Curvelets and Wave Equations

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Agenda

1. Linear wave equations: important basic facts

2. Localization in phase-space

3. Representation of wave propagators

4. What's wrong with wavelets or ridgelets

5. Numerical analysis

The wave equation

Linear hyperbolic equation with non-uniform coefficients. Find $u(t,x)$ such that

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\Delta u \qquad x \in \mathbb{R}^2, \quad t > 0,$$

$$u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x).$$

Examples:

- Acoustics
- Seismics
- Electromagnetism

For now, take $c(x)$ very smooth.

Geometrical optics

Hamiltonian system for sound/light rays:

$$\begin{cases} \dot{x}(t) = H_\xi(x(t), \xi(t)), & x(0) = x_0 \\ \dot{\xi}(t) = -H_x(x(t), \xi(t)) & \xi(0) = \xi_0 \end{cases}$$

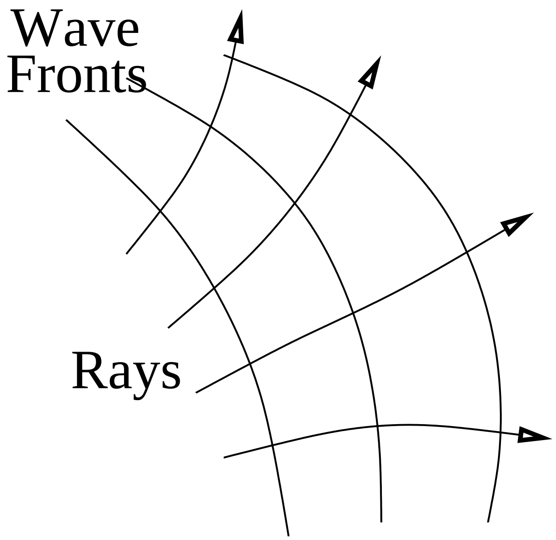
In our case, $H(x, \xi) = c(x)|\xi|$,

$$\begin{cases} \dot{x}(t) = \ c(x(t)) \frac{\xi(t)}{|\xi(t)|}, & x(0) = x_0 \\ \dot{\xi}(t) = -\nabla c(x(t))|\xi(t)| & \xi(0) = \xi_0 \end{cases}$$

Also called bicharacteristics.

Propagation of singularities

Fact: Singularities propagate along rays.



Observation: $u(t,x)$ does not appear in the Hamiltonian system.

$\Rightarrow$ Can predict locations of singularities in advance.

$\Rightarrow$ Dynamics of singularities does not depend on the large scale behavior of u, not even on the type of singularities.

Question: blessing for numerics ?

Multiscale Dream for Computations

small scales (high frequency)	Geometrical optics
Large Scales (low frequency)	Finite differences

Eigenfunctions of the wave equation

Homogeneous medium: $\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$. Assume $u(t,x) = U(x)T(t)$, then

- $T''(t) = -\omega^2 T(t) \qquad \Rightarrow \qquad T(t) = \text{const} \times e^{\pm i\omega t}$,
- $c^2 \Delta U = -\omega^2 U \qquad \Rightarrow \qquad U(x) = \text{const} \times e^{ik\cdot x}$,
  where $k \in \mathbb{R}^2$ and $c|k| = \omega$.

$$\begin{aligned} \text{Complex exponentials} \quad &= \text{(improper) eigenfunctions of the laplacian} \\ &= \text{(uncountable) complete orthogonal family} \end{aligned}$$

$$u(t,x) = \int f_+(k) e^{ik\cdot x} e^{i\omega t} d^2 k + \int f_-(k) e^{ik\cdot x} e^{-i\omega t} d^2 k$$

with

$$f_\pm(k) = \frac{1}{8\pi^2} (\hat{u}_0(k) \pm \frac{\hat{u}_1(k)}{i|k|})$$

Eigenfunctions of the wave equation

Inhomogeneous medium: $\frac{\partial^2 u}{\partial t^2} = c^2(x)\Delta u$. Assume $u(t,x) = U(x)T(t)$, then

- $T''(t) = -\omega^2 T \qquad \Rightarrow \qquad T(t) = \text{const} \times e^{\pm i\omega t}$,
- $\Delta U = -\frac{\omega^2}{c^2(x)} U \qquad \Rightarrow \qquad$ hard !

Geometrical optics: high frequency approximation $U(x) = e^{i\phi(x,k)}$, where $k \in \mathbb{R}^2$, ϕ varies smoothly and $\phi(x, \lambda k) = \lambda \phi(x,k)$

$$\Rightarrow |\phi(x,k)| \leq \text{const} \times |k|.$$

Dispersion relation: $\omega = \overline{c}|k|$.

