The Important Role of Mathematics in the Digital Twin Revolution

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Green Family Public Lecture

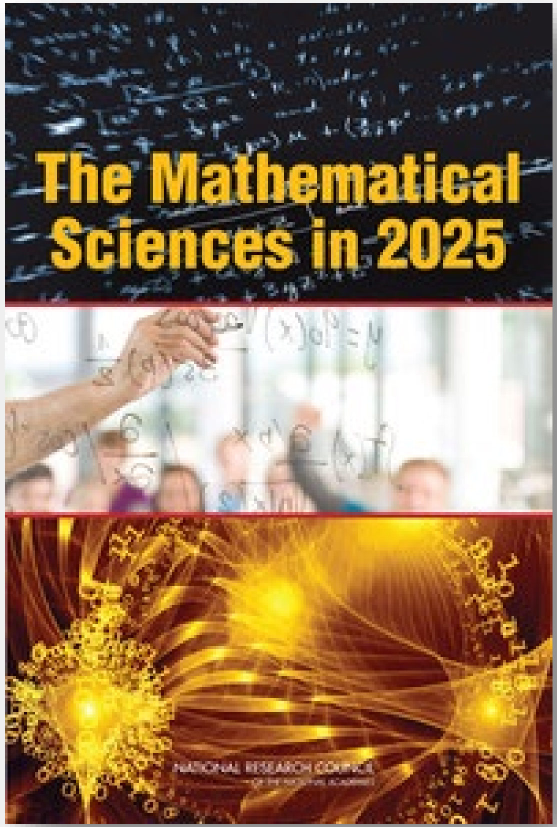
Institute for Pure and Applied Mathematics

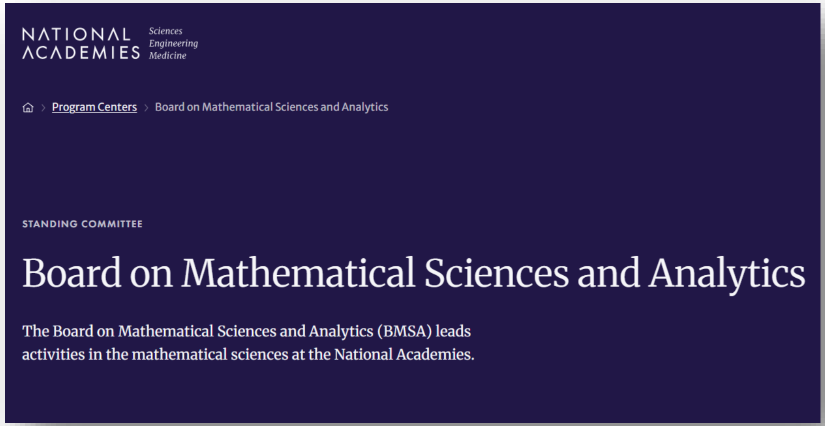
April 13, 2026

ODEN INSTITUTE
FOR COMPUTATIONAL ENGINEERING & SCIENCES

“ *Mark is an unrelenting and perpetually inspiring leader. There is truly nobody like him. His dedication to advancing and ensuring the long-term vitality of the mathematical sciences has helped elevate the community in innumerable ways.*

