Seismic imaging & processing with Curvelets and more ...

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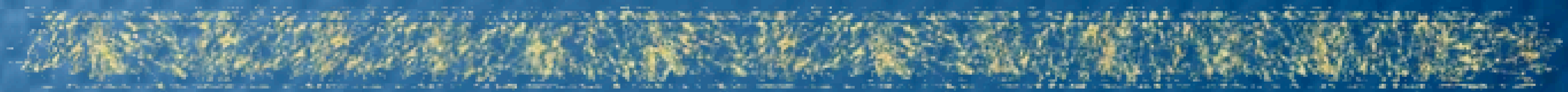


joint work with: Bernabe, Moghaddam, Dupuis, Hennenfent

thanks to: Candes, Demanet, Do, Sacchi, Rickett

Imaging & processing

reflectors	singularities	wavelet	wavelet
stacking	integration	normal operator	amplitude corrections
migration	adjoint FIO	Born approx	single scattering
LS migration	Pseudo inverse	Kirchoff migration	Generalized Radon
stratigraphy	geometry	Hamiltonian flow	Ray tracing
scale exponent	lithology	?	?

Research program

How **to improve seismic images?**

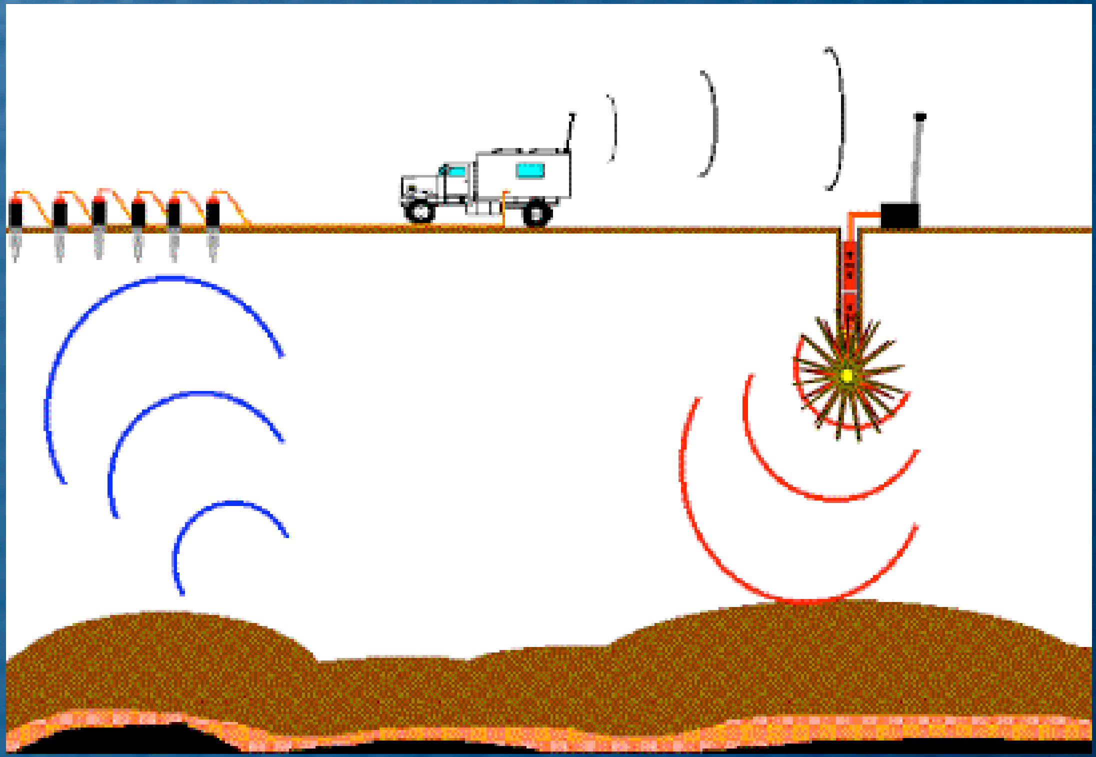
What **is in the image?**

Why **is it in the image?**

Curvelet imaging & processing talk is mostly devoted to the ***'How'.***

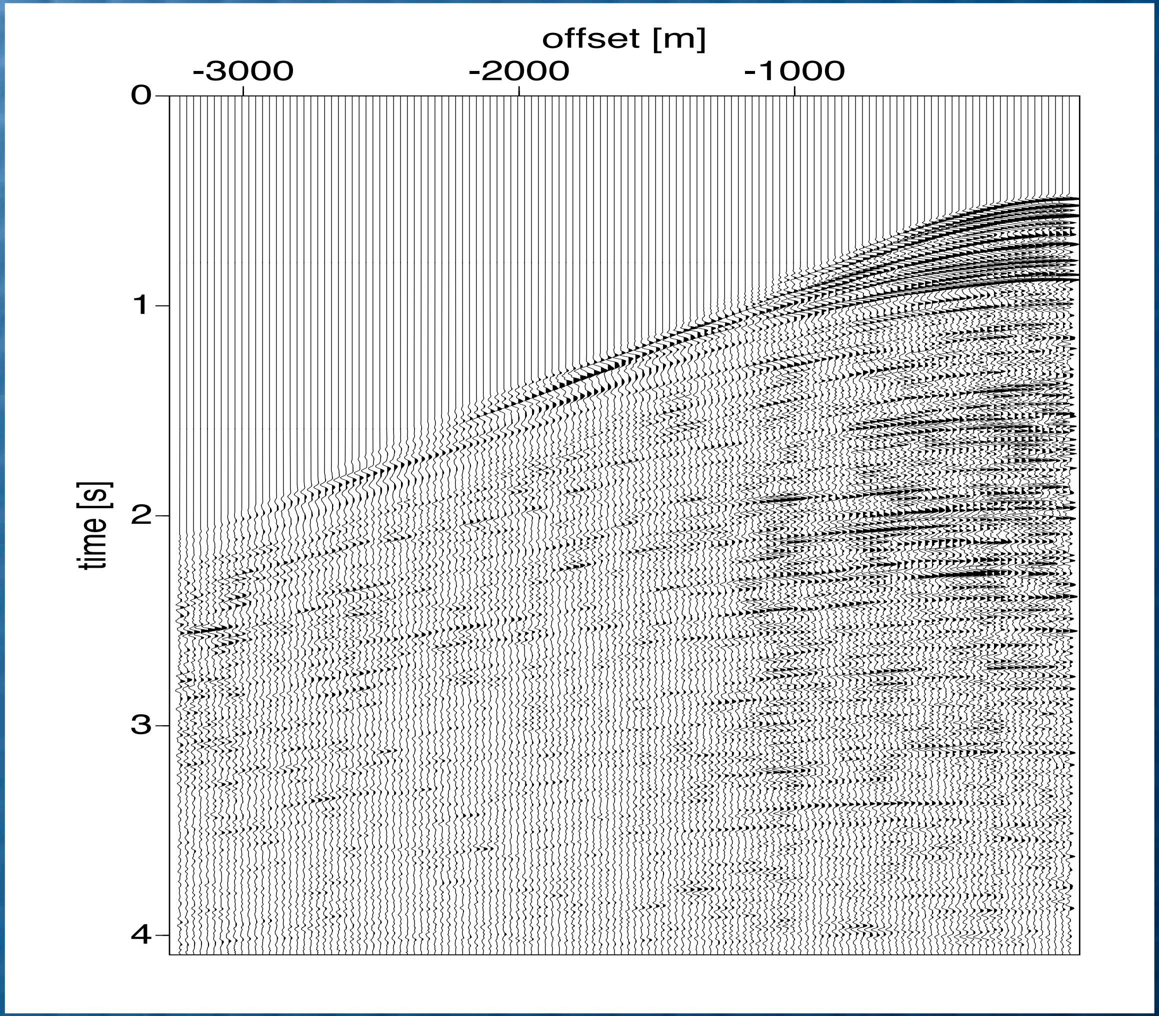
Seismic imaging

- active source
- 2-D acquisition manifold at surface
- huge 5-D data sets
- Many "noise" sources



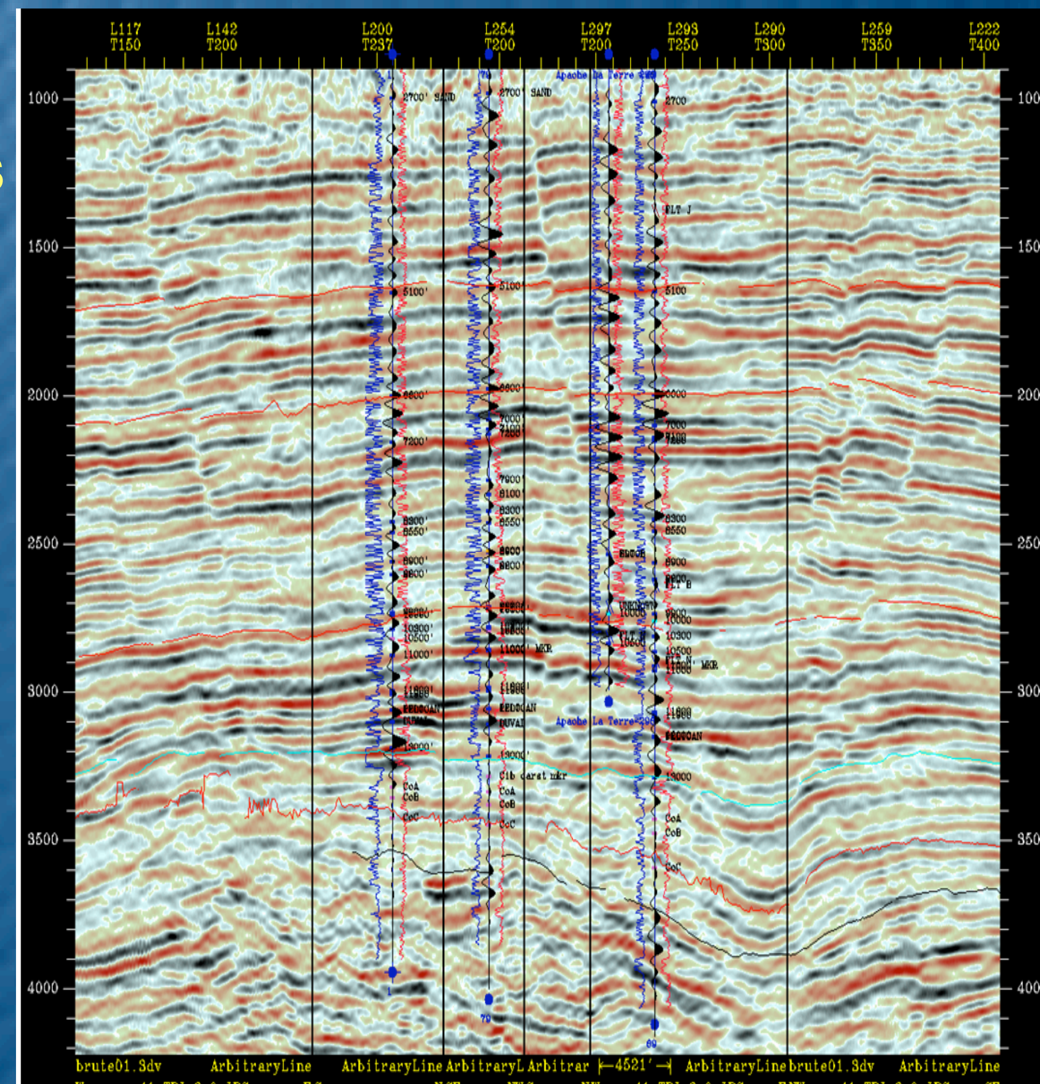
Seismic imaging

- bandwidth limitation
- limited data acquisition
- dispersion
- complex with wave fronts
- noise is coherent, e.g. different components solution PDE look alike



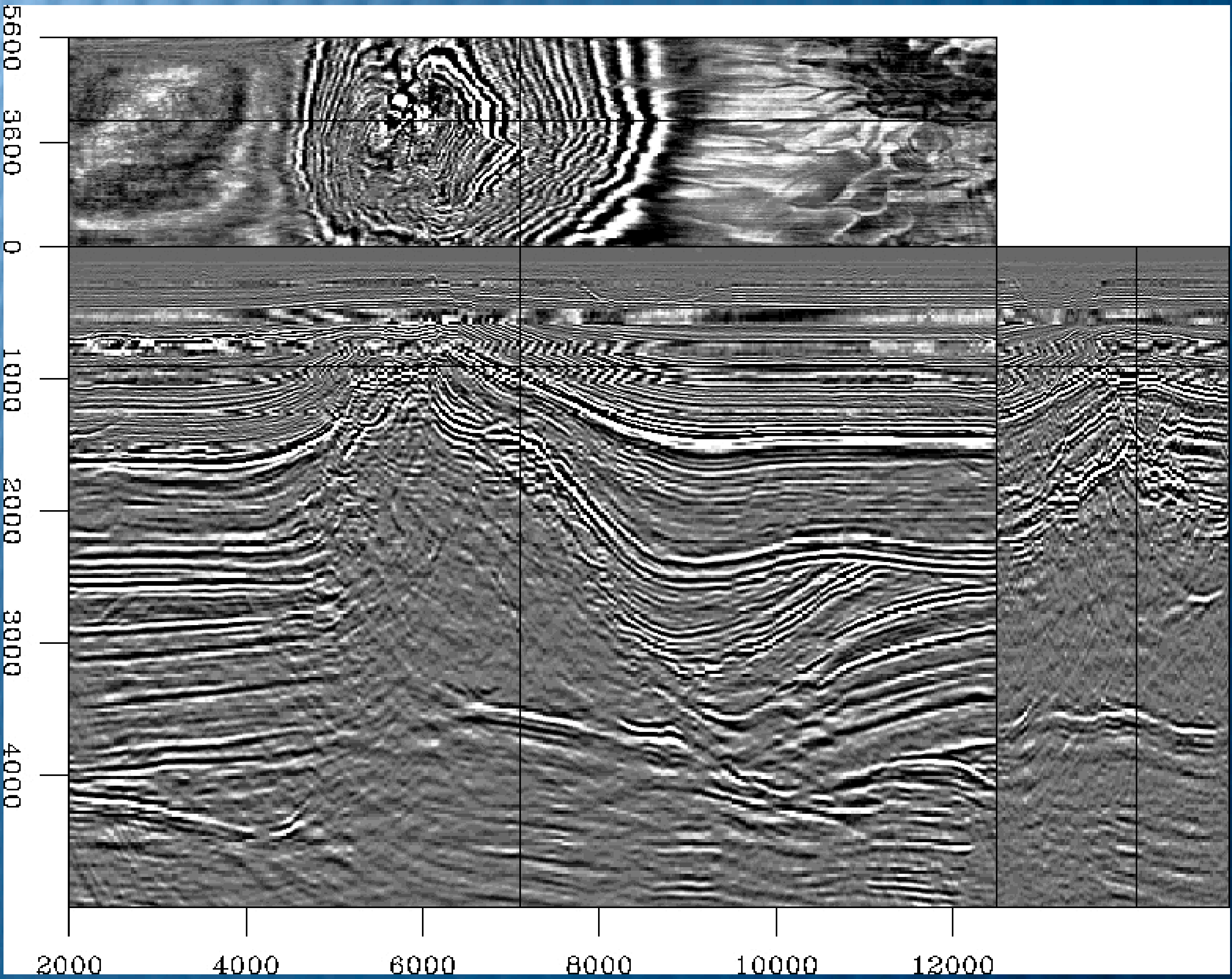
Seismic imaging

- looking for filamental structures
- artifacts look like real reflectors
- hyperbolic PDE’s with singular coeff.
- solution operators are unforgiving
- matrices are full
- little dynamic range
- “dull”, things look way too similar
- sedimental records fractal-like

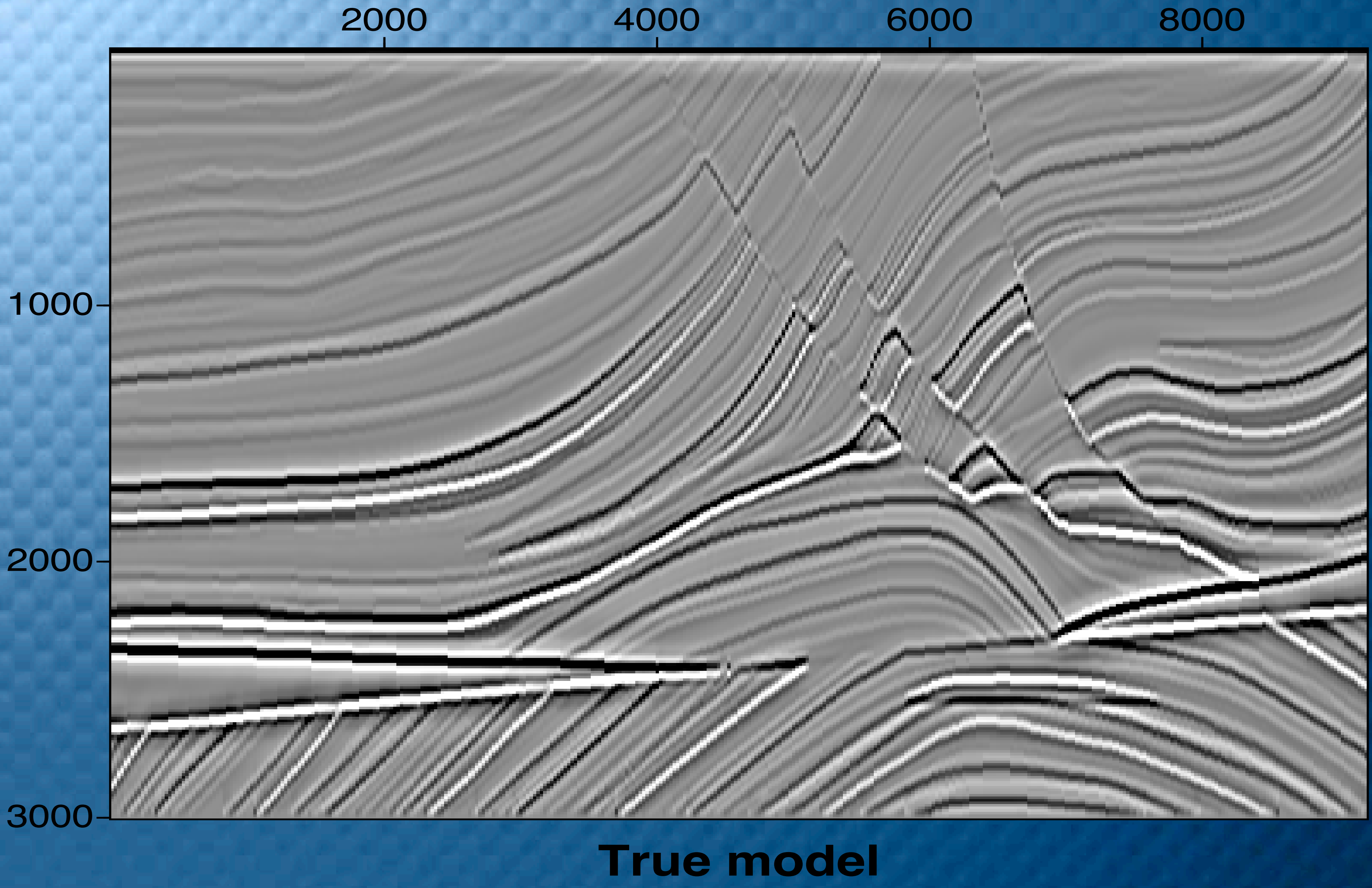


Seismic imaging

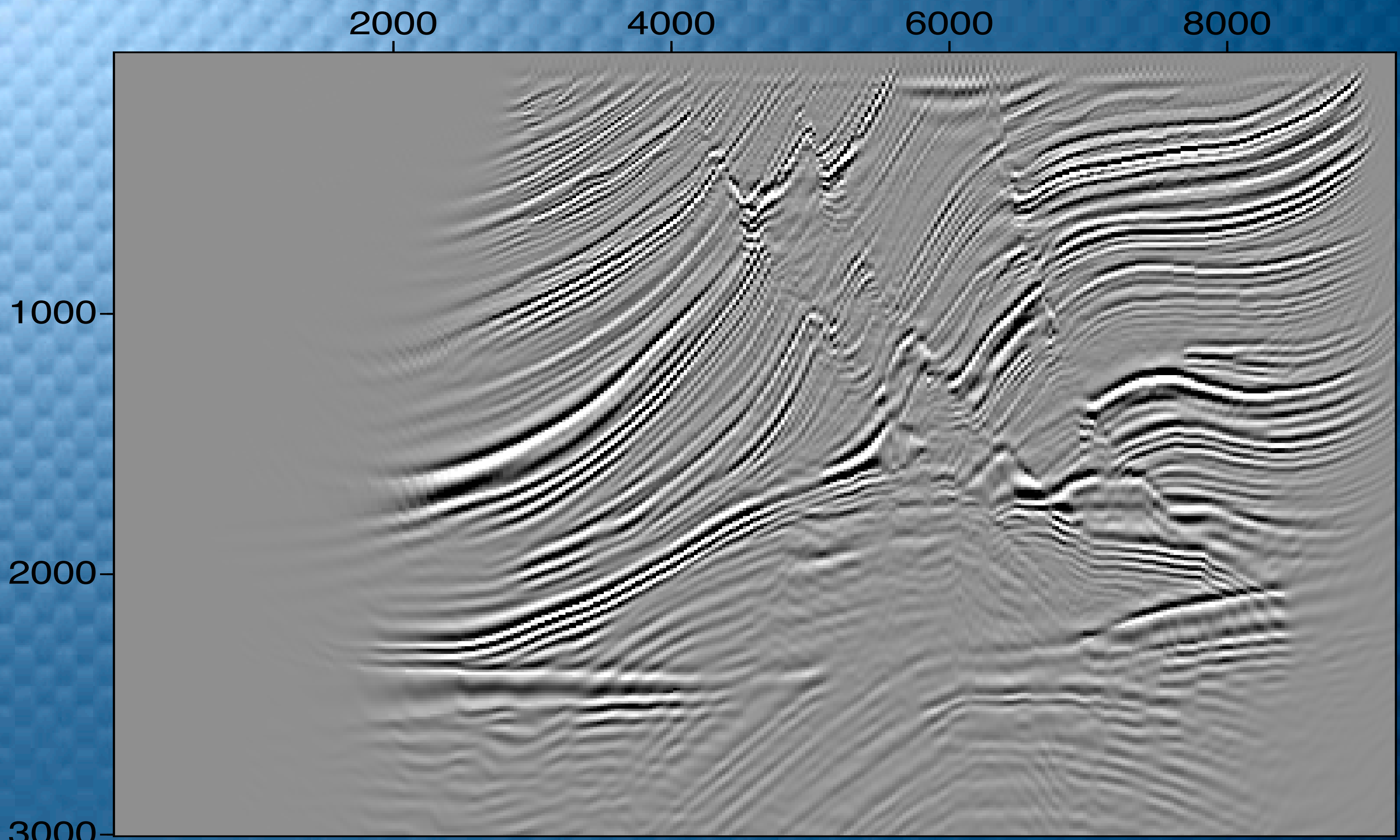
- **Singularities are preserved**
- **Physics ok?**
- **Able to create images**
- **Deal with noise**
- **Compress operators (Smit, Candes, Douma,- & M)**



Synthetic example



Normal operator



Noise-free image

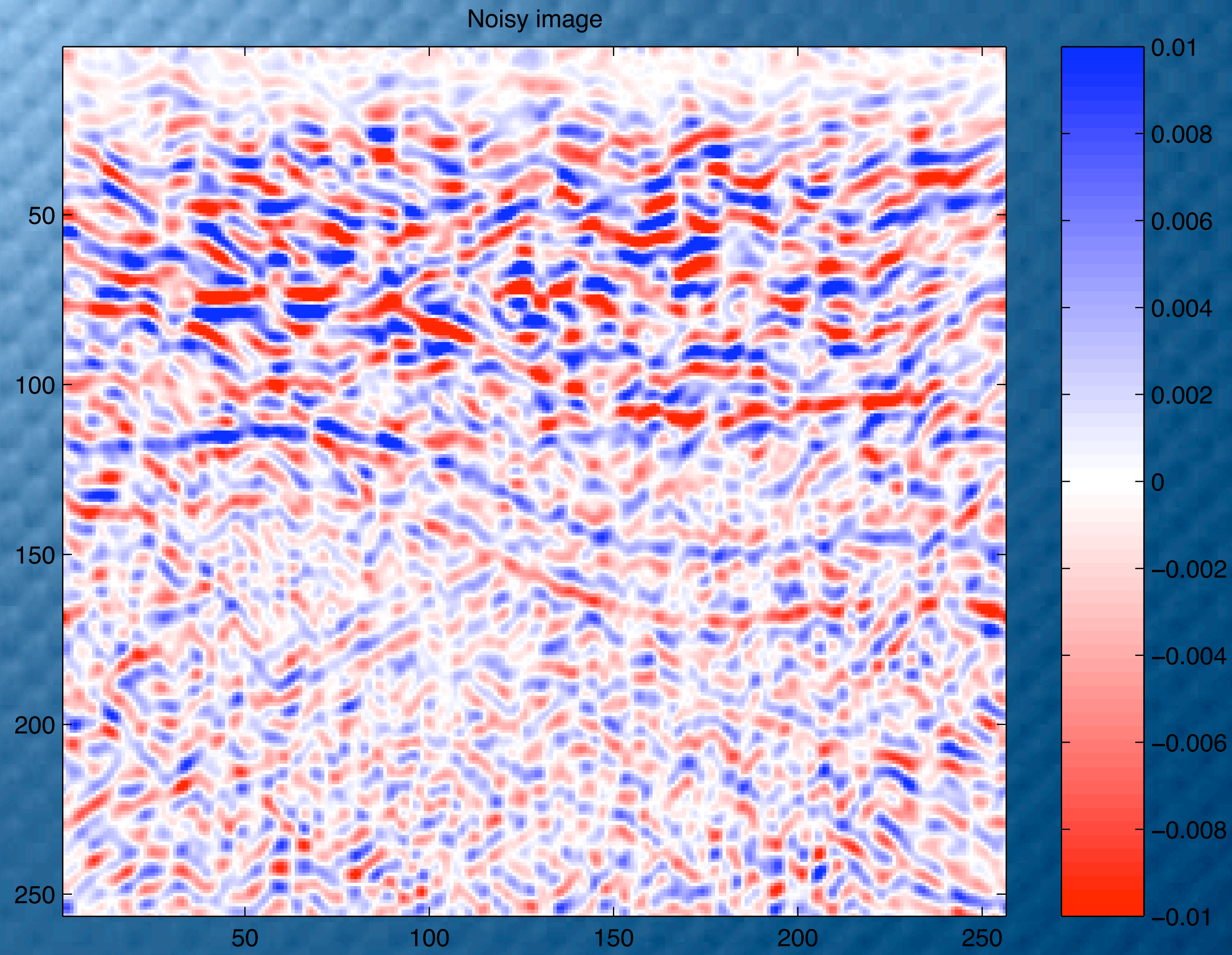
Seismic imaging

We are in the business of

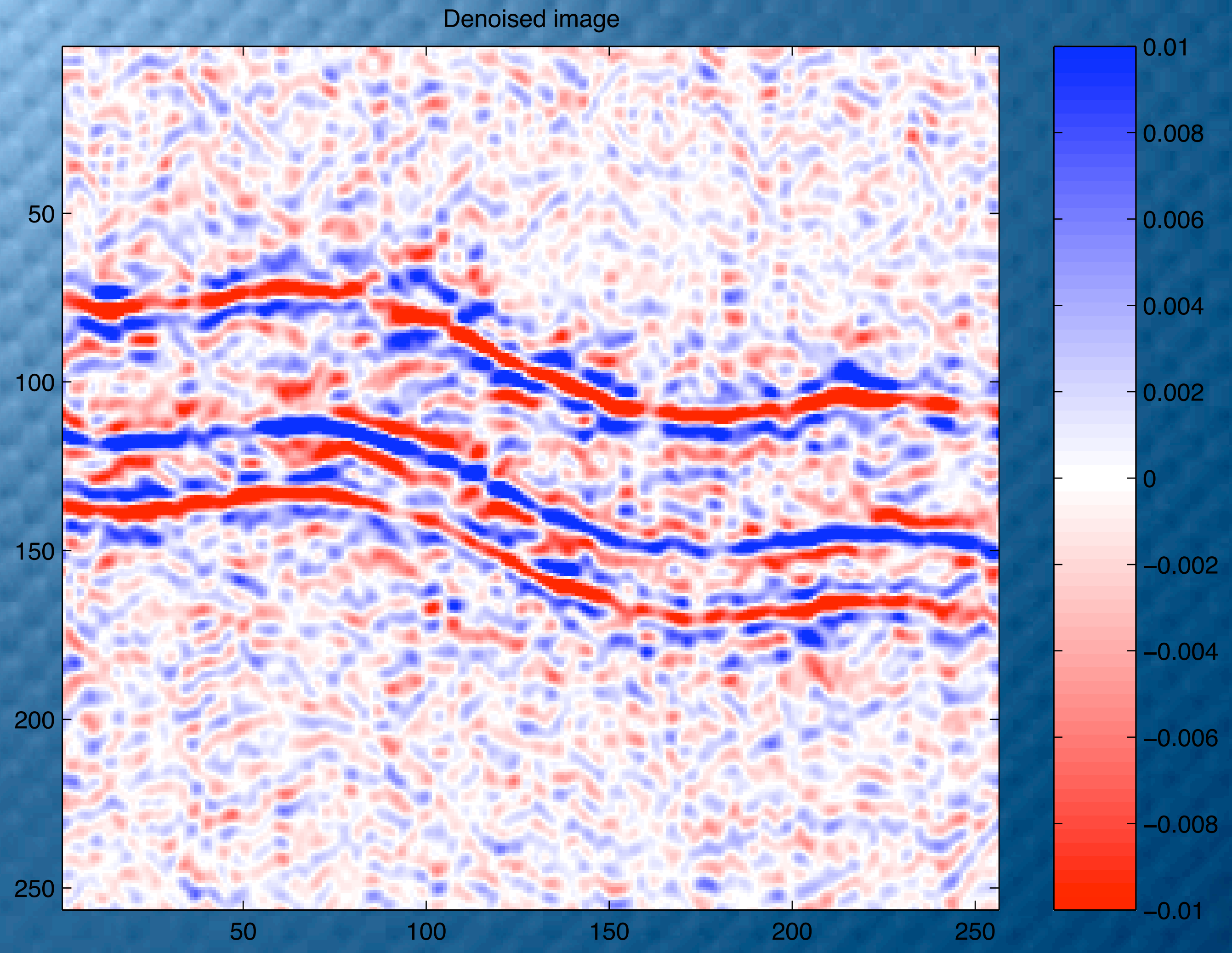
- ***locating the major singularities from bandwidth limited data***
- ***characterization of the major singularities by effective exponents***

In the presence of noise ... Lots of it!