Eigenfunctions of the wave equation

Geometrical optics: high frequency approximation $U(x) = e^{i\phi(x,k)}$.
Use in $c^2(x)\Delta U = -\omega^2 U$:

$$c^2(x)\Delta_x(e^{i\phi}) = -c^2(x)(|\nabla_x\phi|^2 + \Delta_x\phi)e^{i\phi} = -\omega^2 e^{i\phi}$$

Highest order contribution in $|k|$: Eikonal equation

$$|\nabla_x\phi(x,k)| = \frac{\omega}{c(x)}$$

One such equation for each $\frac{k}{|k|}$. For example, impose the boundary condition $\phi(x,k) = 0 = x \cdot k$ along $x = \alpha k^\perp$.

Eigenfunctions of the wave equation

Functions $\{e^{i\phi(x,k)}\}_k$ $\simeq$ (improper) eigenfunctions of the wave equation

$\simeq$ (uncountable) complete orthogonal family

Example: if

$$u_0(x) = \sum_{\text{some } k} c_k e^{i\phi(x,k)} \qquad \text{and} \qquad u_1(x) = \sum_{\text{some } k} d_k e^{i\phi(x,k)},$$

then

$$u(t,x) \simeq \sum_{\text{some } k} f_+(k) e^{i\phi(x,k)} e^{i\omega t} + \sum_{\text{some } k} f_-(k) e^{i\phi(x,k)} e^{-i\omega t},$$

with, again,

$$f_\pm(k) = \frac{1}{8\pi^2}(c_k \pm \frac{d_k}{i|k|})$$

Numerics: fast marching / fast sweeping

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From multi-frequency to multi-scale

Idea: localize each $e^{i\phi(x,k)}$ into wave packets

$$e^{i\phi(x,k)}e^{-i\omega t} \to a(t, x - x_0, k)e^{i\phi(x,k)}e^{-i\omega t}$$

where for each x_0, $a(t, x - x_0, k)$ is a slowly varying amplitude function. For now, assume a is uniformly bounded in k.

$$\frac{\partial^2}{\partial t^2}(ae^{i(\phi-\omega t)}) = [\frac{\partial^2 a}{\partial t^2} + 2i\frac{\partial a}{\partial t}\omega - \omega^2 a]e^{i(\phi-\omega t)}$$

$$\Delta_x(ae^{i(\phi-\omega t)}) = [\Delta_x a + 2i\nabla_x a \cdot \nabla_x \phi + i\delta_x \phi a - |\nabla_x \phi|^2 a]e^{i(\phi-\omega t)}$$

Equate like-powers of $|k|$,

- $O(|k|^2)$ terms: $|\nabla\phi(x, k)| = \frac{\omega}{c(x)}$
- $O(|k|)$ terms: $\frac{\partial a}{\partial t} + \frac{\nabla\phi}{|\nabla\phi|} \cdot \nabla a = f(a, \phi)$

$\nabla\phi$ orthogonal to WF $\Rightarrow$ amplitude convected along rays.

Spatial localization

Goal: simplify $\phi(x,k)$ by localizing it about x_0, by means of a.

$$\phi(x,k) = \phi(x_0,k) + (x-x_0)\cdot\nabla_x\phi(x_0,k) + R_2,$$

$$|R_2| \le \frac{1}{2}\sup_{x\in\text{supp }a}\left(|x-x_0|^2|D_x^2\phi(x,k)|\right)$$

Approximate linearization if R_2 is small on the (essential) support of a.

Requirement: $R_2 = O(1)$, uniformly in $|k|$.

Since $\phi(x,k) = O(|k|)$, we need $|x-x_0| \le \text{const} \times \sqrt{\frac{1}{|k|}}$, or

$$\text{ess supp } a(0,x-x_0,k) \subset B_{x_0}(\text{const } \times \frac{1}{\sqrt{|k|}})$$

Problem: sharp localization of $a(0,x,k)$ for big $|k| \Rightarrow$ dispersion.

Frequency localization

Goal: make sure the amplitude still propagates along rays by localizing it in frequency. Put $\xi_0 = \nabla_x \phi(x_0, k)$.

$$\mathcal{F}_x(a(0, x, k)e^{i(x\cdot\xi_0)}) = \hat{a}(0, \xi - \xi_0, k).$$

Example: $c(x) = 1$.

$$u(t, x) = \int \hat{a}(0, \xi - \xi_0, k)e^{ix\cdot\xi}e^{-i\omega t}d^2\xi + \text{similar},$$

with $\omega = c|\xi|$. Dispersion is due to the nonlinearity of $|\xi|$.

$$|\xi| = \xi_1 + S_2$$

Frequency localization

$$S_2 = |\xi| - \xi_1 = \xi_1(\sqrt{1 + \frac{\xi_2^2}{\xi_1^2}}) - \xi_1$$

$$\leq \text{const} \times \xi_1(1 + \frac{1}{2}\frac{\xi_2^2}{\xi_1^2} - 1)$$

$$\leq \text{const} \times \frac{\xi_2^2}{\xi_1}$$

Approx linearization if S_2 is small on the (essential) support of $\hat{a}$.