Dr. Michelle Schwalbe, National Academies





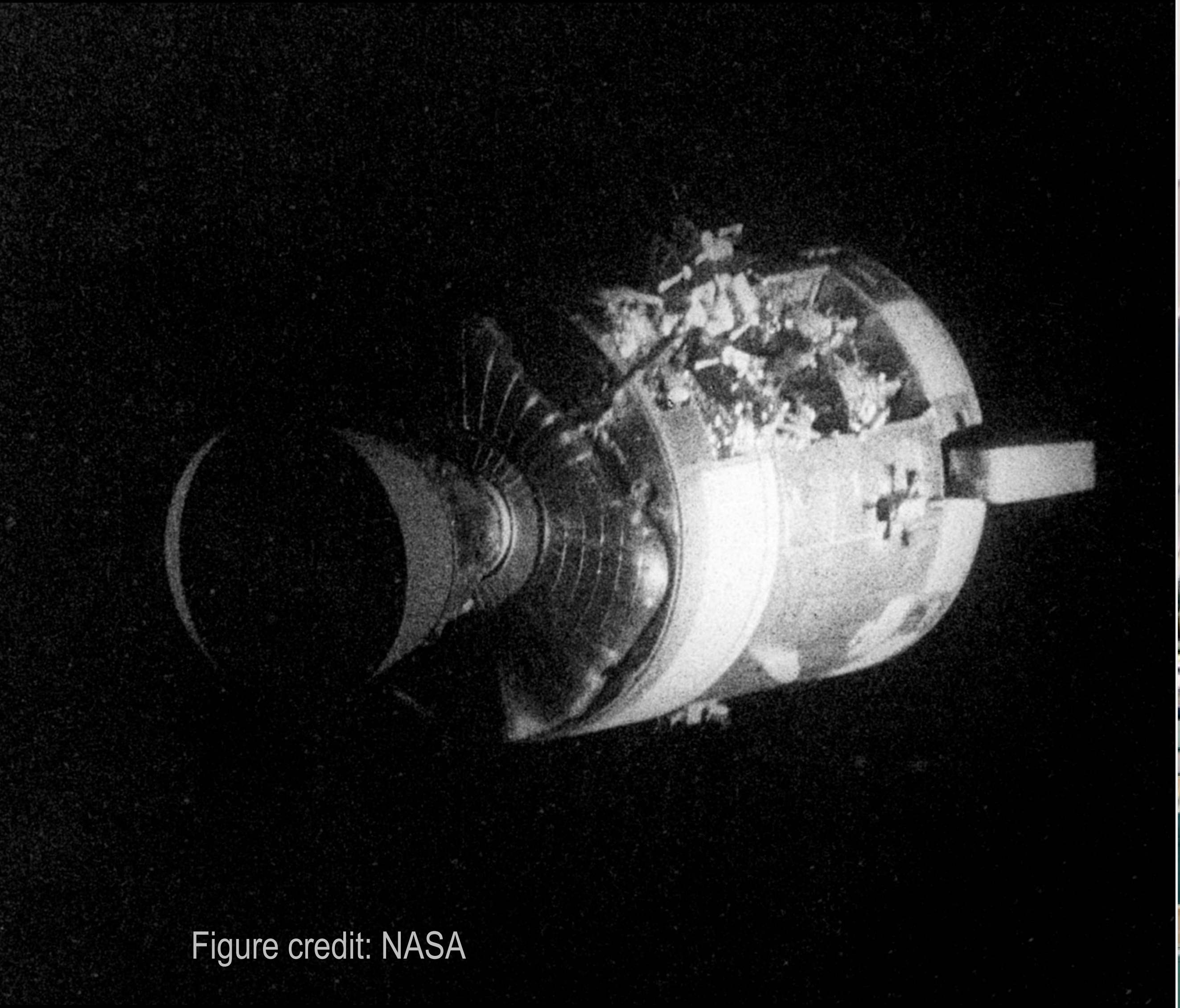
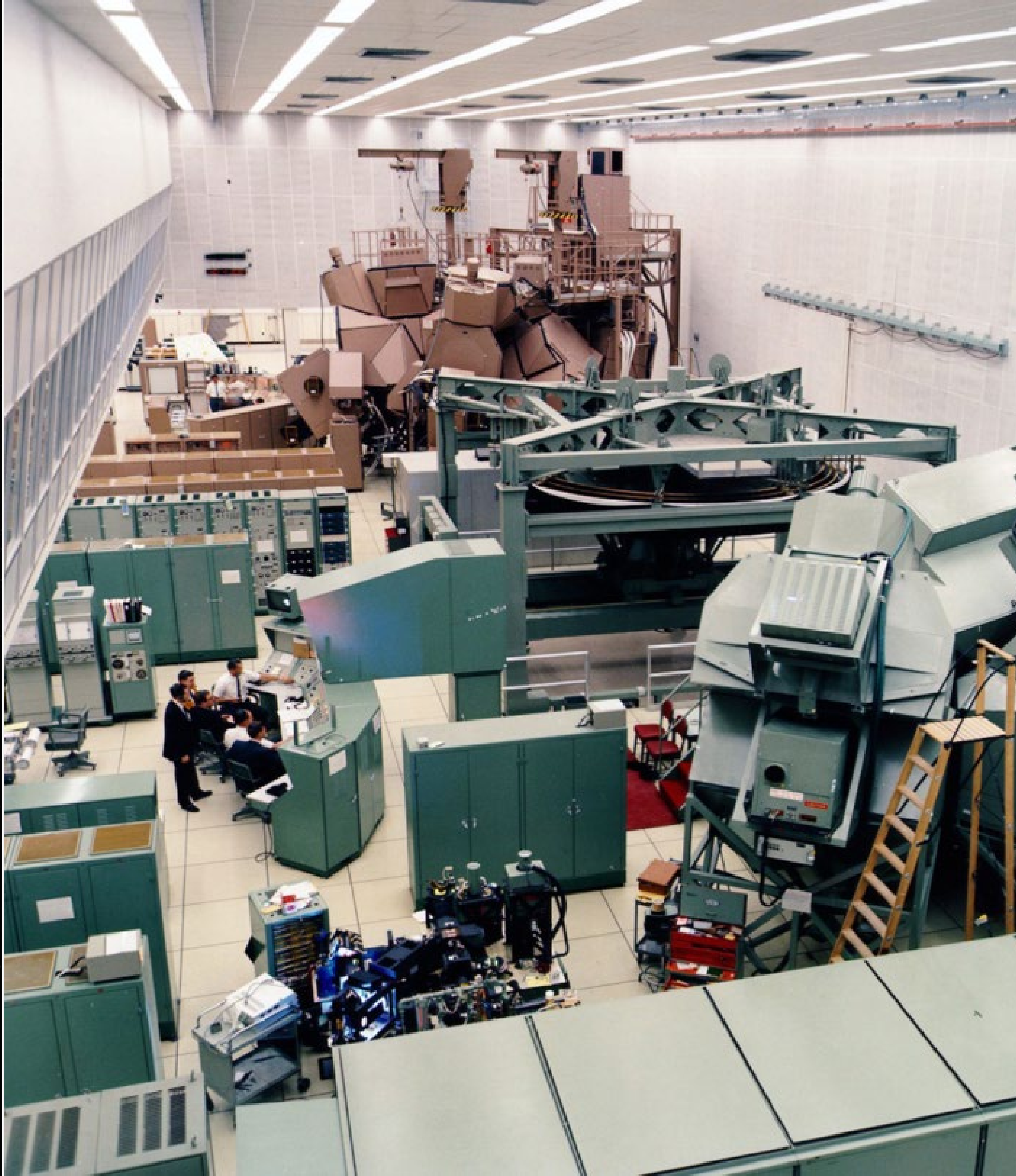