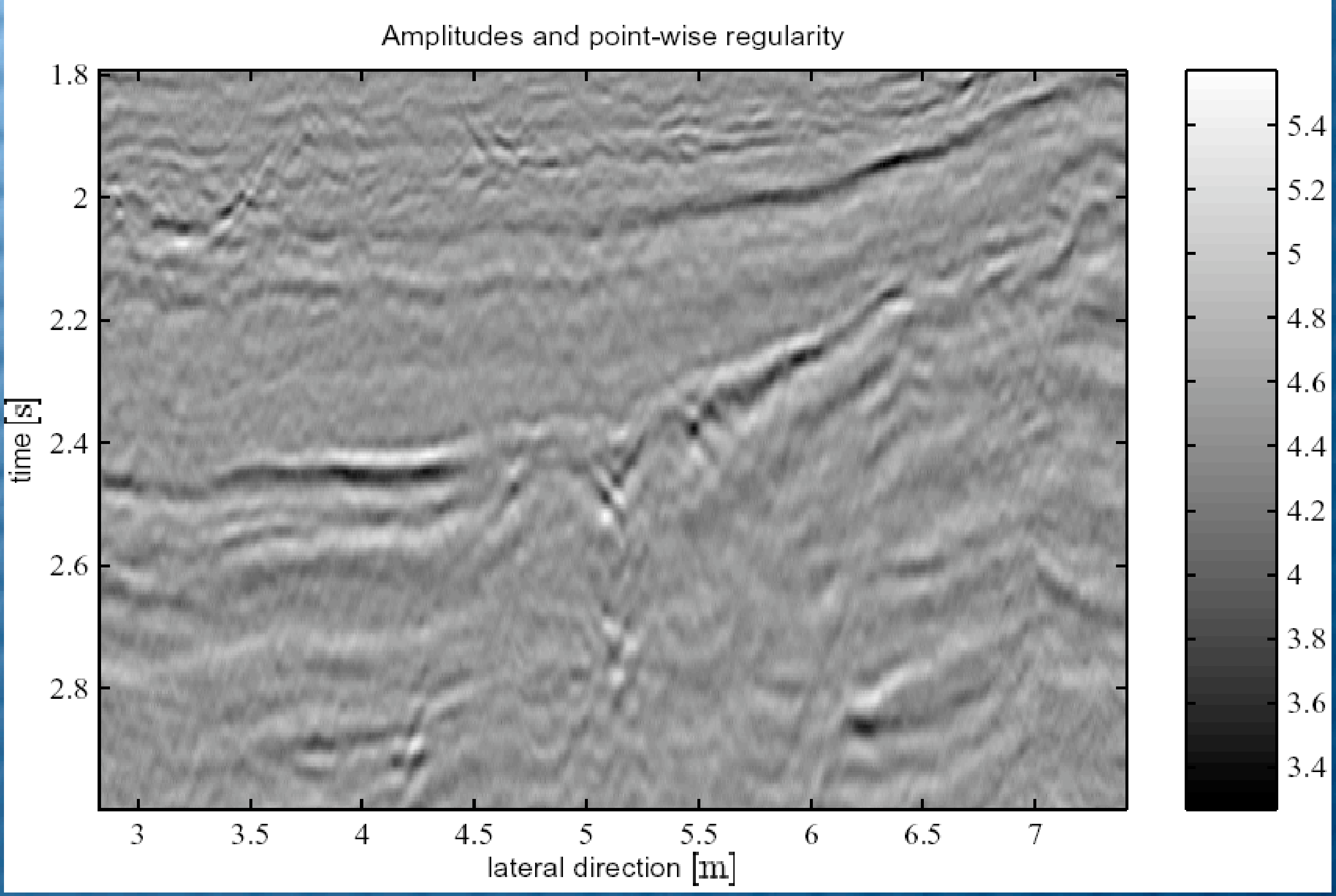
Seismic imaging



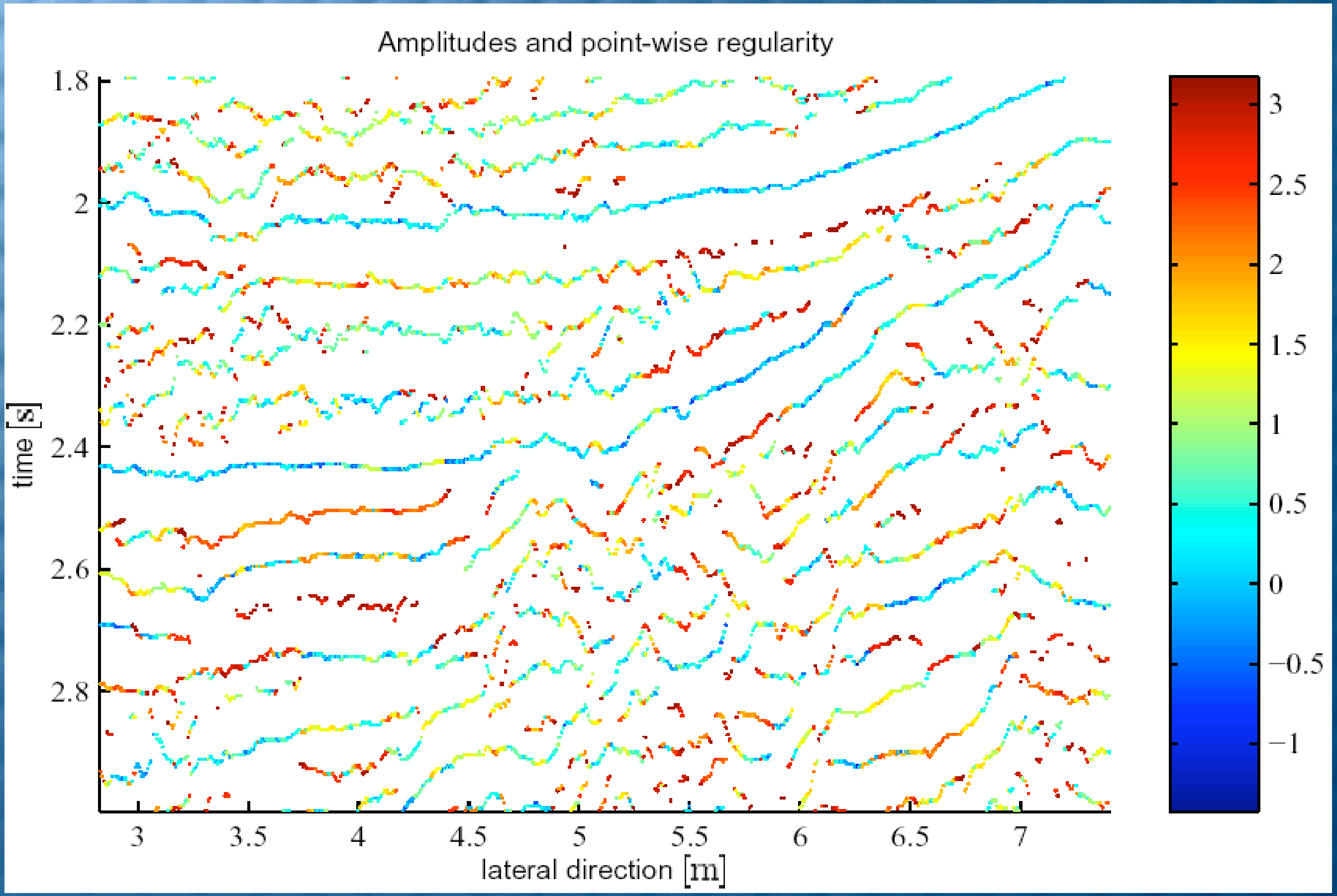
Optimal imaging



Characterization

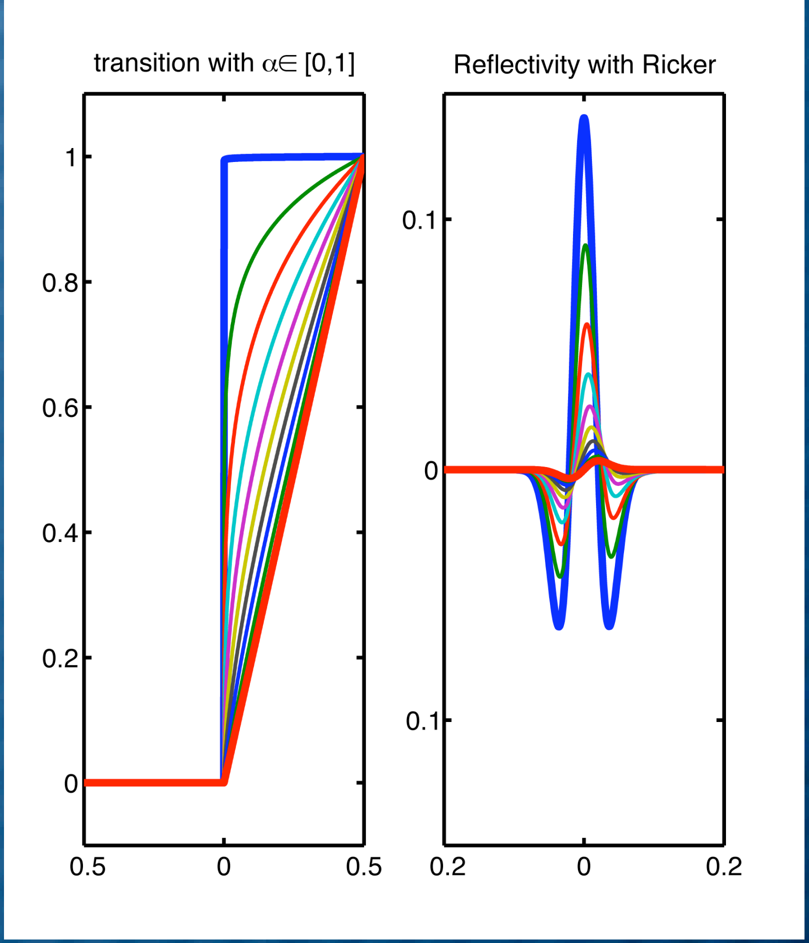


Characterization



Characterization

- between **step - ramp**
- controls ***abruptness***
- amplitude
- waveform
- fractional splines
- geological meaning



Seismic imaging

Singularity-preserved imaging:

- **Exploit redundancy seismic data**
- **Use *non-adaptive* basis-functions**
  - **local, sparse & well-behaved**
  - **non-lin. estimation by thresholding**
- **Approximate the normal/Cov operator**

Seismic imaging

Forward model (physics for the acoustic case):

$$Pu := \left(\frac{1}{c(x)^2} \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right) u(x,t) = f(x,t)$$

- **second order hyperbolic PDE**
- **interested in singularities of the coeff.**

Forward model

Linearized scattering (high-freq. Born):

$$d(r,s,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega (i\omega)^2 \exp(-i\omega t)) \int \mathrm{d}x G_0(r,x,\omega) m(x) G_0(x,s,\omega)$$

with

$$G_0(r,x,\omega) \approx (-i\omega)^{(n-3)/2} \left(\frac{-i\omega}{\pi}\right)^{n/2} \int \mathrm{d}\mathrm{p}_r a(r,\mathrm{p}_r,\omega;x) \exp i\omega\psi(r,\mathrm{p}_r;x)$$

Mind you this is single scattering ...

Seismic imaging

Imaging aims to reconstruct the position of singularities on the model.

An operator is an imaging operator when

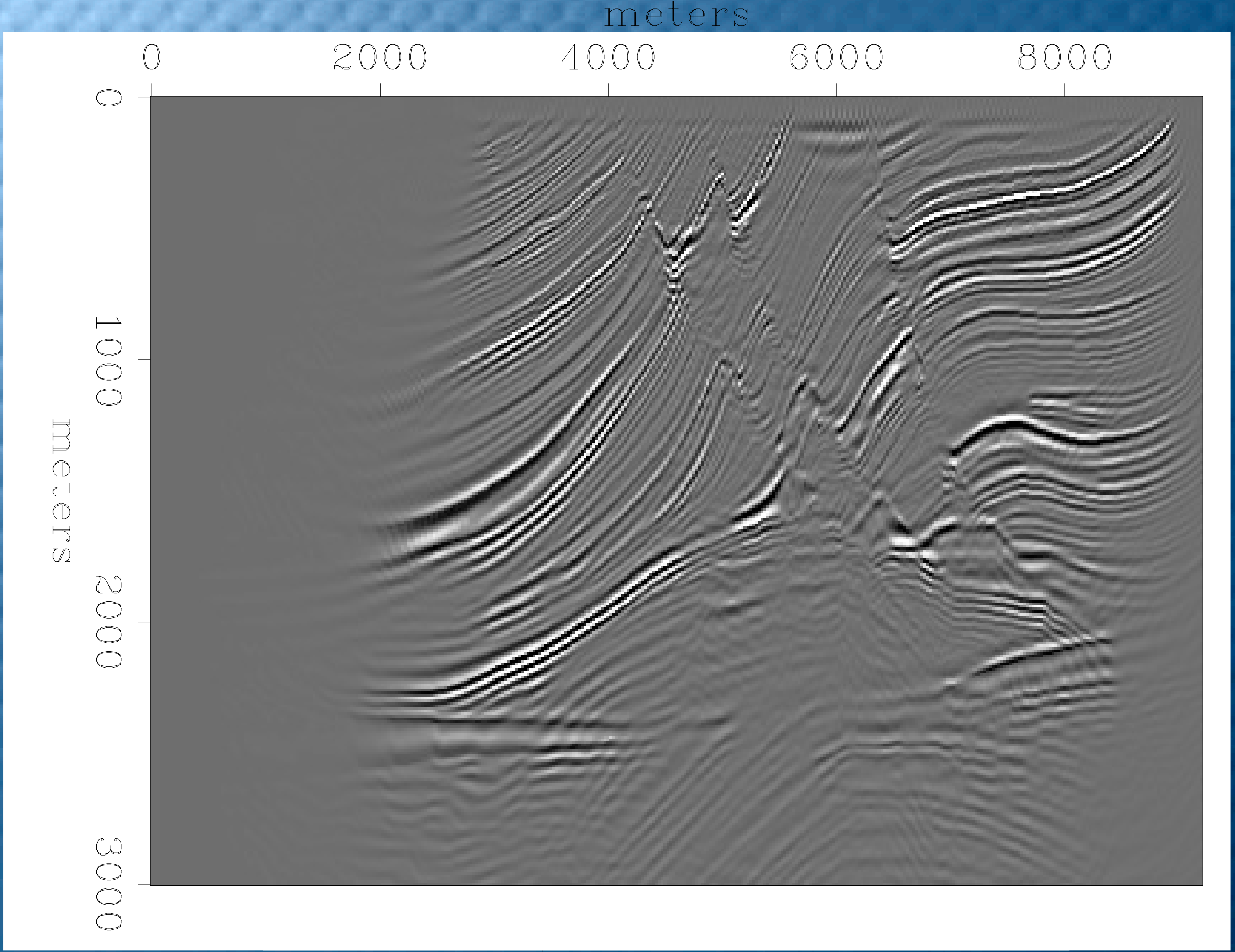
$$\mathbf{K}^T\mathbf{K}\cdot = (\Psi\mathrm{DO})\cdot$$

- depends on the **illumination (0-space)**
- corresponds to the formal **adjoint**

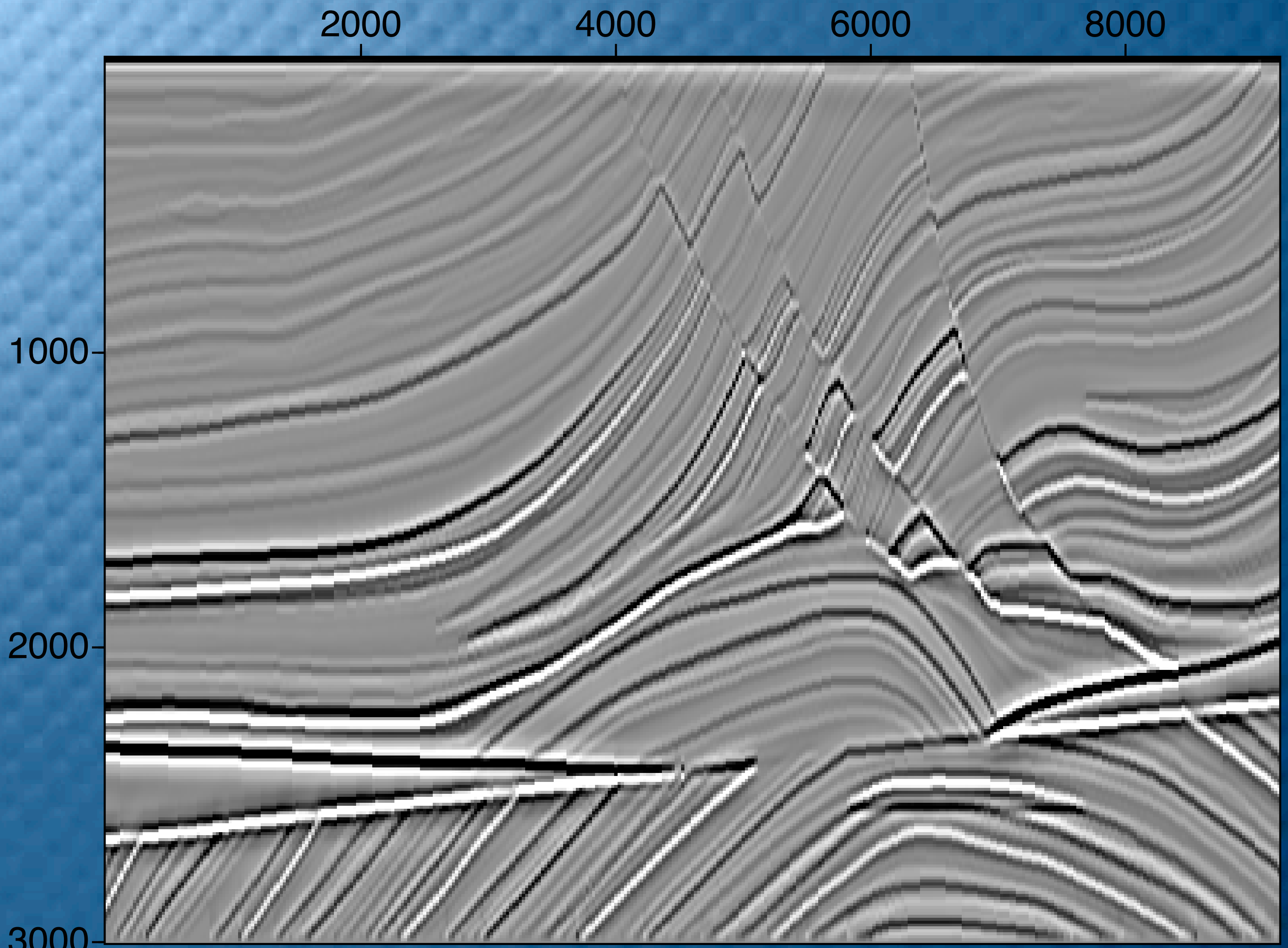
"Preserves" the singularities ...

Stolk 2002, ten Kroode 1997, de Hoop 2000, 2003

Normal operator



Examples



True model

Mapping

Scattering & adjoint are FIO's (Gen. Radon):

$$Tf(x) = \int e^{i\Phi(x,\zeta)} a(x,\zeta) \hat{f}(\zeta) \mathrm{d}\zeta$$

- **transports singularities**

Normal operator is elliptic ψDO

$$Tf(x) = \int e^{i2\pi x\cdot\zeta} a(x,\zeta) \hat{f}(\zeta) \mathrm{d}\zeta$$

- **preserves singularities**

Beylkin 1984. Stolk 2002, ten Kroode 1997, de Hoop 2000, 2003

Seismic imaging

More-Penrose pseudo inverse

- $$\mathbf{m} = \underbrace{\left(\mathbf{K}^*\mathbf{K}\right)^{-1}}_{\text{elliptic }\Psi\text{DO}} \overbrace{\mathbf{K}^*\mathbf{d}}^{\text{FIO}}$$
- when ψDO ⇔ normal operator invertible
- singularities are imaged / preserved
- **assumed noise free!!!**

Stolk 2002, ten Kroode 1997, de Hoop 2000, 2003

Seismic *vs* medical imaging

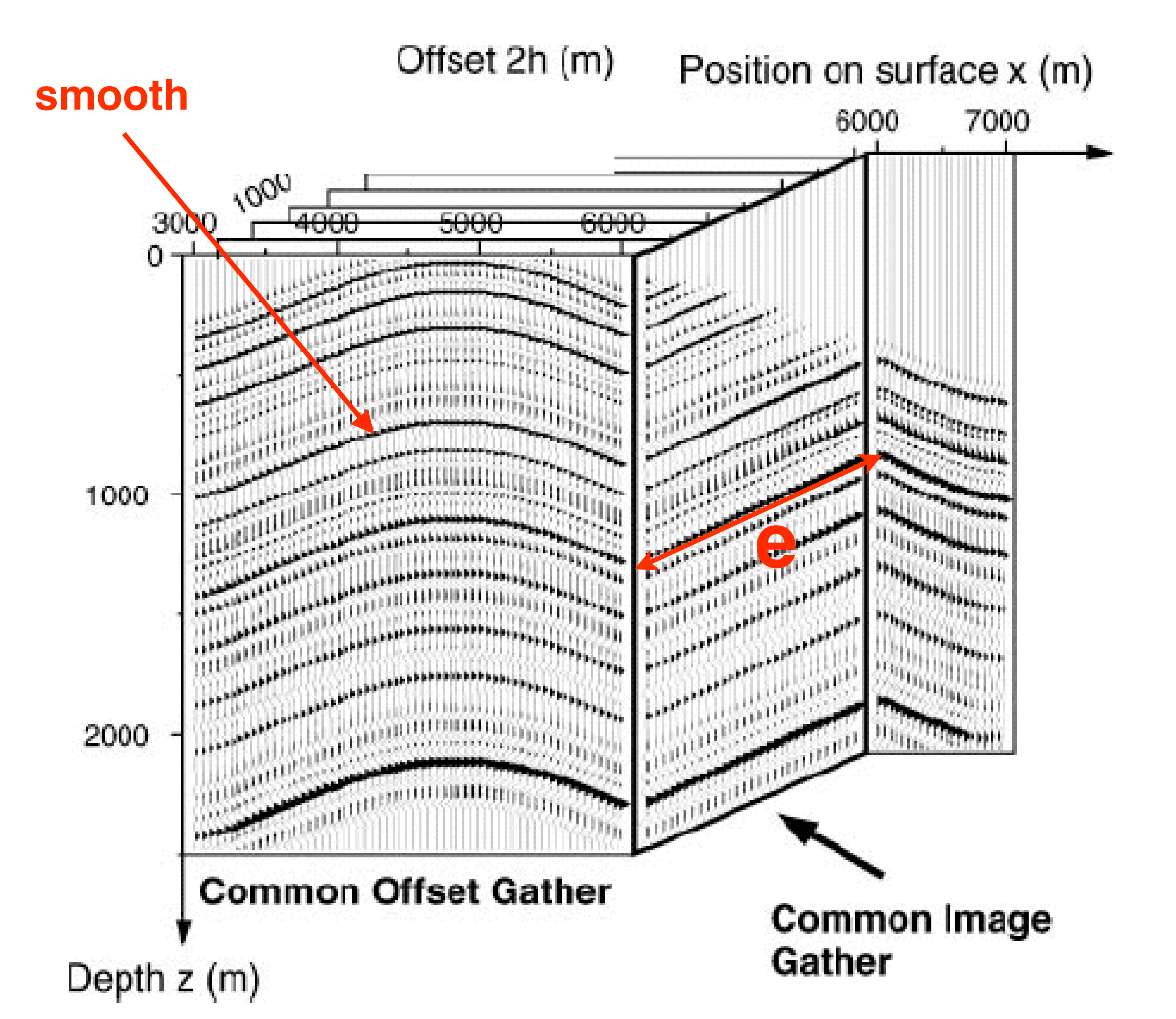
	measure	Normal operator	inverse Normal operator	+	-
"medical" Radon	broadband intensity	$\widehat{\mathbf{K}^*\mathbf{K}}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert^{-1}$	$\widehat{\mathbf{K}^*\mathbf{K}}^{-1}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert$	broadband & non-lin. estimation	high. freq. noise
Gen. Radon	bandwidth limited + phase	$\widehat{\mathbf{K}^*\mathbf{K}}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert$	$\widehat{\mathbf{K}^*\mathbf{K}}^{-1}(\mathbf{k}) \sim \lvert\mathbf{k}\rvert^{-1}$	redundancy & normal operator	bandwidth limitation & recon. moduli smooth part

Seismic imaging

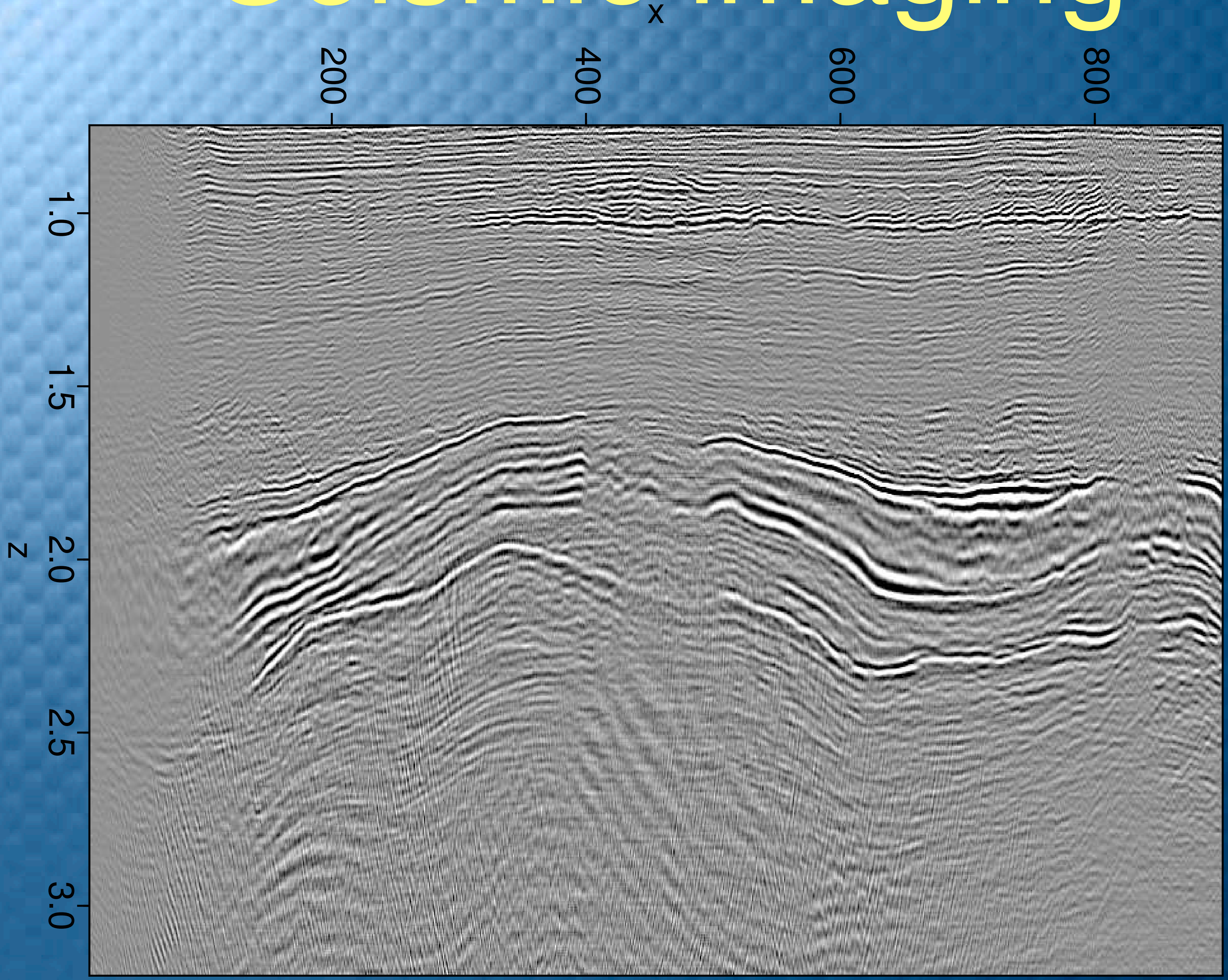
Singularity-preserved imaging:

- **Exploit *redundancy* = *smoothness* in seismic data (- & M)**
- **Use *non-adaptive* basis-functions (-)**
  - **local, sparse & well-behaved**
  - **non-lin. estimation by thresholding**
- **Approximate the normal operator (-)**

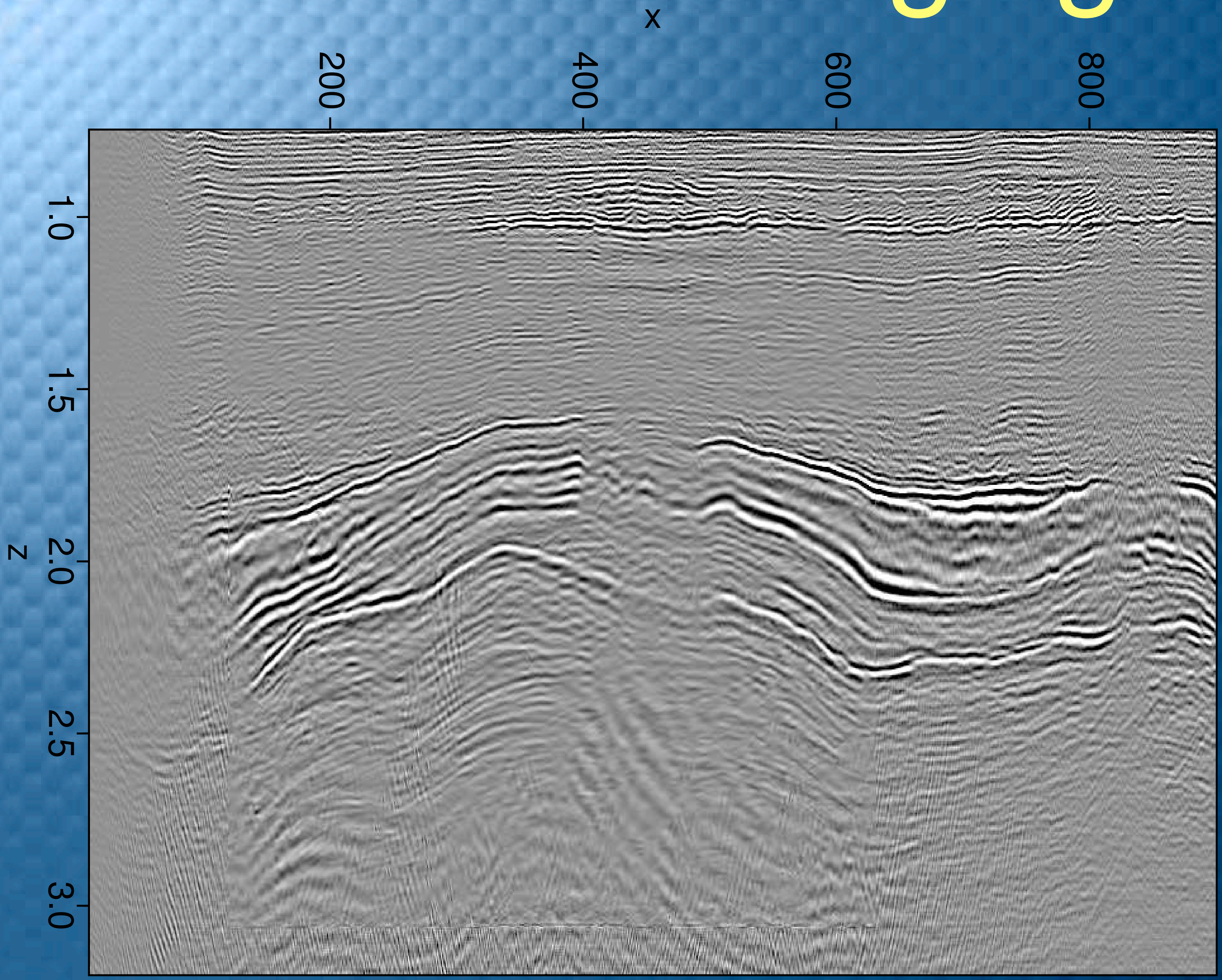
Seismic imaging



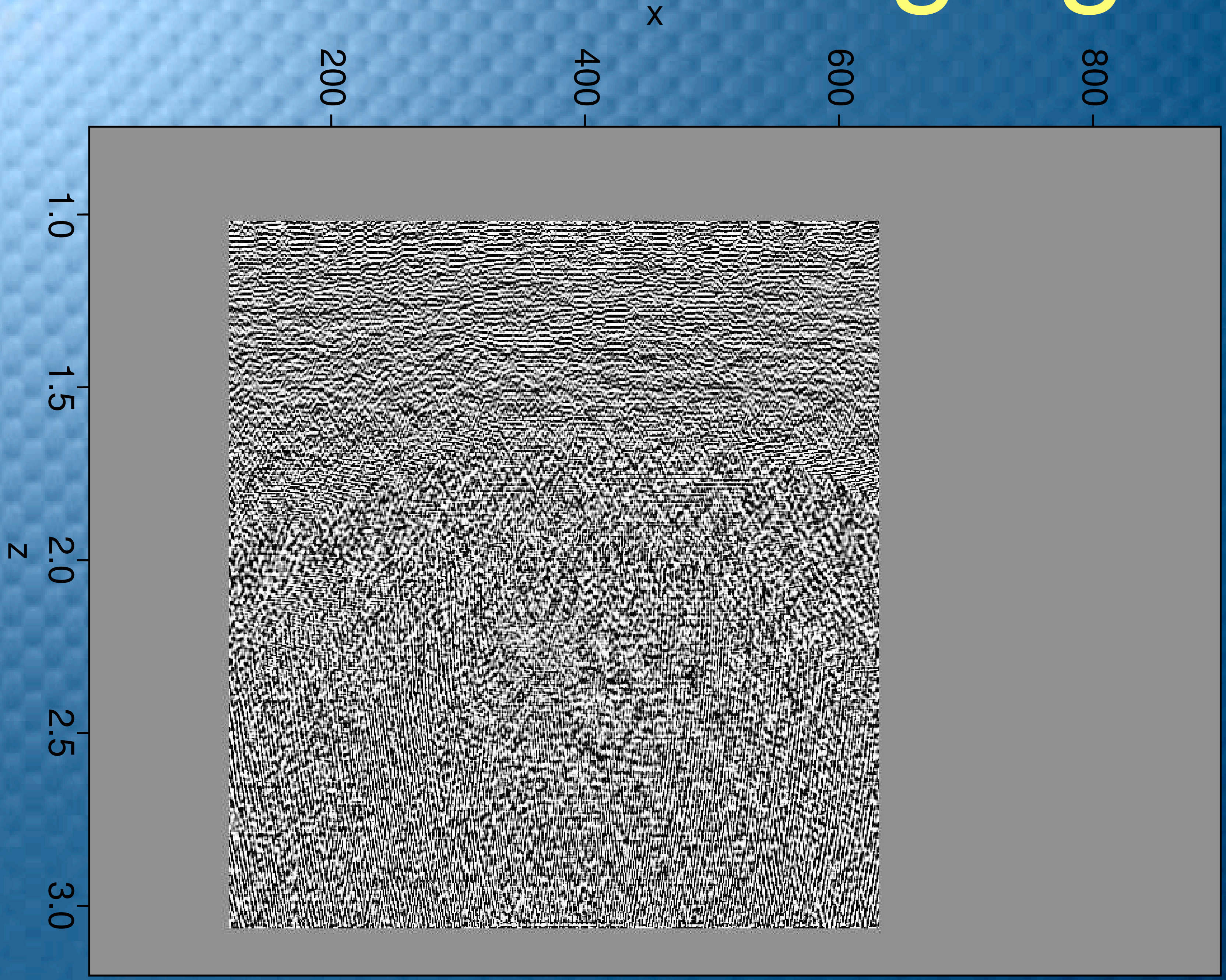
Seismic imaging



Seismic imaging



Seismic imaging



Today's focus

Generic inverse problems:

- **signal separation: what is** $\mathrm{s}_i \text{ for } i = 1 \cdots n$
- $$\mathrm{d} = \mathrm{s}_1 + \cdots + \mathrm{s}_n + \mathbf{n}$$
- **indirect problem: what is** m

$$\mathrm{d} = \mathrm{Km} + \mathrm{n}$$

Develop the appropriate solution strategy

- **noise sources look "alike"**
- **imaging operator with non-trivial 0-space**

Variational problems

Colored noise separation:

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} \frac{1}{2} \|\mathbf{C}_n^{-1/2}(\mathbf{d} - \mathbf{m})\|_2^2 + J(\mathbf{m})$$

with covariance

$$\mathbf{C}_n \equiv \mathbf{E}\{\mathbf{n}\mathbf{n}^T\}$$

and both m, n **related to PDE**

Variational problems

Inverse scattering problem

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} \frac{1}{2}\|\mathbf{d} - \mathbf{K}\mathbf{m}\|_2^2 + J(\mathbf{m})$$

with scattering operator a $1/4(n-1)$ **FIO**

$$Tf(x) = \int e^{i\Phi(z,\zeta)} a(x,\zeta)\hat{f}(\zeta)\mathrm{d}\zeta$$

solution operator PDE (ten Kroode, de Hoop, Stolk, Smit, Demanet).