Requirement: $S_2 = O(1)$, uniformly in $|k|$.

Since $\phi(x, k) = O(|k|)$, we need $\xi_2^2 \leq$ const $\times |k|$, or

$$\text{ess supp } \hat{a}(0, \xi - \xi_0, k) \subset C_0(\xi_0, \text{ const } \times \frac{1}{\sqrt{|k|}})$$

Compatibility with the uncertainty principle

- Spatial constraint: particle interpretation

$$\text{ess supp } a(0, x - x_0, k) \subset B_{x_0}(\text{const } \times \frac{1}{\sqrt{|k|}})$$

- Frequency constraint: wave interpretation

$$\text{ess supp } \hat{a}(0, \xi - \xi_0, k) \subset C_0(\xi_0, \text{ const } \times \frac{1}{\sqrt{|k|}})$$

Then, for $u(0, x, k) = a(0, x, k)e^{ix\cdot\xi_0}$,

$$u(t, x, k) \simeq u(0, U(t)(x - x(t)) + x_0, k)$$

Uncertainty:(size in x) $\times$ (size in ξ) $\geq$ const, along each direction.

References

- Wave packets and Fourier integral operators: Cordoba and Fefferman (78)
- Wave atoms: FBI transform in microlocal analysis: Bros and Iagolnitzer (75)
- Dyadic-parabolic partitioning in Fourier: Second dyadic decomposition in harmonic analysis: Fefferman (73), Seeger, Sogge and Stein (93)
- Curvelets: Smith (97, 98), Candès and Donoho (02, 04)

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Principles of harmonic analysis

The set of wave packets $a(0, x - x_0, k)e^{ix\cdot\xi_0}$ is largely over-complete.

Curvelets: possible to extract a subfamily $\varphi_\mu(x)$, $\mu = (j, n_1, n_2, \ell)$, so that

- Tight frame: $\|u\|^2 = \sum_\mu |\langle u, \varphi_\mu\rangle|^2$
- Expansions: $u = \sum_\mu \langle u, \varphi_\mu\rangle \varphi_\mu$
- Almost orthogonality: for all $M > 0$,
  $|\langle \varphi_\mu, \varphi_{\mu'}\rangle| \leq C_M \cdot \omega(\mu, \mu')^{-M}$,

where ω is the dyadic-parabolic distance in phase-space:

$$\omega(\mu, \mu') = 2^{|j-j'|} \left(1 + 2^{\min\{j,j'\}} d(\mu, \mu')\right),$$

$$d(\mu, \mu') = |\theta_\mu - \theta_{\mu'}|^2 + |x_\mu - x_{\mu'}|^2 + |\langle \frac{\xi_\mu}{|\xi_\mu|}, x_\mu - x_{\mu'}\rangle|$$

The Ingredients of Curvelets

Curvelets are indexed by 4 integers $\mu = (j, n_1, n_2, \ell)$:

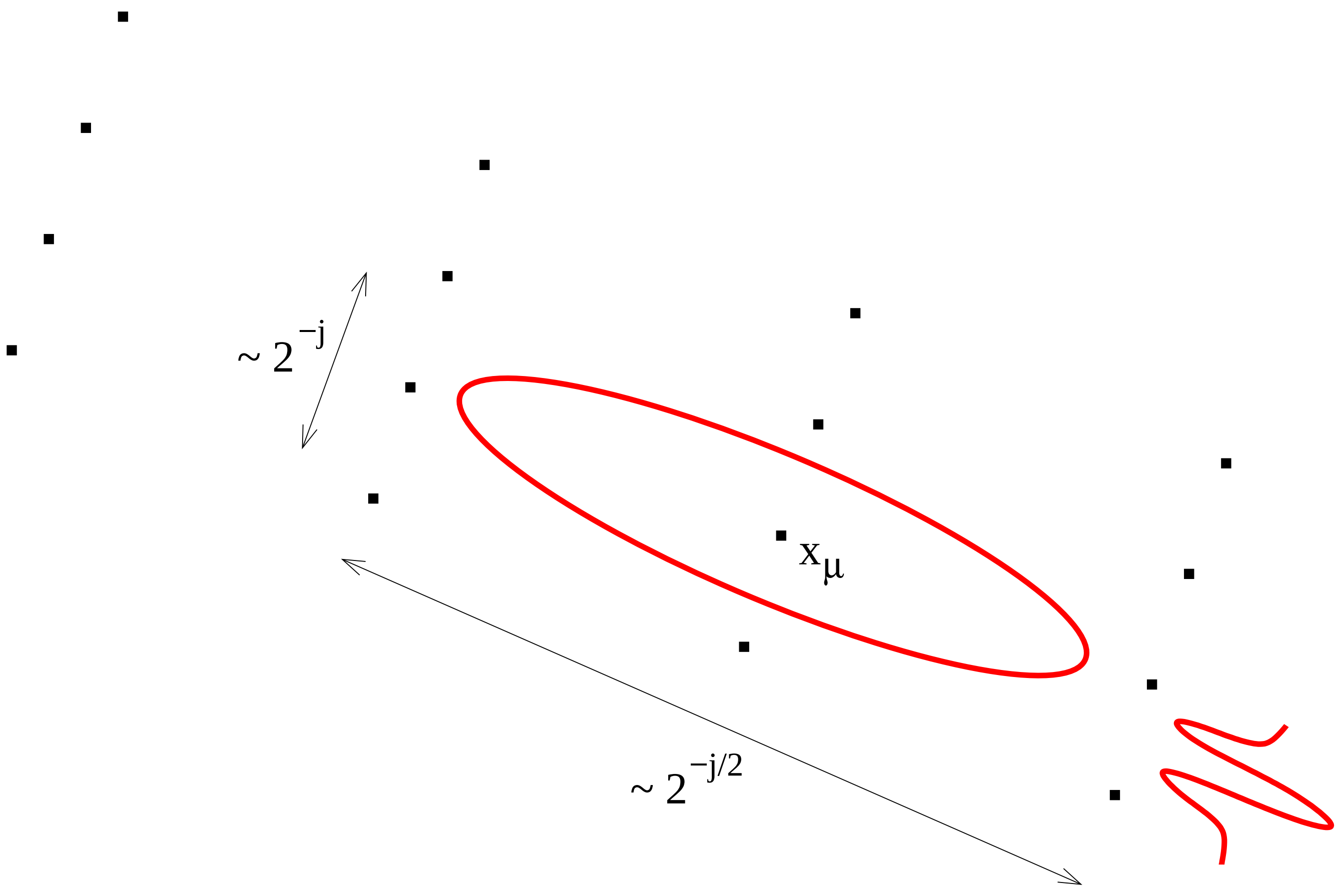
$$\varphi_\mu(x) = 2^{3j/4}\varphi(D_j R_{\theta_{j\ell}} x - n)$$

