Comparing Point Clouds

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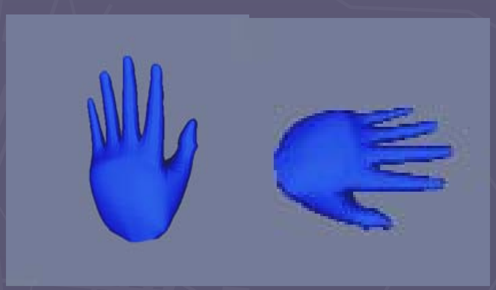
What is this?

- Suppose we have two surfaces X and Y and (dense) Point Clouds $\mathbb{X}_m \subset X$ and $\mathbb{Y}_{m'} \subset Y$
- How can we measure similarity between X and Y based on measures taken over $\mathbb{X}_m$ and $\mathbb{Y}_{m'}$?
- X and Y are similar $\Longleftrightarrow$ $\mathbb{X}_m$ and $\mathbb{Y}_{m'}$ are similar ?

Similarity (?)

- For example:

  Rigid Similarity



- Another Case:

  General Isometries (Bends)

Inspired by Elad-Kimmel

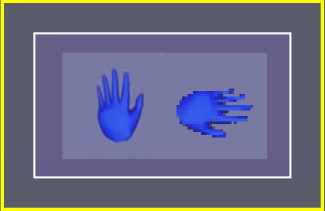
Reality...

- We define a measure of *similarity* between the underlying surfaces $D_c(X,Y)$
- And another (related) measure of *similarity* between the Point Clouds $D_d(\mathbb{X}_m,\mathbb{Y}_{m'})$
- Also , it often happens that we cannot compute $D_d(\mathbb{X}_m,\mathbb{Y}_{m'})$ exactly, instead we obtain an approximate value $\widehat{D}_d(\mathbb{X}_m,\mathbb{Y}_{m'})$

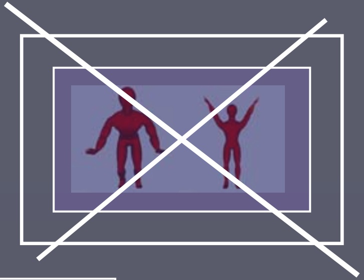
We want to have control:

$$A\left(D_c(X,Y)-\alpha\right)\leq \widehat{D}_d(\mathbb{X}_m,\mathbb{Y}_{m'})\leq B\left(D_c(X,Y)+\beta\right)$$

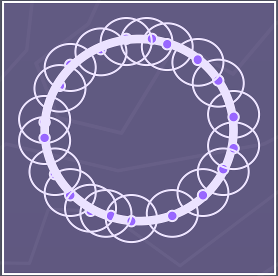
α and β should depend on how well $\mathbb{X}_m$ and $\mathbb{Y}_{m'}$ sample X and Y respectively.



Rigid Similarity (I)

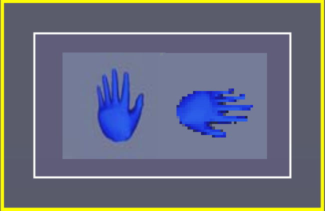


- $D_c = D_d = D$ and $\boxed{D(Z,Z') := \inf_\Phi d_{\mathcal{H}}(Z,\phi(Z'))}$ for Z and Z' compact sets in R^3 where $\Phi \in O(3)$.
  One has Triangle Inequality...
- One has $D(X,\mathbb{X}_m) \leq r$ if $X \subset \cup_{x\in\mathbb{X}_m} B(x,r)$
  $D(Y,\mathbb{Y}_{m'}) \leq r'$ if $Y \subset \cup_{y\in\mathbb{Y}_{m'}} B(y,r')$
  r and r' are **Covering Radii**..

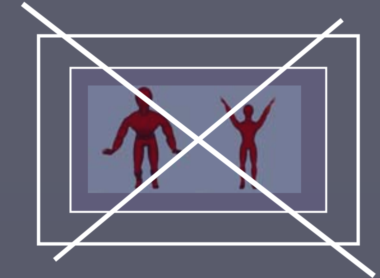


- Therefore

$$|D(X,Y) - D(\mathbb{X}_m,\mathbb{Y}_{m'})| \leq r + r'$$



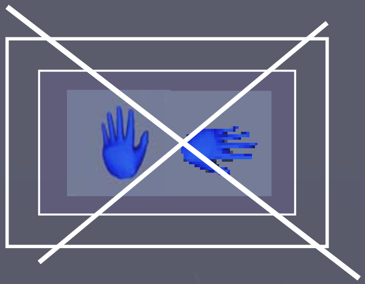
Rigid Similarity (II)