Figure credit: NASA



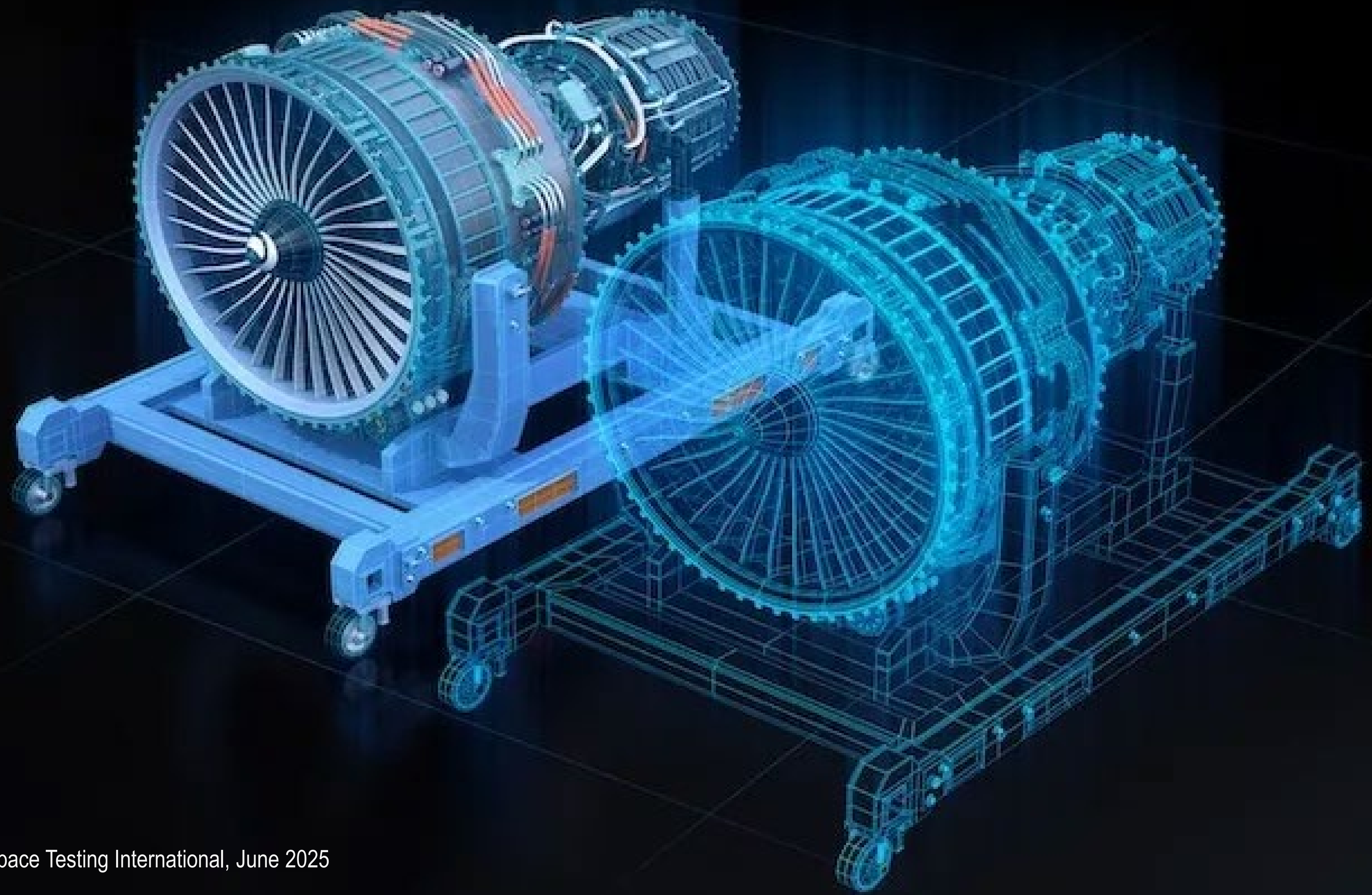
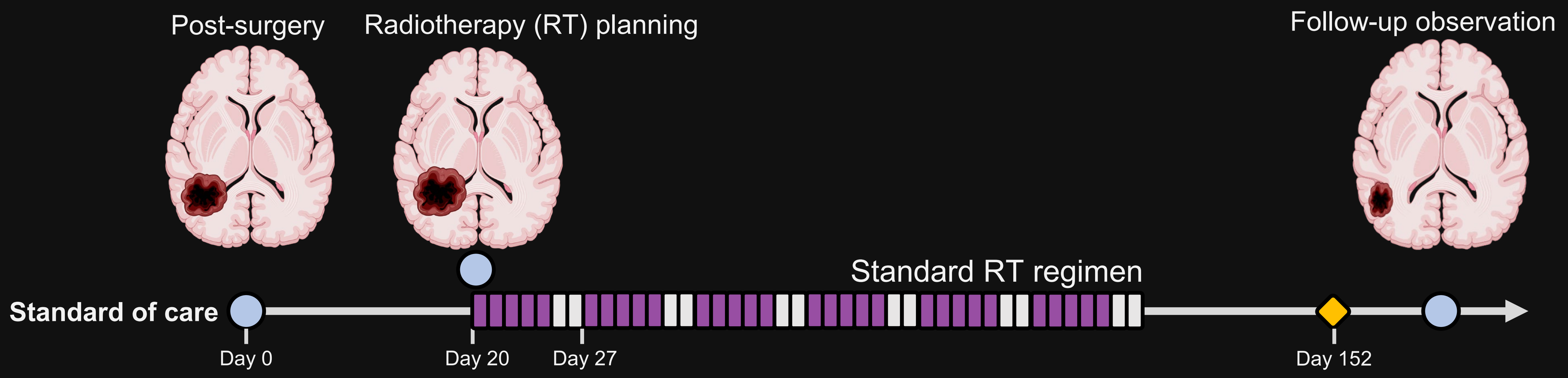