Variational problems

Divide-conquer-approach:

1. **Replace quadratic Tikhonov *regularization* (Starck, Daubechies, etc) by l^p-sparseness on the basis**
   - **optimal NLA, diagonal thresholding**
2. **Non-linear *synthesis* that brings out wavefronts**
   - **constrained optimization (Starck, Candes)**

Variational problems

Step 1: left- or right-precondition

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{F}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_{p,w}$$

with

$$\|\mathbf{x}\|_{p,w} = \left[\sum_{\mu \in \mathcal{M}} w_\mu |\langle m, \varphi_\mu \rangle|^p\right]^{1/p}$$

- **imposes sparseness on the model coeff.**
- **corrects for colored noise through precond**

Variational problems

Step 2:

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} J(\mathbf{m}) \quad \text{s.t.} \quad |\mathbf{A}\mathbf{m} - \hat{\mathbf{x}}|_\mu \leq \mathbf{e}_\mu$$

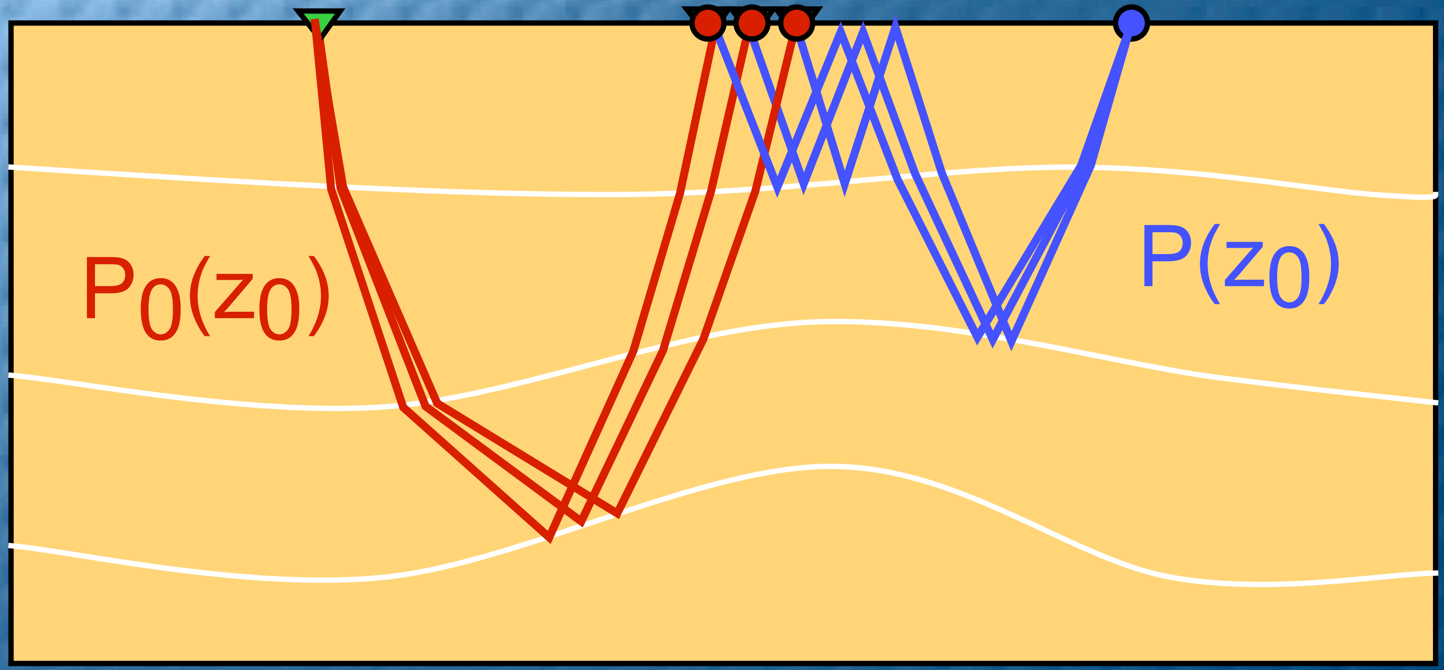
with

A **contains the Curvelet transform**

e_μ **vector with tolerances**

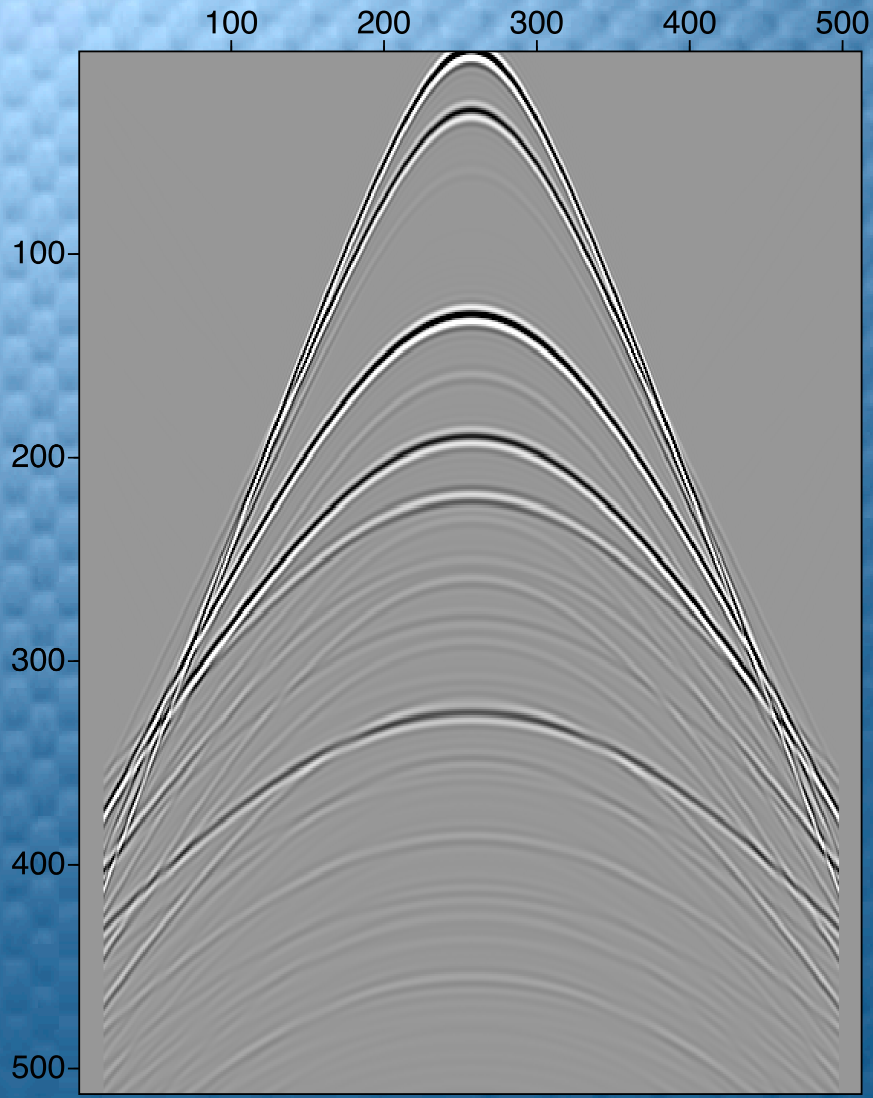
- **sparseness & continuity on the model via** $J(\mathbf{m})$
- **inverts preconditioning and normal operator**

Surface multiple elimination

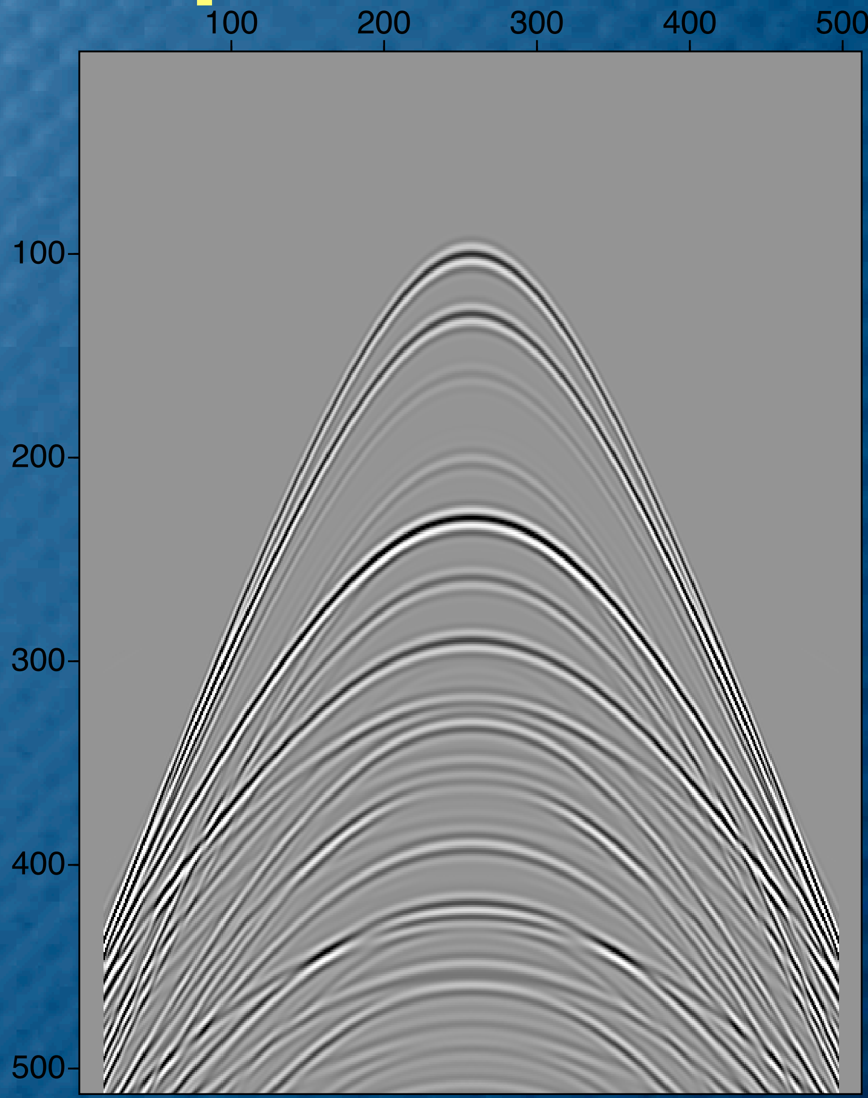


Multiple prediction: convolution along surface

1-D example

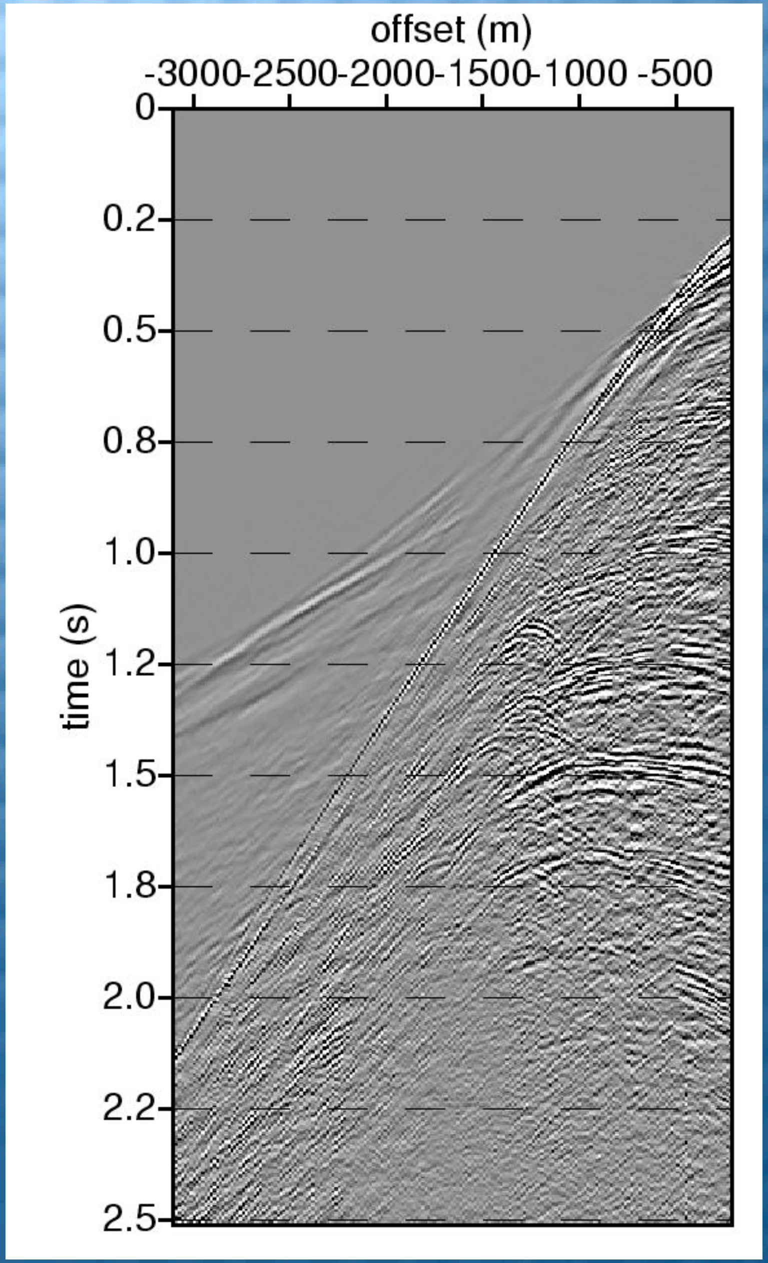


Primaries

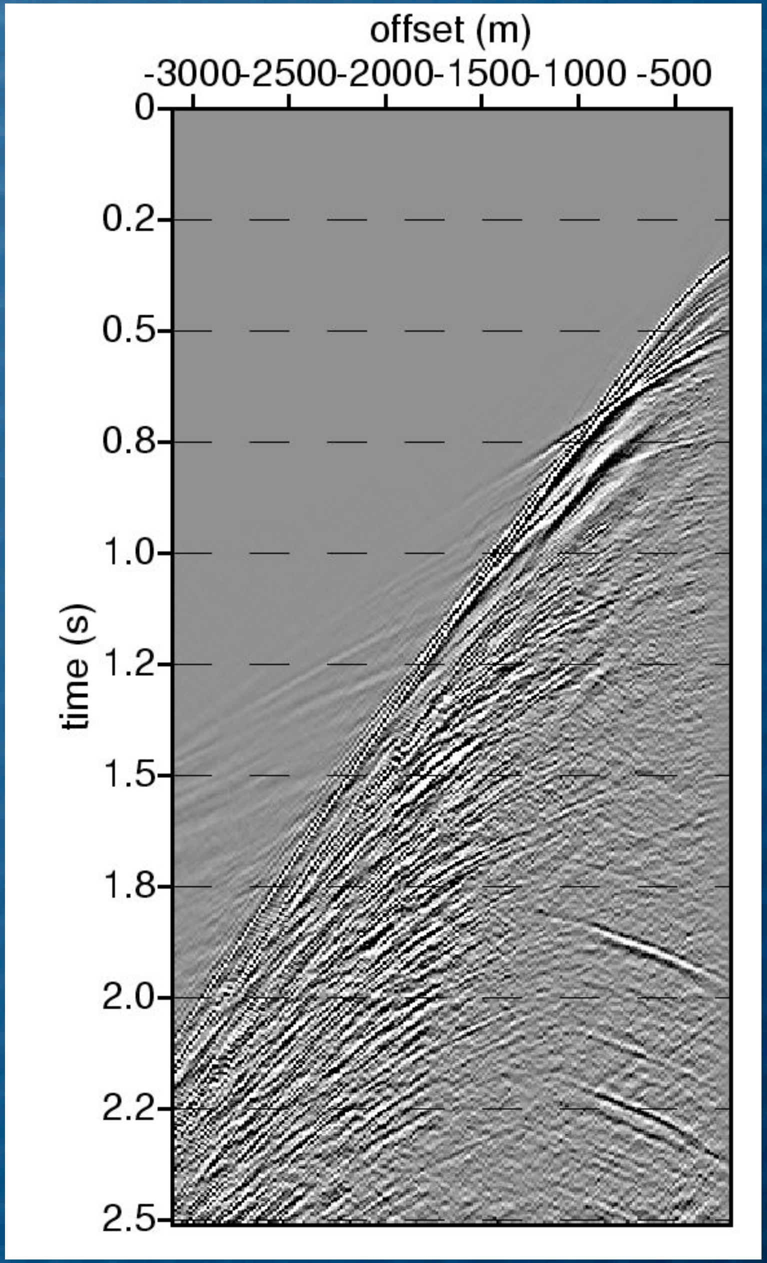


Multiples

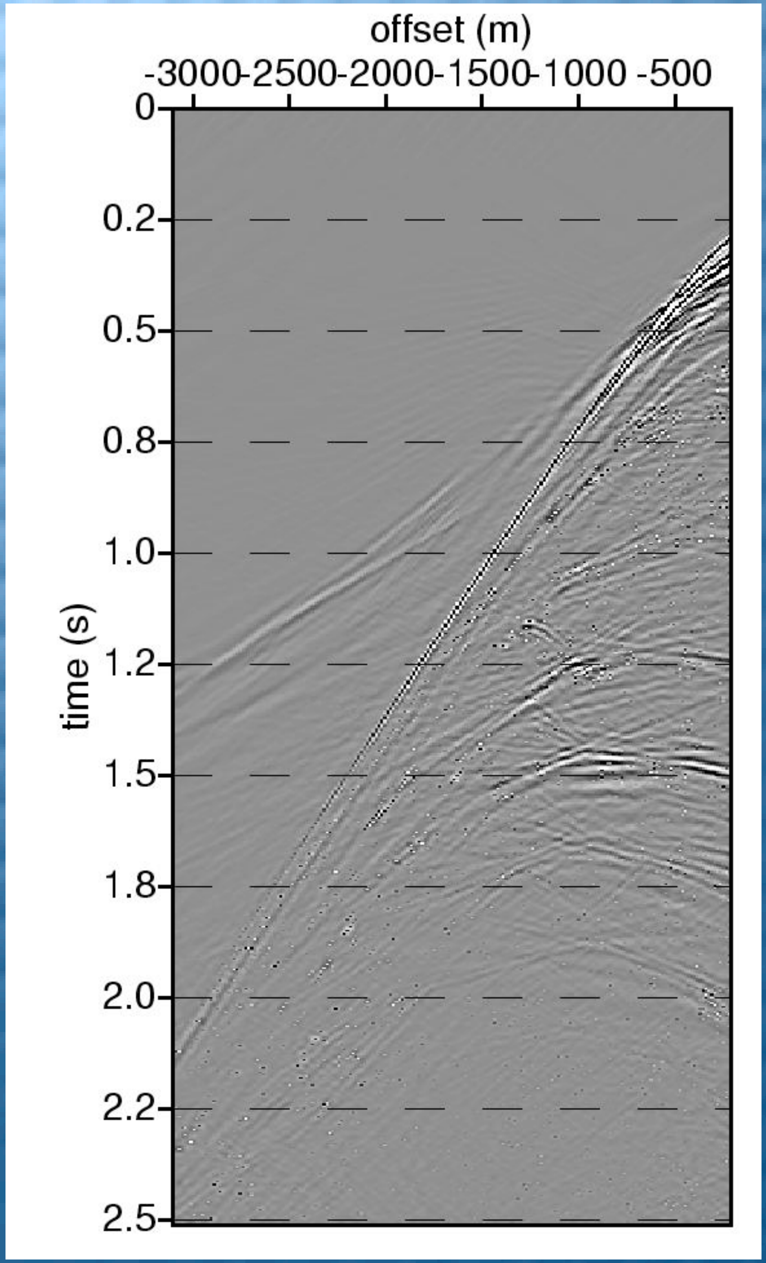
Appetizer



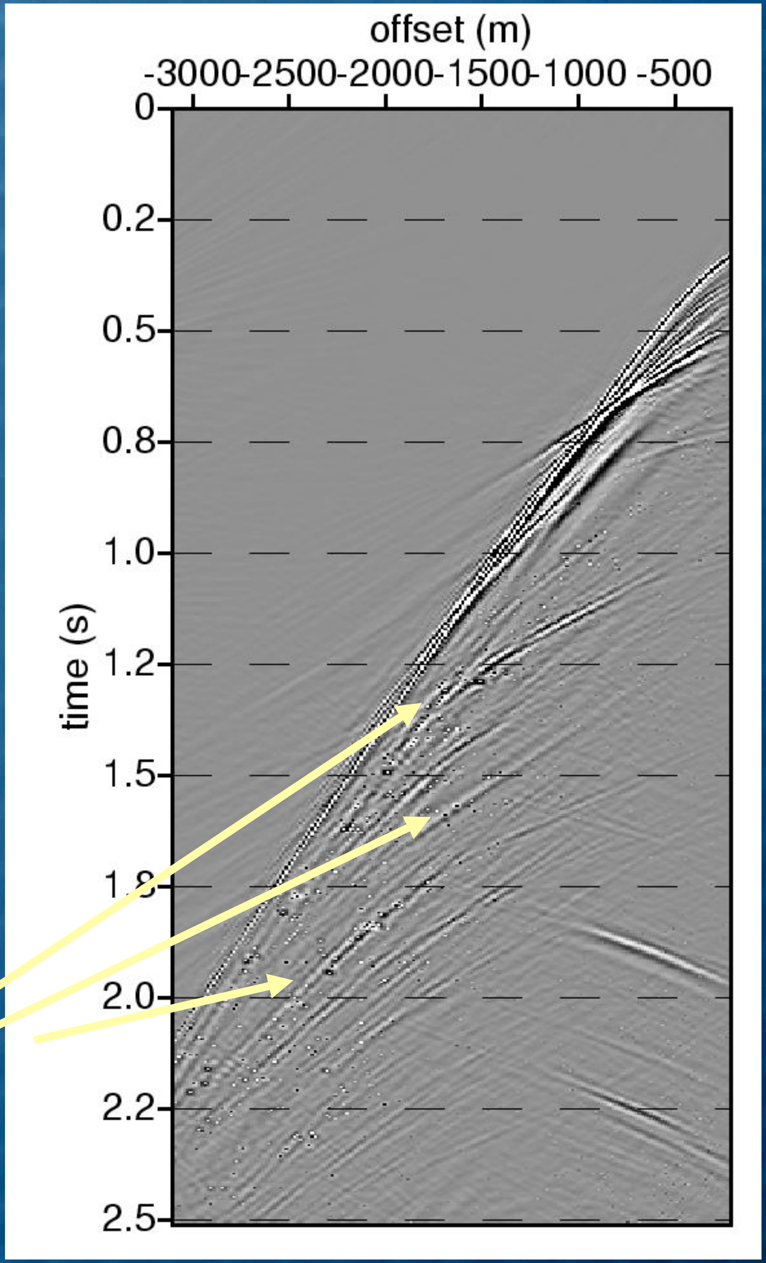
Output
SRME
multi-L2
subtraction



Multiple suppression with curvelets



Output curvelet filtering with stronger threshold



Preserved primaries

L2 adaptive subtraction

Matched filter:

$$\overbrace{\hat{\mathbf{n}}}^{\text{denoised}} : \min_{\mathbf{\Phi}} = \| \underbrace{\mathbf{d}}_{\text{noisy data}} - \overbrace{\mathbf{\Phi}}^{\text{matched filter}} \overset{t}{*} \underbrace{\mathbf{m}}_{\text{pred. noise}} \|_p$$

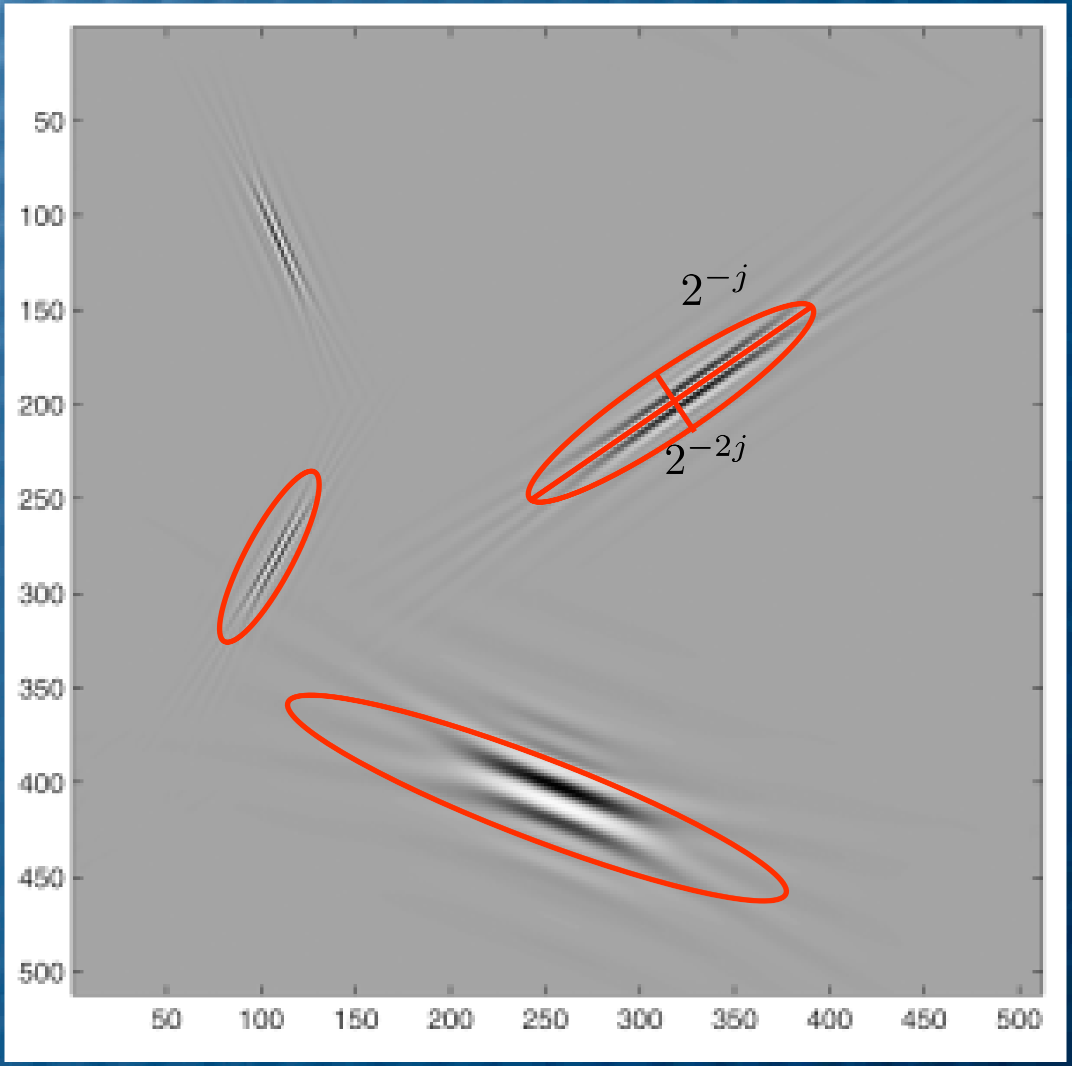
- **p=1 enhances sparseness**
- **residue is the denoised data**
- **risk of over fitting**

Loose primary reflection events ...

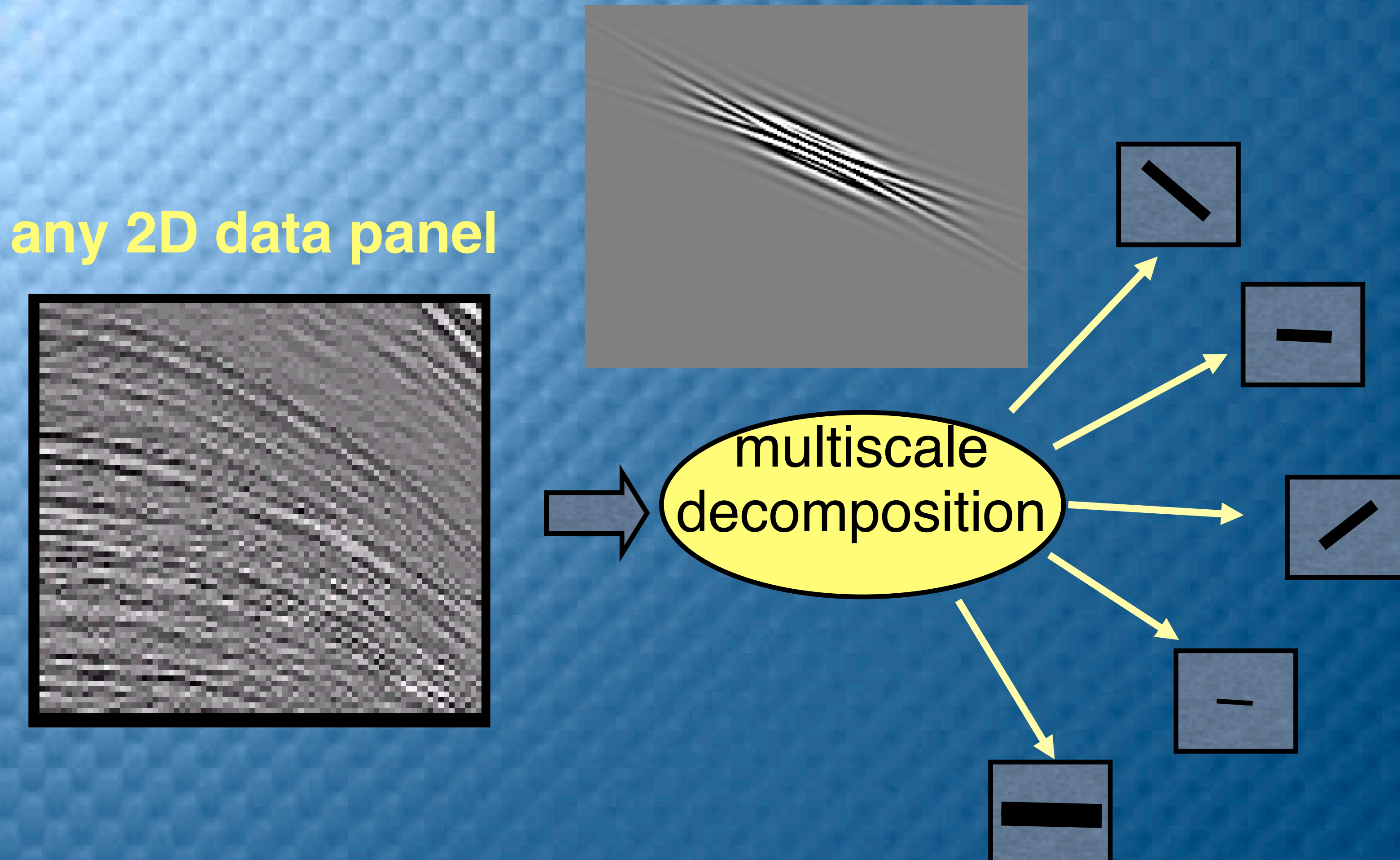
Guitton& Verschuur'04

Why curvelets

- Nonseparable
- Local in 2-D space
- Local in 2-D Fourier
- Anisotropic
- Multiscale
- Almost orthogonal
- Tight frame
- Optimal



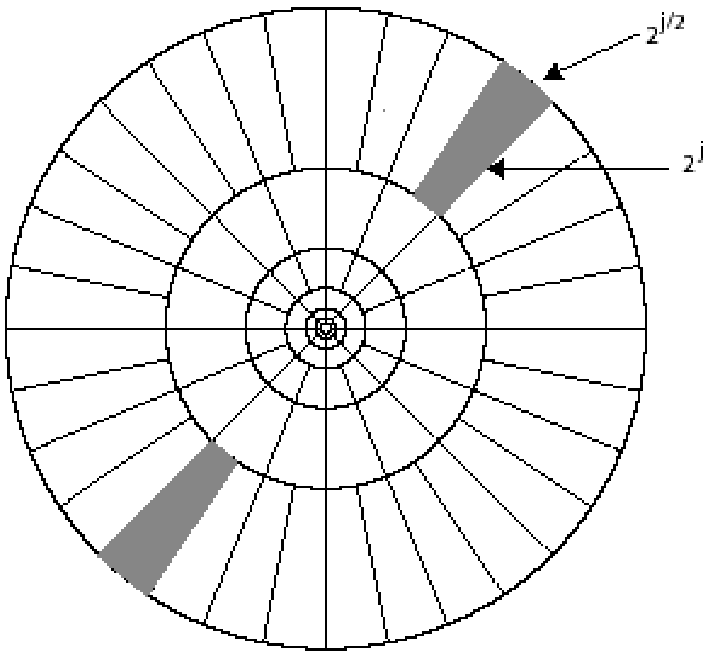
Why Curvelets



- **Almost orthogonal decomposition into multiscale basis functions with local frequency and local dip properties**
- **Natural basis for wave equations**
- **Consist of plane wavelets invariant under migration**

Why curvelets

$$W_j = \{\zeta, \quad 2^j \le |\zeta| \le 2^{j+1}, |\theta - \theta_J| \le \pi \cdot 2^{\lfloor j/2 \rfloor}\}$$



second dyadic partitioning

source: Candes'01, Stein '90

Fourier/SVD/KL

$$||f - \tilde{f}_m^F|| \propto m^{-1/2}, \ m \to \infty$$

Wavelet

$$||f - \tilde{f}_m^W|| \propto m^{-1}, \ m \to \infty$$

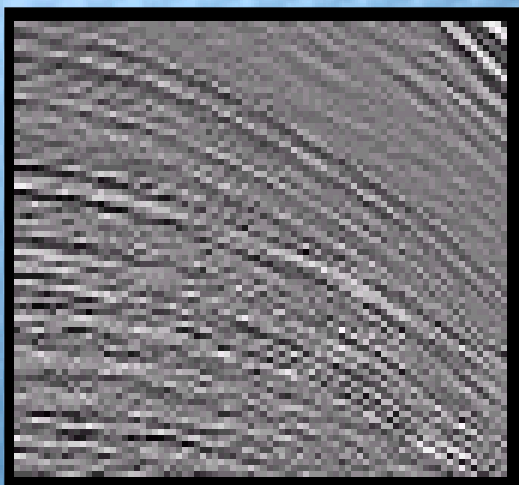
Optimal data adaptive

$$||f - \tilde{f}_m^A|| \propto m^{-2}, \ m \to \infty$$

Close to optimal Curvelet

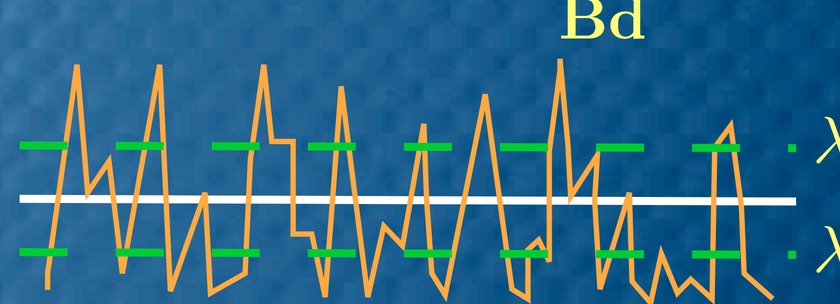
$$||f - \tilde{f}_m^C|| \le C \cdot m^{-2}(\log m)^3, \ m \to \infty$$

Why Curvelets



Input data with multiples

Curvelet transform



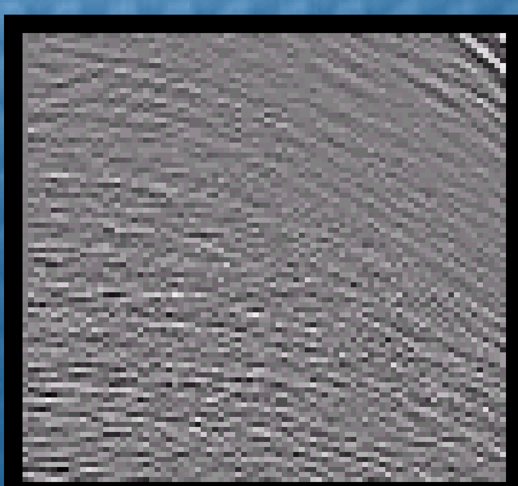
Curvelet coeff. data

Threshold

$$\Theta_\lambda\left(\mathbf{Bd}\right)$$

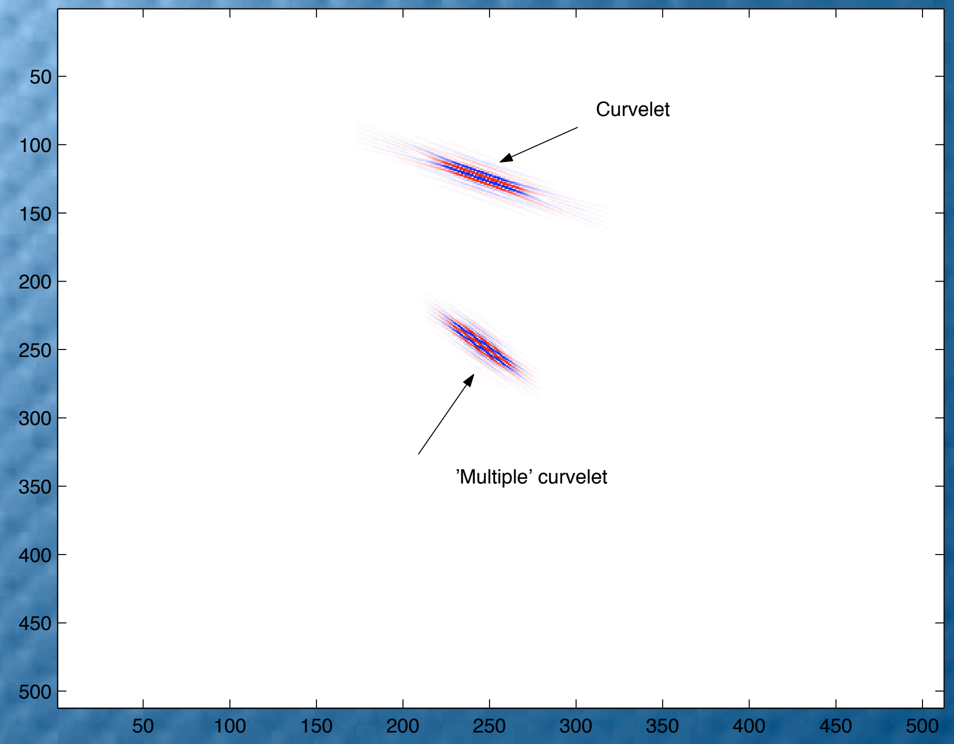
Inv. curvelet transform

denoised Curvelet coeff.