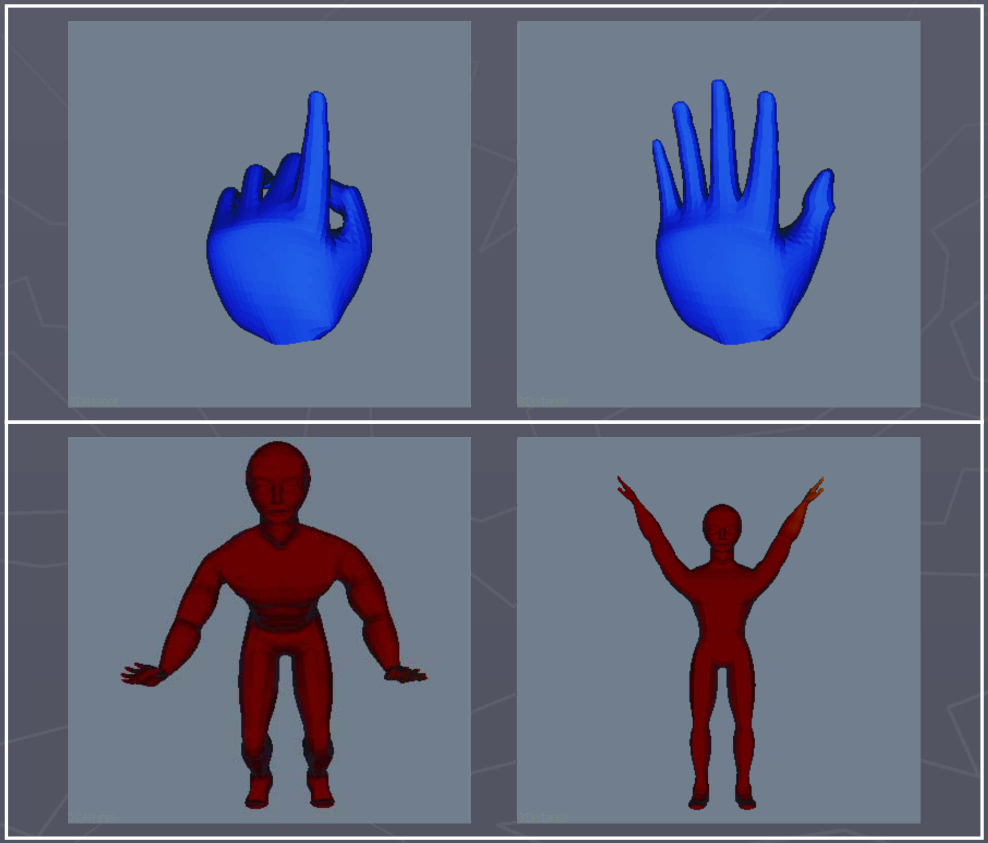
- Using the method of Goodrich et al (Approximate geometric pattern matching under rigid motions), we find an approximate value for the discrete measure which is within a maximum distance from the true discrete measure, hence we obtain (control !):

$$D_c(X,Y)-(r+r') \le \widehat{D}_d(\mathbb{X}_m,\mathbb{Y}_{m'}) \le 10\,\left(D_c(X,Y)+(r+r')\right)$$

Bending Invariance (I)



- Now, what if we want to allow for bends ??
  (Isometric Transformations)

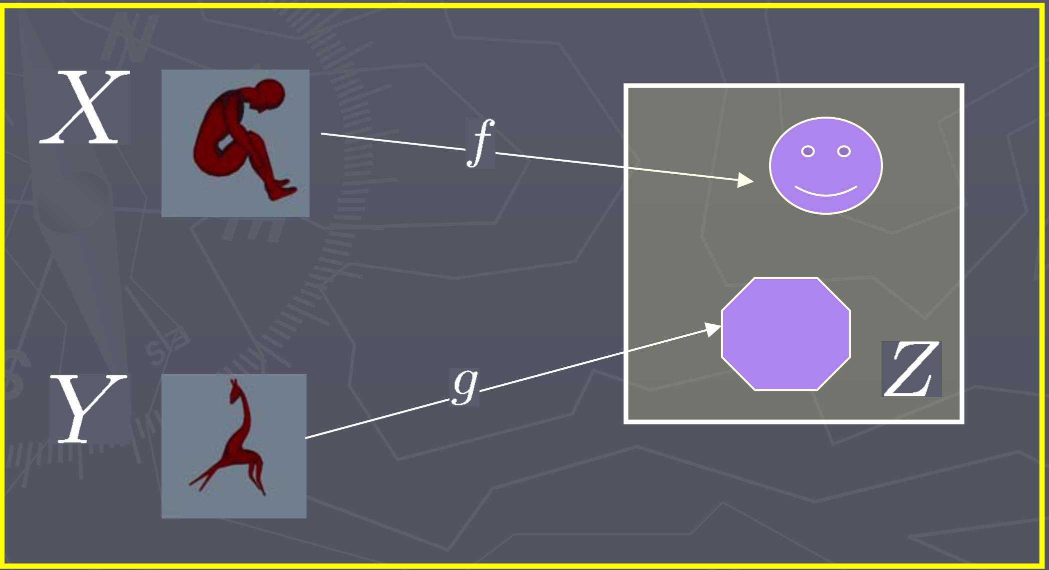


Bending Invariance (II)