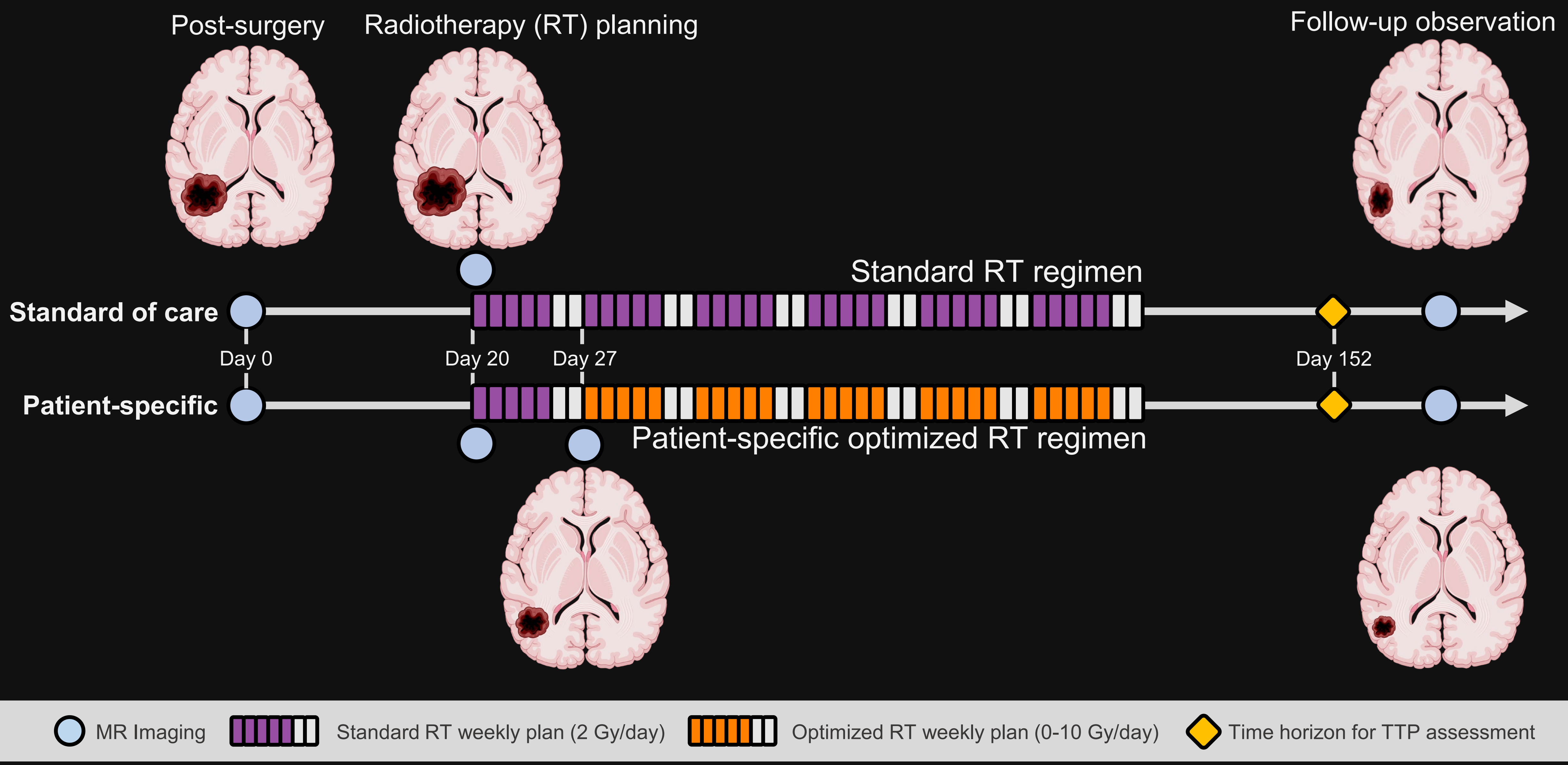
Figure credit: Aerospace Testing International, June 2025