with

$$D_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix} \quad \text{Parabolic scaling}$$

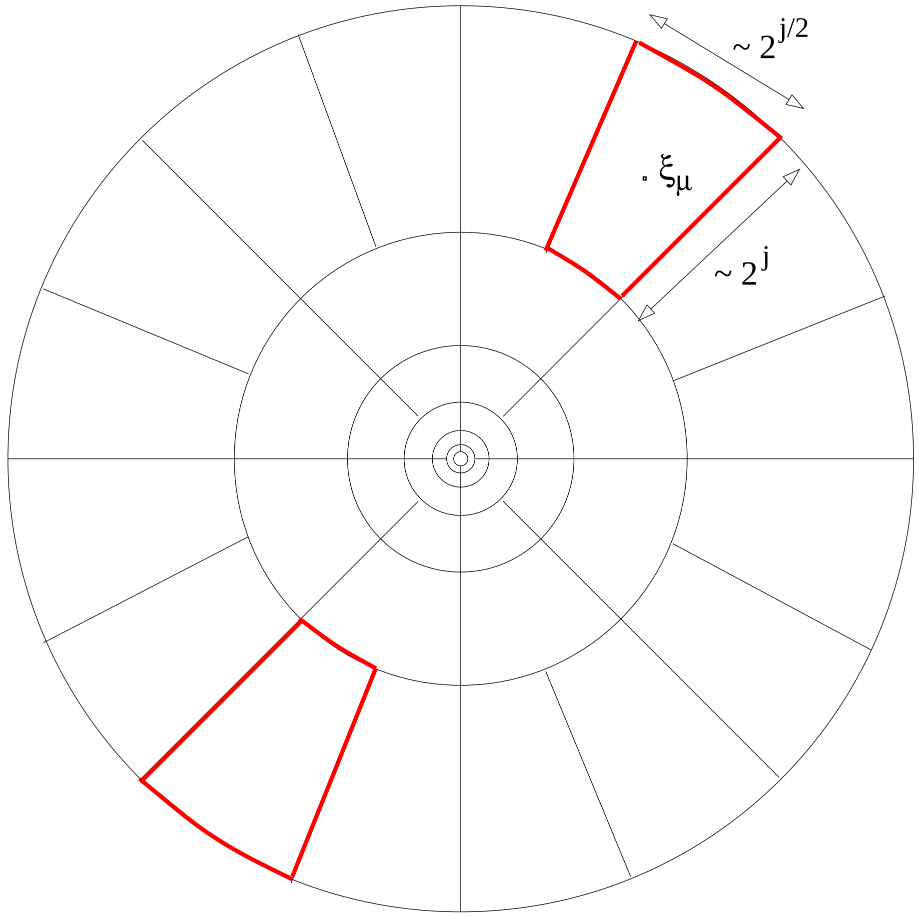
$$\theta_{j\ell} \sim \ell 2^{-j/2}$$

Tiling the x space



The red ellipse sketches the essential support of a curvelet ϕ_μ.