- We use Gromov-Hausdorff distance: let X and Y be compact Metric Spaces, then we use

$$d_{\mathcal{GH}}(X,Y) := \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(X,Y)$$

$f : X \to Z$, $g : Y \to Z$ are *isometric embeddings*



$$(D_c = D_d = d_{\mathcal{GH}})$$

Bending Invariance (III)

- This is truly a measure of metric similarity:

$$d_{\mathcal{GH}}(X,Y) = \inf_{\substack{\phi : X \to Y \\ \psi : Y \to X}} \sup_{x \in X, y \in Y} \frac{1}{2} |d_X(x, \psi(y)) - d_Y(y, \phi(x))|$$

- Properties ... (triangle inequality +)

(1) $d_{\mathcal{GH}}(X,Y) = 0 \Leftrightarrow X$ and Y are **isometric**.

(2) Let $\mathbb{X}_m$ a r-covering of X and $\mathbb{Y}_{m'}$ a r'-covering of Y. Then

$$|d_{\mathcal{GH}}(X,Y) - d_{\mathcal{GH}}(\mathbf{X}_m, \mathbf{Y}_{m'})| \leq r + r'$$

Bending Invariance (IV)

- Then, if X and Y were sampled finely enough (r+r' is "small") we'd be able to say things about the Continuous World based on Discrete Observations, and reciprocally.
- The first idea is trying to compute $d_{\mathcal{GH}}(\mathbf{X}_m, \mathbf{Y}_{m'})$

which depends on the Distance Matrices corresponding to both Point Clouds, $D_{\mathbb{X}_m} = \left(d_X(x_i, x_j)\right)$ and $D_{\mathbb{Y}_{m'}} = \left(d_Y(y_i, y_j)\right)$

$$d_{\mathcal{GH}}(\mathbb{X}_m, \mathbb{Y}_{m'}) = \min_{P,Q} \max_{\{1 \le i \le m, 1 \le j \le m'\}} \frac{1}{2} |d_X(x_i, y_{Q_j}) - d_Y(y_j, x_{P_i})|$$

where $P : \{1, \ldots, m\} \rightarrow \{1, \ldots, m'\}$ and
$Q : \{1, \ldots, m'\} \rightarrow \{1, \ldots, m\}$

Bending Invariance (V)

- But ... it looks too complex...there are too many transformations (P&Q) to try....
- Let's reduce the complexity just a bit...
- Take m=m', P a permutation and $P = Q^{-1}$
- Then we have a few more tools to deal with this Matching Problem
- Define, for

$$\mathbb{X} = \{x_1, \ldots, x_m\} \text{ and } \mathbb{Y} = \{y_1, \ldots, y_m\}$$

$$d_{\mathcal{I}}(\mathbb{X}, \mathbb{Y}) := \min_{\pi \in \mathcal{P}_n} \max_{1 \le i,j \le n} \frac{1}{2} |d_X(x_i, x_j) - d_Y(y_{\pi_i}, y_{\pi_j})|$$

Bending Invariance (VI)