Filtered input data

Curvelet 'Multiples'



- **Almost diagonalize Green's functions (Candes & Demanet '04)**
- **Natural basis for wave equations**
- **Invariant under convolution, i.e. 'multiple multiple' = curvelet-like**
- **Curvelets sense *local* dip and *local* frequency content and can *discriminate* on these properties**

Our approach

Near diagonalization Covariance operator

Approximate prediction multiples

Left precondition: $\mathrm{d} = \mathrm{m} + \mathrm{n}$

into $\mathbf{y} = \mathbf{x} + \boldsymbol{\epsilon},\ \mathbf{y} \equiv \boldsymbol{\Gamma}^{\dagger}\mathbf{B}$

with $\Gamma^2 \equiv \mathrm{diag}\{\mathrm{diag}\{\mathbf{B}\mathbf{n}\mathbf{n}^T\mathbf{B}^T\}\}$

such that $\mathbf{C}_\epsilon \cdot \approx \mathbf{I}\cdot$

Our approach

Simply set $\mathbf{w} = \lambda \,\text{with}\, \lambda \in [3, 5]$

and $\Gamma = |\mathbf{B}\mathbf{n}_p|$

Solve

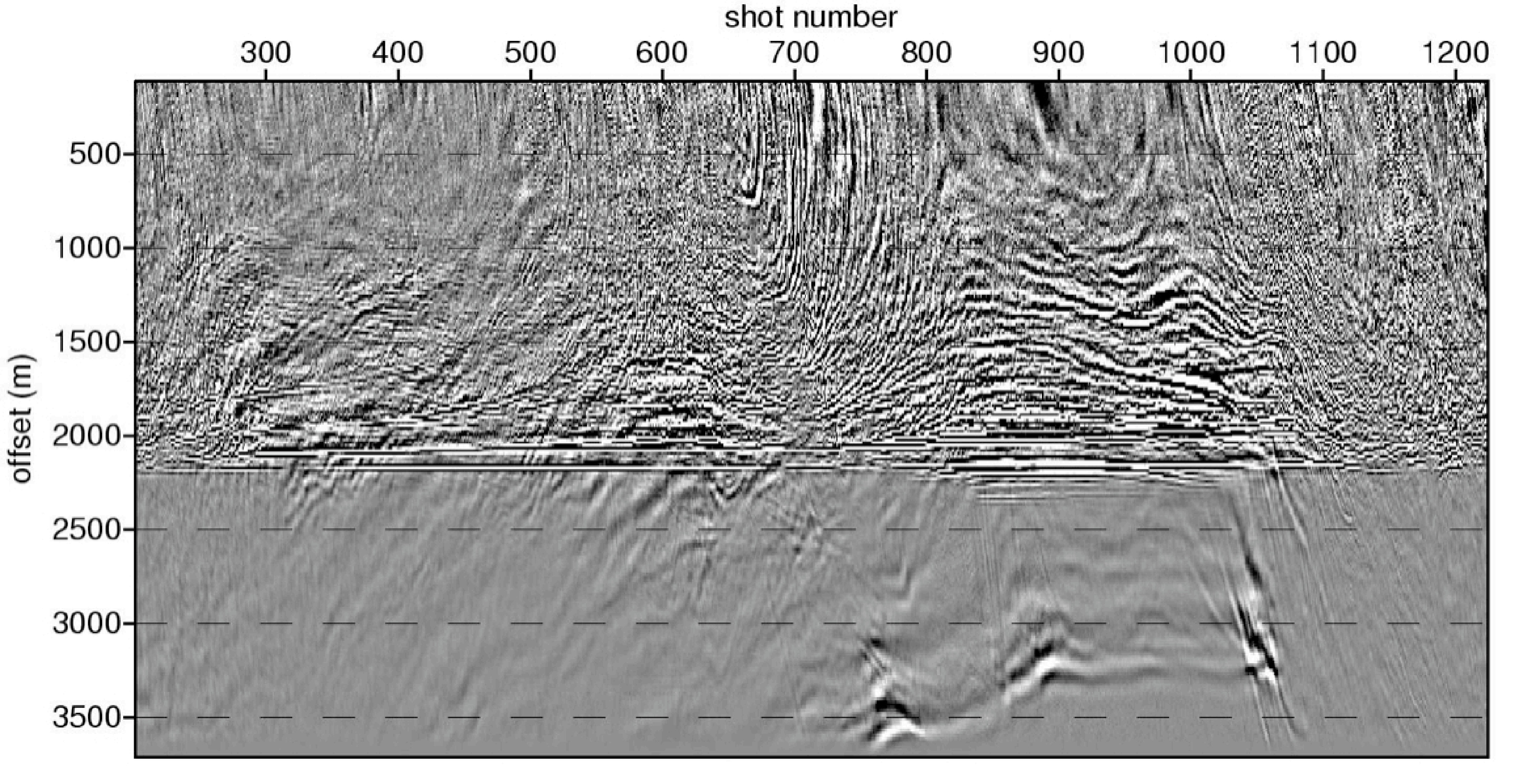
$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{I}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_{p=1,w}$$

with soft thresholding

$$\hat{\mathbf{m}} = \Gamma\mathbf{B}^T S_\lambda(\underbrace{\mathbf{y}}_{\Gamma^\dagger \mathbf{B}\mathbf{d}}) = \mathbf{B}^T S_{\lambda\Gamma}(\mathbf{B}\mathbf{d})$$

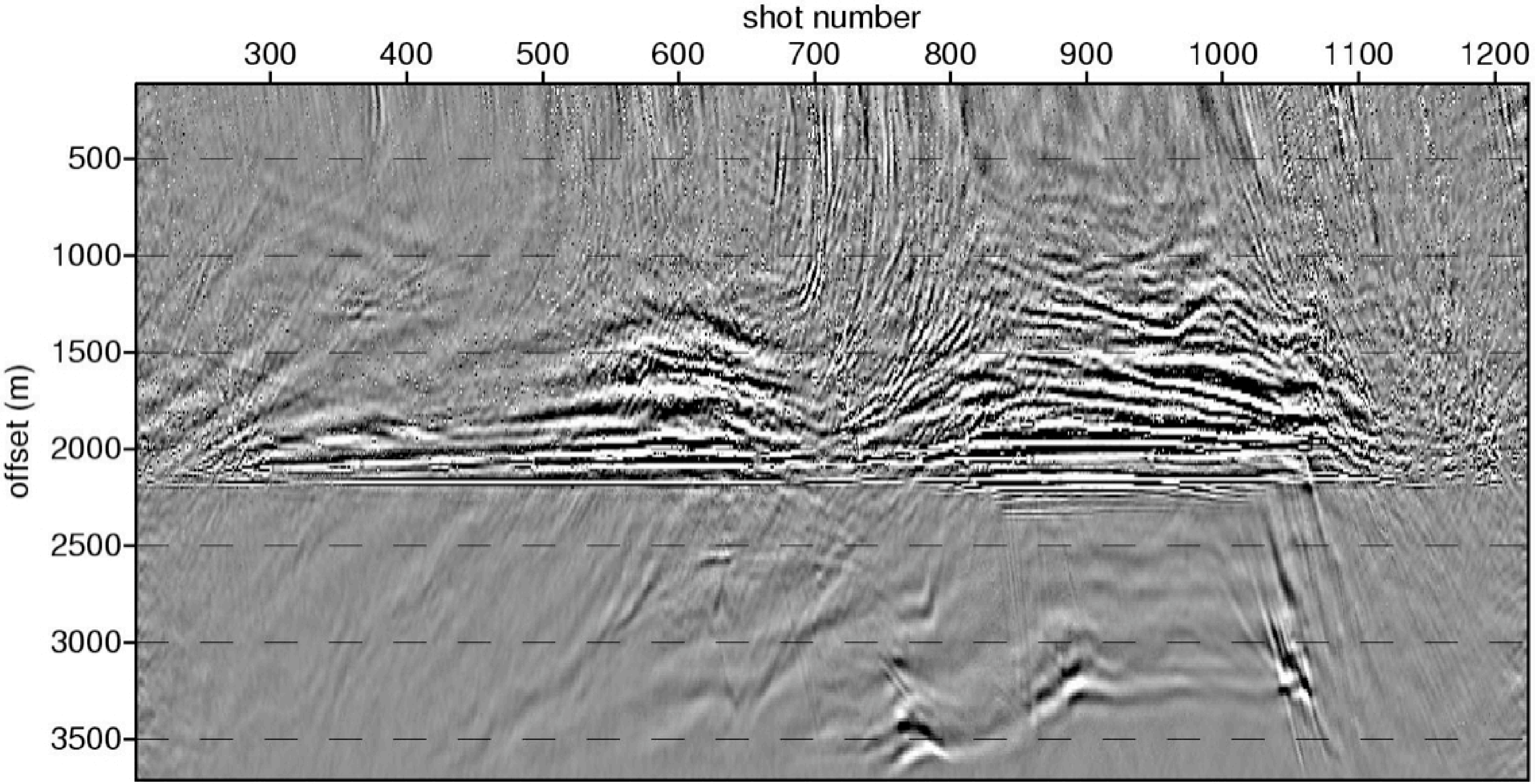
Time slices

L2



Time slices

curvelet

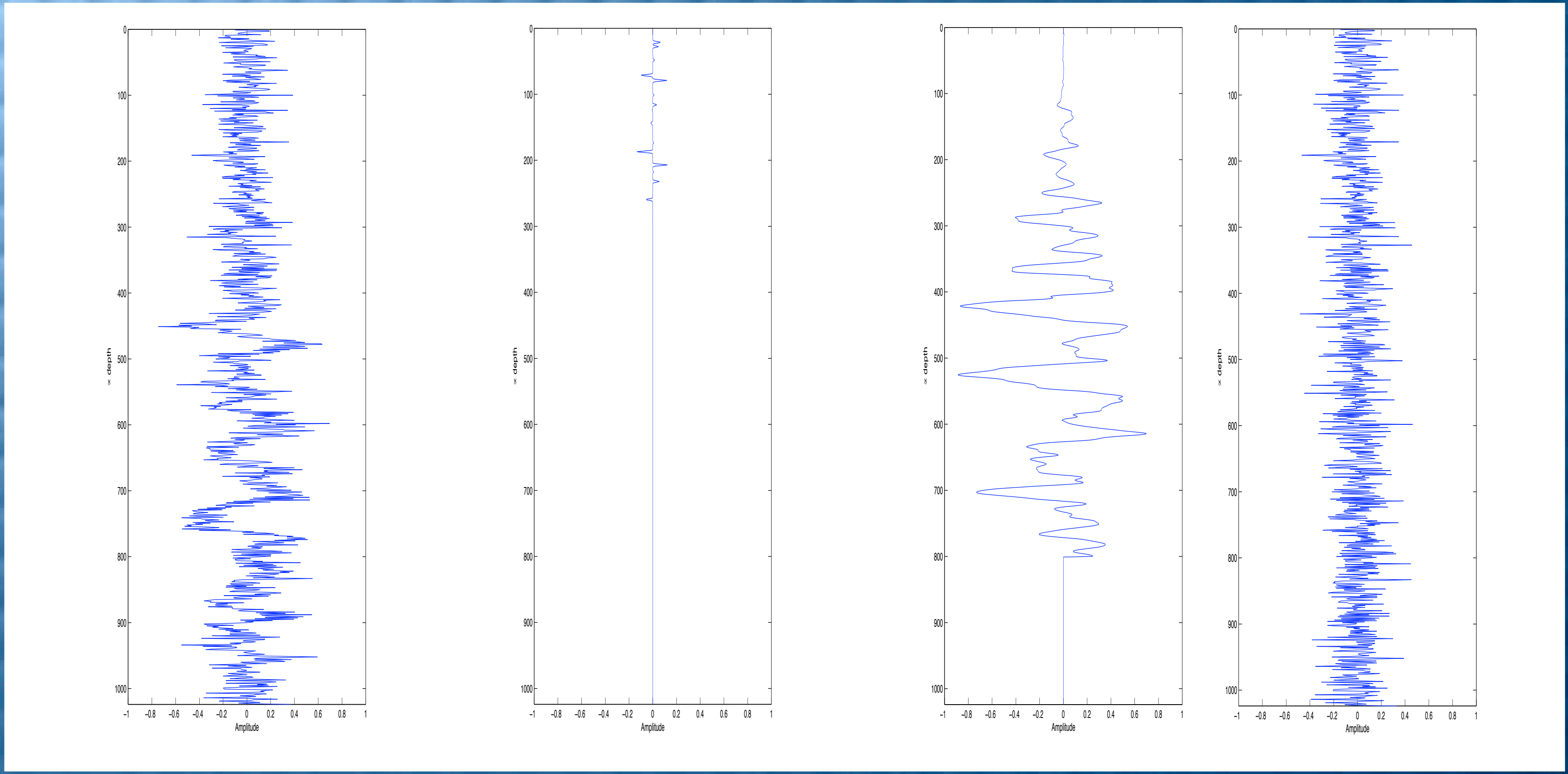


Inverse problems

Extend to include operators?

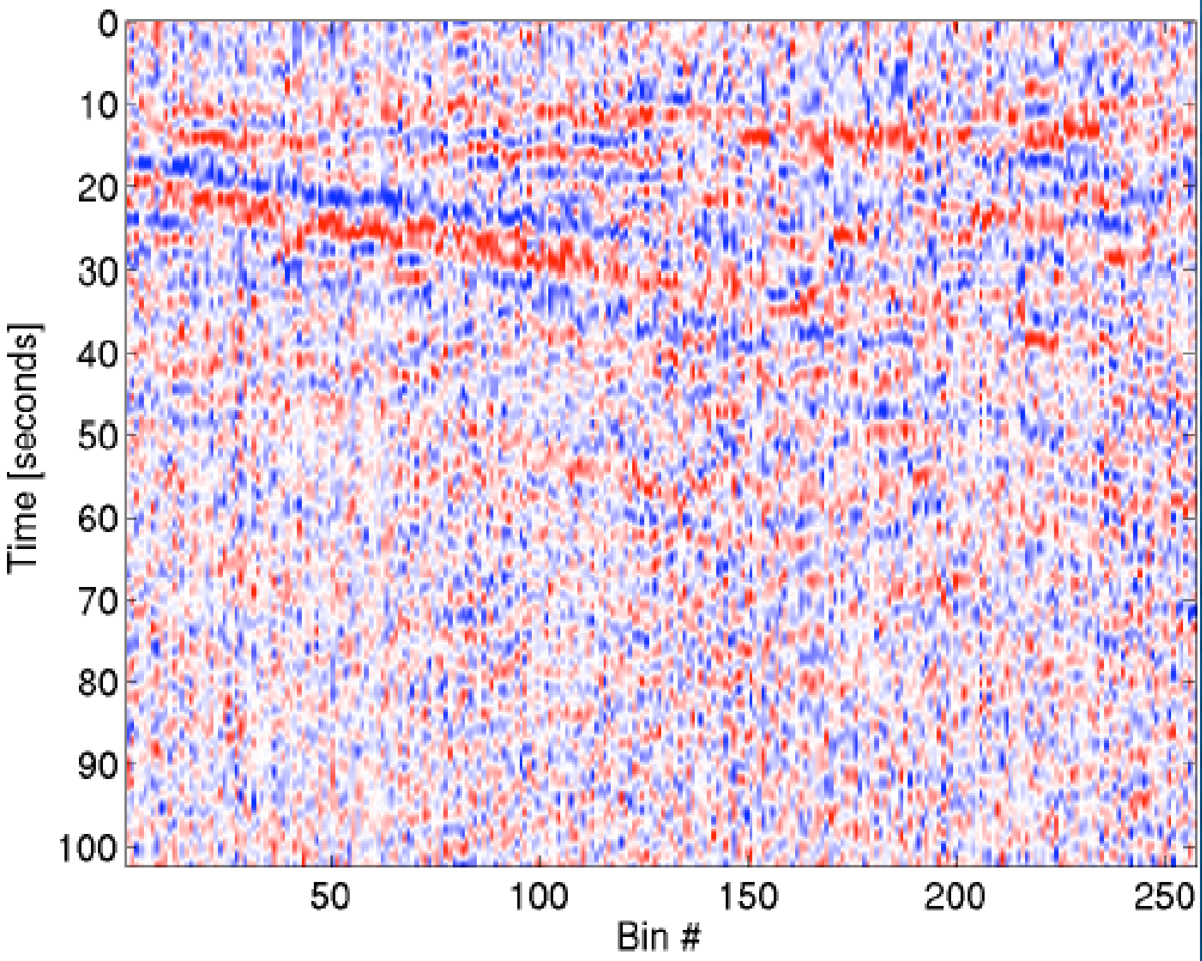
- **sparse basis model does *not* diagonalize operator (e.g. convolution)**
  - **Forward (Neelami)**
  - **iterated thresholding (Daubechies)**
- **sparse basis nearly *diagonalizes* operator**
  - **Quasi-SVD/Wavelet-Vaguelette (Donoho, Lee, Candes, Kalifa)**

Tele-seismic deconvolution



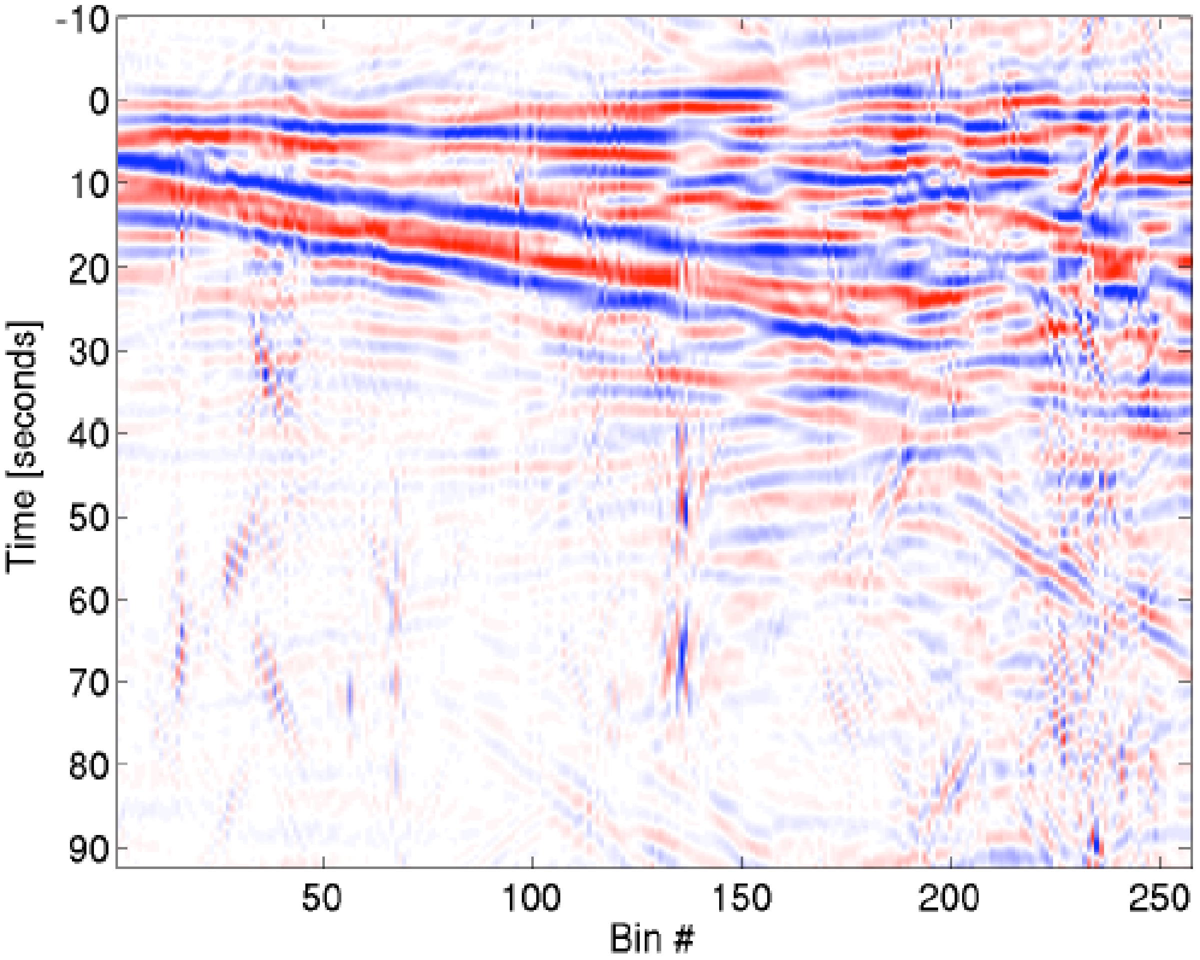
$$Data = Green\ function * Source + Noise$$

Real Data



Water level deconvolution + LP 0.4 Hz
(conventional method)

Real Data



Wiener filter deconvolution + denoising + LP 0.4 Hz

Seismic imaging

Forward model:

$$\overbrace{\mathbf{d}}^{\text{data}} = \underbrace{\mathbf{K}}_{\text{scat. oper.}} \overbrace{\mathbf{m}}^{\text{model/refl.}} + \underbrace{\mathbf{n}}_{\text{noise}}$$

Imaging:

$$\underbrace{\mathbf{K}^T\mathbf{d}}_{\text{noisy \& blurry image}} = \overbrace{\mathbf{K}^T\mathbf{K}\mathbf{m}}^{\text{"burry" refl.}} + \underbrace{\mathbf{K}^T\mathbf{n}}_{\text{col. noise}}$$

- **remove $\mathrm{K}^T\mathrm{K}$ = inversion (Sacchi, Schuster)**
- **L2 migration/migration deconvolution**

Operators

$$\hat{\mathbf{m}} = \overbrace{\left(\mathbf{K}^T\mathbf{K}\right)^{-1}}^{\Psi\mathrm{DO}} \underbrace{\mathbf{K}^T}_{\mathrm{FIO}}\mathbf{d}$$

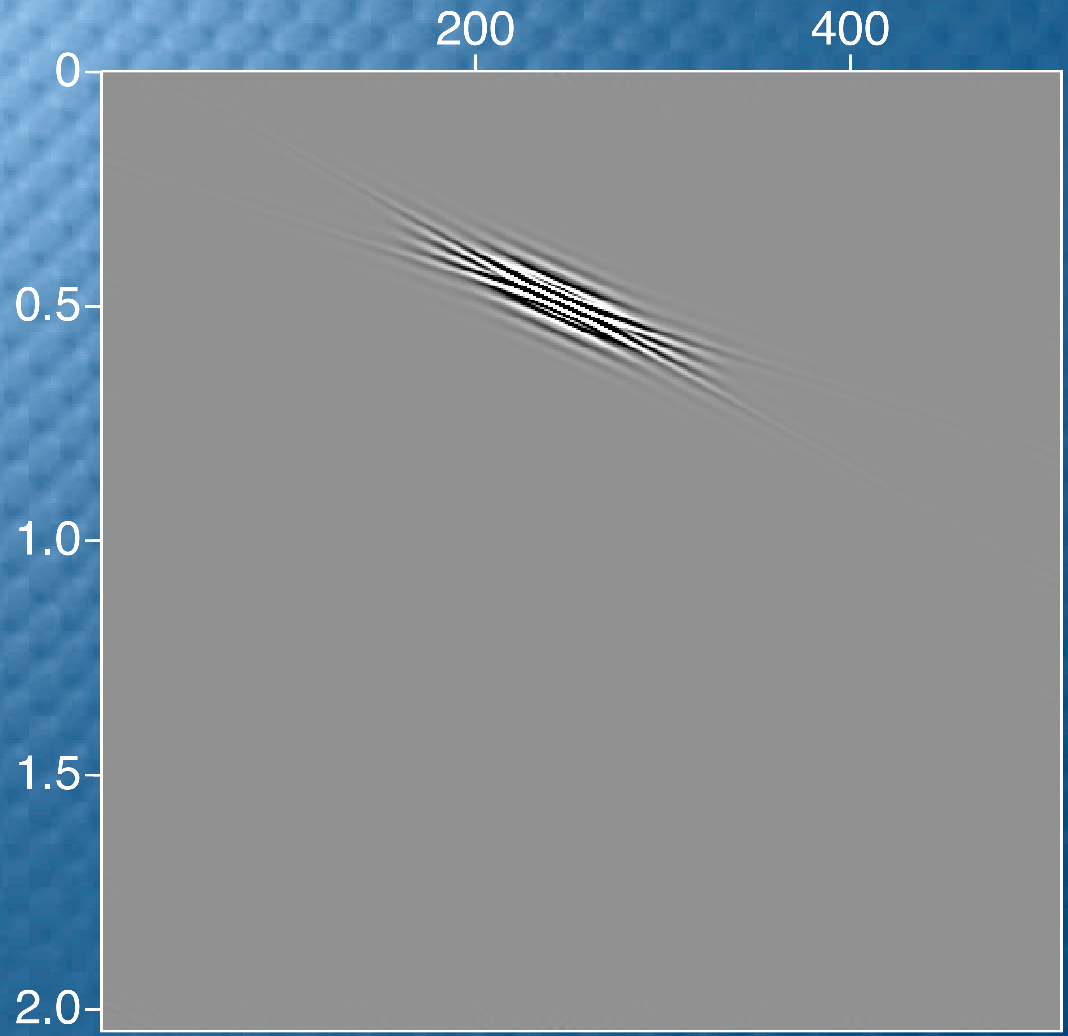
L2 amp.