Tiling the ξ space



In red, the essential frequency support of a curvelet ϕ_μ.

Curvelet representation of the wave group

Hyperbolic system of m equations: find $u \in \mathbb{R}^m$,

$$\frac{\partial u}{\partial t} = \sum_k A_k(x) \frac{\partial u}{\partial x_k} + B(x)u, \qquad u|_{t=0} = u_0.$$

Introduce vector-valued curvelets $\boldsymbol{\varphi}_{\mu\nu} = \varphi_\mu \mathbf{e}_\nu$ where $\mathbf{e}_\nu = (0 \ldots 1 \ldots 0)$.

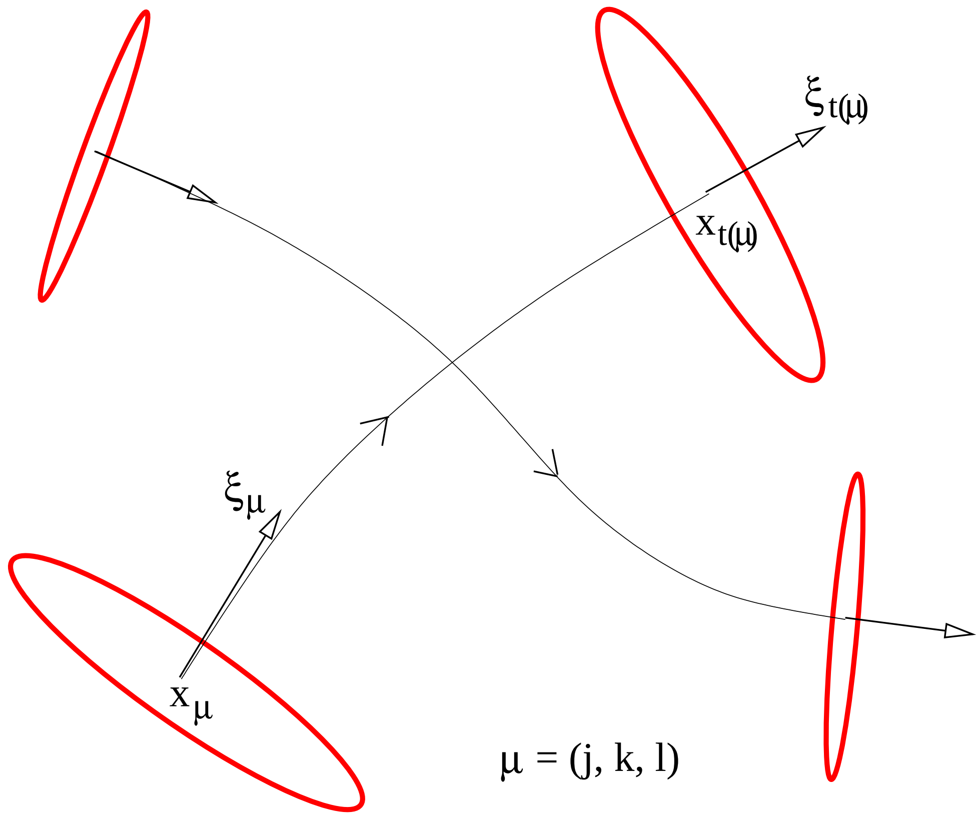
$$\begin{array}{ccc} u_0 & \xrightarrow{E(t)} & u(x,t) = \sum_{\mu\nu} c_{\mu\nu}(t) \boldsymbol{\varphi}_{\mu\nu}(x) \\ \downarrow & & \uparrow \\ c_{\mu\nu}(0) & \xrightarrow{E(\mu,\nu;\mu',\nu')(t)} & c_{\mu\nu}(t) \end{array}$$

with

$$c_{\mu\nu}(t) = \langle u(t), \boldsymbol{\varphi}_{\mu\nu} \rangle, \qquad E(\mu,\nu;\mu',\nu')(t) = \langle E(t) \boldsymbol{\varphi}_{\mu'\nu'}, \boldsymbol{\varphi}_{\mu\nu} \rangle.$$

Why curvelets ?

$E(t)$ acting on a curvelet $\varphi_\mu \mathbf{e}_\nu$ essentially moves it rigidly along each of the m Hamiltonian flows.



$$\begin{cases} \dot{x} = (H_\nu)_\xi(x,\xi) \\ \dot{\xi} = -(H_\nu)_x(x,\xi) \end{cases}$$

Implications

We observed:

1. Coherence: $E(t)\varphi_\mu \mathbf{e}_\nu \simeq \sum_{k=1}^m c_k \varphi_{t_k(\mu)} \mathbf{e}_k$

2. Almost orthogonality: $\langle \varphi_\mu \mathbf{e}_\nu, \varphi_{\mu'} \mathbf{e}_{\nu'} \rangle \simeq \delta_{\mu\mu'} \delta_{\nu\nu'}$