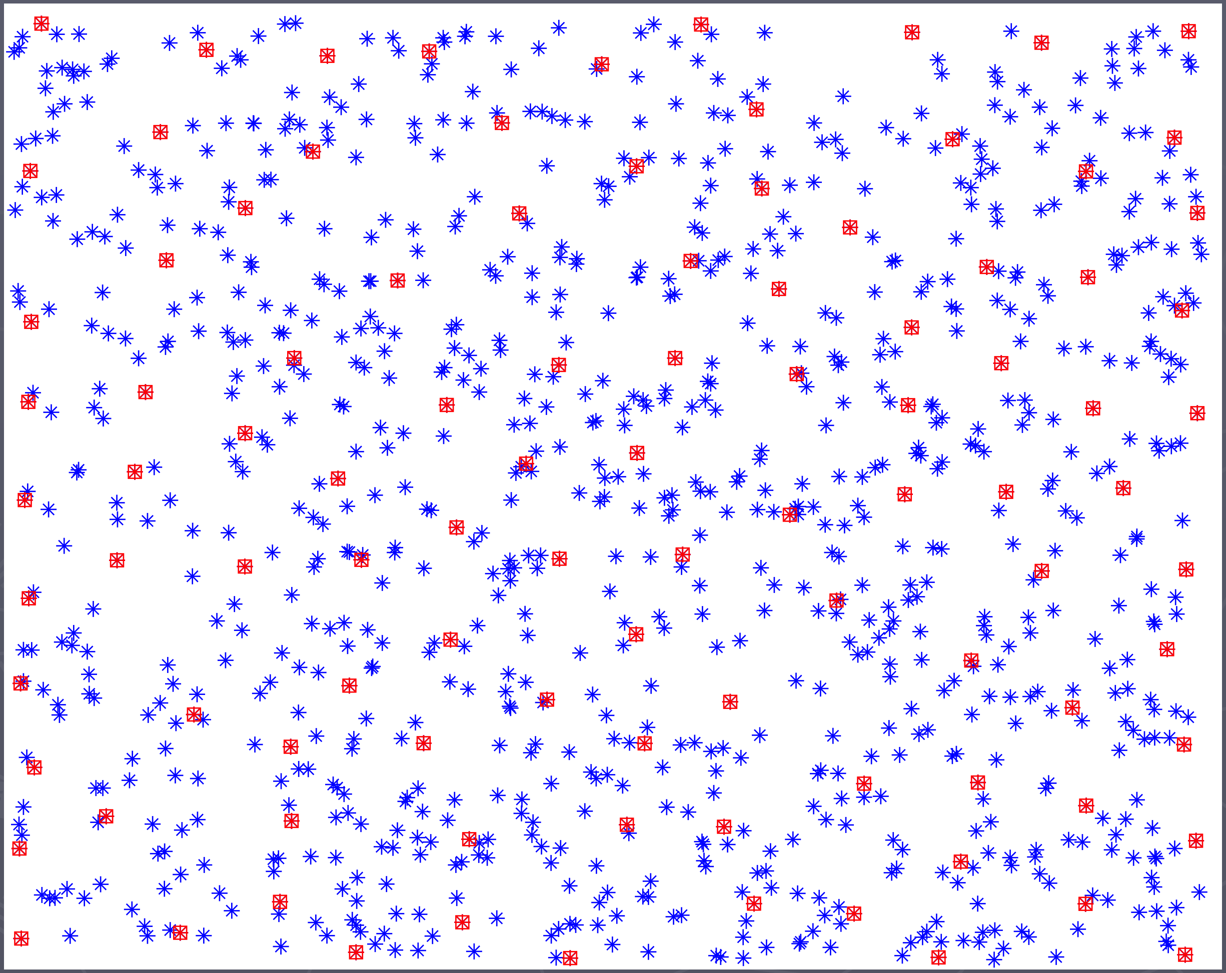
- Obviously, $d_{\mathcal{GH}}(\mathbb{X},\mathbb{Y}) \leq d_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$
- Then, we take a roundabout way. We consider representative subsets (nets) of each of the input Point Clouds ($n \ll m, m'$)

$$N_{X,n}^{(R,s)} \subset \mathbb{X}_m \text{ and } N_{Y,n}^{(R',s')} \subset \mathbb{Y}_{m'}$$

$N_{Y,n}^{(R',s')}$ is a R'-covering of X its points are s'-separated

$N_{X,n}^{(R,s)}$ is a R-covering of X its points are s-separated

Bending Invariance (VII)



$\mathbb{X}_m$ is in blue and $N_{X,n}^{(R,s)}$ in red

Bending Invariance (VIII)

- Hence,

$$d_{\mathcal{GH}}(X,Y) \leq R + R' + \; d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, N_{Y,n}^{(R',s')}\right)$$

- Then, if the RHS is small, X and Y will be similar.

- What about the other implication? if X and Y are similar, will we see this through any nets

$$N_{X,n}^{(R,s)} \subset \mathbb{X}_m \; \underline{\text{and}} \; N_{Y,n}^{(R',s')} \subset \mathbb{Y}_{m'}$$

?

Bending Invariance (IX)

- Basically, by the definition of $d_{\mathcal{GH}}(X,Y)(=\eta)$ given $N_{X,n}^{(R,s)} \subset \mathbb{X}_m \subset X$ one can find a subset of points $\widehat{\mathbb{Y}}_n = \{\hat{y}_1, \ldots, \hat{y}_n\}$ in Y (not in $\mathbb{Y}_{m'}$!) with

$$d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, \widehat{\mathbb{Y}}_n\right) \leq \eta$$

- We could try to use the triangle inequality:

$$\begin{aligned} d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, N_{Y,n}^{(R',s')}\right) &\leq d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, \widehat{\mathbb{Y}}_n\right) + d_{\mathcal{I}}\left(N_{Y,n}^{(R',s')}, \widehat{\mathbb{Y}}_n\right) \\ &\leq \eta + \text{small}(R, R', s, s') \end{aligned}$$

Bending Invariance (X)