	ΨDO	FIO	d & m
Wavelets	✗	✗	✗
Curvelets	✔	✔	✔

Theorem from Candes & Demanet '04:

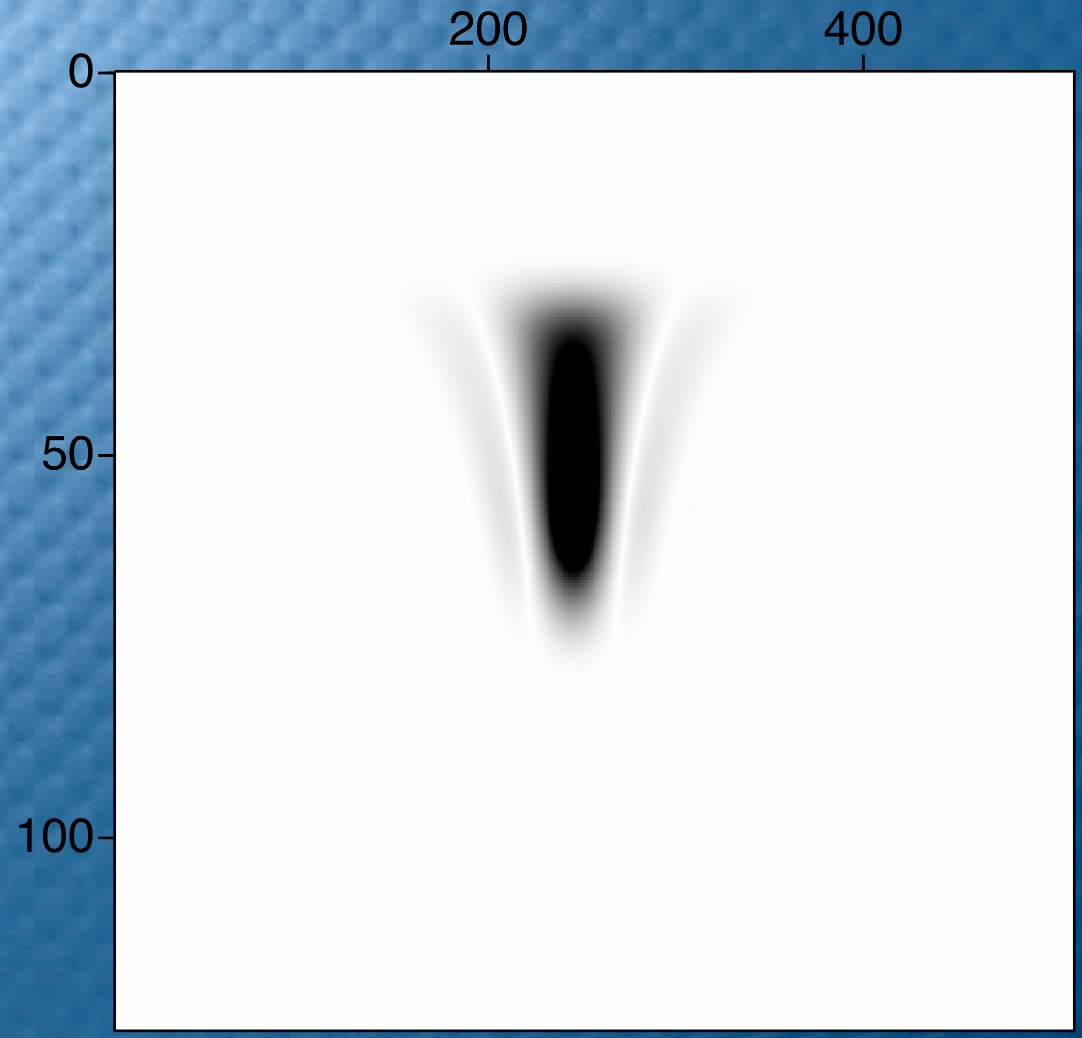
$$\|\mathbf{K}^T\mathbf{d} - \mathbf{K}^T_{\mathrm{trunc.}}\mathbf{d}\|_2 \leq C(\#\,\mathrm{per\ col.})^{-M}\,\mathrm{for\ each}\ M$$

Constrained optimization



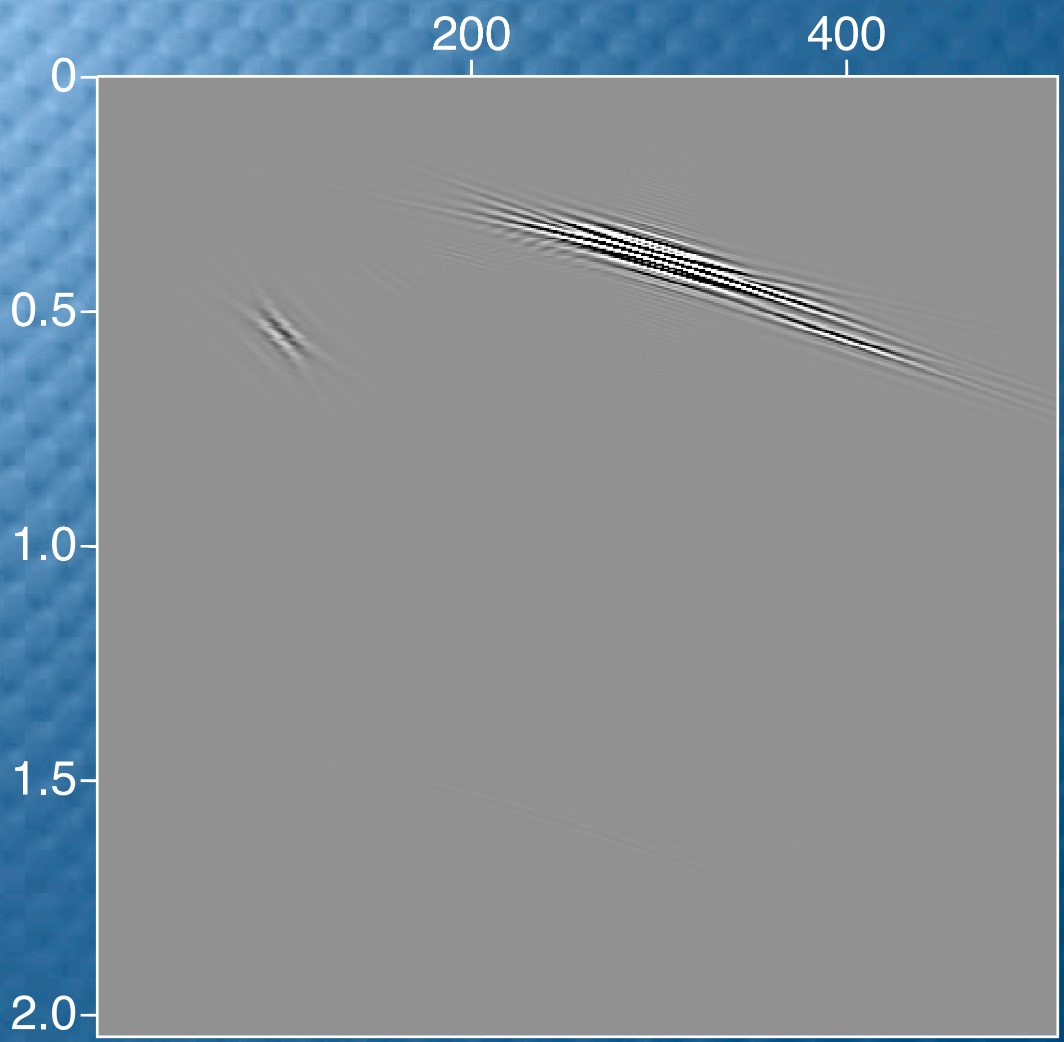
A Curvelet

Constrained optimization



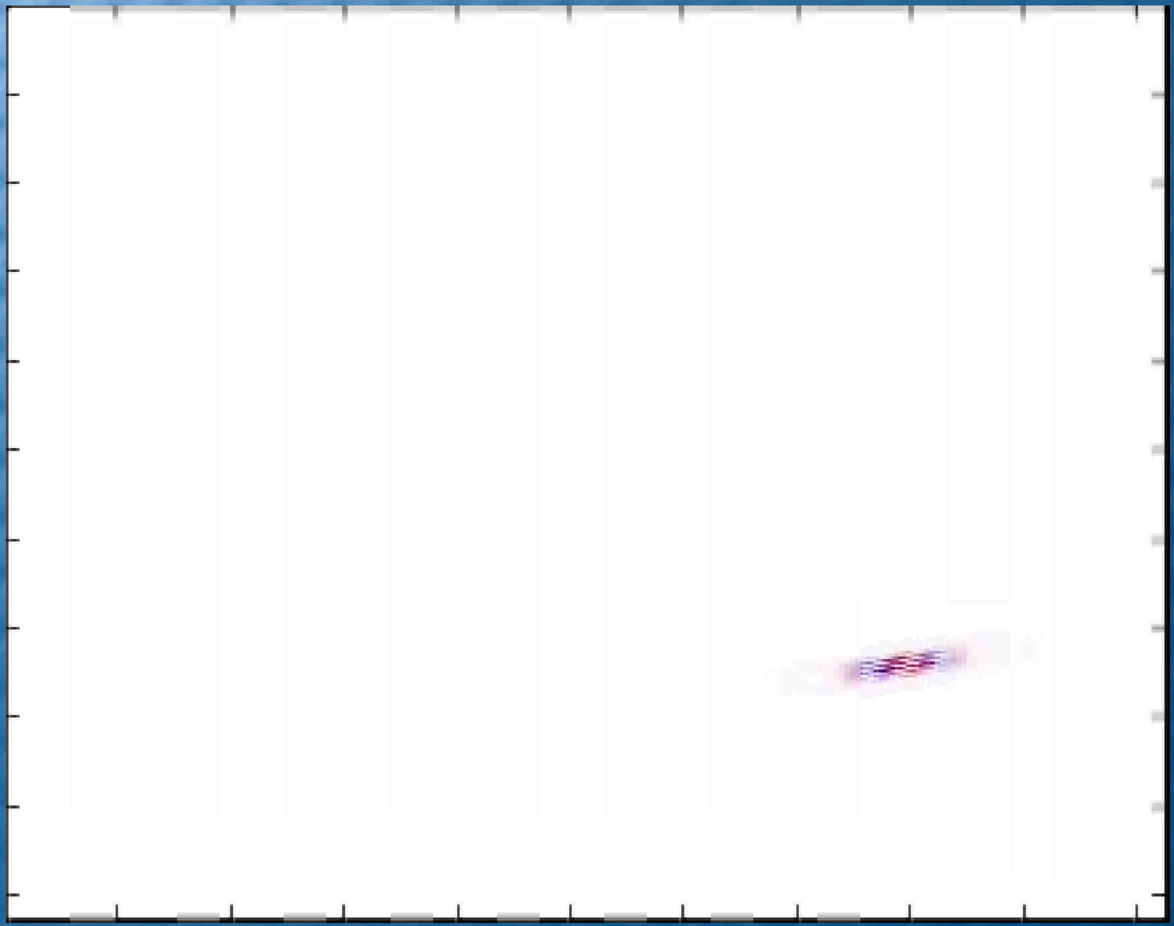
Curvelet in FK-domain

Constrained optimization

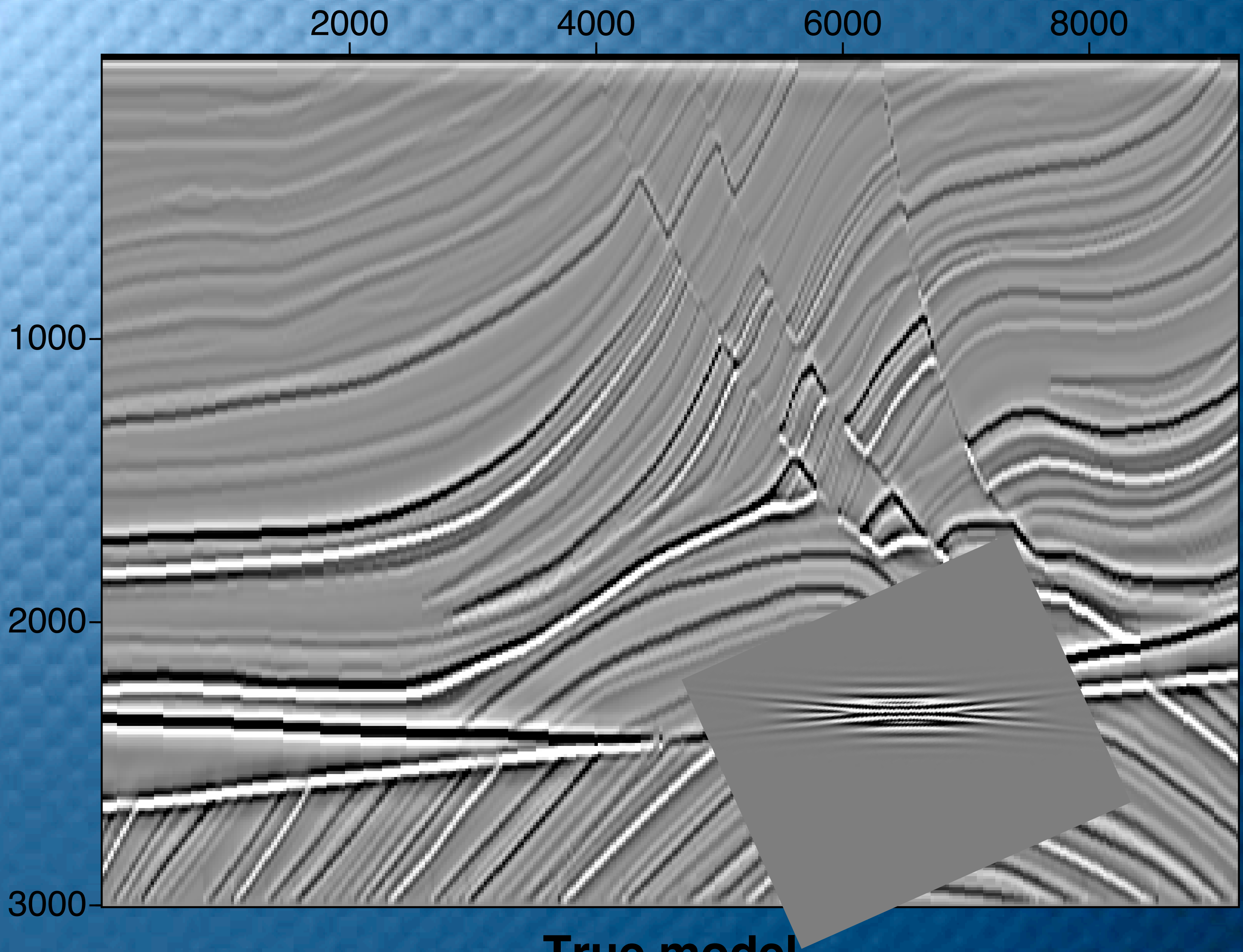


Demigrated Curvelet

Cool!

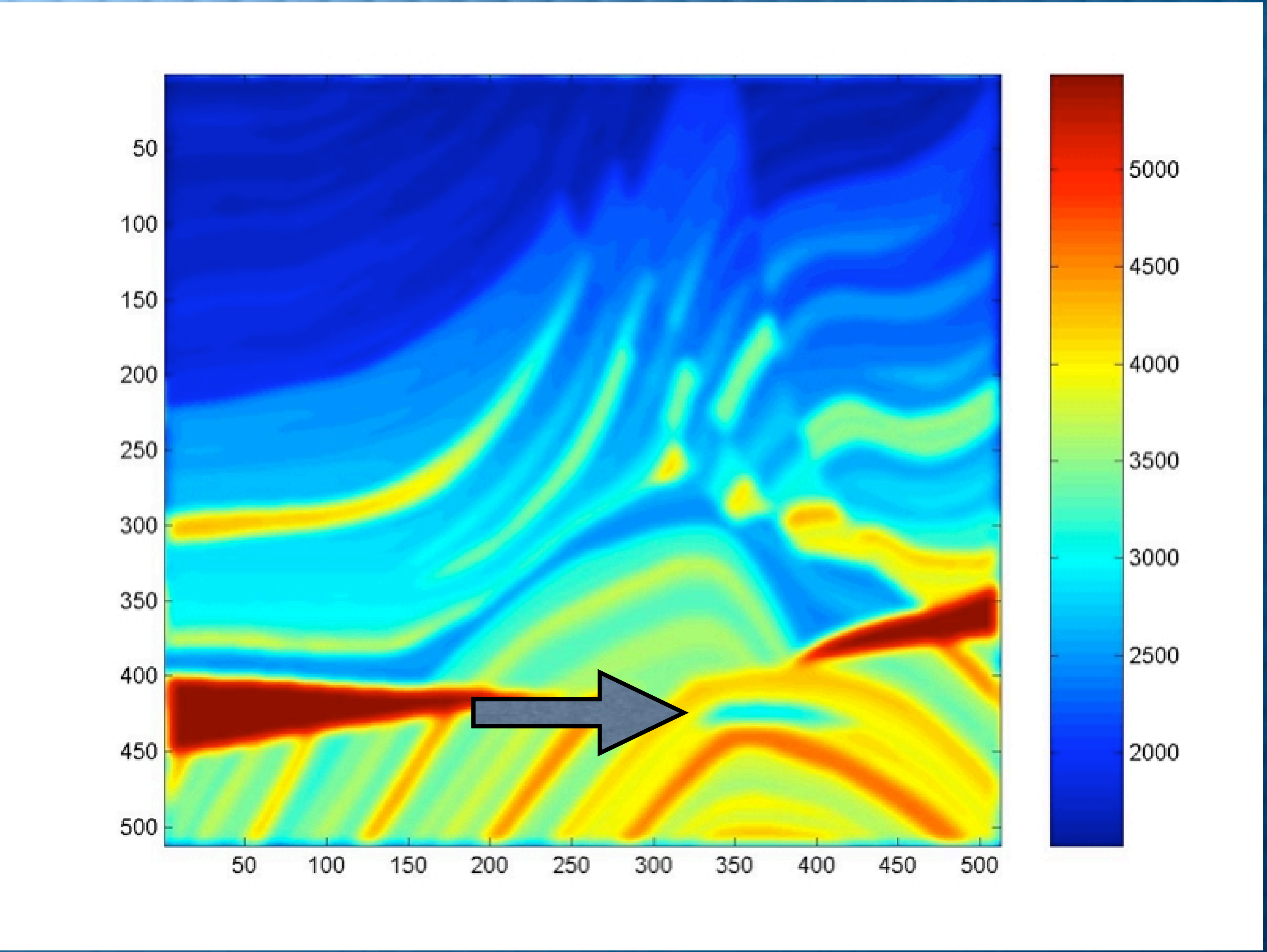


Examples

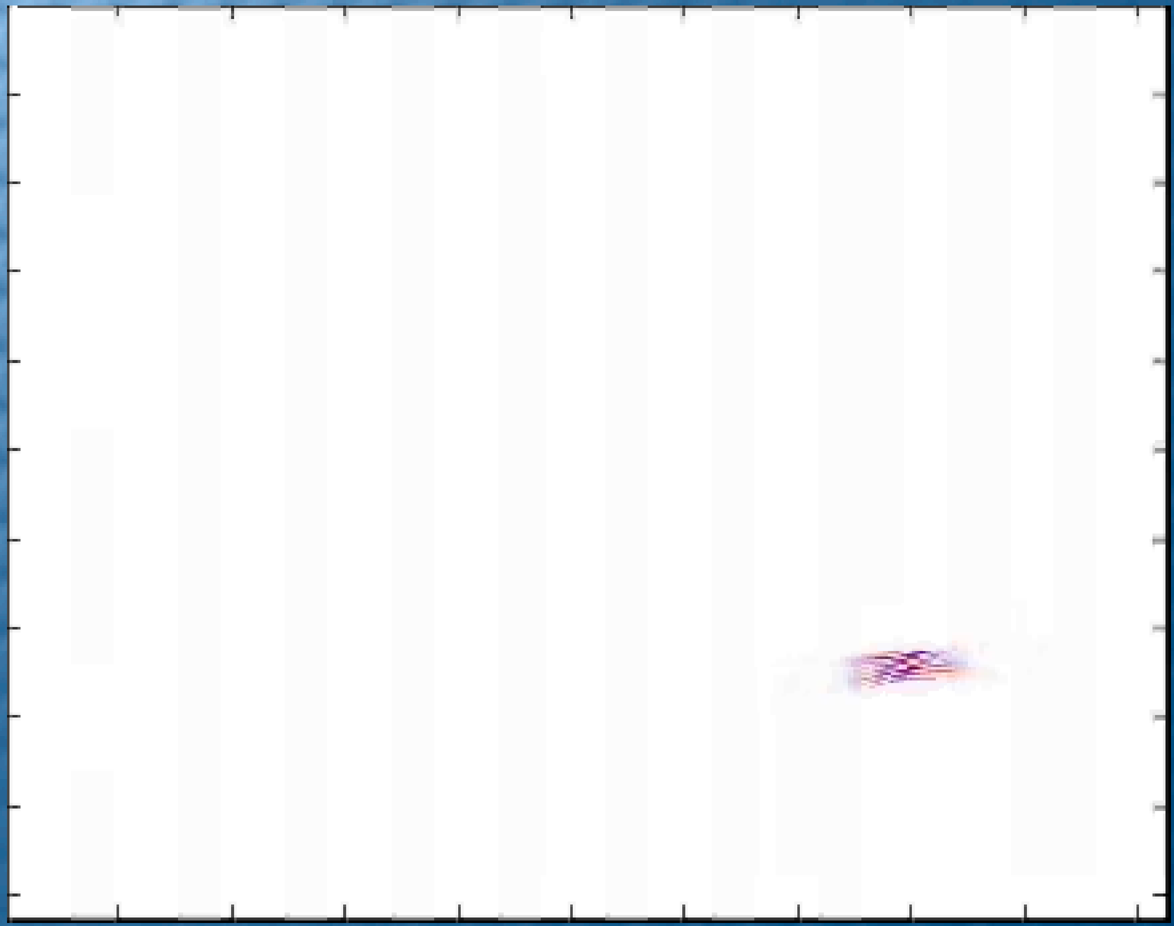


True model

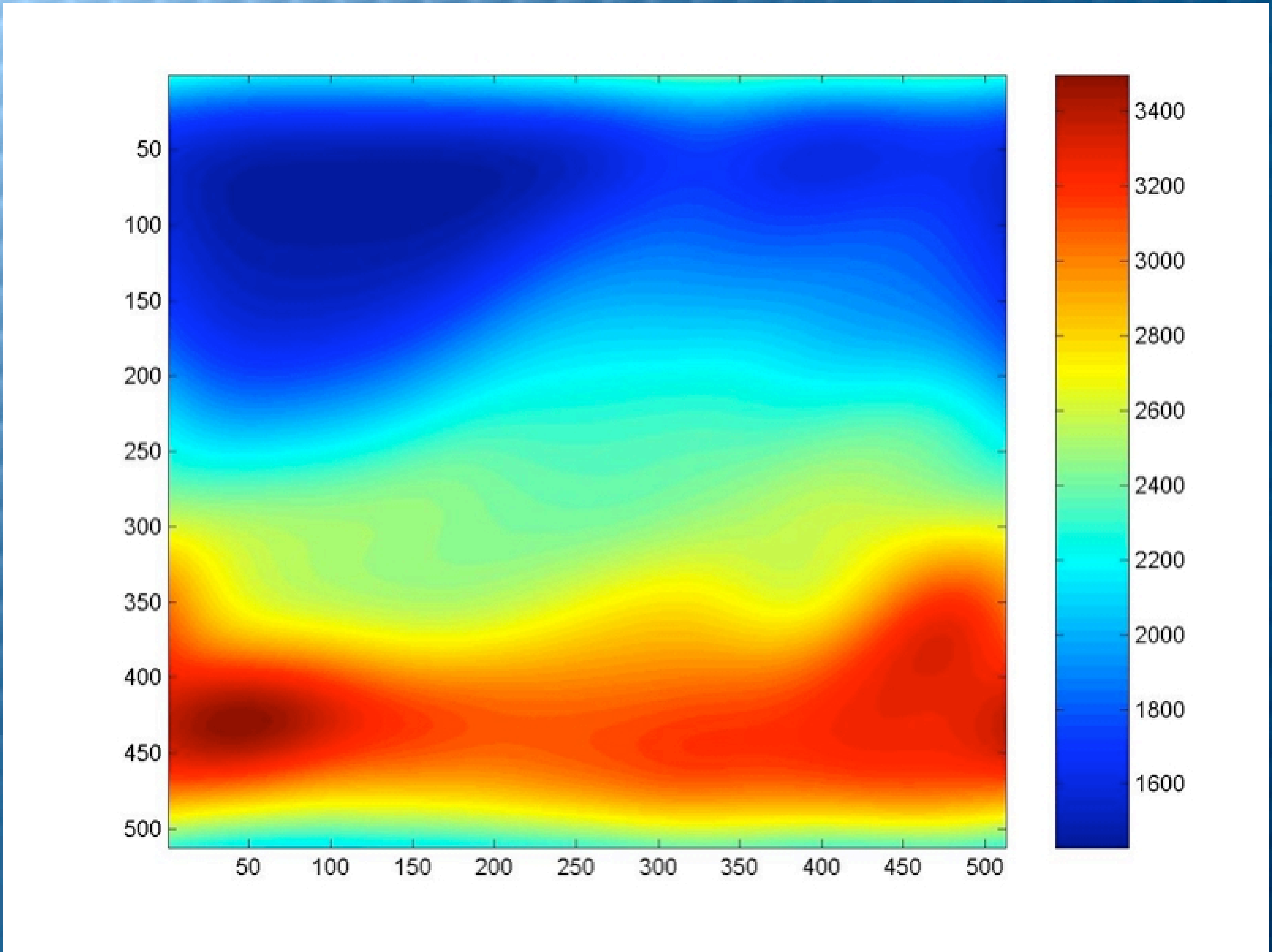
Horror movie I



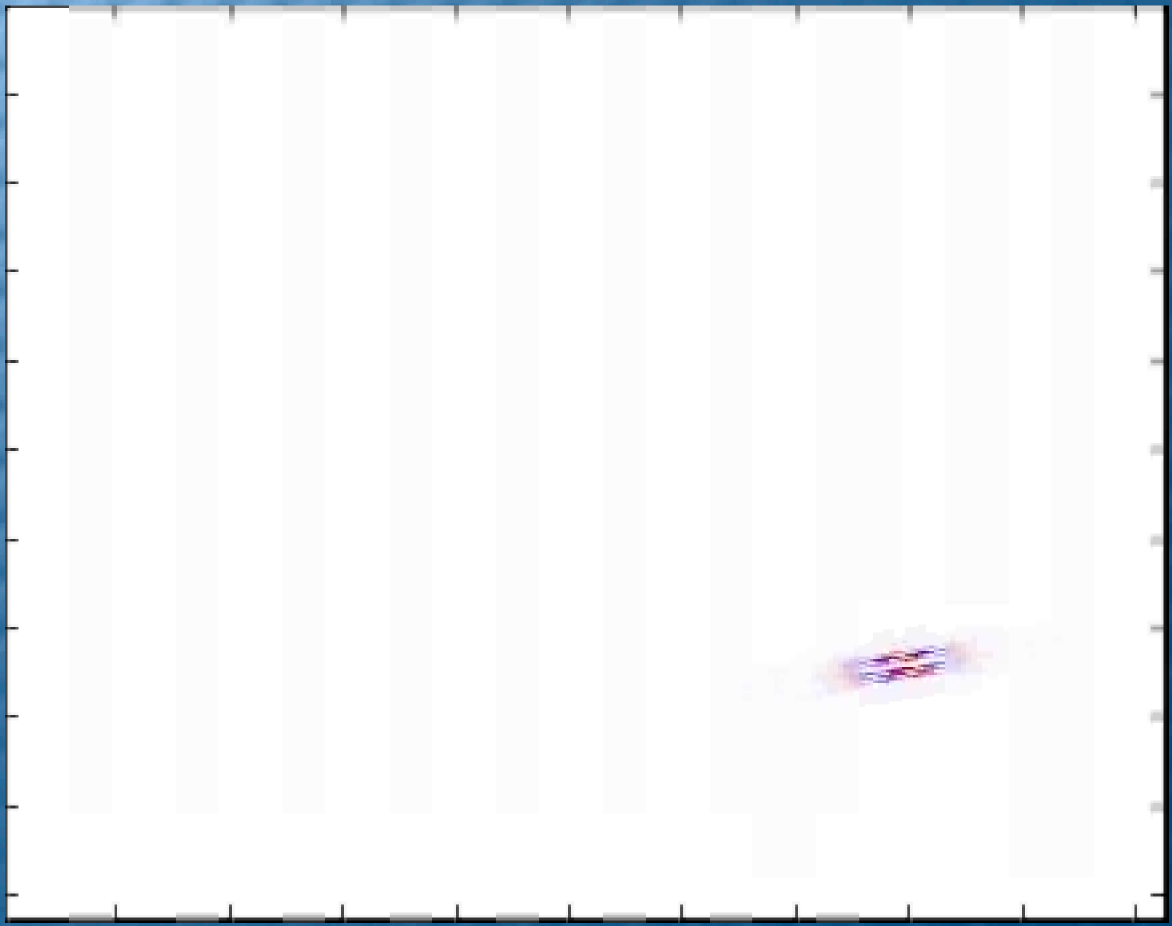
Horror movie I



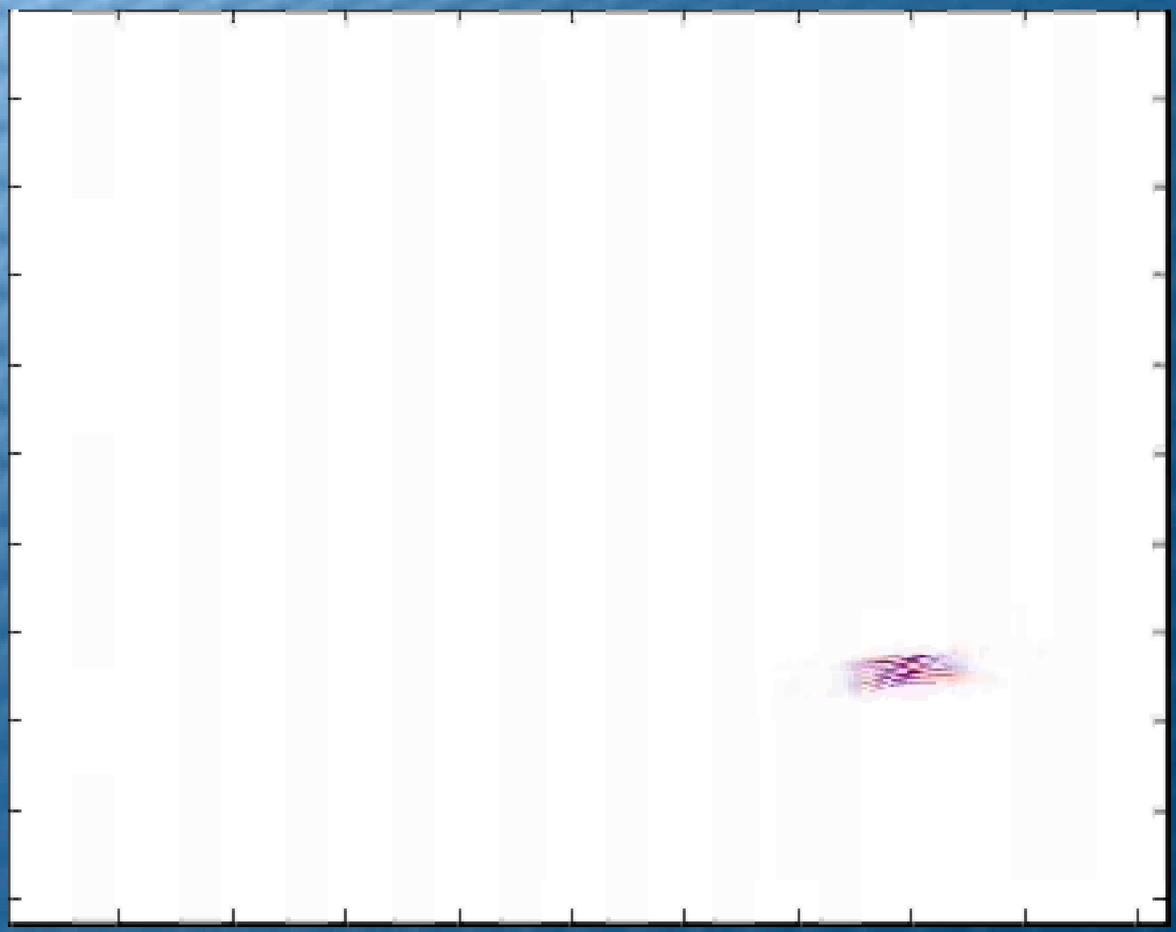
Horror movie II



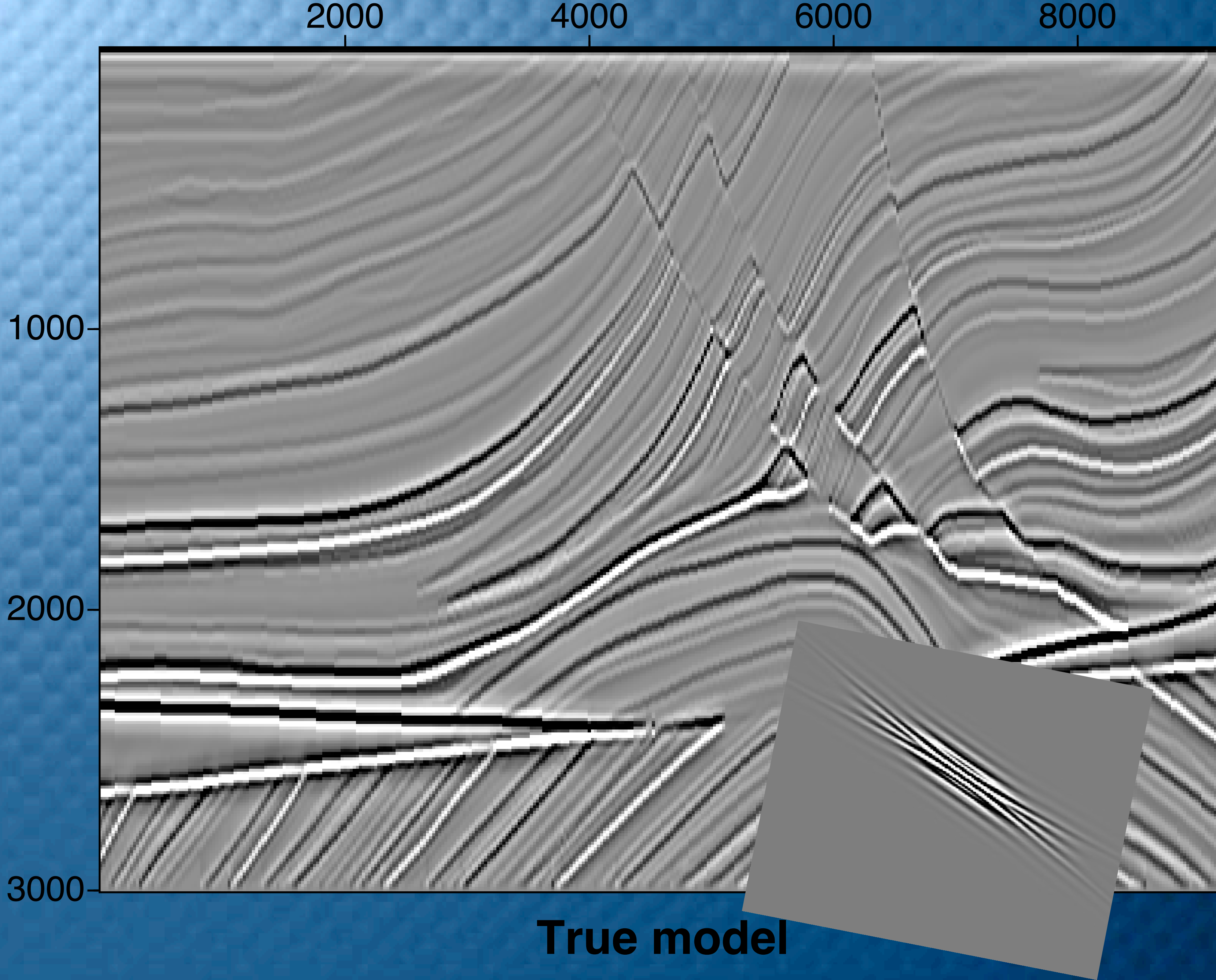
Horror movie II



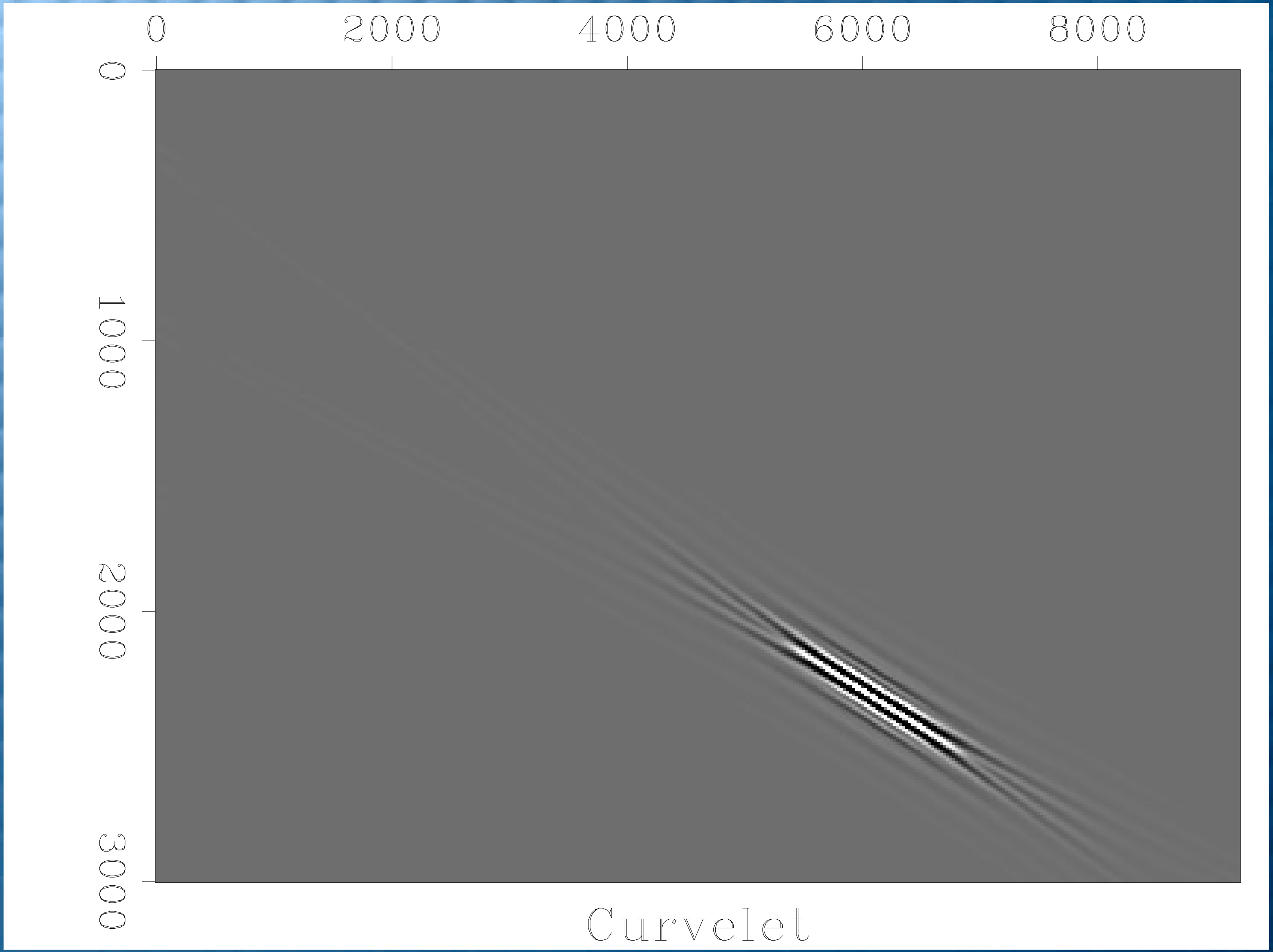
Still hope for now ... or is this too strong