Radiotherapy Regimens for High-Grade Gliomas



MR Imaging Standard RT weekly plan (2 Gy/day) Time horizon for TTP assessment

Radiotherapy Regimens for High-Grade Gliomas



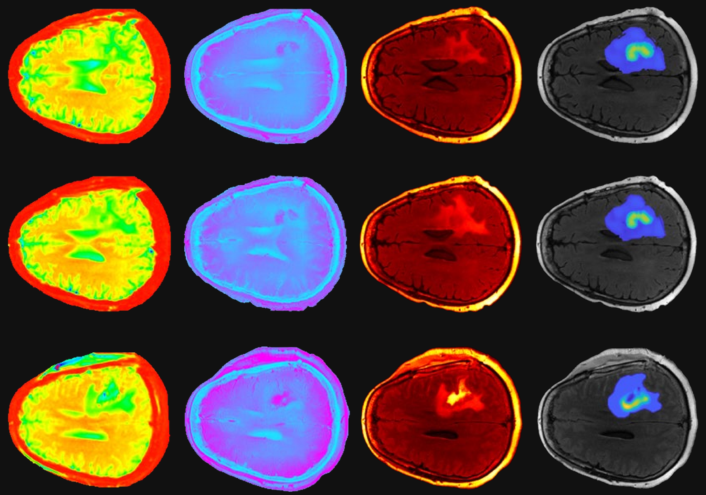
MODELS + DATA

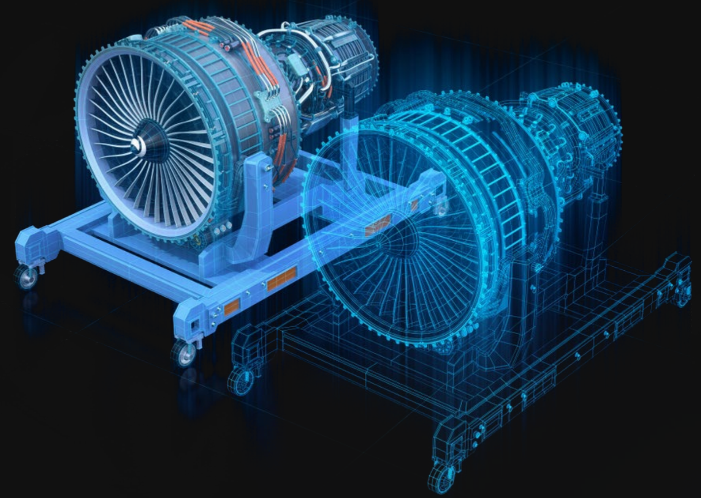
DATA ASSIMILATION

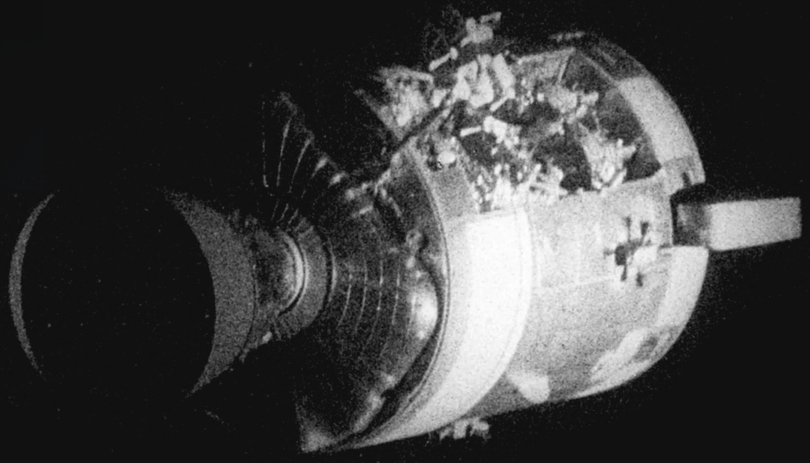
PREDICTION

DECISION & ACTION

Figure credit: David Hormuth & Thomas Yankeelov
Oden Institute, Center for Computational Oncology

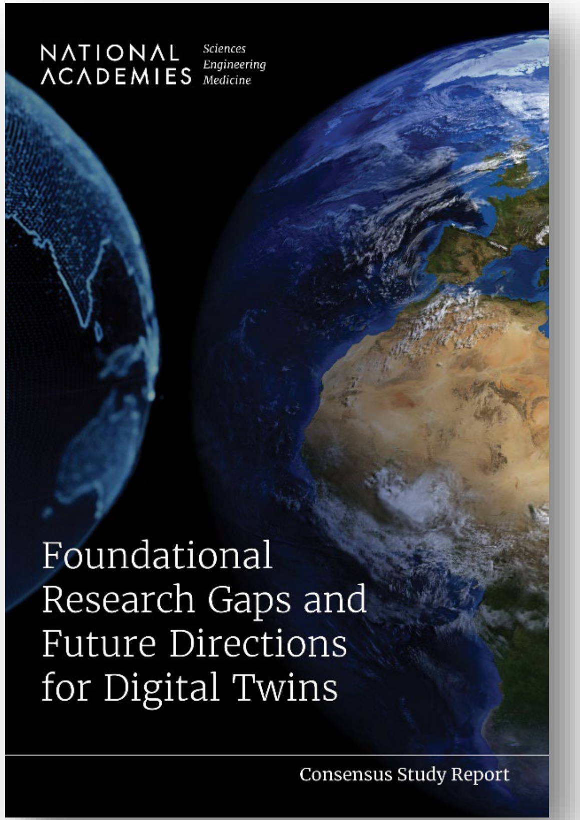






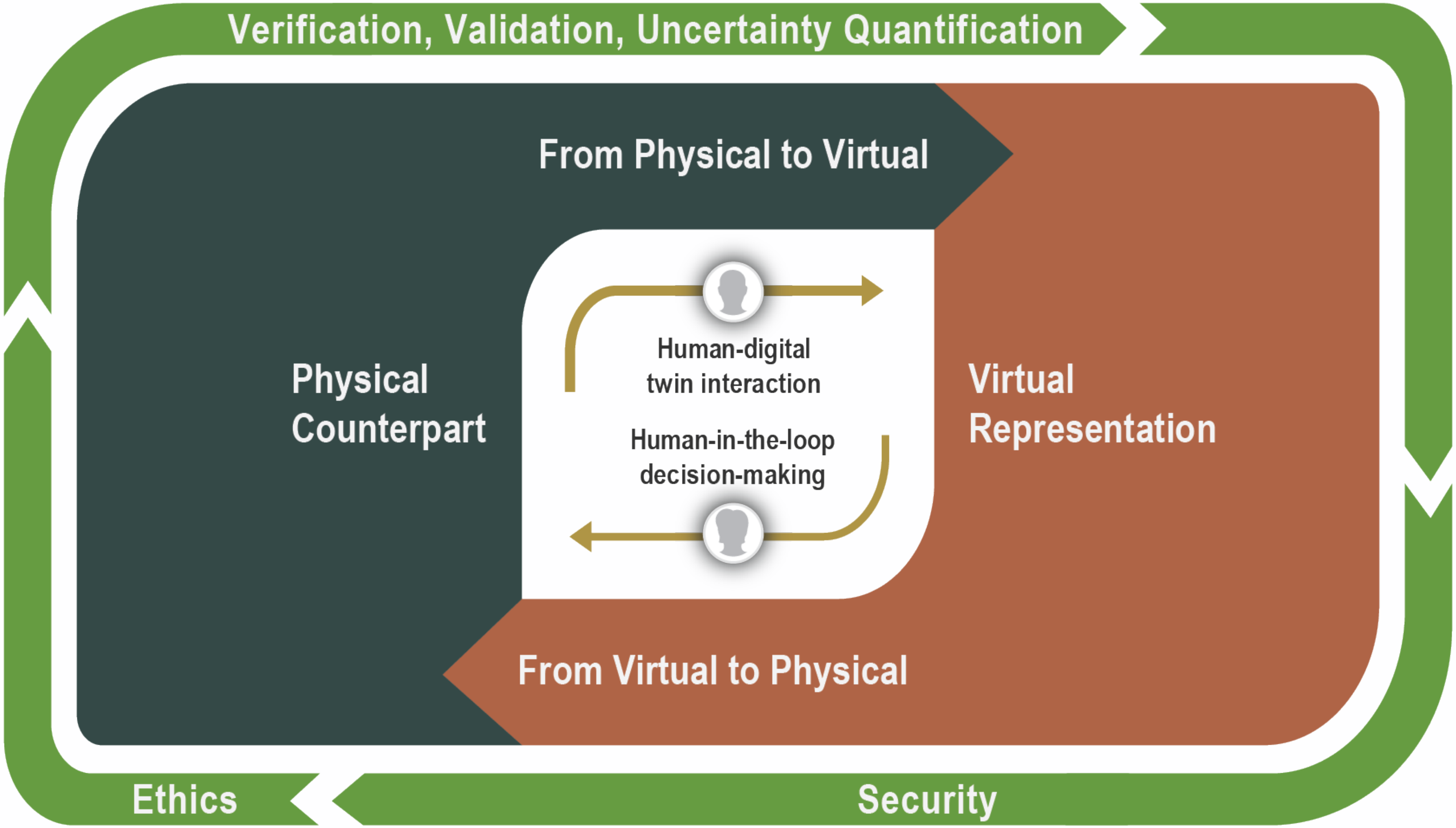
Figures credit: NASA, Aerospace Testing International