- But what we want to bound by $\mathsf{small}(R, R', s, s')$ is a combinatorial-metric distance between two different nets of the same metric space, there are counterexamples in some cases.
- But we can approach this probabilistically: we will model the point clouds as i.i.d. (uniform) samples from the surfaces

Bending Invariance (XI)
-idea for a theorem-

- We can try deal with

$$d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, N_{Y,n}^{(R',s')}\right) \leq d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, \widehat{\mathbb{Y}}_n\right) + d_{\mathcal{I}}\left(N_{Y,n}^{(R',s')}, \widehat{\mathbb{Y}}_n\right)$$

in a more relaxed way:

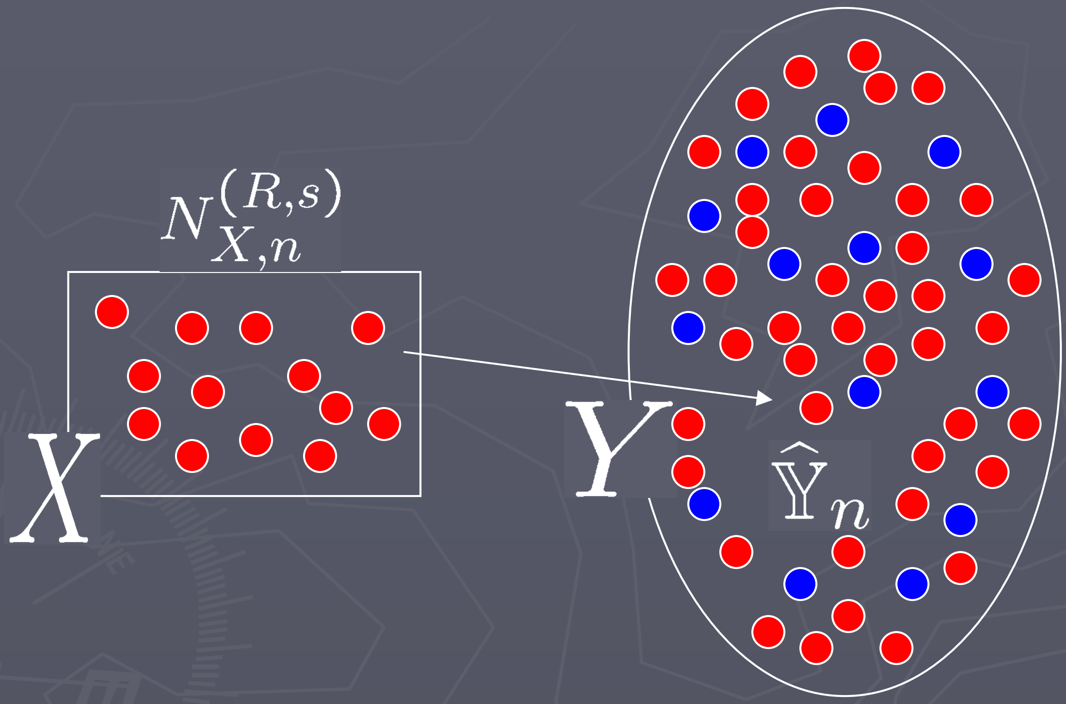
- Break symmetry...
- Allow for $N_{Y,n}^{(R',s')}$ to be chosen from $\mathbb{Y}_{m'}$:

$$d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, N_{Y,n}\right) \leq d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, \widehat{\mathbb{Y}}_n\right) + d_{\mathcal{I}}\left(N_{Y,n}, \widehat{\mathbb{Y}}_n\right)$$

Choose it so as to make the last term small....

Bending Invariance (XII)