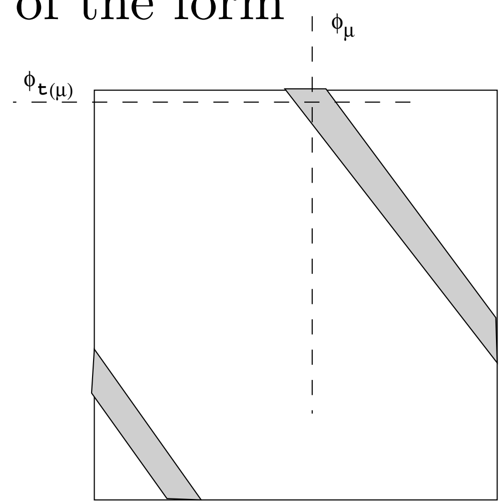
Consequence: the matrix $E(\mu, \nu; \mu', \nu')(t) = \langle E(t)\varphi_{\mu'} \mathbf{e}_{\nu'}, \varphi_\mu \mathbf{e}_\nu \rangle$ has very small entries except for m shifted diagonals.

$\Rightarrow$ Sparse and Well-organized.

Main result (Candès, _)

$$\frac{\partial u}{\partial t} = \sum_k A_k(x) \frac{\partial u}{\partial x_k} + B(x)u, \qquad u|_{t=0} = u_0.$$

Take $A_k(x)$ and $B(x)$ smooth. Assume the eigenvalues of the matrix $\sum_k A_k(x)\xi_k$ have fixed multiplicity uniformly in x and ξ. Then for all $t > 0$ the matrix $E(\mu, \nu; \mu', \nu')(t) = \langle E(t)\phi_{\mu'}\mathbf{e}_{\nu'}, \phi_\mu \mathbf{e}_\nu \rangle$ is a sum of m matrices of the form



Main result (Candès, _)

Away from the shifted diagonals, the elements decay faster than any negative power of the index. Let (a_n) be any row or column, re-order it in decreasing order, $|a|_(n)$. Then for all $N > 0$, $|a|_(n) \leq C_N \cdot n^{-N}$.

Let $t_k(\mu)$, $k = 1, \ldots, m$ be the m hamiltonian correspondences of subscripts. Then, under the same assumptions, for every $N > 0$, there exists a constant $C_N > 0$ so that

$$|\langle E(t)\phi_{\mu'}\mathbf{e}_{\nu'}, \phi_\mu \mathbf{e}_\nu\rangle| \leq C_N \sum_{k=1}^{m} \omega(\mu, t_k(\mu'))^{-N}.$$

Implications

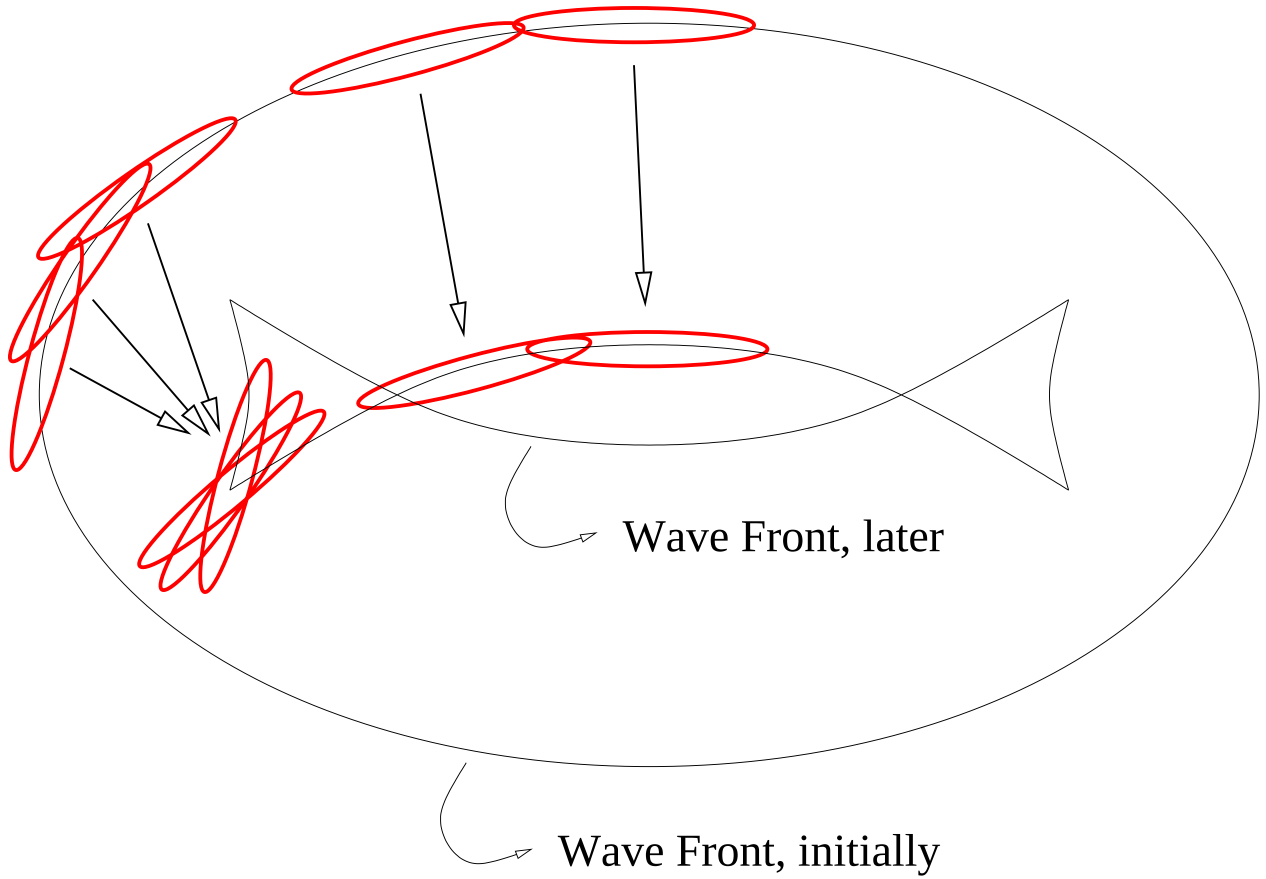
Corollary: The curvelet matrix of $E(t)$ maps boundedly ℓ_p onto itself, for all $p > 0$.