> *A digital twin is a set of virtual information constructs that mimics the structure, context, and behavior of a natural, engineered, or social system (or system-of-systems), is dynamically updated with data from its physical twin, has a predictive capability, and informs decisions that realize value. The bidirectional interaction between the virtual and the physical is central to the digital twin.*



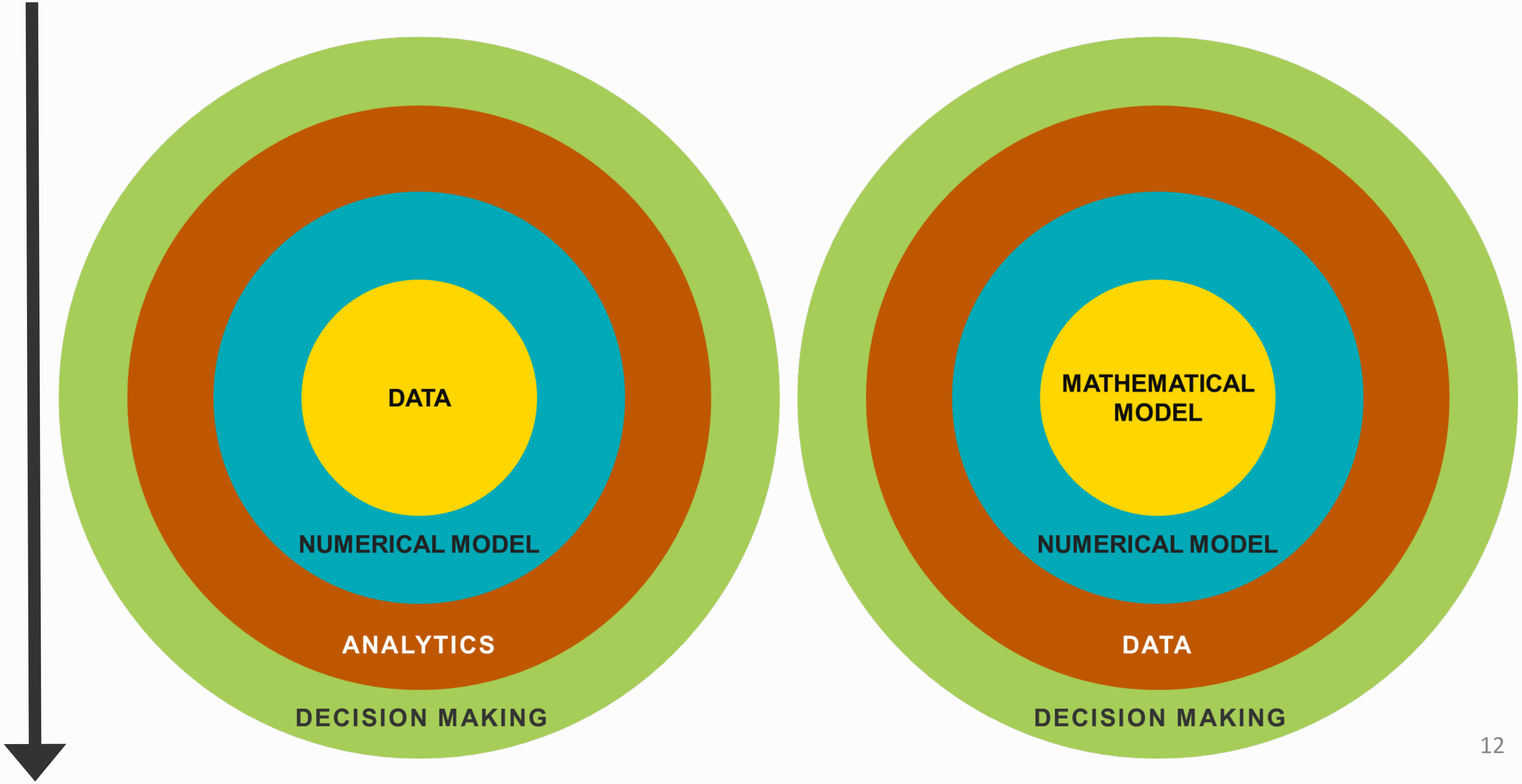
National Academies Study on Foundational Research Gaps and Future Opportunities for Digital Twins (2024)

Digital Twins provide a new mathematical paradigm for integrating data, models and decisions



National Academies Study on Foundational Research Gaps and Future Opportunities for Digital Twins (2024)

Conceptualizing a Digital Twin



The Mathematics of Digital Twins

MODELS + DATA

DATA ASSIMILATION

PREDICTION

DECISION & ACTION

$$\frac{\partial y}{\partial t} + \mathcal{R}(y, m, u) = 0 \quad \text{in } \Omega \times \left(t_0, t_f\right)$$
$$y\big|_{t=0} = y_0$$

Given model parameters m,
decision variables u,
and initial conditions y_0 :

Solve the governing physical equations to determine the spatiotemporal evolution of the state y.