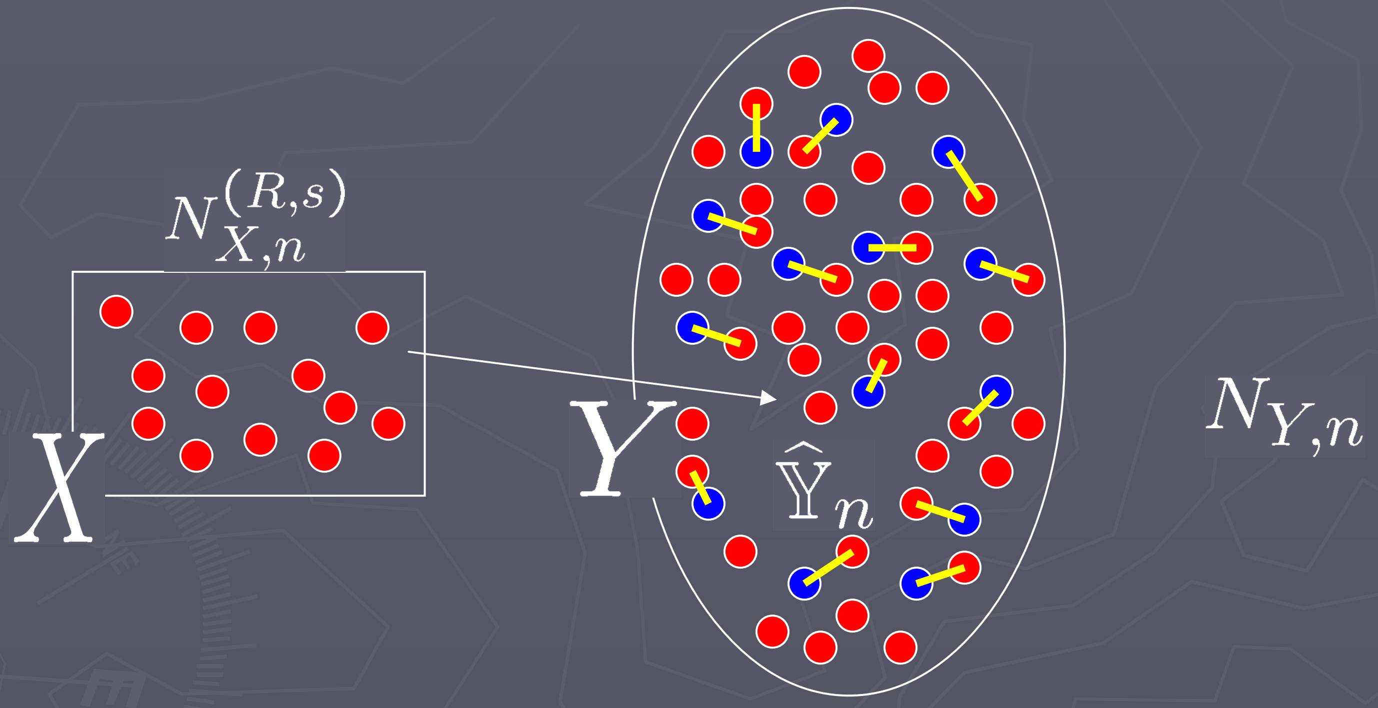
-idea for a theorem-



$N_{Y,n}$???

In red, given points. In blue: $\widehat{\mathbb{Y}}_n$

Bending Invariance (XII)
-idea for a theorem-



In red, given points. In blue: $\widehat{\mathbb{Y}}_n$

Bending Invariance (XIII)
(we need to use probabilities)

Theorem Let X and Y compact submanifolds of $\mathbb{R}^d$. Let $N_{X,n}^{(r,s)}$ be a covering of X with separation s such that for some positive constant c, $s - 2d_{\mathcal{GH}}(X,Y) > c$. Then, given any number $p \in (0,1)$, there exists a positive integer $m = m_n(p)$ such that if $\mathbf{Y}_m = \{y_k\}_{k=1}^m$ is a sequence of $i.i.d.$ points sampled uniformly from Y, we can find, with probability at least p, a set of n different indices $\{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ such that

$$d_{\mathcal{I}}(N_{X,n}^{(r,s)}, \{y_{i_1}, \ldots, y_{i_n}\}) \leq 3\, d_{\mathcal{GH}}(X,Y) + r$$

Idea of the Proof (I)

- We know that if $d_{\mathcal{GH}}(X,Y) \leq \eta$ then we can find $\widehat{\mathbb{Y}}_n = \{\hat{y}_1, \ldots, \hat{y}_n\} \subset Y$ such that

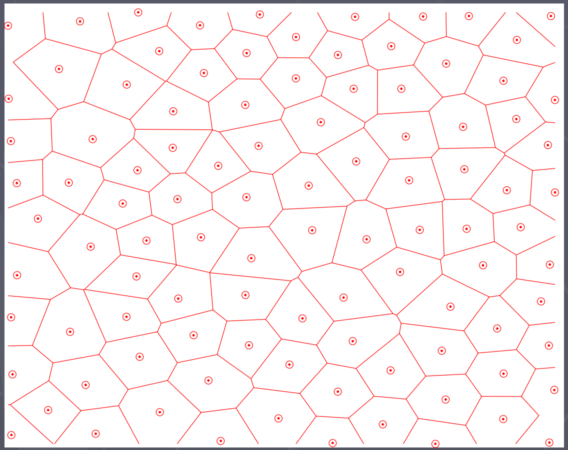
$$d_{\mathcal{I}}\left(N_{X,n}^{(R,s)}, \widehat{\mathbb{Y}}_n\right) \leq \eta$$