Corollary: Truncate to keep scales $\geq 1/N \Rightarrow E_N(t)$. Truncate $E_N(t)$ to m shifted bands of width $B \Rightarrow T_{B,N}$. Then for all $M > 0$,

$$\|T_N - T_{B,N}\|_2 \leq C_M \cdot B^{-M}.$$

Matrix compression !

Shrinking ellipse



Note : there is more than geometric optics in this picture !

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Other dilations ?

Take a general dilation

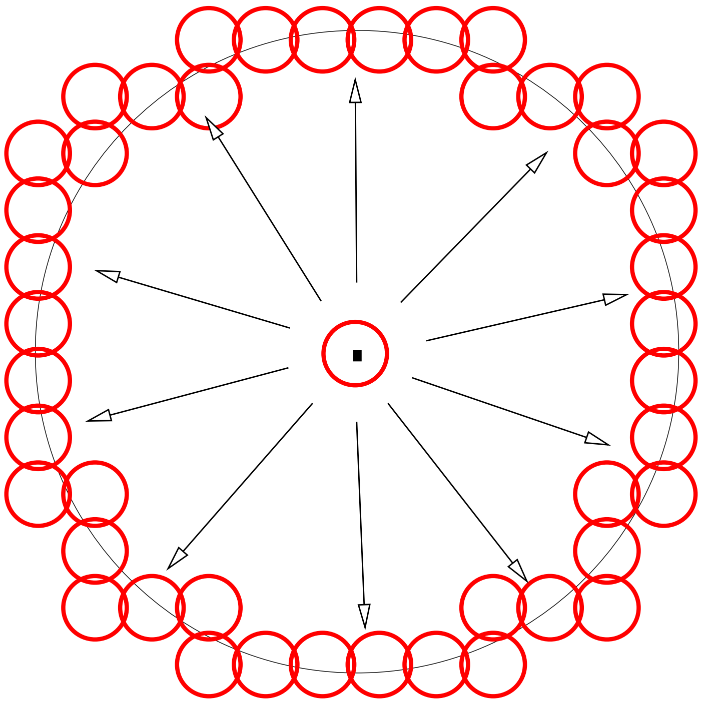
$$D_j^\alpha = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{\alpha j} \end{pmatrix}$$

Examples :

- $\alpha = 1$: Wavelets
- $\alpha = 1/2$: Curvelets
- $\alpha = 0$: Ridgelets

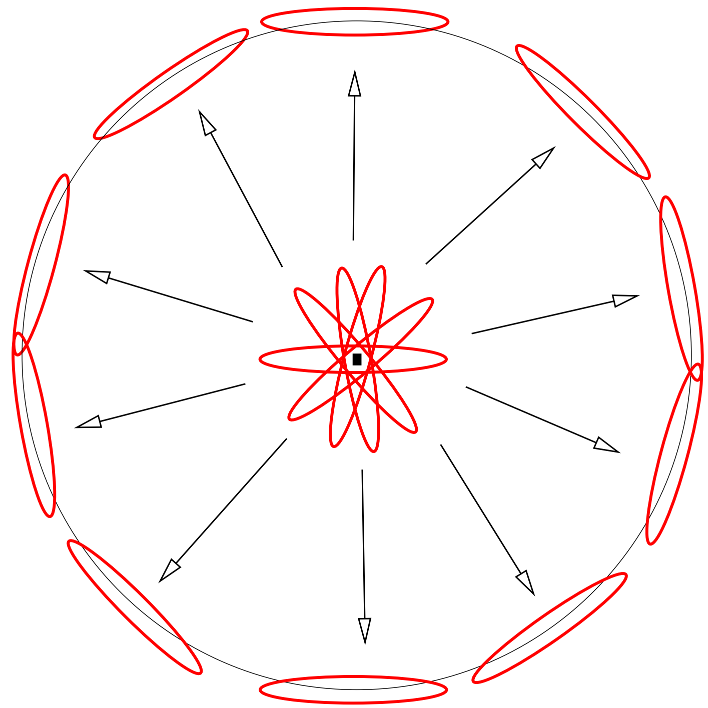
Wavelets don't work !