"The Forward Problem"

MODELS + DATA

DATA ASSIMILATION

PREDICTION

DECISION & ACTION

$$\frac{\partial y}{\partial t} + \mathcal{R}(y, m, u) = 0 \quad \text{in } \Omega \times \left(t_0, t_f\right)$$

$$y\Big|_{t=0} = y_0$$

$$\mathbf{o}^{(i)} = \mathcal{B}\Big(y(t_i; m, u)\Big) + \epsilon^i, \quad i = 1, \ldots, N_{obs}$$

State y at time t_i generates noisy observational data $\mathbf{o}^{(i)}$

MODELS + DATA

DATA ASSIMILATION

PREDICTION

DECISION & ACTION

$$\frac{\mathrm{d}\pi_{\mathrm{post}}}{\mathrm{d}\pi_{\mathrm{pr}}} \propto \pi_{like}(\mathbf{o} \mid m, u)$$

$$\min_m \underbrace{\frac{1}{2}\sum_{i=1}^{N_{obs}} \left\| \mathcal{B}\big(y(t_i; m, u)\big) - \mathbf{o}^{(i)} \right\|^2_{\Gamma^{-1}_{\mathrm{noise}}}}_{\text{data misfit}} + \underbrace{\Phi_{\mathrm{reg}}(m)}_{\text{regularization}}$$

MODELS + DATA

DATA ASSIMILATION

PREDICTION

DECISION & ACTION

Parameter-to-QoI map

$$\mathcal{Q}(m,u) \coloneqq Q(y(m,u))$$

Posterior predictive distribution

$$\mathcal{Q}(\cdot,u)_{\#}\pi_{\text{post}}$$

MODELS + DATA

DATA ASSIMILATION

PREDICTION

DECISION & ACTION

Optimization under uncertainty

$$\min_{u \in \mathcal{U}_{\text{ad}}} \rho_{m \sim \pi_{\text{post}}} [\mathcal{Q}(m,u)] + \mathcal{P}(u)$$

MODELS + DATA

↓

DATA ASSIMILATION

↓

PREDICTION

↓

DECISION & ACTION

↓

$$\frac{\partial y}{\partial t} + \mathcal{R}(y, m, u) = 0 \quad \text{in } \Omega \times \left(t_0, t_f\right)$$

$$\mathbf{o}^{(i)} = \mathcal{B}\big(y(t_i; m, u)\big) + \epsilon^i, \quad i = 1, \dots, N_{obs}$$

$$\frac{\mathrm{d}\pi_{\mathrm{post}}}{\mathrm{d}\pi_{\mathrm{pr}}} \propto \pi_{like}(\mathbf{o} \mid m, u)$$

$$\mathcal{Q}(\cdot, u)_{\#}\pi_{\mathrm{post}}$$

$$\min_{u \in \mathcal{U}_{\mathrm{ad}}} \rho_{m \sim \pi_{\mathrm{post}}}\left[\mathcal{Q}(m, u)\right] + \mathcal{P}(u)$$

Digital Twins are enabled by math advances

Example:
Cancer Patient Digital Twin

Mechanistic Model (Forward Problem)

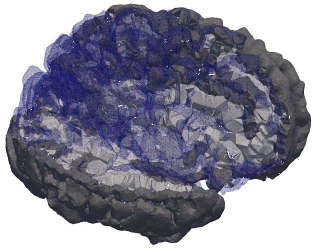
Phenomenological modeling focuses on two “hallmarks” [Hanahan & Weinberg]

- Infiltration of tumor into healthy tissues
- Proliferation of existing tumor

$$\frac{\partial y}{\partial t} - \underbrace{\nabla \cdot (e^{m_d}\, \nabla y)}_{\text{infiltration}} - \underbrace{e^{m_\kappa} y(1-y)}_{\text{proliferation}} = f(y) \quad \text{in } \Omega \times (t_0, t_f)$$

$$y(\boldsymbol{x}, t_0) = y_0 \quad \text{in } \Omega$$

$$\nabla y \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times (t_0, t_f)$$



$y(t, \boldsymbol{x})$ tumor volume fraction

$m_d(\boldsymbol{x})$ log diffusivity field

$m_\kappa(\boldsymbol{x})$ proliferation rate

$f(y)$ treatment

Mechanistic Model (Forward Problem)

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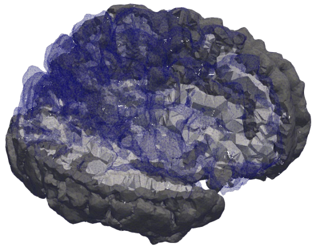


