- Convex relaxations to the ℓ_0 minimization problem exist, e.g.
  - $\hookrightarrow$ LASSO: $\text{minimize} \ \|\boldsymbol{\Theta}(\boldsymbol{x})\boldsymbol{\xi} - \dot{\boldsymbol{x}}\|_2^2 + \lambda\|\boldsymbol{\xi}\|_1$
  - $\hookrightarrow$ Basis Pursuit Denoising, etc.
- Alternatively, one can use greedy algorithms to solve the ℓ_0 minimization problem, e.g.
  - $\hookrightarrow$ Orthogonal Matching Pursuit (Mallat *et al.*, 198?)
  - $\hookrightarrow$ Forward Regression Orthogonal Least Squares (Chen & Billings, 1987)
  - $\hookrightarrow$ Hard-thresholding least-squares (Brunton *et al.*, 2015).
- FROLS is my personal favorite. Routinely used to identify discrete-time NARMAX models.
  - $\hookrightarrow$ Good understanding of its sparsity-promoting properties and easy to code.

SINDy

Extending SINDy: adding (convex) constraints

- Because SINDy aims to minimize a convex function, one can use all the tools from convex optimization theory.
- In particular, one can enforce linear equality and convex inequality constraints

$$
\begin{aligned}
\underset{\boldsymbol{\xi}}{\text{minimize}} \ & \|\boldsymbol{\Theta}(\boldsymbol{x})\boldsymbol{\xi} - \dot{\boldsymbol{x}}\|_2^2 + \lambda\|\boldsymbol{\xi}\|_1 \\
\text{subject to } & \boldsymbol{C}\boldsymbol{\xi} = \boldsymbol{d} \\
& \boldsymbol{h}(\boldsymbol{\xi}) \leq 0.
\end{aligned}
$$

- These can be used to enforce physical constraints in the identification procedure, e.g.
  - ↪ Energy-preserving nonlinearities (Loiseau & Brunton, 2018),
  - ↪ Some form of symmetry in the unknown equations (more on this in a few slides),
  - ↪ Invariant quantities (e.g. Hamiltonian systems' first integral).

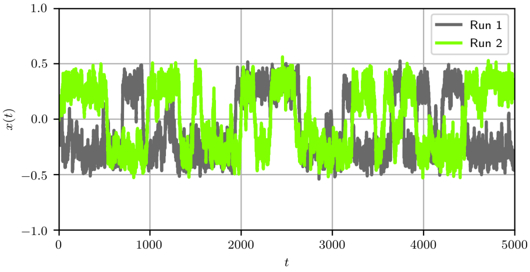
SINDy

Extending SINDy: stochastic systems

- SINDy-like framework has been proposed for stochastic ODE

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})\mathrm{d}t + \boldsymbol{g}(\boldsymbol{x})\mathrm{d}W$$

- Relies on Kramers-Moyal averaging and its link to the Fokker-Planck equation.
  - ↪ See Boninsegna *et al.*, *J. Chem. Phys.*, 2018.
- Currently limited to low-dimensional system due to the curse of dimensionality.



SINDy

Extending SINDy: MANDy, a tensor-train formulation

$$\boldsymbol{\Theta}(\boldsymbol{x}) = \begin{bmatrix} 1 & x & y & z & x^2 & xy & xz & y^2 & yz & z^2 \end{bmatrix}$$

$$\Updownarrow$$

$$\boldsymbol{\Theta}_{TT}(\boldsymbol{x}) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ y \\ y^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}$$

- If $\boldsymbol{x} \in \mathbb{R}^n$ and one consider $\boldsymbol{\Theta}(\boldsymbol{x})$ as the library of polynomials function, then

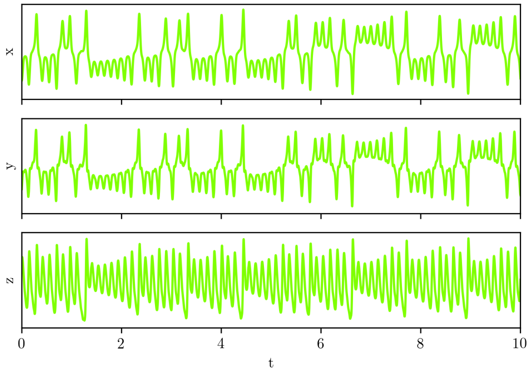
$$\dim(\boldsymbol{\Theta}) = \binom{n+d-1}{d}$$

- It rapidly becomes intractable to form it explicitely.
- For certain classes of functions (poly. included), $\boldsymbol{\Theta}(\boldsymbol{x})$ admits a tensor-train decomposition.
  - $\hookrightarrow$ See Gelss *et al.*, arXiv 1809.02448, 2019.

SINDy

Application to the chaotic thermosyphon

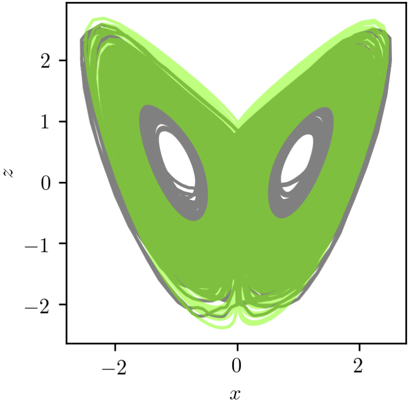
- Training data:
  - ↪ Time-series of the DMD modes' amplitudes.
- Library of funtions:
  - ↪ Polynomials in x, y and z up to quadratic.
- Feature selection:
  - ↪ Forward Regression Orthogonal Least Squares (FROLS)



Training time series.