Wavelets



Turn ℓ^p, $p < 1$, into ℓ^1.

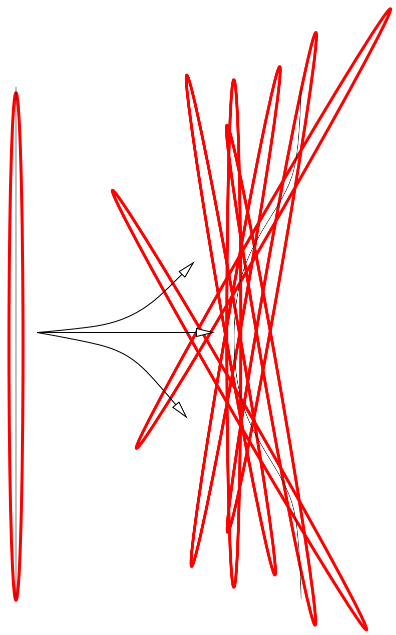
Curvelets



Turn ℓ^p into ℓ^p.

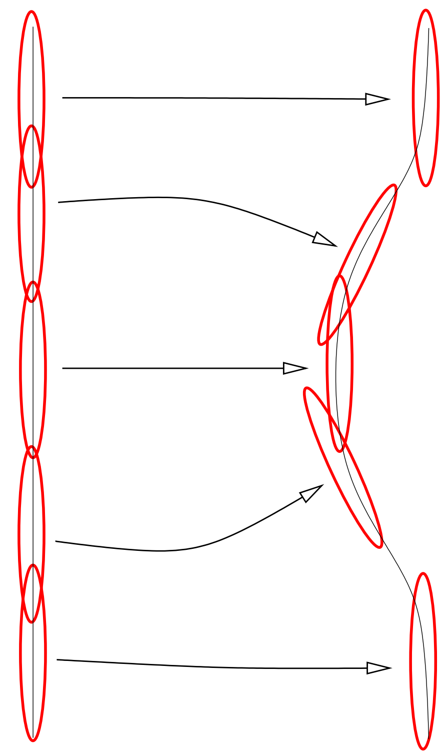
Ridgelets don't work !

Ridgelets



Turn ℓ^p, $p < 1$, into ℓ^1.

Curvelets



Turn ℓ^p into ℓ^p.

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Algorithm

Say $\partial_t u = Au$.

Idea: build the whole propagator $E^{curv}_{B,N}(t)$, in the curvelet domain.

1. Initialization: Obtain $\tilde{A}^{curv}_{B,N}$, then
$$\tilde{E}^{curv}_{B,N}(\Delta t) = I + \Delta t\ \tilde{A}^{curv}_{B,N}$$

2. Recursion: Forecast where the biggest elements are going to be, then
$$\tilde{E}^{curv}_{B,N}(2^n \Delta t) = \mathrm{Trunc}_{B,N}[\tilde{E}^{curv}_{B,N}(2^{n-1}\Delta t)]^2$$

3. Terminate: At time $T = 2^K \Delta t$,
$$\alpha = \tilde{E}^{curv}_{B,N}(T)\alpha_0, \qquad \text{where } \alpha_0 = \{\langle u_0, \boldsymbol{\varphi}_{\mu,\nu}\rangle\},$$
and inverse transform, $\tilde{u}(t) = C^{-1}\alpha$.

Performance (Candès, _)

- Assume u_0 is bandlimited and comes from N samples on a square grid.
- Assume the domain of dependence of u_0 is essentially included in the computational domain.

Fix $\epsilon > 0$. Then an approximate solution satisfying

$$\|u(t) - \tilde{u}(t)\|_2 \leq \epsilon$$

can be obtained by the above algorithm in $C_{\epsilon,\delta} N^{1+\delta}$ operations, for all $\delta > 0$ and $C_{\epsilon,\delta} \to \infty$ when $\epsilon, \delta \to 0$.

Multiscale dream comes true

- Curvelets provide a correct decomposition of phase-space for hyperbolic equations.

- Rotations and the parabolic scaling are essential.

- Curvelets are a natural generalization of wavelets to 2-D and n-D.

Future directions

- Numerics, including oscillatory integrals
- Reflection, refraction and diffraction of wave packets
- Dispersive equations

http://www.acm.caltech.edu/~demanet