And $\widehat{\mathbb{Y}}_n$ is a

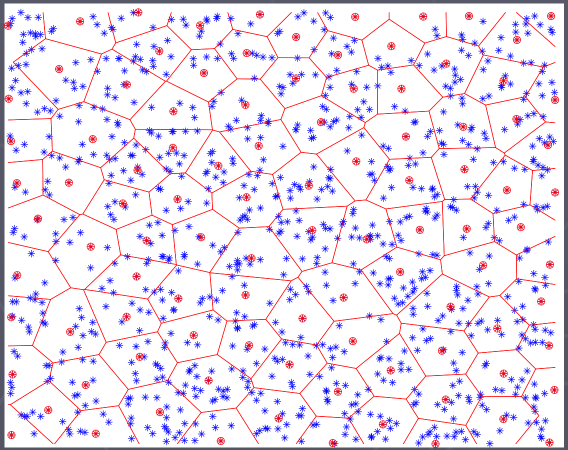
$(R+2\eta)$-covering of Y, and also $(s-2\eta)$ separated as well, separation is <u>important</u>

Idea of the Proof (II)

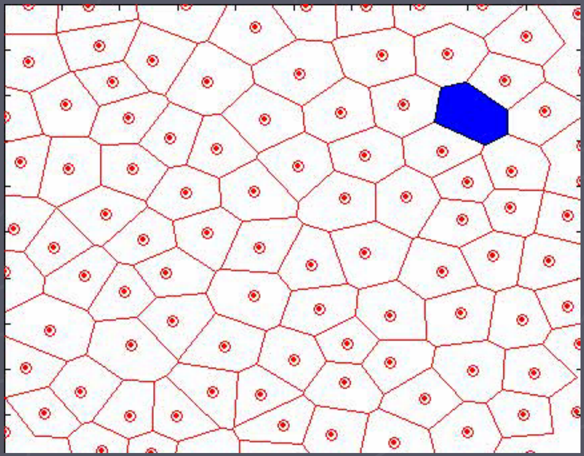
- Coupon Collecting... Occupancy of Voronoi Cells of $\widehat{\mathbb{Y}}_n$ are "coupons" I want to collect



Voronoi Diagram



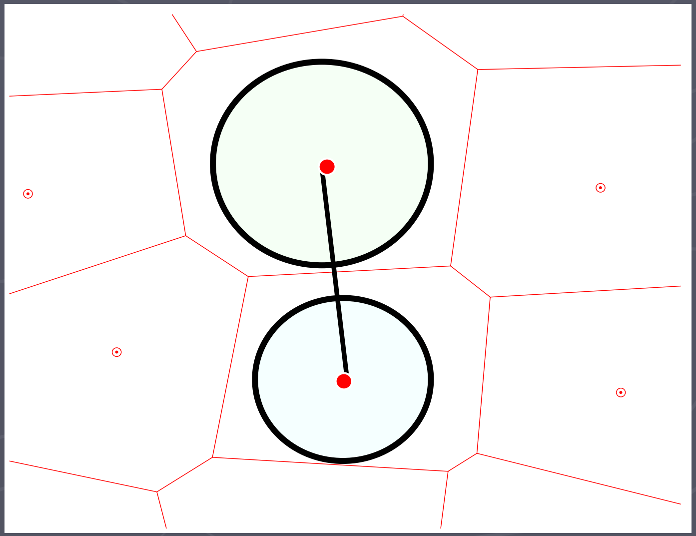
Voronoi Diagram + Point Cloud



Seen as Sequential Sampling

Idea of the Proof (III)

- Large separation ($(s - 2\eta) = c > 0$) is important because it somehow controls the area of Voronoi Cells, small area means small probability of occupancy...we don't want that!



In practice...

- In practice we compute a symmetrical quantity

$$d_{\mathcal{F}}(\mathbb{X}_m, \mathbb{Y}_{m'}) := \max\left(L_X^Y, L_Y^X\right)$$

where

$$L_X^Y := \min_{J_n \subset \{1:m\}} d_{\mathcal{I}}(N_{X,n}^{(r,s)}, \mathbb{Y}_m[J_n])$$

and

$$L_Y^X := \min_{I_n \subset \{1:m\}} d_{\mathcal{I}}(N_{Y,n}^{(r',s')}, \mathbb{X}_m[I_n])$$

For which we have control in probability

Control with Probability

- We obtain (with some probability)