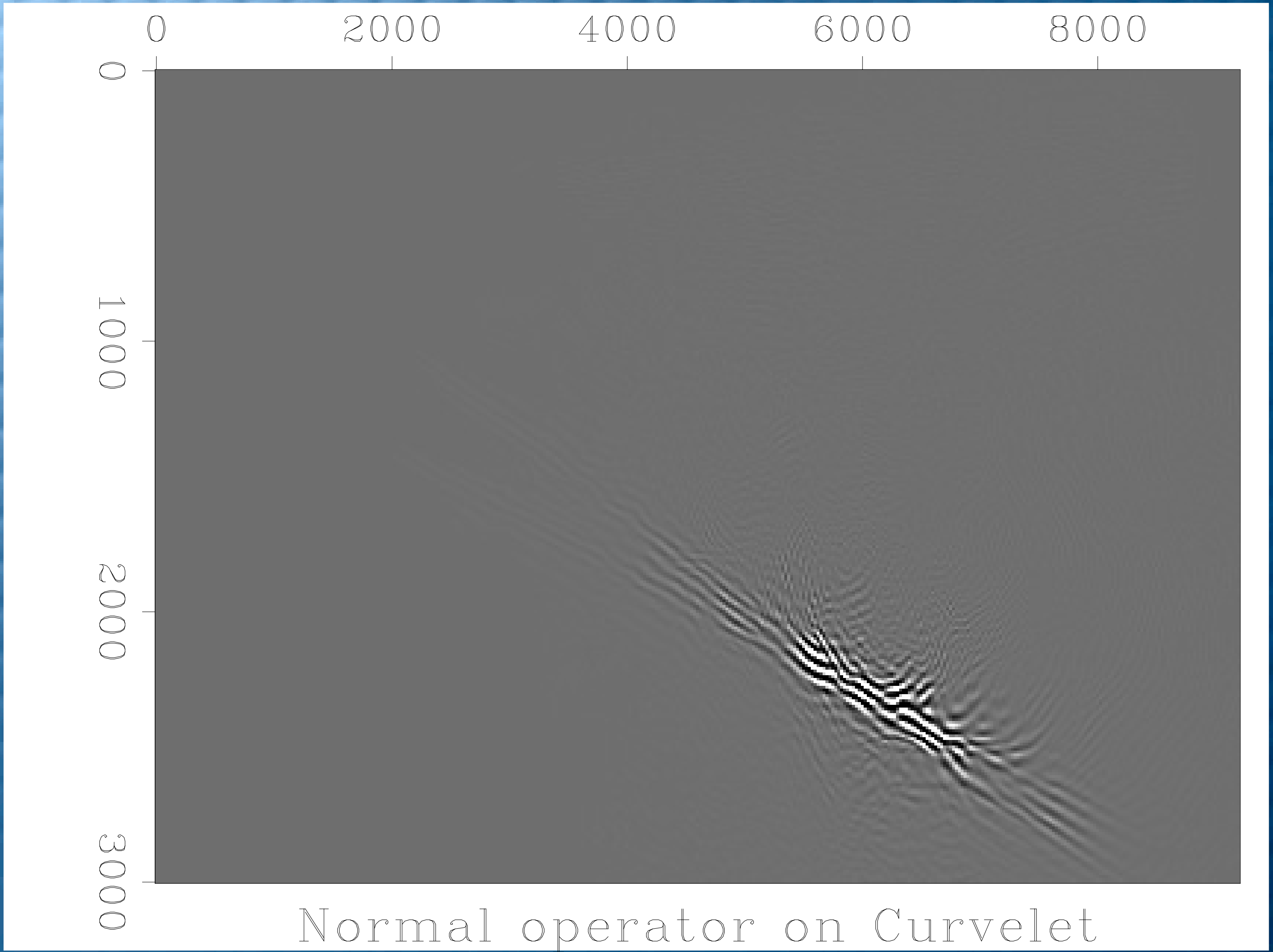
Examples



Operators



Operators

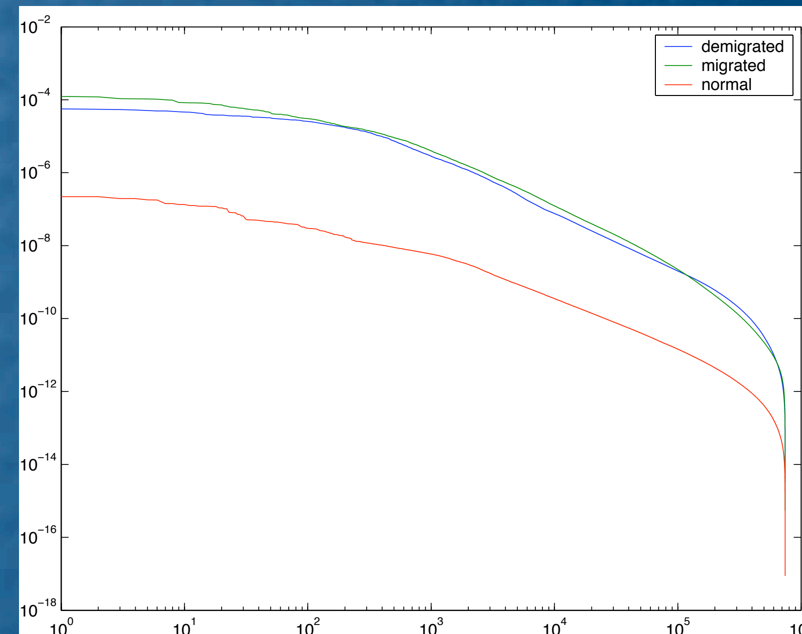

0
2000
4000
6000
8000
0
1000
2000
3000
Normal operator on Curvelet

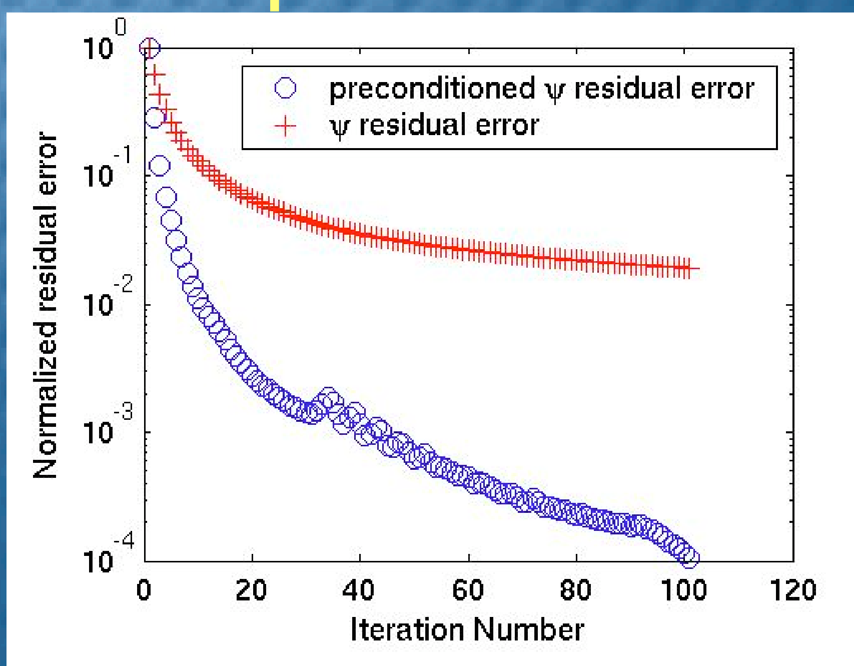
Operators

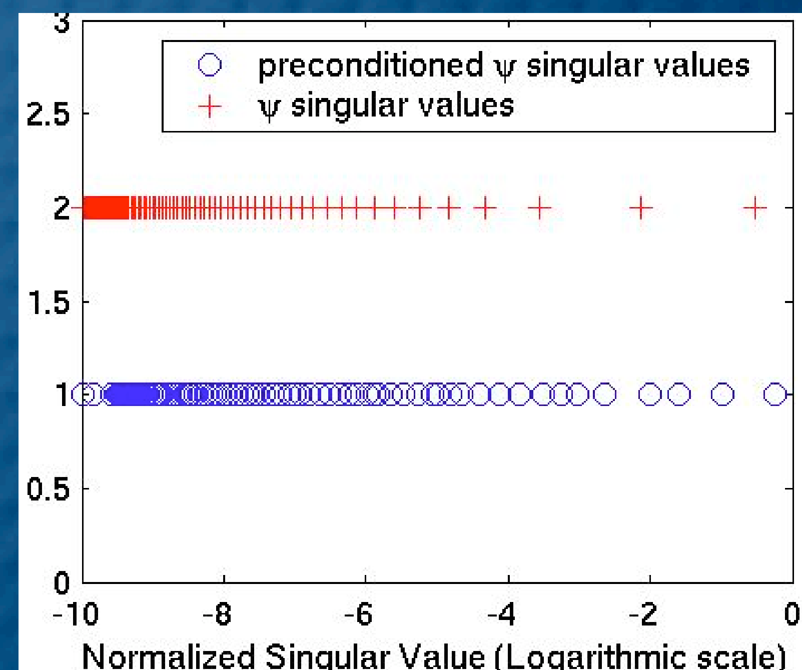
- **Curvelets remain curvelet-like**
- **Almost diagonalize normal/ blurring operator**
- **“Compress” the operators Krylov subspace methods**
- **Ultimate preconditioners**

\# of coeff.

magnitude







Operators

$$
\mathbf{K}^T\mathbf{K} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}
$$

very large and full

- Curvelet normal & inverse operators are sparse
- Square-root diagonal whitens
- Diagonal corrects for amplitudes (like decon)

$$
\mathbf{B}\mathbf{K}^T\mathbf{K}\mathbf{B}^T = \begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}
$$

very large but sparse

$$
\begin{aligned}
\mathbf{C}_{\tilde{n}} &\triangleq \mathbf{E}\{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^T\} \\
&= \mathrm{diag}\left(\mathrm{diag}\left(\mathbf{B}\mathbf{K}^T\mathbf{K}\mathbf{B}^T\right)\right) \\
&\triangleq \mathbf{\Gamma}^2
\end{aligned}
$$

$$
\tilde{\mathbf{n}} \triangleq \mathbf{B}\mathbf{K}^T\mathbf{n}
$$

Seismic imaging

Direct computation of

$$\mathbf{KB}^* = \mathbf{V}^*\mathbf{\Gamma}$$

$$\mathbf{KU}^* = \mathbf{B}^*\mathbf{\Gamma}$$

- is ***prohibitively*** expensive
- requires 1 (de)-migration per basis-function
- Use Monte-Carlo sampling to compute $\mathbf{\Gamma}$
- Use sparse interative matrix solvers

Donoho,Candes, Lucier, Kane, , Kalifa, H.

Seismic imaging

Right precondition:

$$\mathbf{d} = \mathbf{K}\mathbf{M}^{-1}\mathbf{x} + \mathbf{n} \equiv \mathbf{F}\mathbf{x} + \mathbf{n}, \qquad \mathbf{x} \equiv \mathbf{M}\mathbf{m}$$

where

$$\mathrm{M}\cdot = \Gamma \mathrm{B}\cdot,\ \mathrm{M}^{-1}\cdot = \mathbf{B}^T\boldsymbol{\Gamma}^{\dagger}\cdot \text{ and } \langle \mathbf{F}\mathbf{f},\ \mathbf{F}\mathbf{g}\rangle \approx \langle \mathbf{f}, \mathbf{g}\rangle$$

$$\boxed{\begin{aligned} \mathbf{F}^T\mathbf{d} &= \mathbf{F}^T\mathbf{F}\mathbf{x} + \mathbf{F}^T\mathbf{n} \\ \mathbf{y} &= \underbrace{\mathbf{N}}_{\approx \mathbf{I}}\mathbf{x} + \boldsymbol{\epsilon} \end{aligned}}$$

Seismic imaging

Solve for $\mathbf{u}$ **in** $\mathbf{y} = \mathbf{u} + \boldsymbol{\epsilon}, \quad \mathbf{u} = \mathbf{N}\mathbf{x}$

by

$$\hat{\mathbf{u}} = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{u}\|_2^2 + \|\mathbf{x}\|_{p=1,w}$$

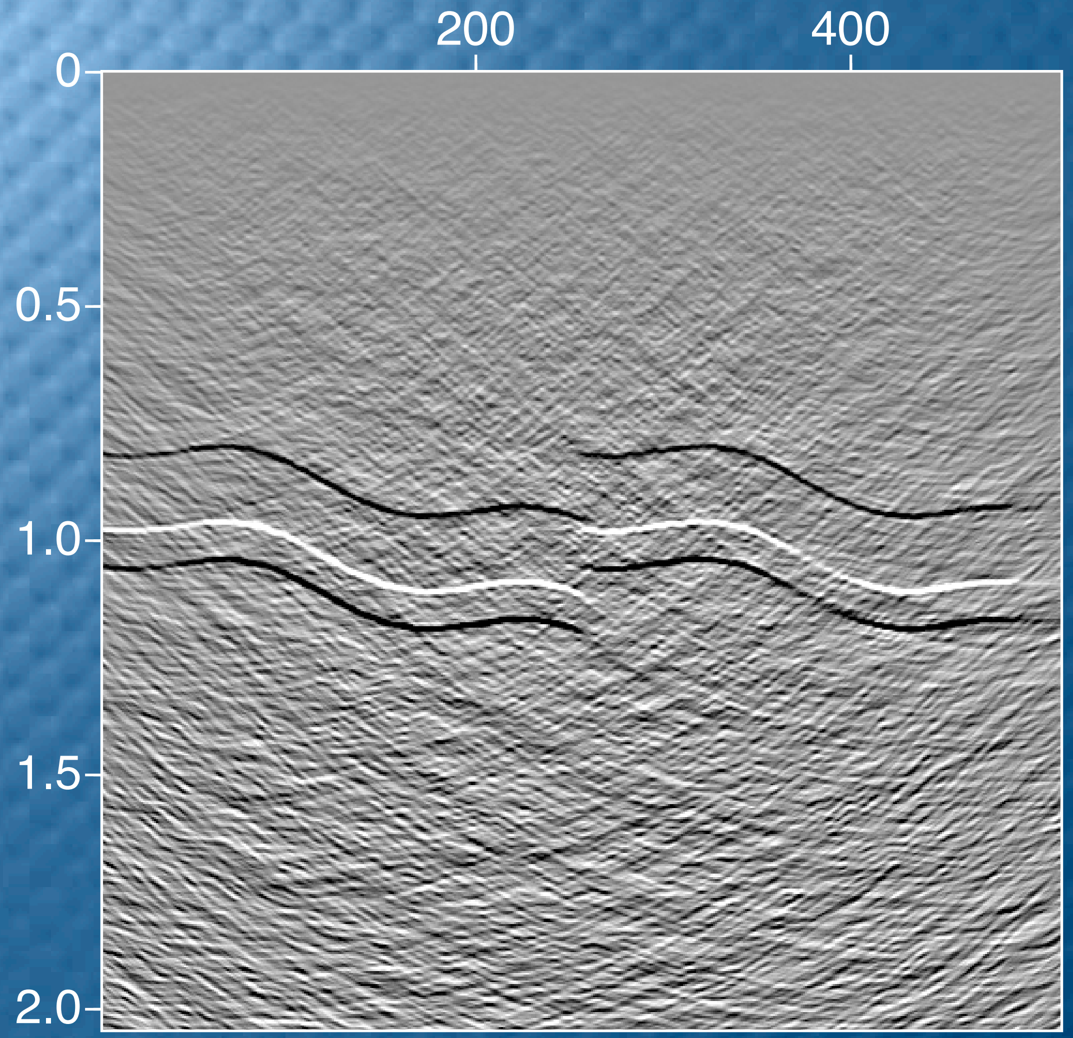
yielding

$$\hat{\mathrm{m}} = \mathbf{B}^T\mathbf{N}^\dagger\boldsymbol{\Gamma}S_\lambda\left(\boldsymbol{\Gamma}^\dagger\mathbf{B}\mathbf{K}^T\mathbf{d}\right) = \mathbf{B}^T\mathbf{N}^\dagger S_{\lambda\boldsymbol{\Gamma}}\left(\mathbf{B}\mathbf{K}^T\mathbf{d}\right)$$

diagonal approximation

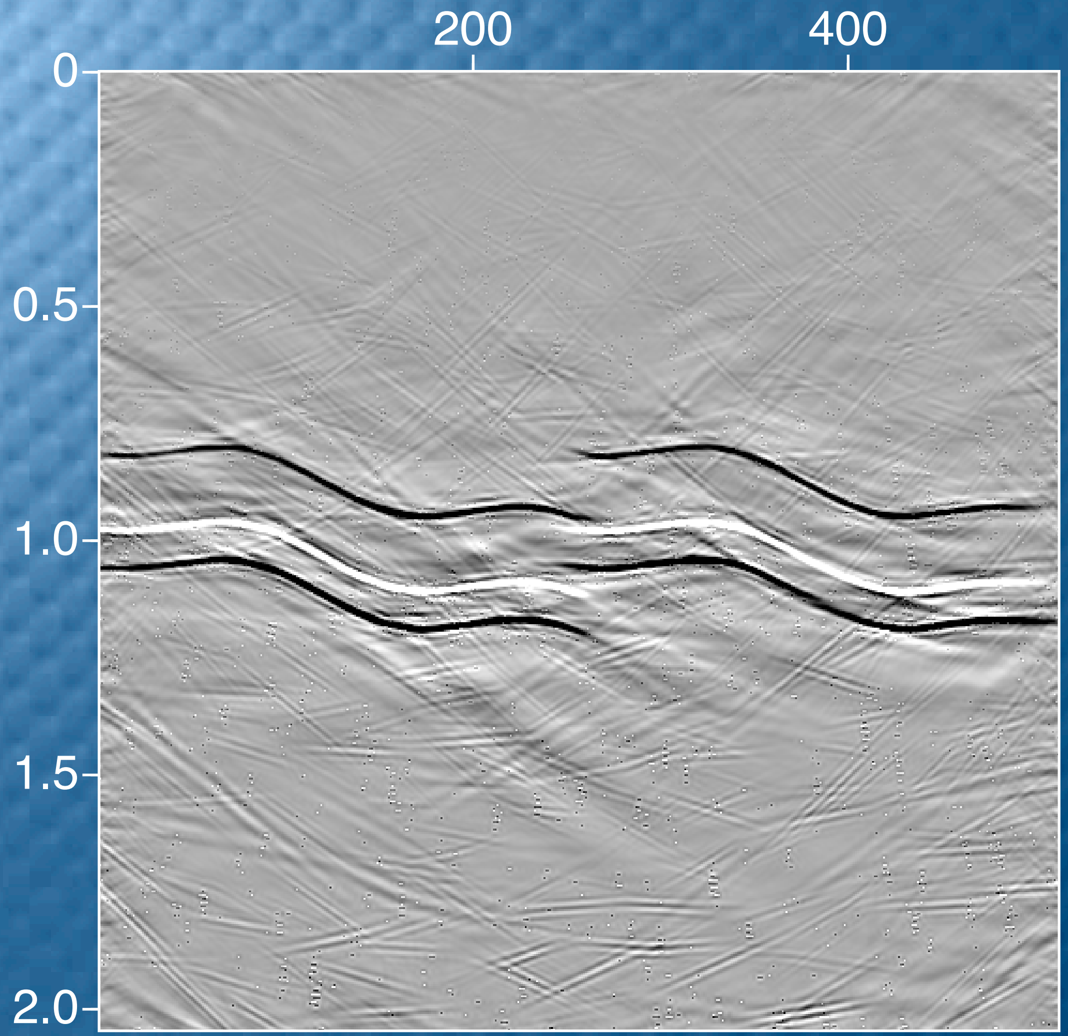
$$\hat{\mathbf{m}}_{\mathrm{diag}} = \left(\boldsymbol{\Gamma}^2\right)^\dagger S_{\lambda\Gamma}\left(\mathbf{B}\mathbf{K}^T\mathbf{d}\right)$$

Examples



Noisy Image

Examples



Denoised after Thresholding

Seismic imaging

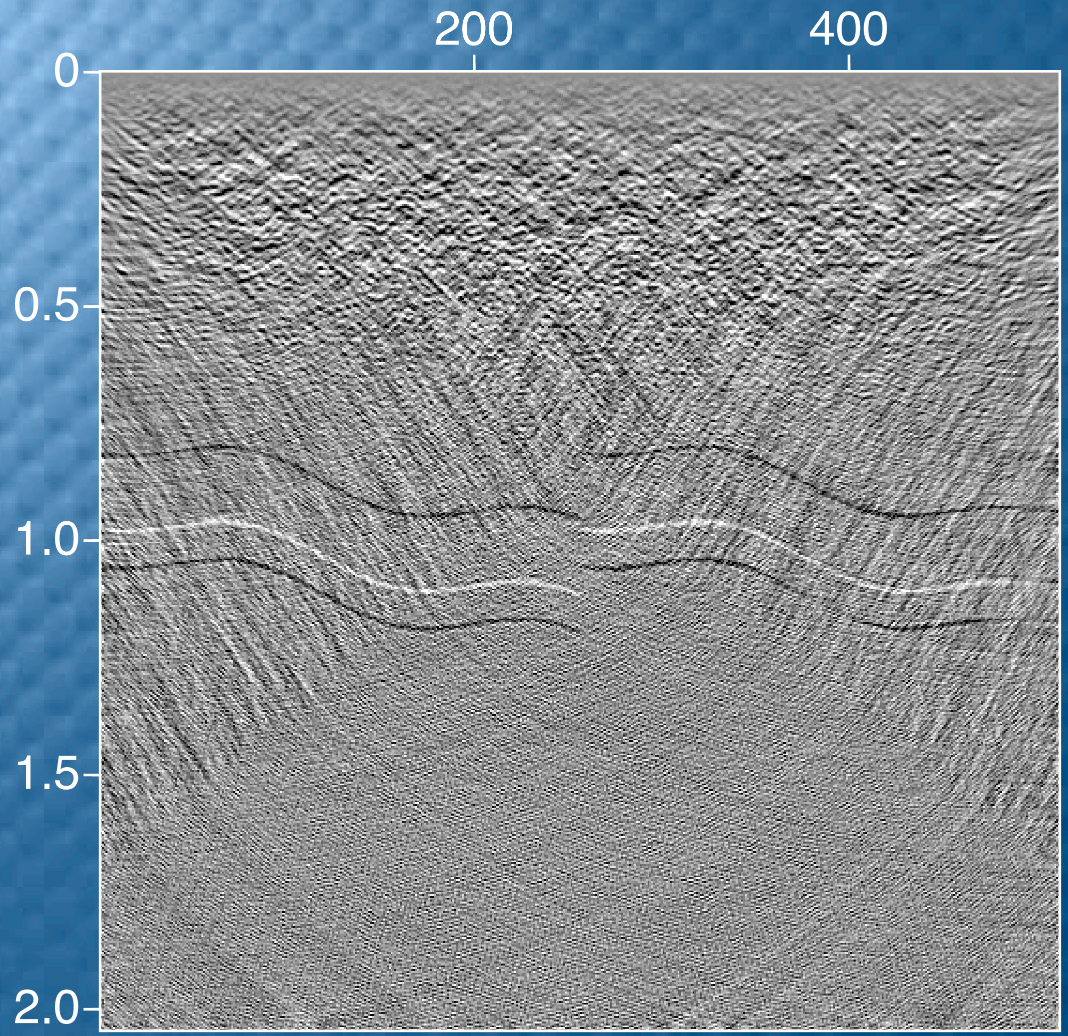
Use initial guess in Krylov-subspace app

$$\mathsf{N}_k \mathsf{Q}_k \cdot = \mathsf{Q}_k \mathsf{T}_k \cdot + \mathsf{R}_k \cdot \quad \text{given} \quad \hat{\mathbf{u}}$$

with

$$\mathsf{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & \ddots & \ddots & \beta_{k-1} \\ 0 & \cdots & & \beta_{k-1} & \alpha_k \end{pmatrix}$$

Examples



Least-squares migrated Image

Optimization

Impose *prior* info *via* constrained opt.

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} J(\mathbf{m}) \quad \text{s.t.} \quad |\mathrm{N}\mathbf{m} - \hat{\mathbf{x}}|_\mu \leq \mathbf{e}_\mu$$

with $J(\mathbf{m}) = \|\mathbf{m}\|_1$

- **Uses augmented Lagrangian (Nocedal and Wright '01)**
- **Initial Lagrangian multipliers by gradient of** $\hat{\mathrm{x}}_0$
- **Uses Steepest Decent and line search**

Optimization

Set the tolerances:

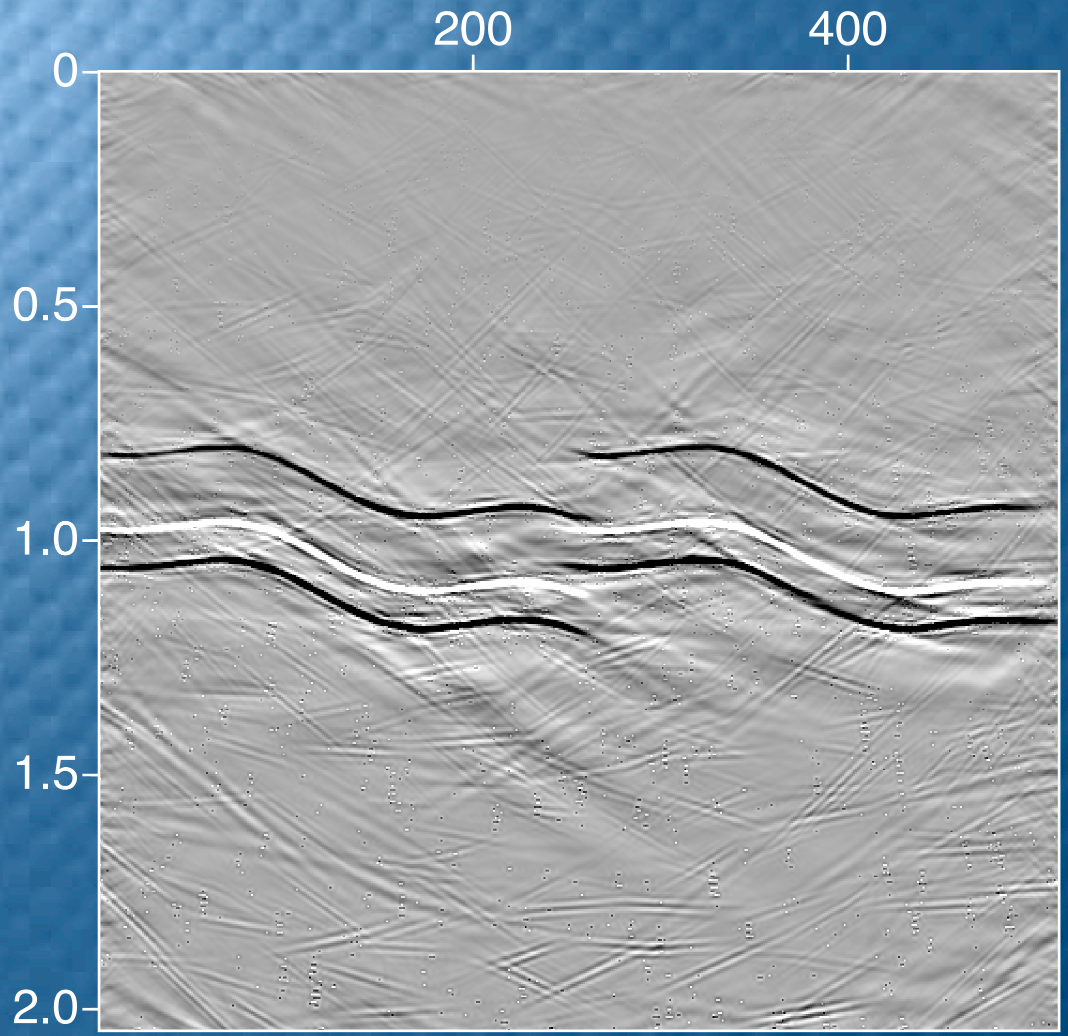
$$e_\mu = \begin{cases} w_\mu & \text{if} \quad |\hat{x}_i|_\mu \quad \geq \quad |w|_\mu \\ \lambda w_\mu & \text{if} \quad |\hat{x}_i|_\mu \quad < \quad |\lambda w|_\mu \end{cases}$$

with (Kalifa)

$w_\mu = \lambda$ for $\mu \in \mathcal{M}_0$ and $w_\mu = \infty$ for $\forall \mu \in \mathcal{M}$ $\wedge\, \mu \notin \mathcal{M}_0$

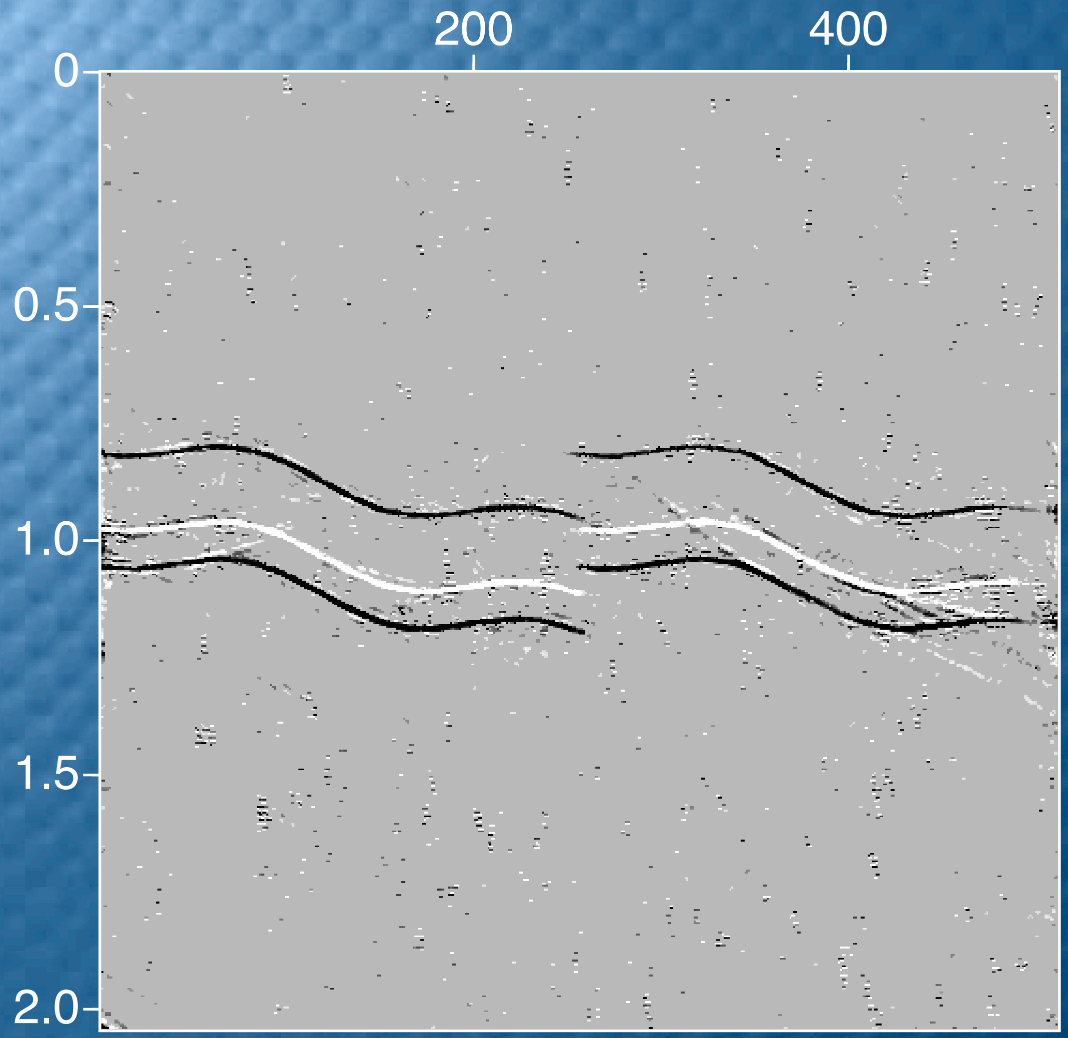
where $\{\mathcal{M}_0 : \quad x_\mu \leq \sup_{f \in \boldsymbol{\mathcal{F}}} |\langle f, \varphi_\lambda \rangle| \}$