SINDy

Baseline unconstrained model



- The identified unconstrained model reads

$$\dot{x} = -92.26x + 93.67y + 8.33xz$$
$$\dot{y} = 12.99x - 30xz - 6.84yz$$
$$\dot{z} = -15.62z + 41.25xy + 5.07y^2 - 42.92$$

- Lorenz-like system with a few additional terms.
  - ↪ Satisifies all the properties listed in Part I.
- Is it possible to further simplify the model ?
  - ↪ Less than 10 parameters to infer.

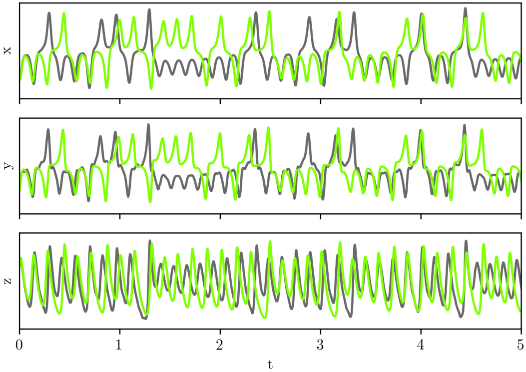
SINDy

Adding constraints

- The sparse constrained model reads

$$\dot{x} = 96.25(y - x) + 9.73xz$$
$$\dot{y} = 14.62x - 35.7xz$$
$$\dot{z} = -17z + 47.12xy - 43.25$$

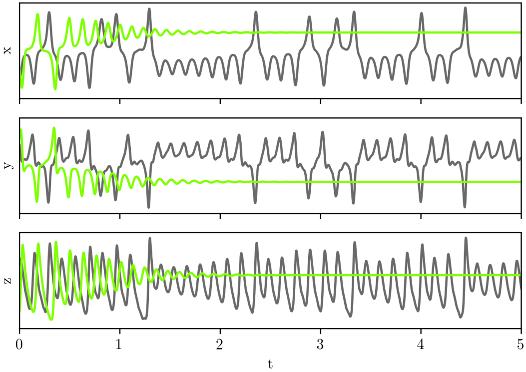
- Constant term in z equation simply corresponds to a shift of the phase space's origin.
- Only one additional coupling term compared to classical Lorenz.

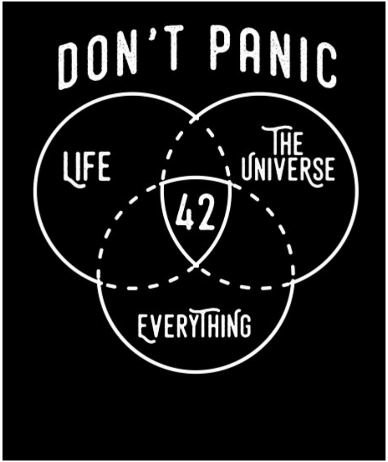


SINDy

What's next?

- Chaotic switching is detrimental for the performances of the thermal exchanger.
  - ↪ Need to apply optimal control theory to stabilize the system.
- So far we have learned $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.
  - ↪ Uncontrolled dynamics.
- We now need to learn $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$.
  - ↪ Use `SINDy with control` to learn the new rhs function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$.
  - ↪ Use feedback control to get $\boldsymbol{u} = \mathcal{K}(\boldsymbol{x})$.





Part IV

Random thoughts on life, the universe and everything

- Summary
- Random thoughts and open problems.

Summary

Dimensionality reduction

- Critical preprocessing step in order to obtain a good low-dimensional embedding.
  - ↪ PCA, DMD or AutoEncoders or time-delayed embedding can be used.
- Unclear at the present time which dimensionality reduction technique provides the best embedding for system identification purposes.
  - ↪ Combining AutoEncoders and SINDy altogether is currently the best approach, albeit computational expensive (Champion *et al.*, ???, 2019).
- DMD and its variants are my personal favorites at the moment.
  - ↪ Relatively low computational cost.
  - ↪ Appears to make the linear terms as simple as possible.
  - ↪ Possible connection to normal form theory albeit unexplored currently.

Summary

System Identification

- SINDy is pretty versatile framework for continuous-time nonlinear system identification.
  - ↪ Generalization of NARMAX.
  - ↪ Easy to extend for input-ouput (controlled systems) and stochastic systems or to add (certain) physical constraints.
- Finding a good metric for cross-validation is a crucial step.
  - ↪ Basic one step ahead prediction is not sufficient.
  - ↪ Autocorrelation function, probability density functions and Lyapunov exponents are better candidates for chaotic systems.
- Still unclear at the present time how to properly handle latent variables.
  - ↪ Differential embedding is good candidate (although the data should not be noisy).

Summary

Practical tips for system identification

- Before using SINDy:
  - ↪ Find a similar/easier already solved problem to gain some physical insights from it.
  - ↪ Find a good low-dimensional embedding or set of measurements. (Hardest part probably!)
  - ↪ Conduct exploratory analysis of your data to get an idea of what info. it really contains.
- While using SINDy:
  - ↪ Incrementally increase the number of functions in $\boldsymbol{\Theta}(\boldsymbol{x})$ starting with polynomials.
  - ↪ Look at the unconstrained least-squares solutions to give you an idea of the possible sparsity pattern.
  - ↪ Try different sparsity-promoting techniques to assess the robustness of your identification.
  - ↪ Add constraints if needed.
- Once the system has been identified:
  - ↪ Cross-validate it using appropriate metrics.
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Thank you for your attention.

Any questions?