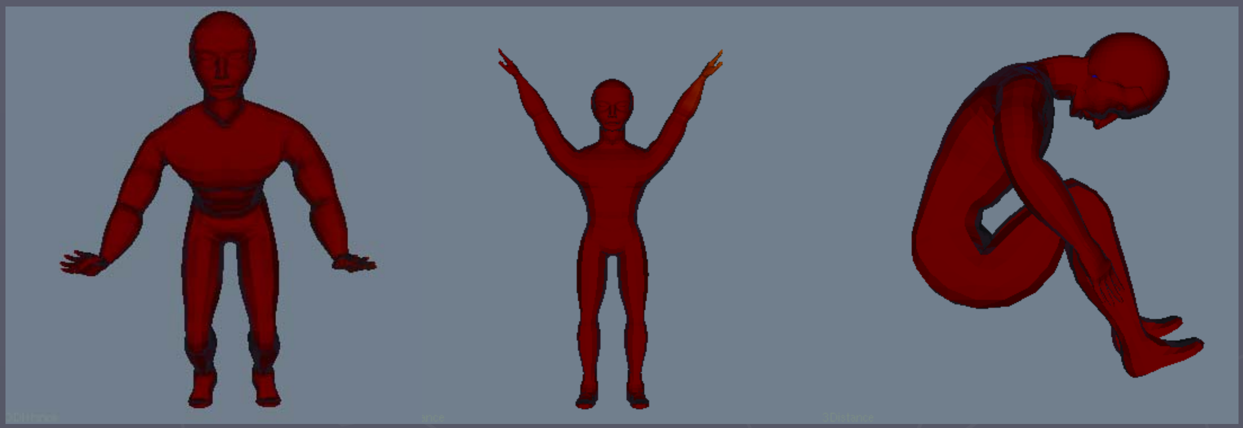
$$
\begin{aligned}
d_{\mathcal{GH}}(\mathbb{X}_m, \mathbb{Y}_{m'}) &\leq d_{\mathcal{F}}(\mathbb{X}_m, \mathbb{Y}_{m'}) \\
&+ \min\left(d_{\mathcal{H}}^X(\mathbb{X}_m, \mathbb{X}_m[I_n]),\, d_{\mathcal{H}}^Y(\mathbb{Y}_{m'}, \mathbb{Y}_{m'}[J_n])\right) \\
&+ \max(r, r')
\end{aligned}
$$

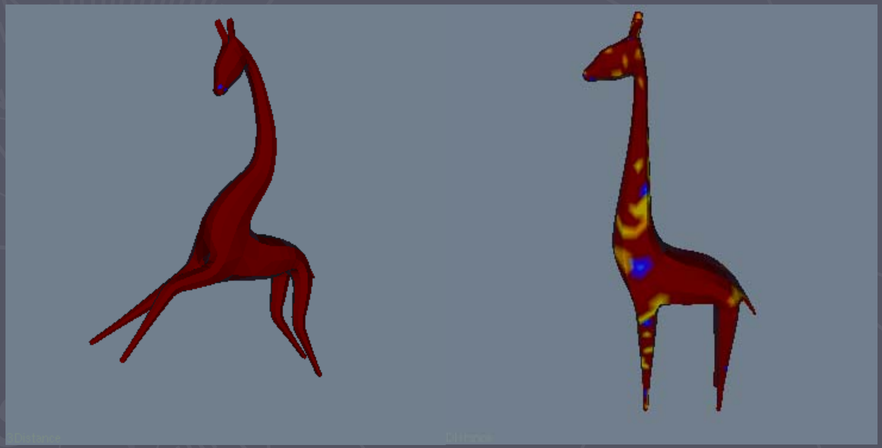
- Where the second term measures how well the selected sub-point-clouds represent the initial ones...

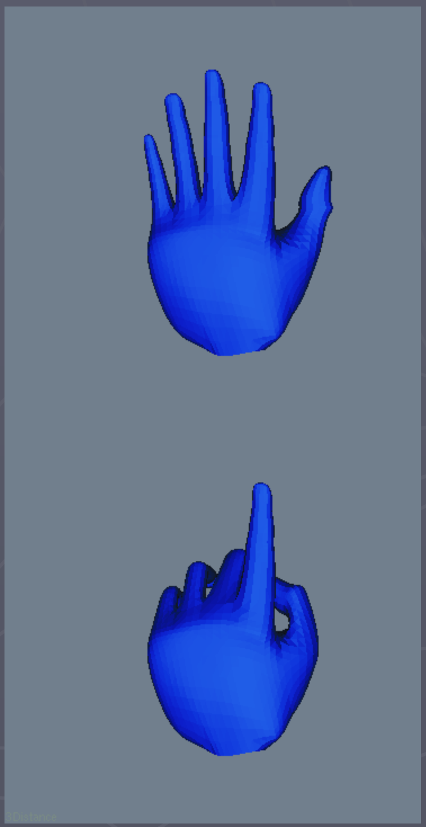
Computational Considerations

- Bounds on the number of sample points needed...depending on the prespecified probability
- Coverings of X and Y found using Farthest Point Sampling
- Geodesic distances for points on X and Y
- Select matching points of X and Y following our theory

Examples







Datasets courtesy of Prof. R. Kimmel
and his group at the Technion.

Conclusions

- Theoretical and computational framework for comparing point clouds under some invariance..
- N-dim (experimentation..)
- The future:
  - Improving computational complexity
  - Other invariance and topology
  - Provably good approximation

- The preprint can be found at: (IMA preprints)

http://www.ima.umn.edu/preprints/apr2004/1978.pdf

Thanks….