Examples



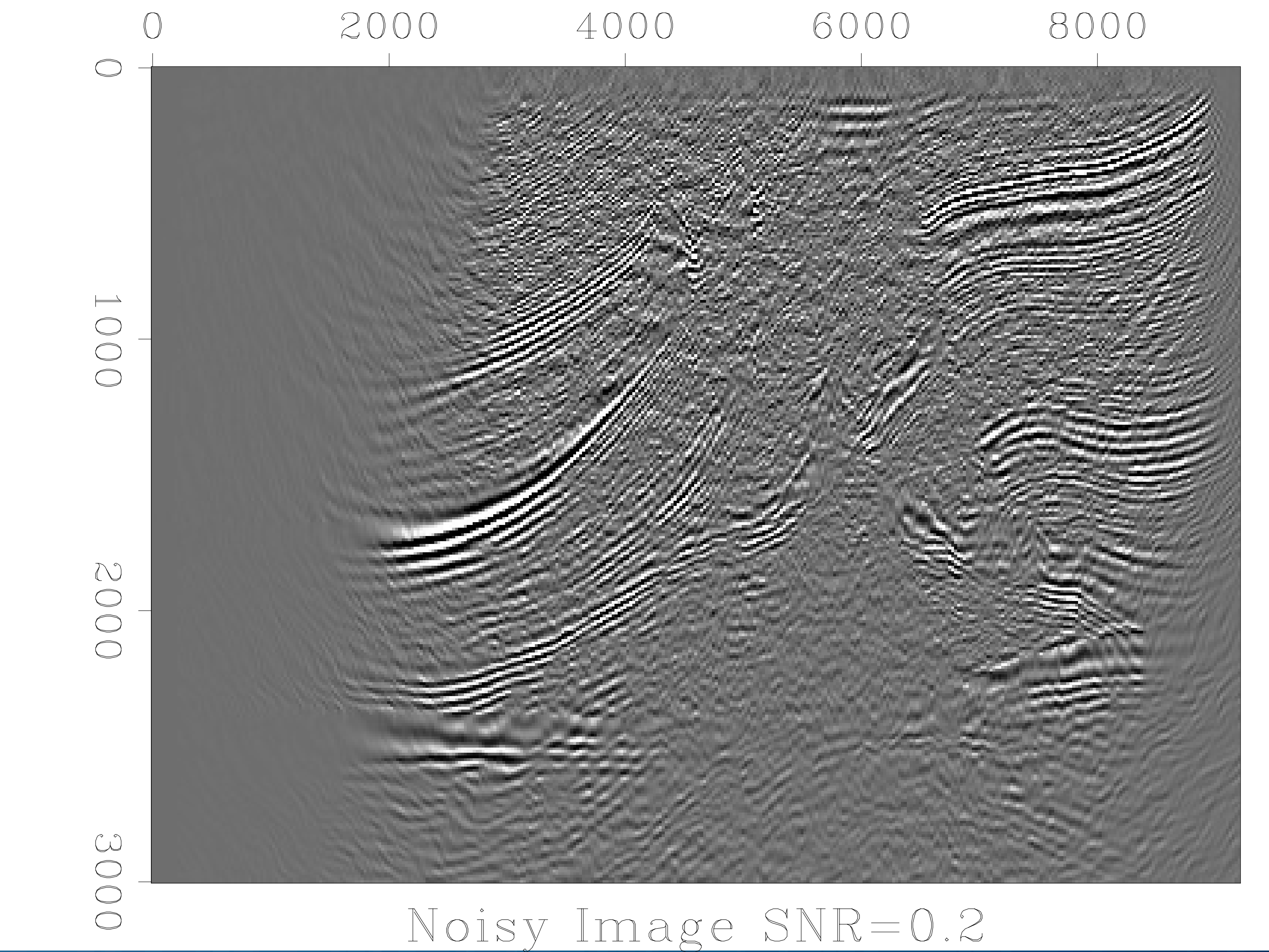
Denoised after Thresholding

Examples

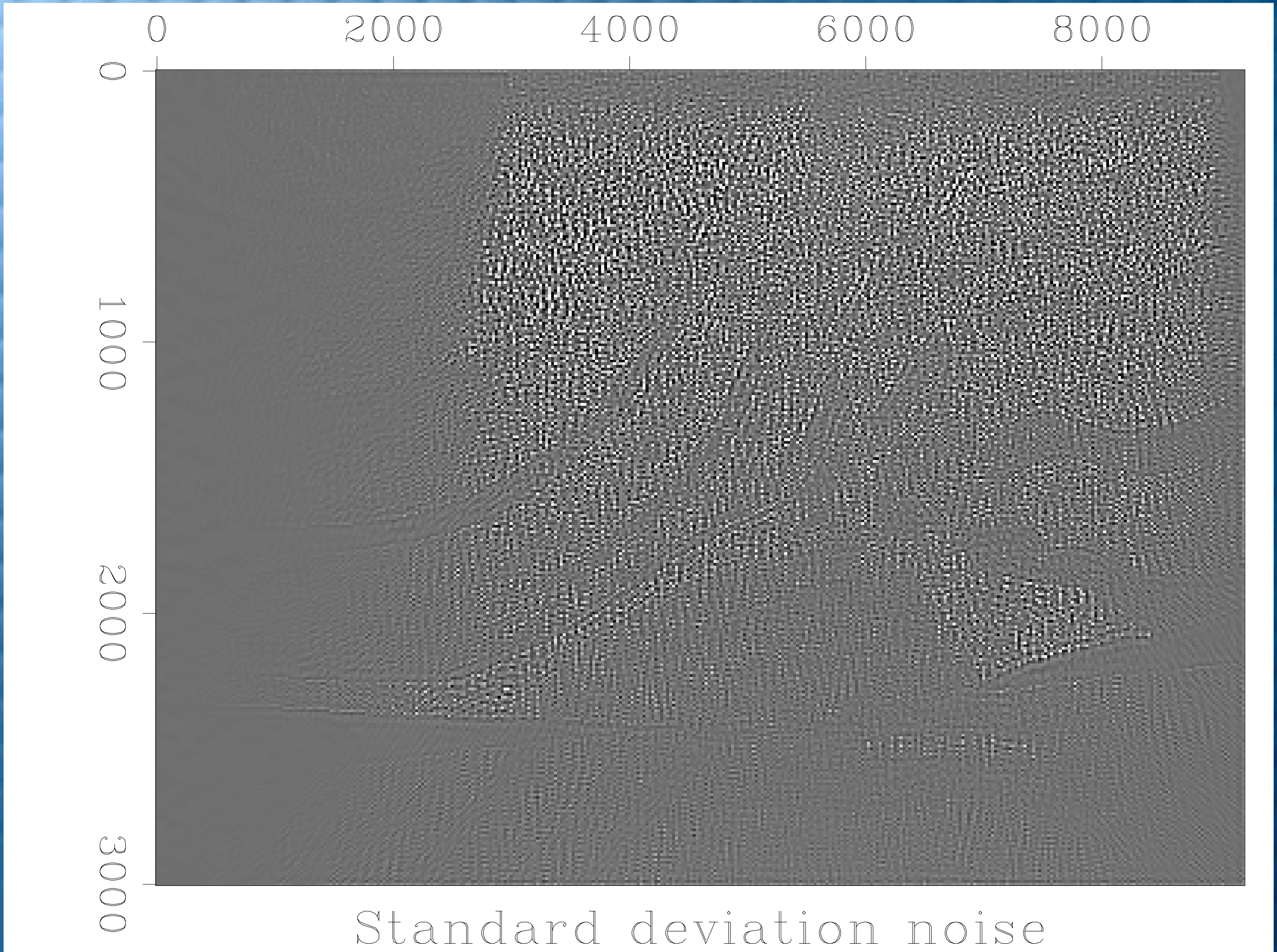


Constrained Optimization

Noisy Marmoussi

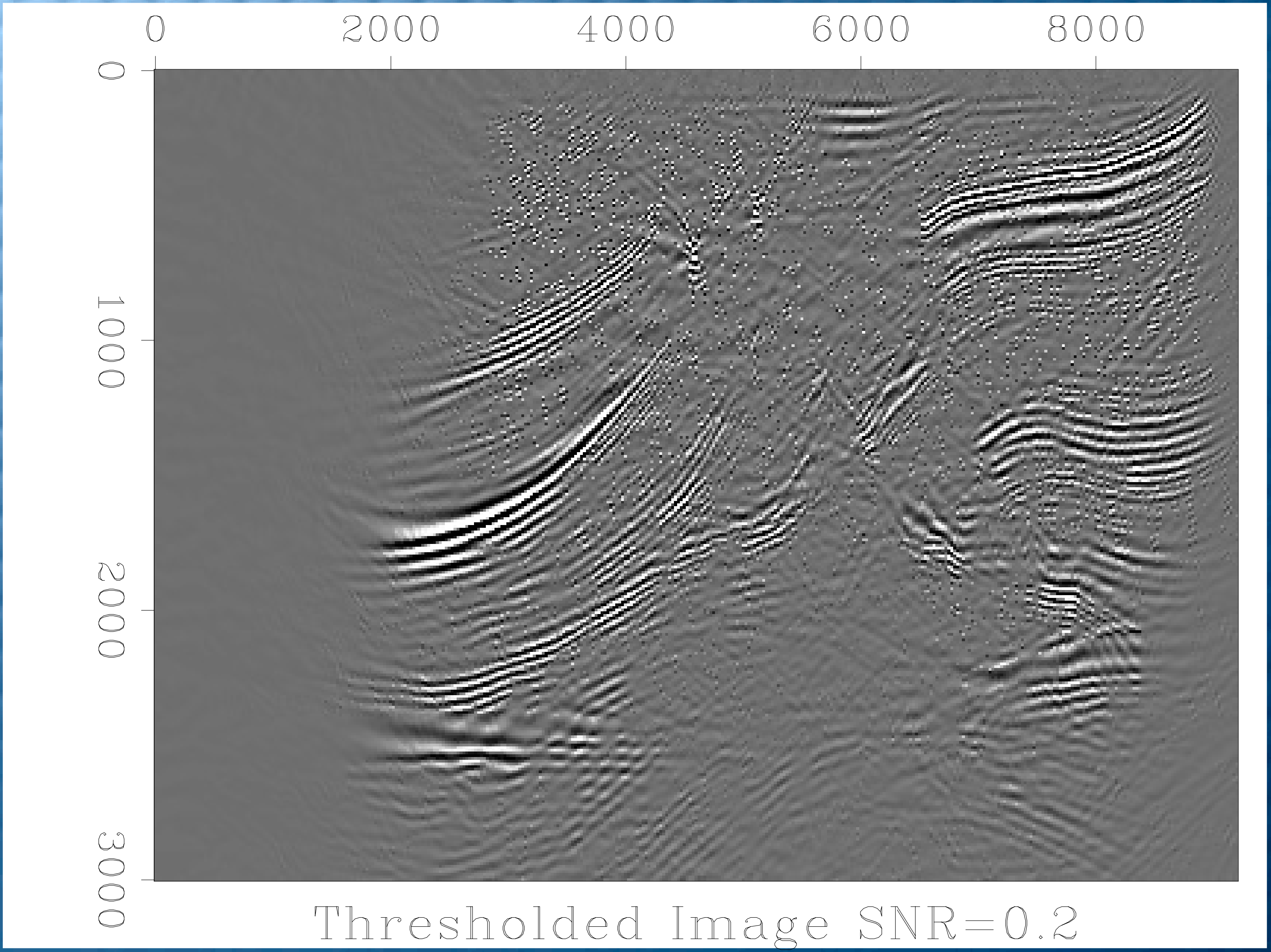


Γ Marmoussi

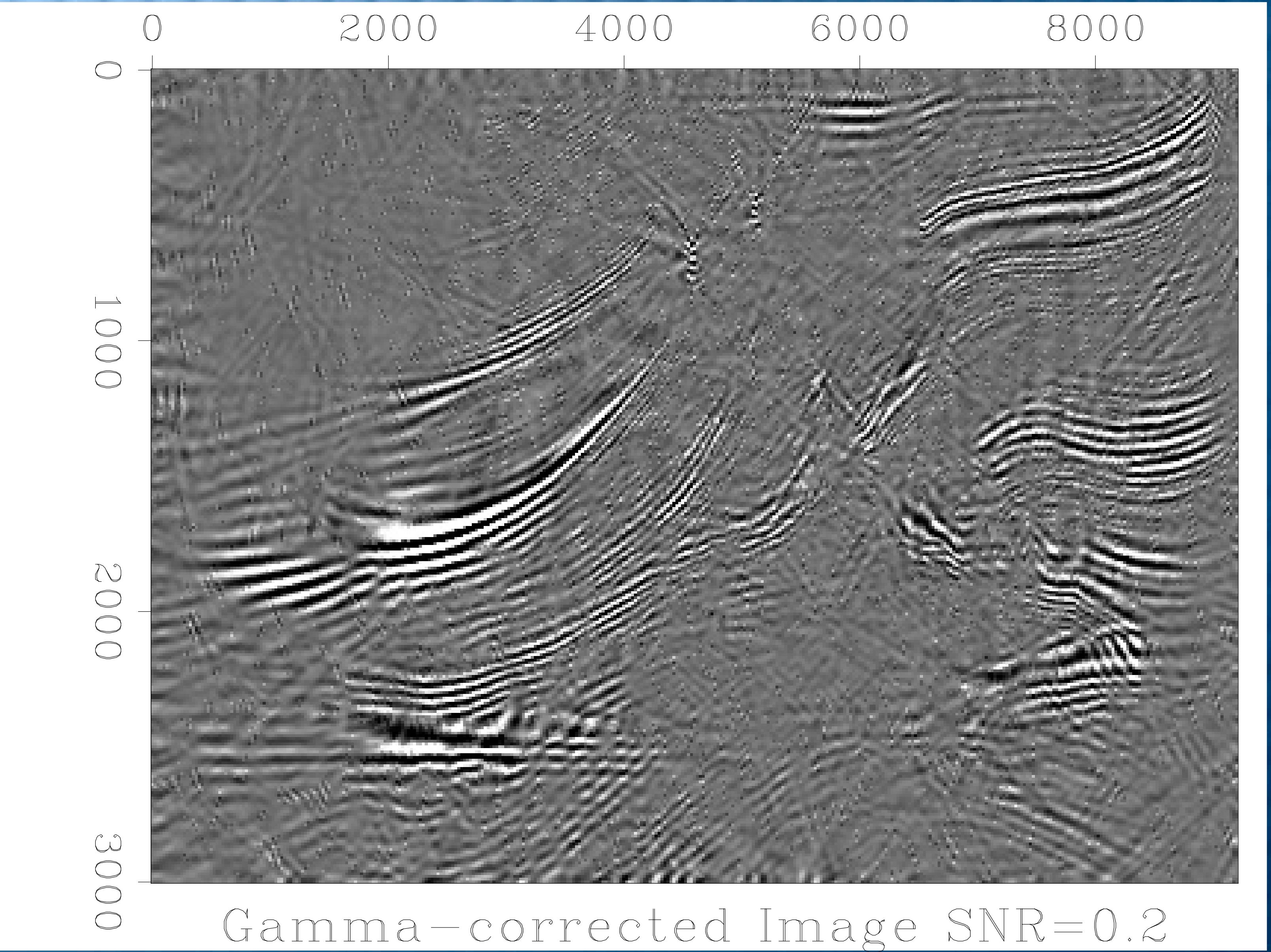


0
2000
4000
6000
8000
0
1000
2000
3000
Standard deviation noise

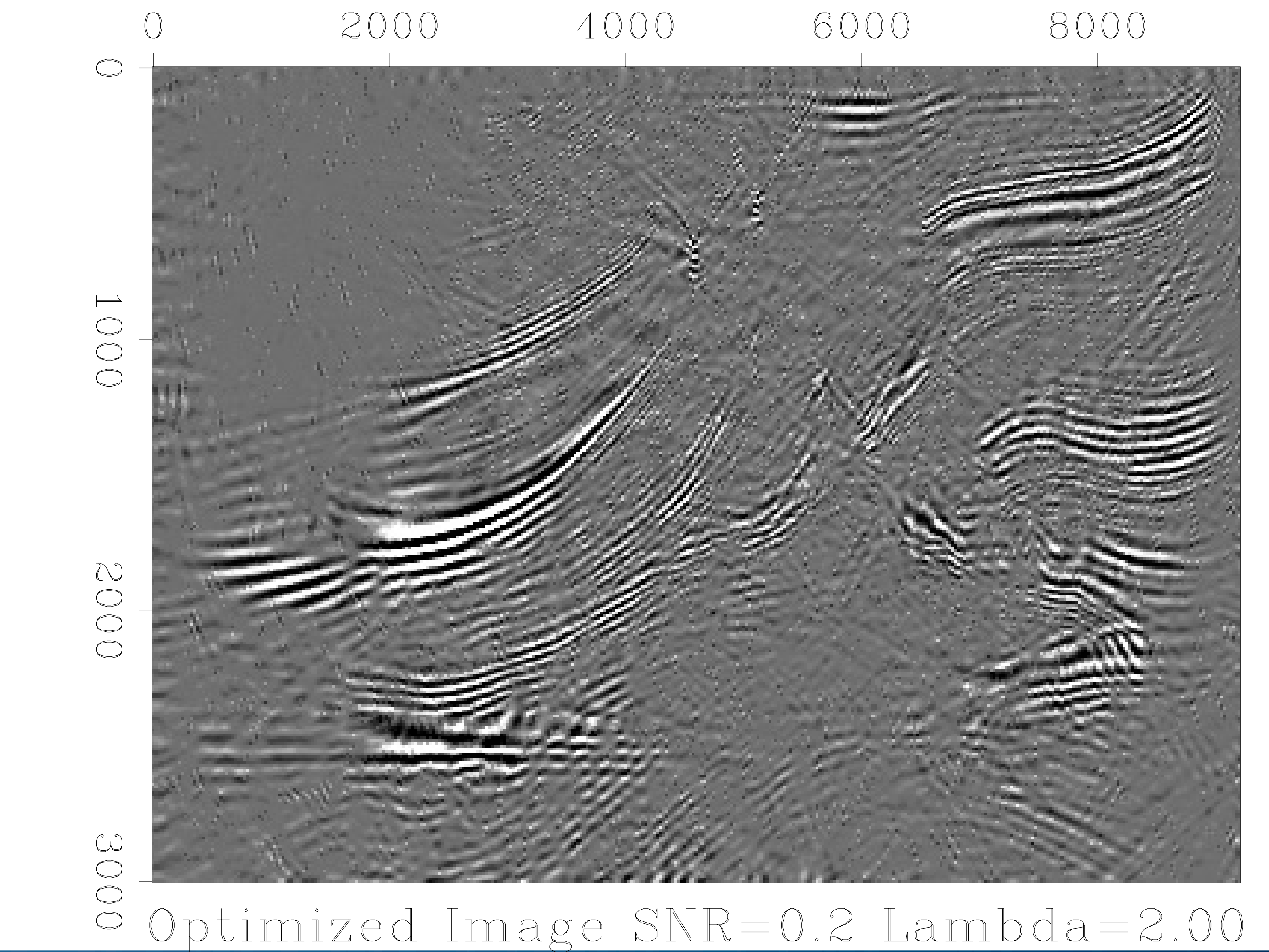
Marmoussi thresholded



Marmoussi corrected



Marmoussi denoised



Variational problems

Remaining issues

- **find appropriate penalty functional for wave-fronts** $J(\mathrm{m}) : \mathrm{m} \notin W^s$
- **exploit redundancy = smoothness**
- **sparse basis on the model**
- **compress FIO's (Douma, M&-)**

Refine norm: anisotropic diffusion

Set

$$J(\mathbf{m}) = \|\mathbf{\Lambda}^{1/2}\nabla\mathbf{m}\|_p$$

with

$$\mathbf{\Lambda} = \frac{1}{|\nabla\bar{c}|^2+2\beta}\left\{\begin{pmatrix}\partial_{x_2}\bar{c}\\ -\partial_{x_1}\bar{c}\end{pmatrix}\begin{pmatrix}\partial_{x_2}\bar{c} & -\partial_{x_1}\bar{c}\end{pmatrix}+\beta\mathbf{I}\right\}$$

- **uses information on background**
- **penalizes fluctuations along reflectors**

Osher, Schertzer

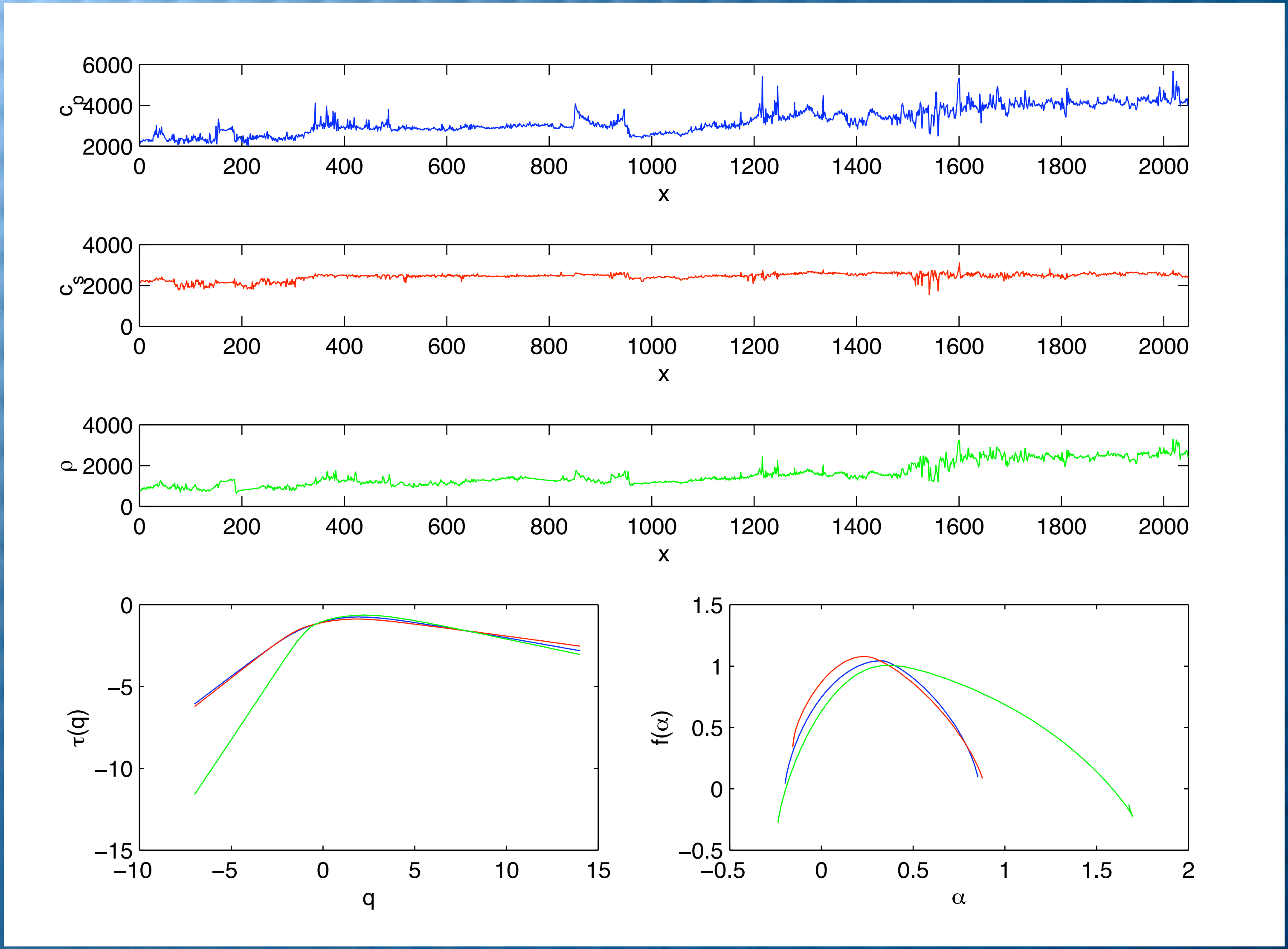
Characterization

Non-adaptive decomposition for estimation.

Data-adaptive decomp. for characterization/ feature extraction:

- **“vertical” earth looks *multifractal* (Muller, Saucier, -)**
- **seismic images are like smoothed *derivatives* ⇔ only singularities reconstructed ...**
- **“phase”associated to reflectors**

Multifractal behavior



Characterization

Singularity characterization:

- **Exploit *waveforms* that carry info on the type of *reflectors***
- **Use *adaptive* redundant dictionary**
  - **localize the *knots* (*stratigraphy*)**
  - **estimate local *regularity* (*lithology*)**

Pattern extraction ....

Characterization

Parametric reflector representation:

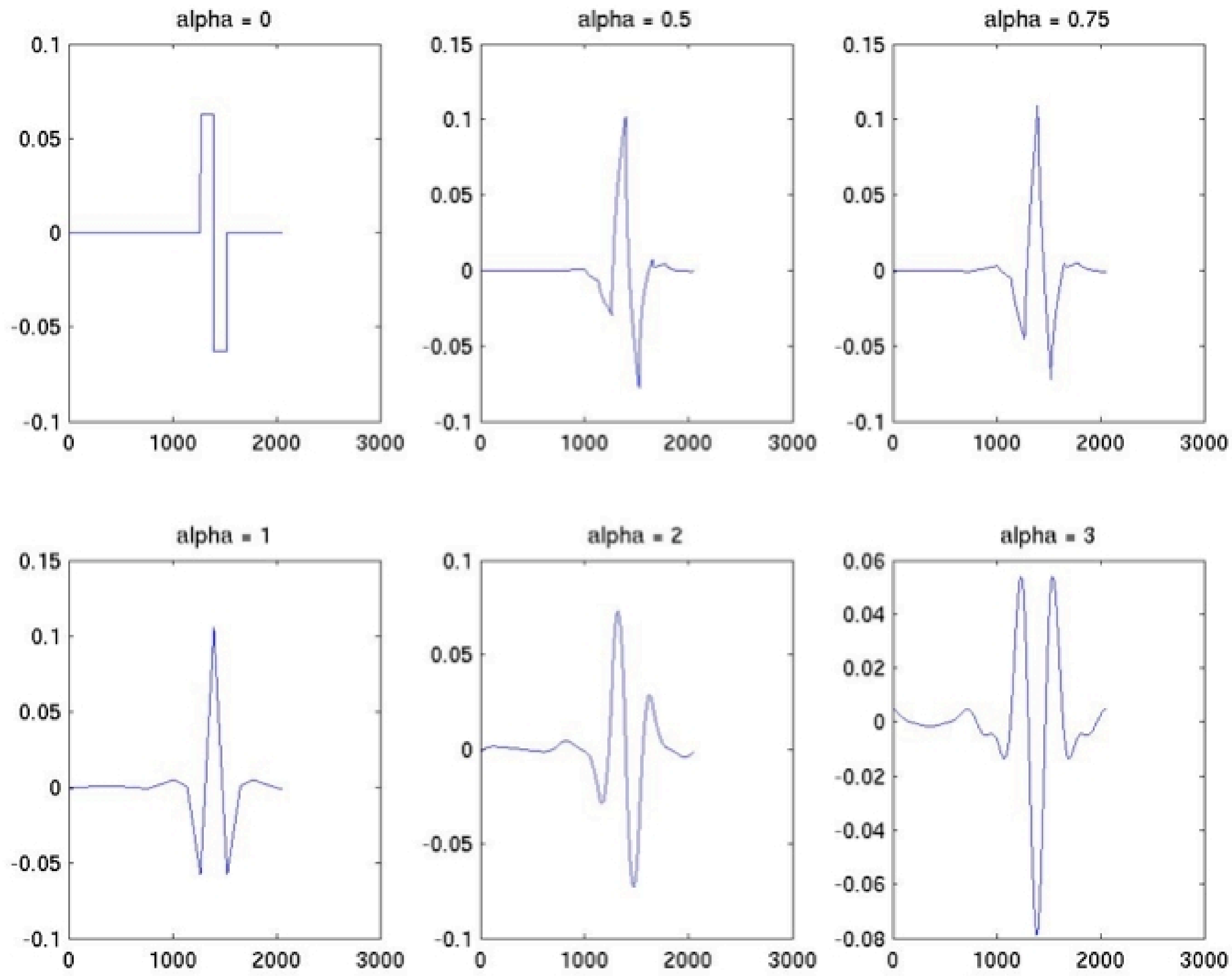
$$\chi_{\pm}^{\alpha}(z) = \begin{cases} 0 & z \lesseqgtr 0 \\ \frac{z^{\alpha}}{\Gamma(\alpha+1)} & z \gtrless 0 \end{cases}$$

Multifractional spline representation:

$$m(z) = \sum_{n \in N} c_{\pm}^{n} \chi_{\pm}^{\alpha_n}(z - z_n).$$

H.'03

Some orthogonal fractional spline wavelets



Characterization

Basis Pursuit (Chen,Starck):

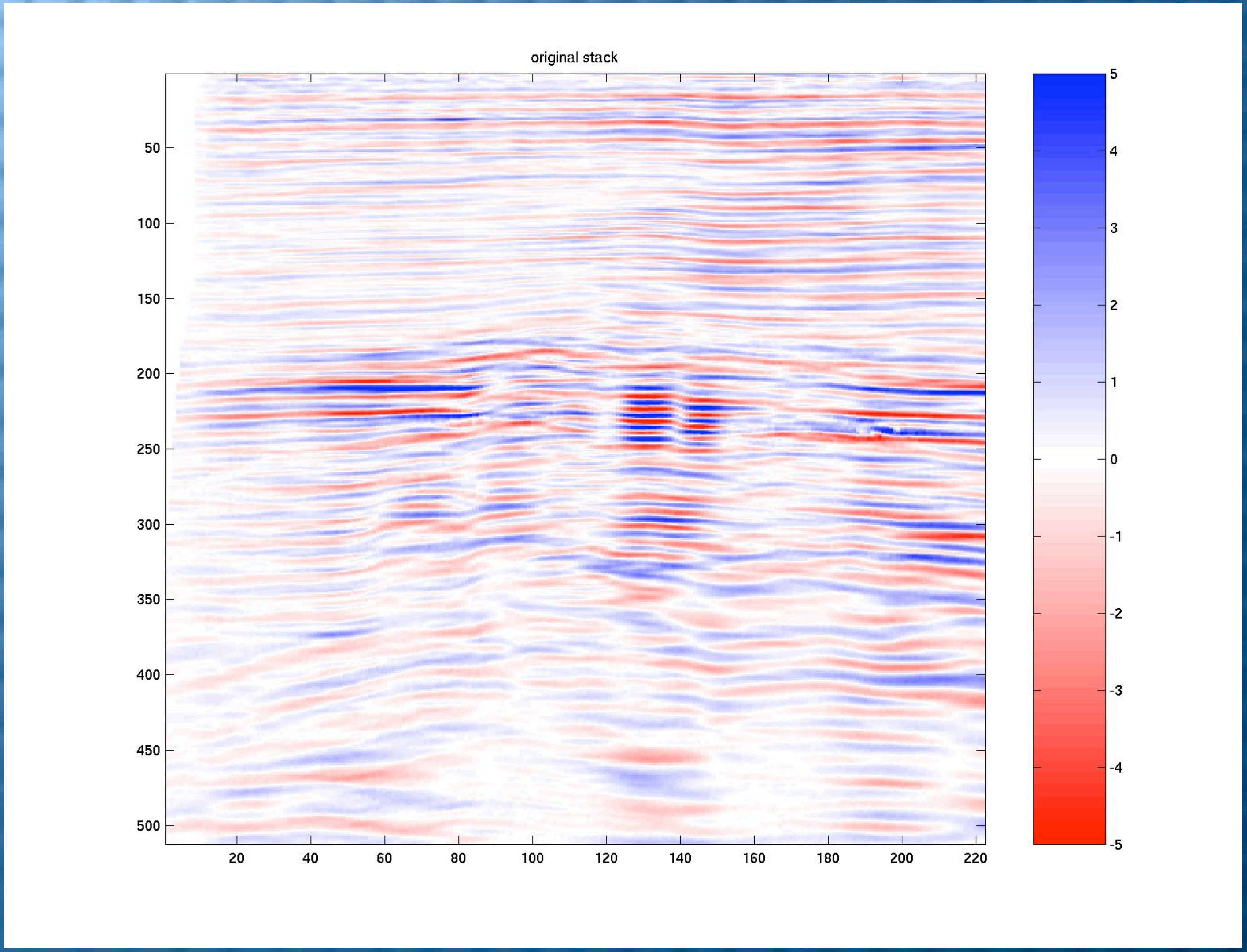
$$\min_{c} \|\mathbf{c}\|_1 \quad \text{subject to} \quad \boldsymbol{\Phi}\mathbf{c} = \mathbf{d}$$

Basis Pursuit Denoising:

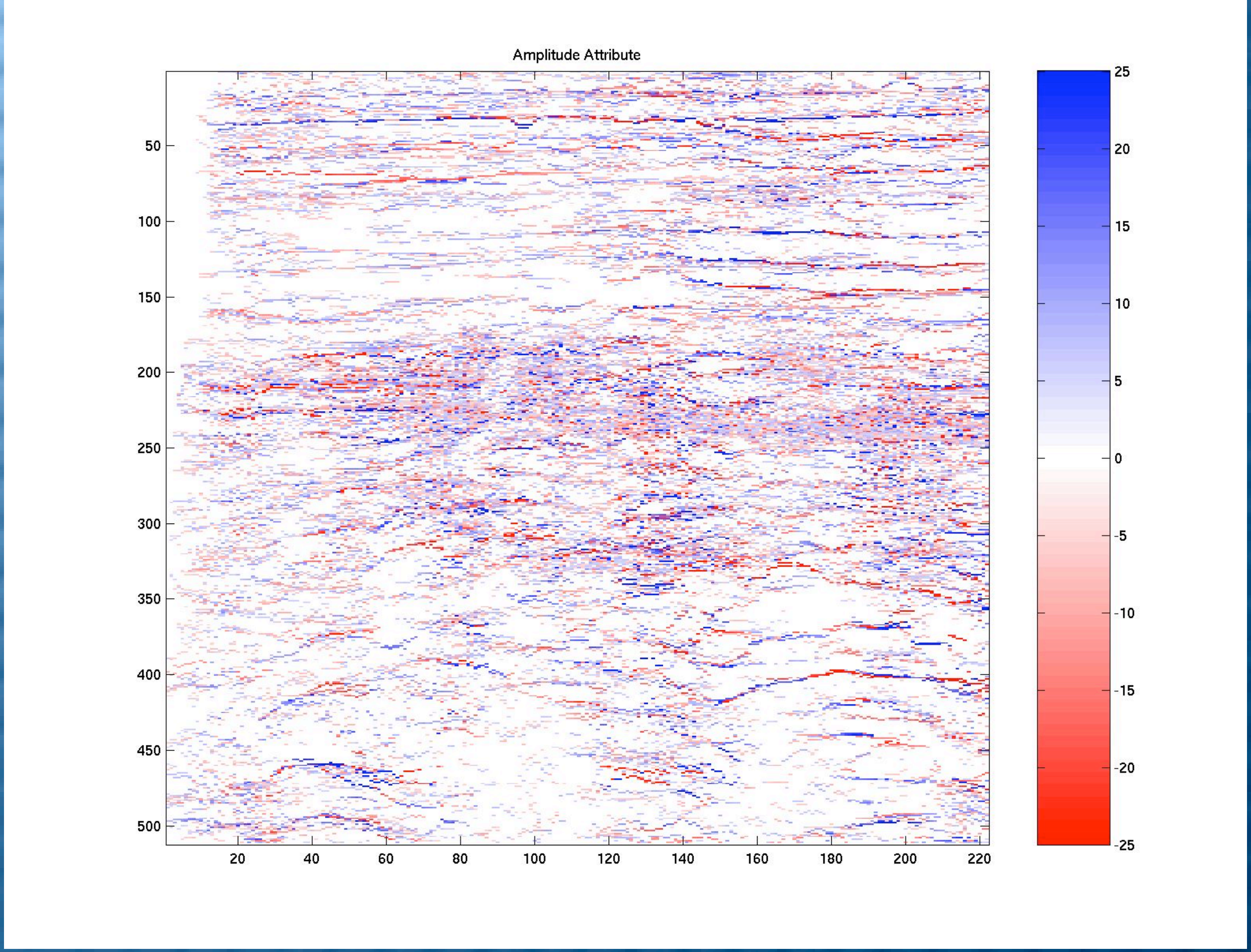
$$\min_{\mathbf{c}} \frac{1}{2}\|\mathbf{d} - \Phi\mathbf{c}\|_2^2 + \nu\|\mathbf{c}\|_1$$

- **Linear programming**
- **super-resolution**

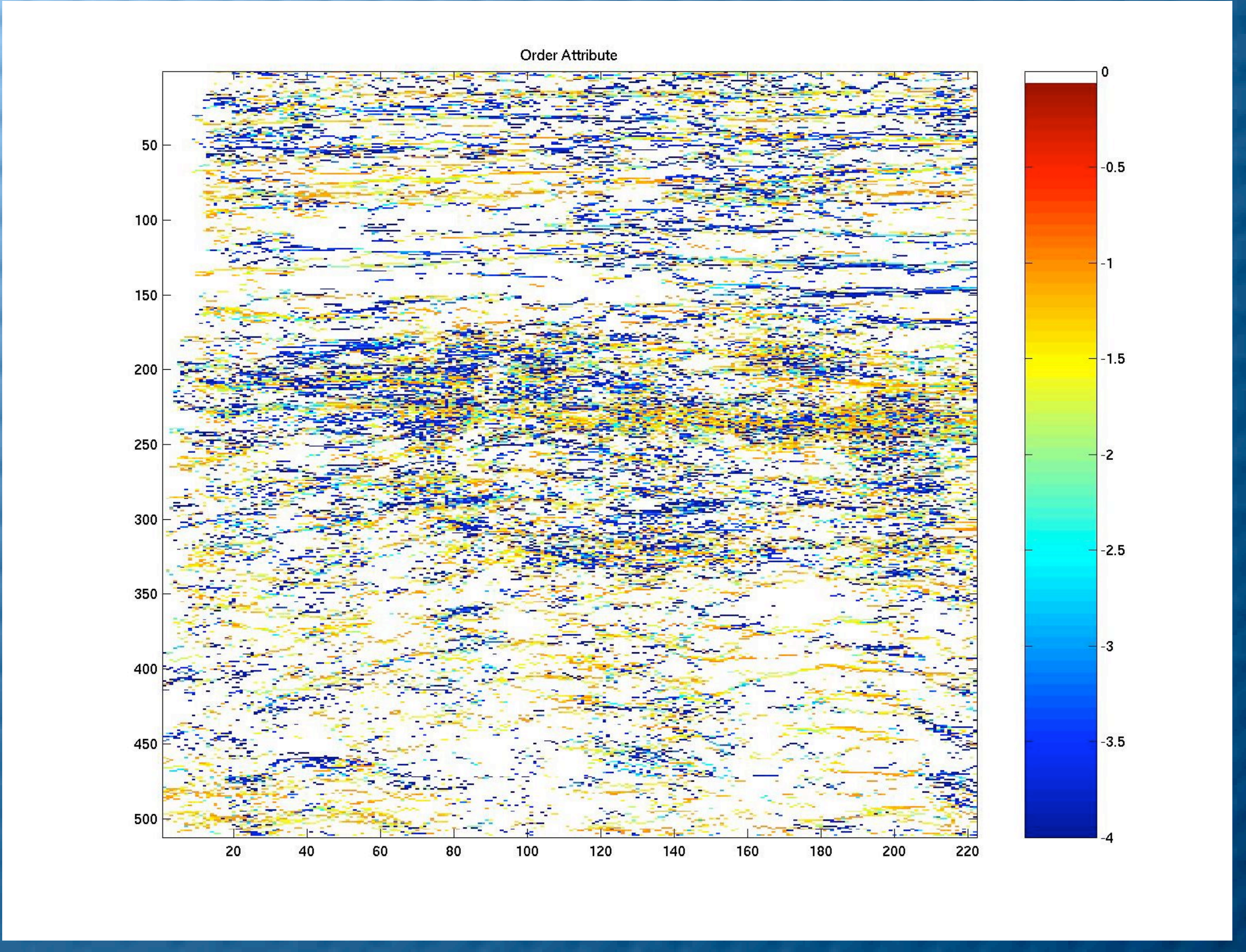
Characterization



Characterization



Characterization



Characterization

Fractional Splines Matching Pursuit:

- **locates the nodes of the splines**
- **their magnitude (”spiky” decon}**
- **their fractional order**
- **their “phase”**

Non-uniqueness is a BIG problem!

Modeling

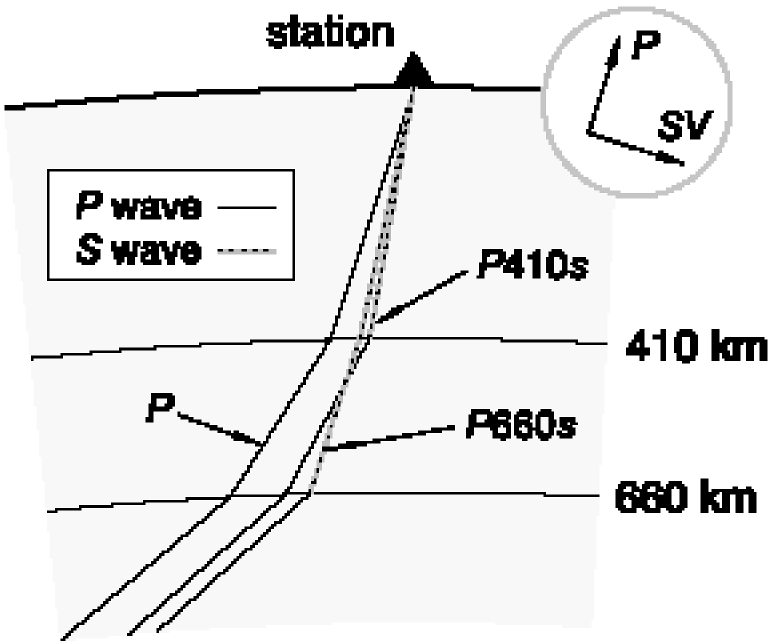
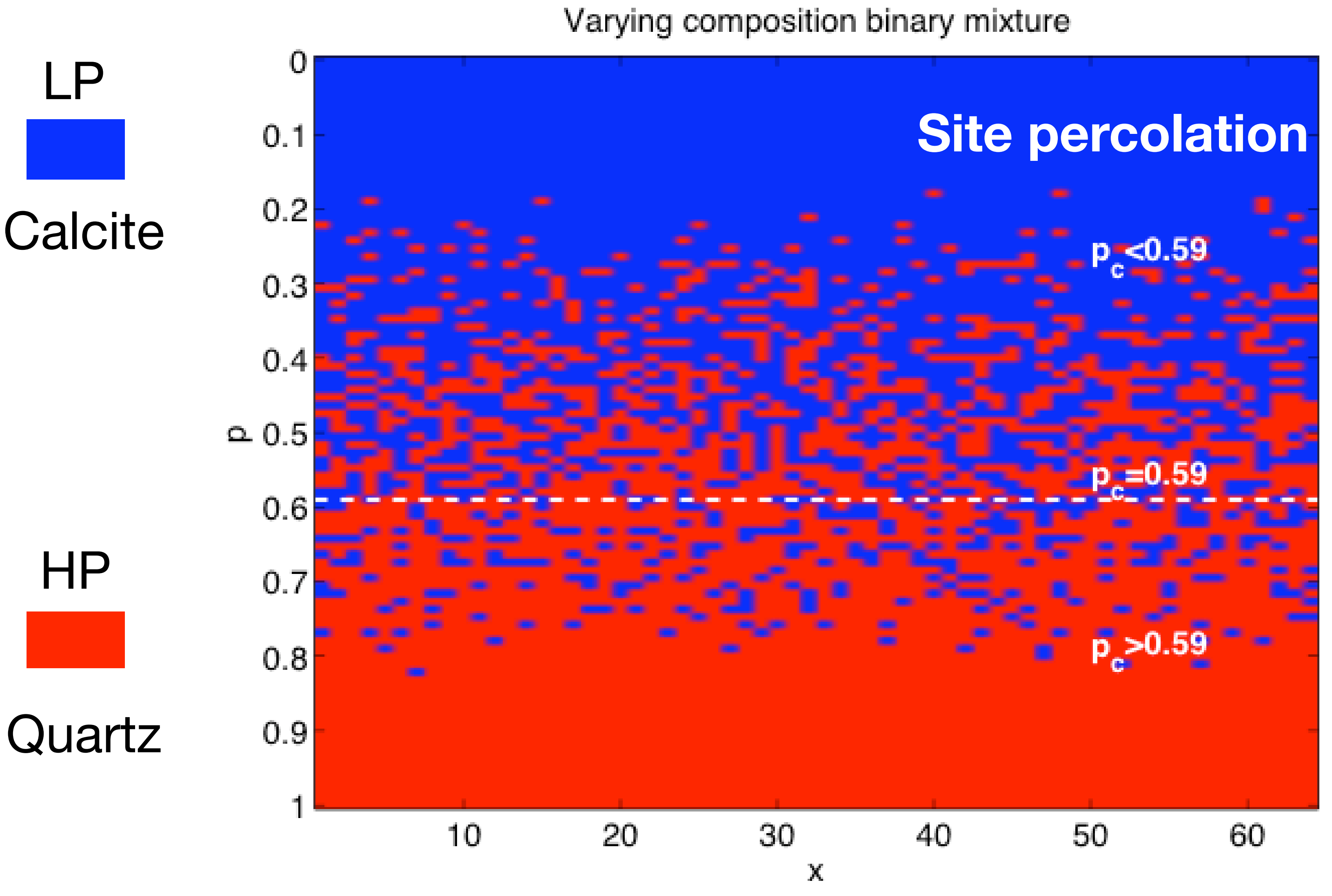


Fig. 1. Schematic ray diagram of the *P*, *P*410*s*, and *P*660*s* waves in a laterally homogeneous mantle. Triangle denotes the seismic station on the Earth's surface. The axes *P* and *SV* (top right) are in the direction of the P and SV waves' particle motion.

Lebedev 2002

Mixture laws for binary mixtures



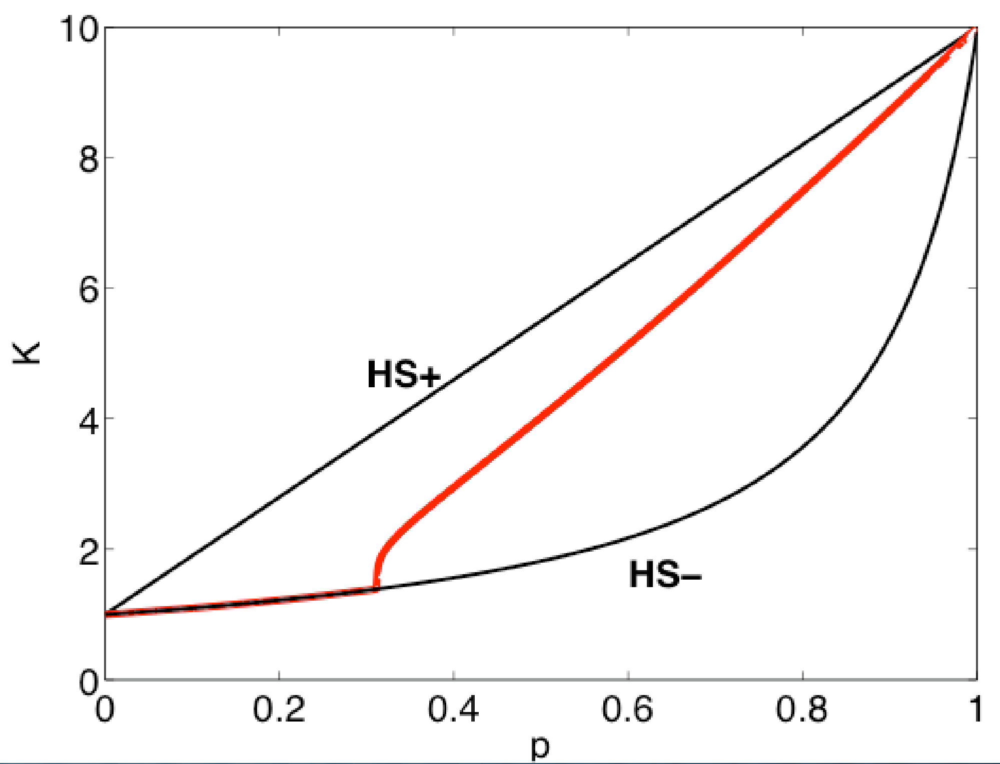
Mixture laws for binary mixtures

Percolation model:

Below p_c, HP is disconnected, use lower bound HS_-

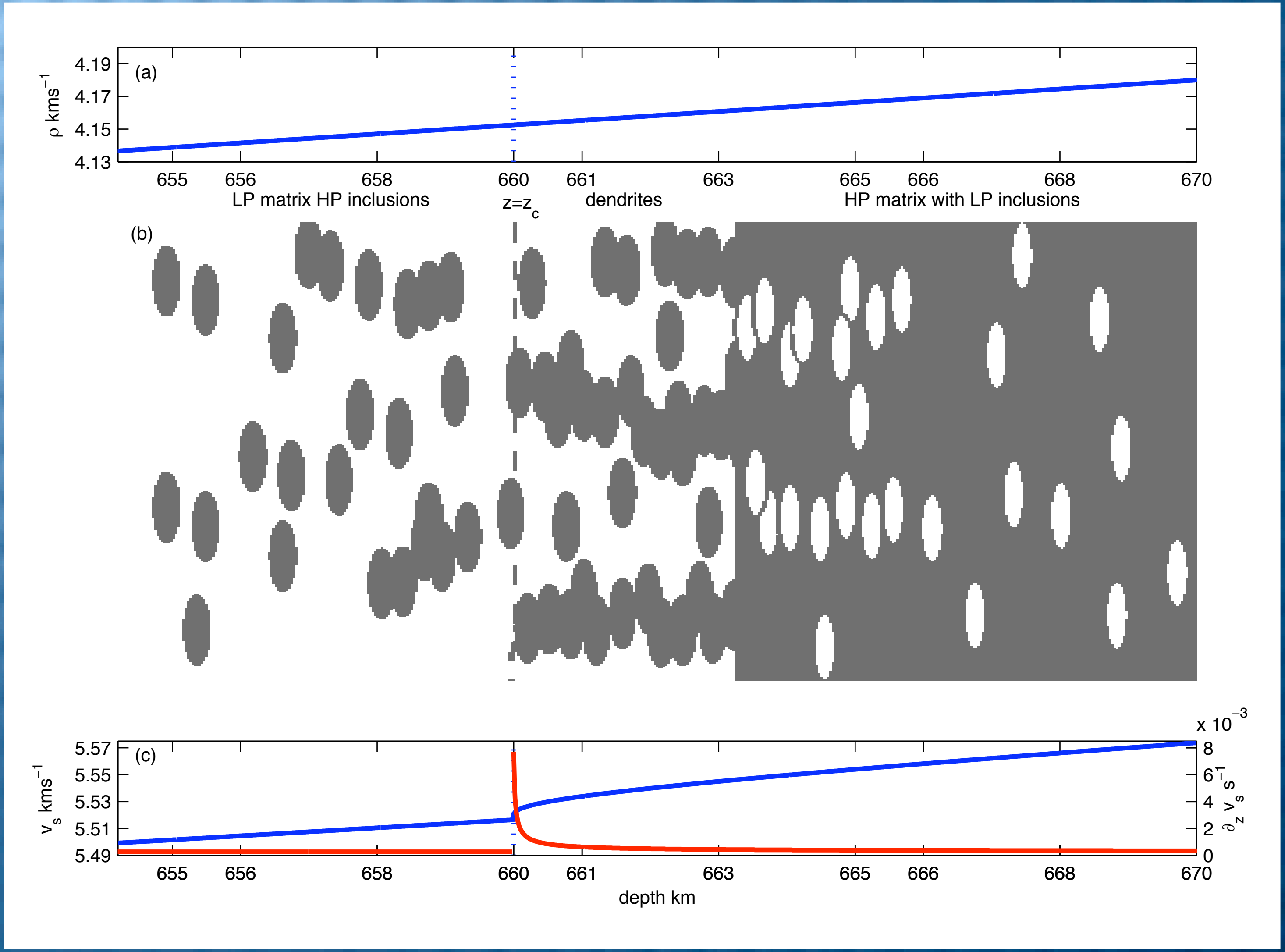
Above p_c, HP is connected, switch to HS_+ with appropriate volume fractions

Switching leads to singularity at $p = p_c$



Connectivity is the ruler of the game ...

Modeling



(a)
ρ kms−1
4.19
4.17
4.15
4.13
655 656 658 660 661 663 665 666 668 670
LP matrix HP inclusions
z=zc
dendrites
HP matrix with LP inclusions
(b)
(c)
x 10−3
vs kms−1
5.57
5.55
5.53
5.51
5.49
8
6
4
2
0
∂z vs s−1
655 656 658 660 661 663 665 666 668 670
depth km

Mixture laws for binary mixtures

Elastic case:

- **Controlled by *connectivity* of stiffest phase**

Transport case:

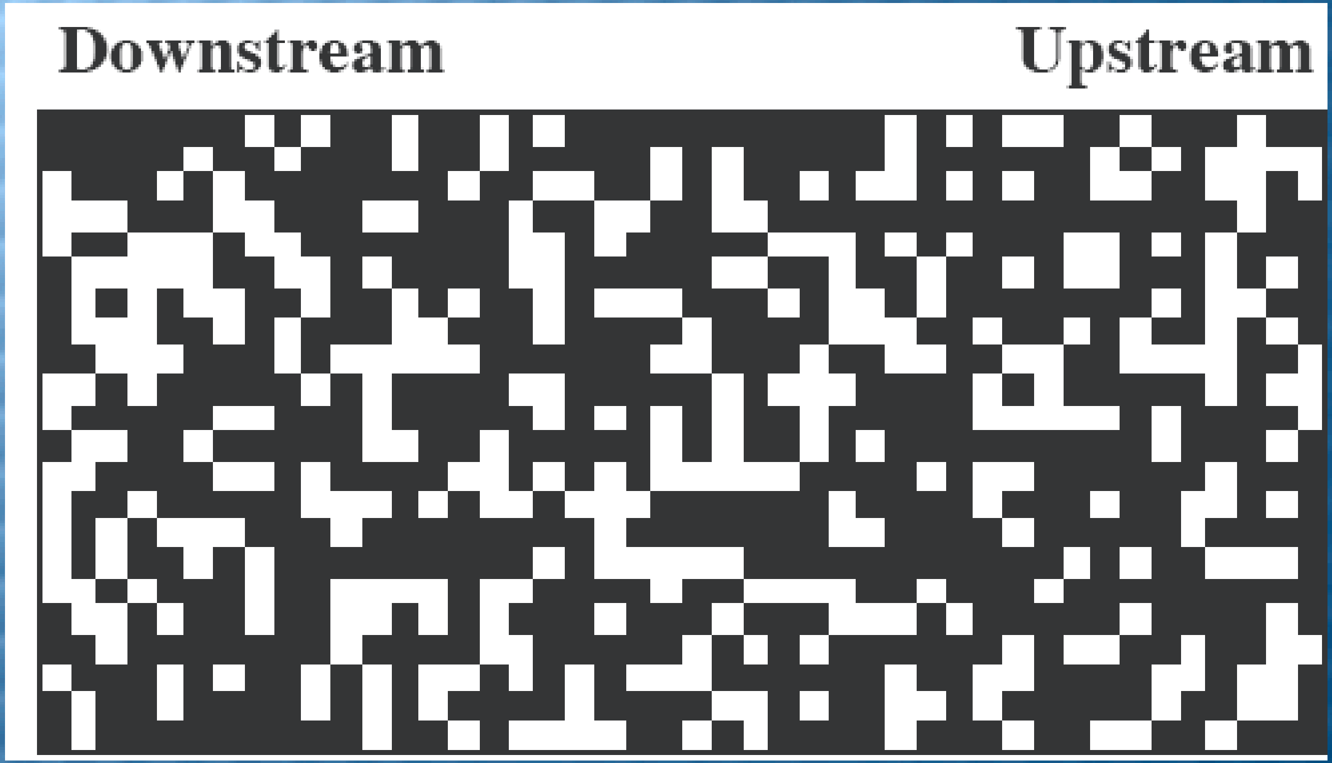
- **Controlled by *connectivity* of high-conductivity phase**

Note: Stiff phase = low-porosity, low-conductivity phase

Numerical setup

Grid up to 50 X 50 X 50

k_{HP}/k_{LP} from ~1 to 10^6

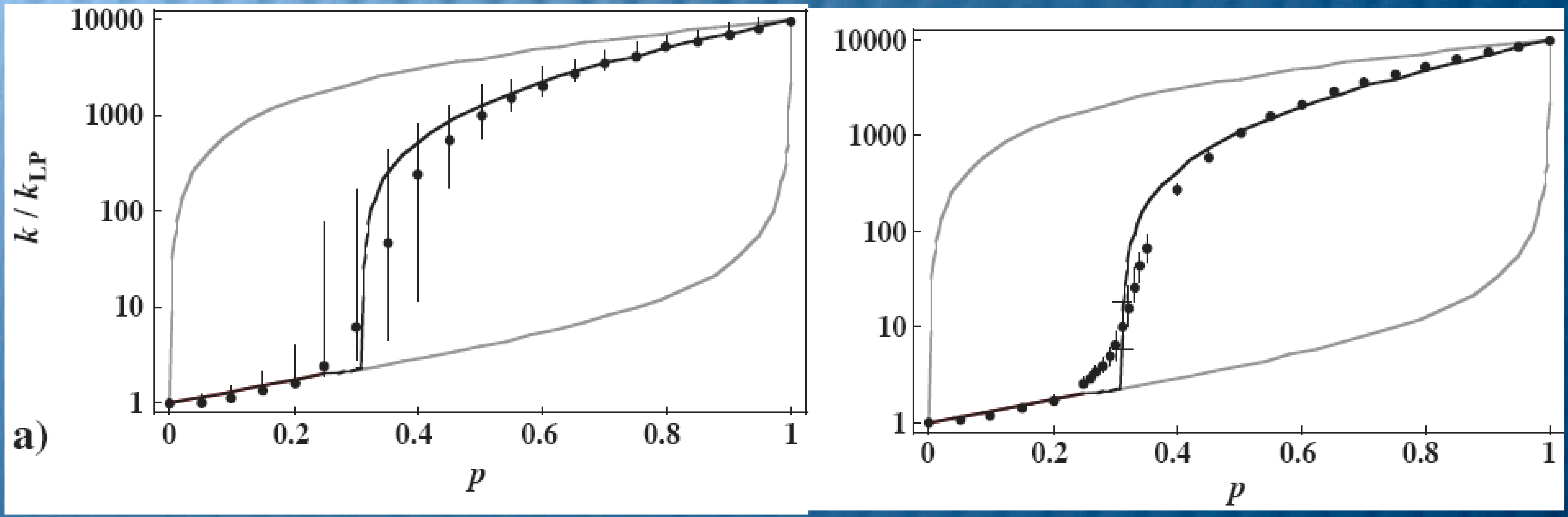


Steady-state flow equation:

$$\frac{\partial}{\partial x}\left(\frac{k}{\mu}\frac{\partial P_p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k}{\mu}\frac{\partial P_p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{k}{\mu}\frac{\partial P_p}{\partial z}\right)=0$$

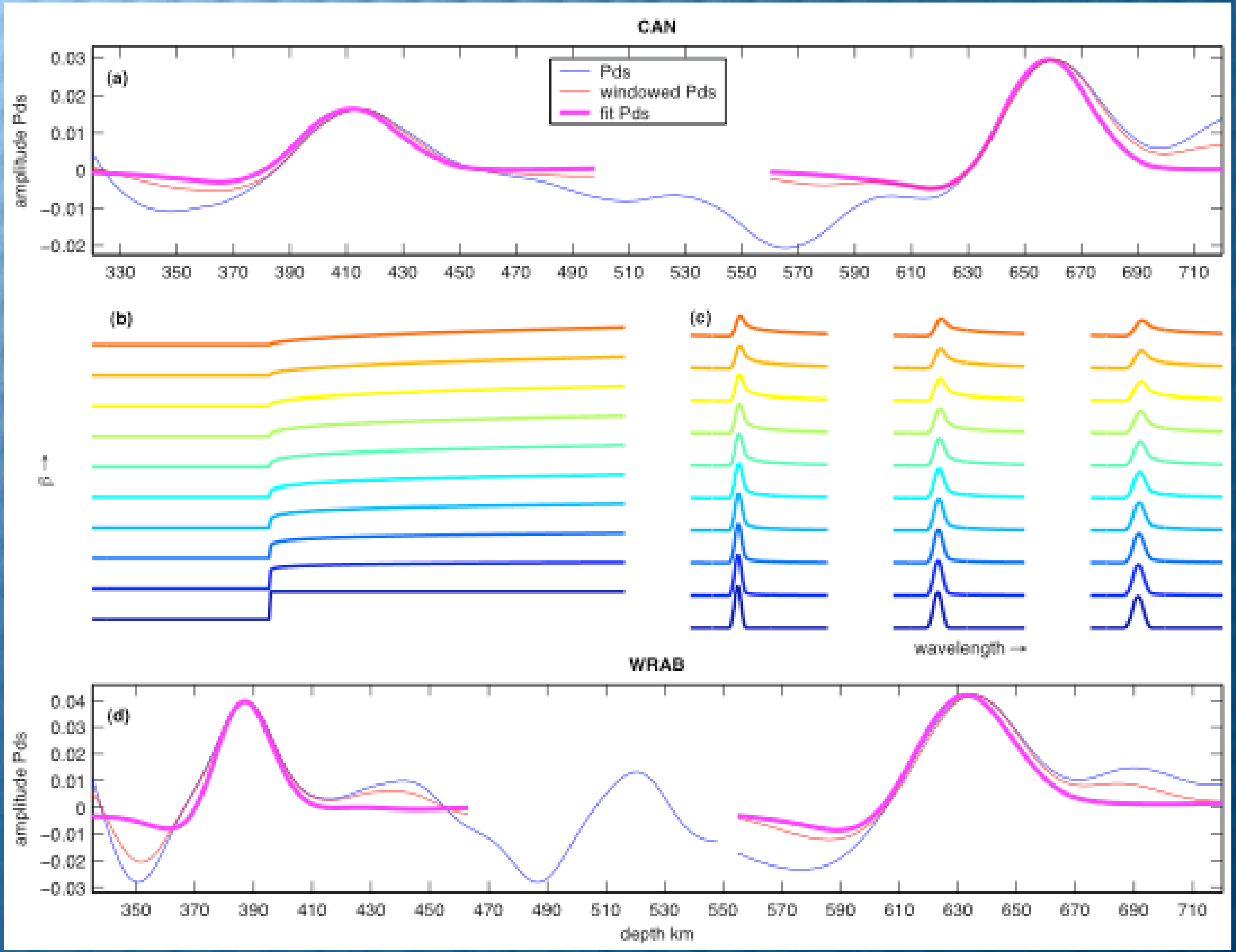
Numerical simulations

10 X 10 X 10 $k_{HP}/k_{LP} = 10^4$ 50 X 50 X 50

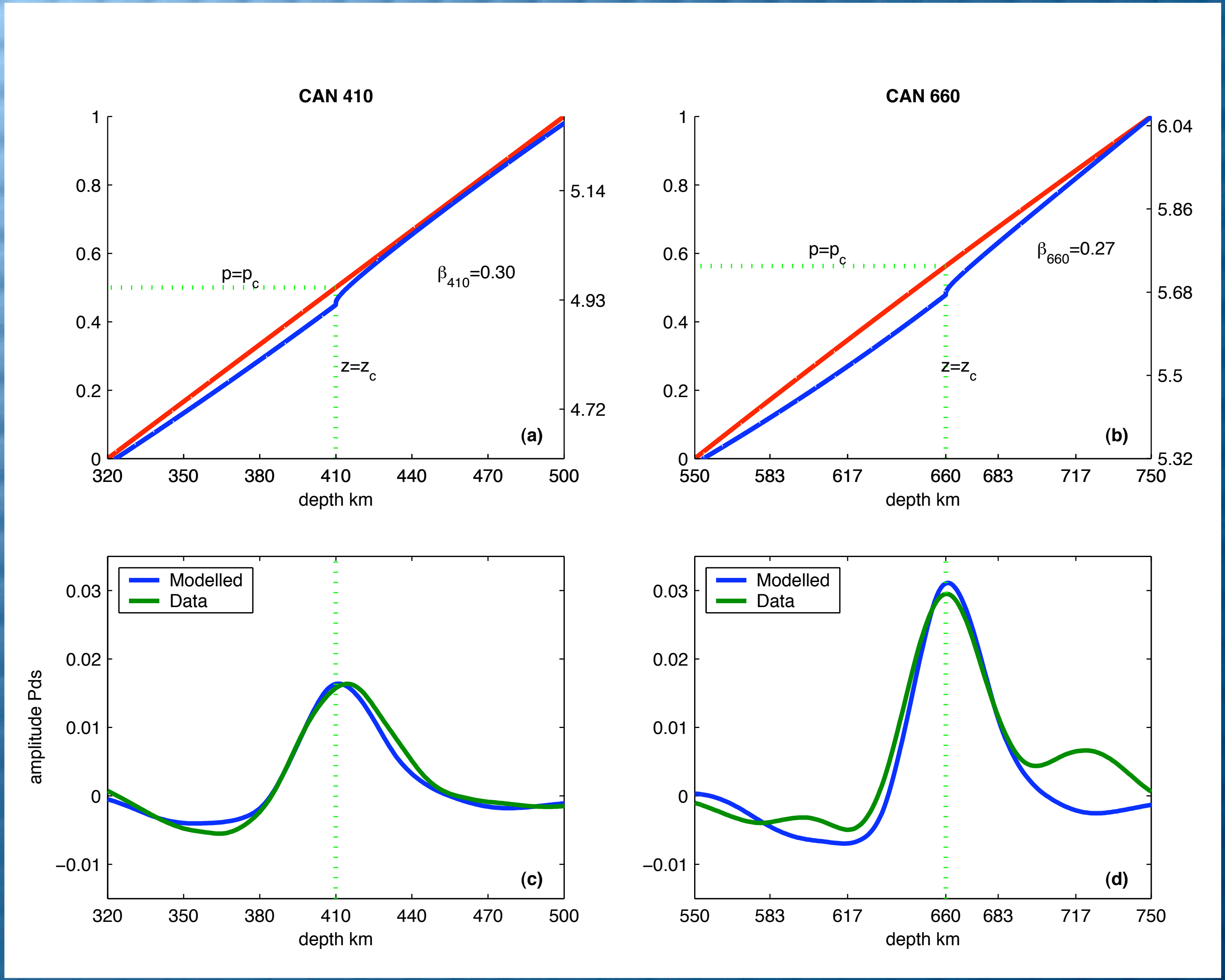


No effective numerical solution. Multiscale diffusion ....?

Modeling


CAN
(a)
Pds
windowed Pds
fit Pds
amplitude Pds
0.03
0.02
0.01
0
−0.01
−0.02
330 350 370 390 410 430 450 470 490 510 530 550 570 590 610 630 650 670 690 710
(b)
β →
(c)
wavelength →
WRAB
(d)
amplitude Pds
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
350 370 390 410 430 450 470 490 510 530 550 570 590 610 630 650 670 690 710
depth km

Modeling



CAN 410
CAN 660
p=p_c
z=z_c
β_410=0.30
β_660=0.27
1
0.8
0.6
0.4
0.2
0
5.14
4.93
4.72
6.04
5.86
5.68
5.5
5.32
(a)
(b)
320 350 380 410 440 470 500
550 583 617 660 683 717 750
depth km
Modelled
Data
amplitude Pds
0.03
0.02
0.01
0
−0.01
(c)
(d)

Observations

Create a *singularity* while composition varies *smoothly*.

Creates a *scale-invariant* non-trivial *singularity*.

Explains *observed seismic waveforms!*

Lots of open math. question w.r.t. *critical mixture models (Percolation).*

Concluding remarks

We do not "fully" understand

- **Curvelets interact with singularities**
- **FIO's with singular coeff.**
- **beyond single scattering**
- **reflection preserving Homogenization**
- **couple medium multiscale properties to the "mono-scale" wavefield**

Concluding remarks

In particular how critical *microscopic* connectivity of binary mixtures is linked to singularities seen by *macroscopic* probing waves ...

- **Percolation phenomena/Geometry/ Critical paths/Fractals**
- **Interaction of these critical "surfaces" with probing wave fronts ...**