$y(\boldsymbol{x}, t)$ tumor volume fraction

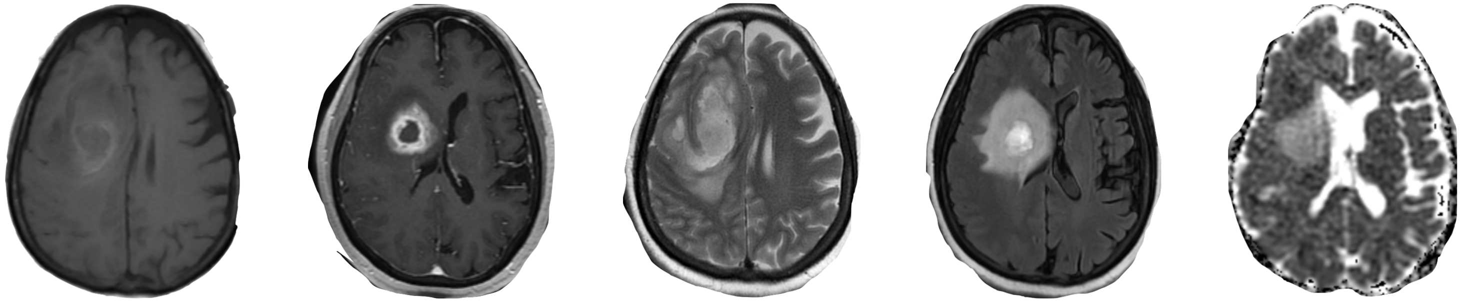
$m_d(\boldsymbol{x})$ log diffusivity field

$m_\kappa(\boldsymbol{x})$ proliferation rate

$f(y)$ treatment

Imaging Data

Multiple modalities used to assess patients



T1
Anatomy

T1-C
Necrosis
Enhancing
Tumor

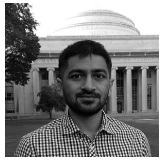
T2
Anatomy

FLAIR
Edema
Non-enhancing
Tumor

ADC
Cellularity

DIGITAL-TWIN-ENABLED CANCER TREATMENT

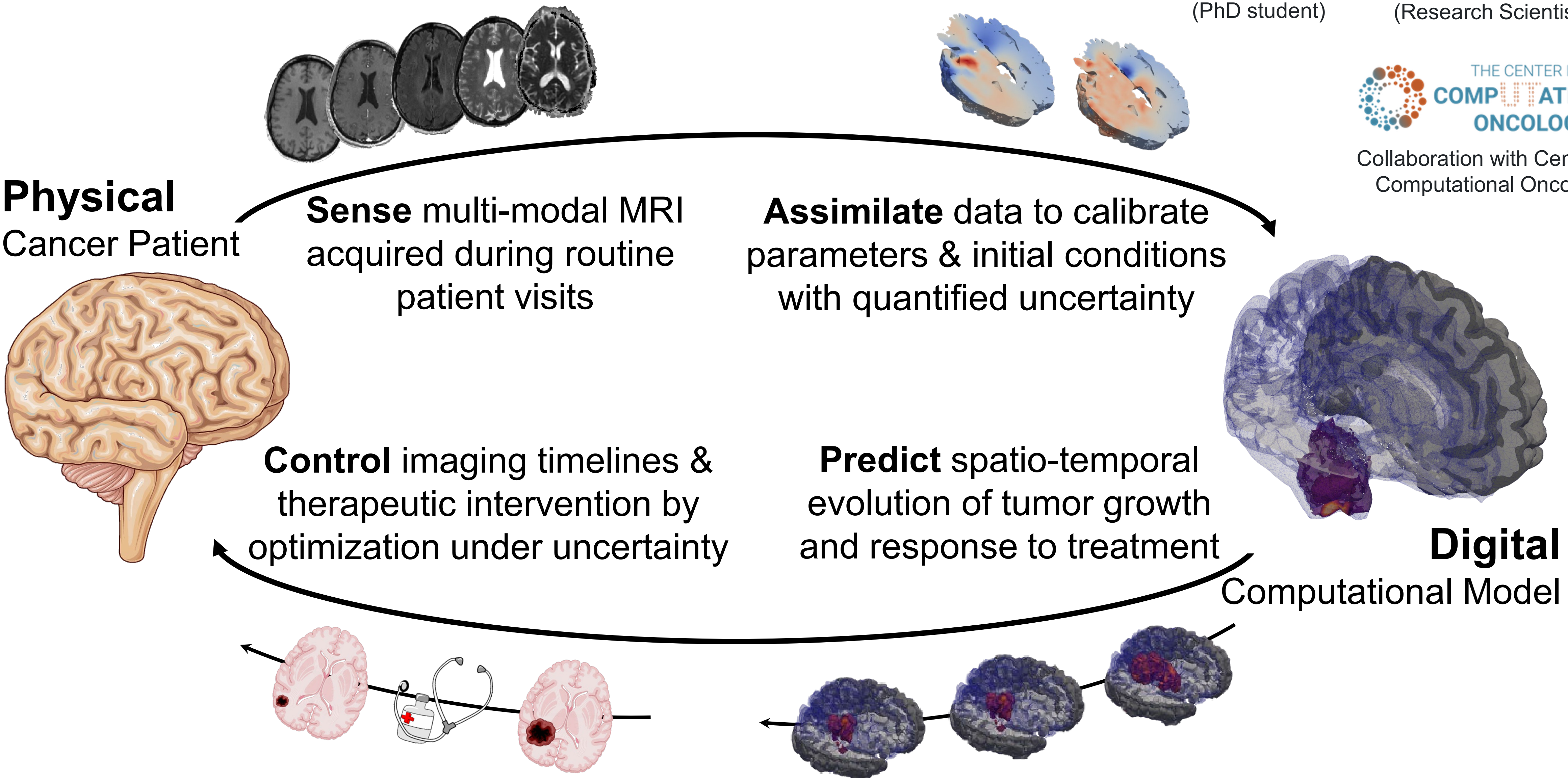
Graham Pash
(PhD student)



Anirban Chaudhuri
(Research Scientist)

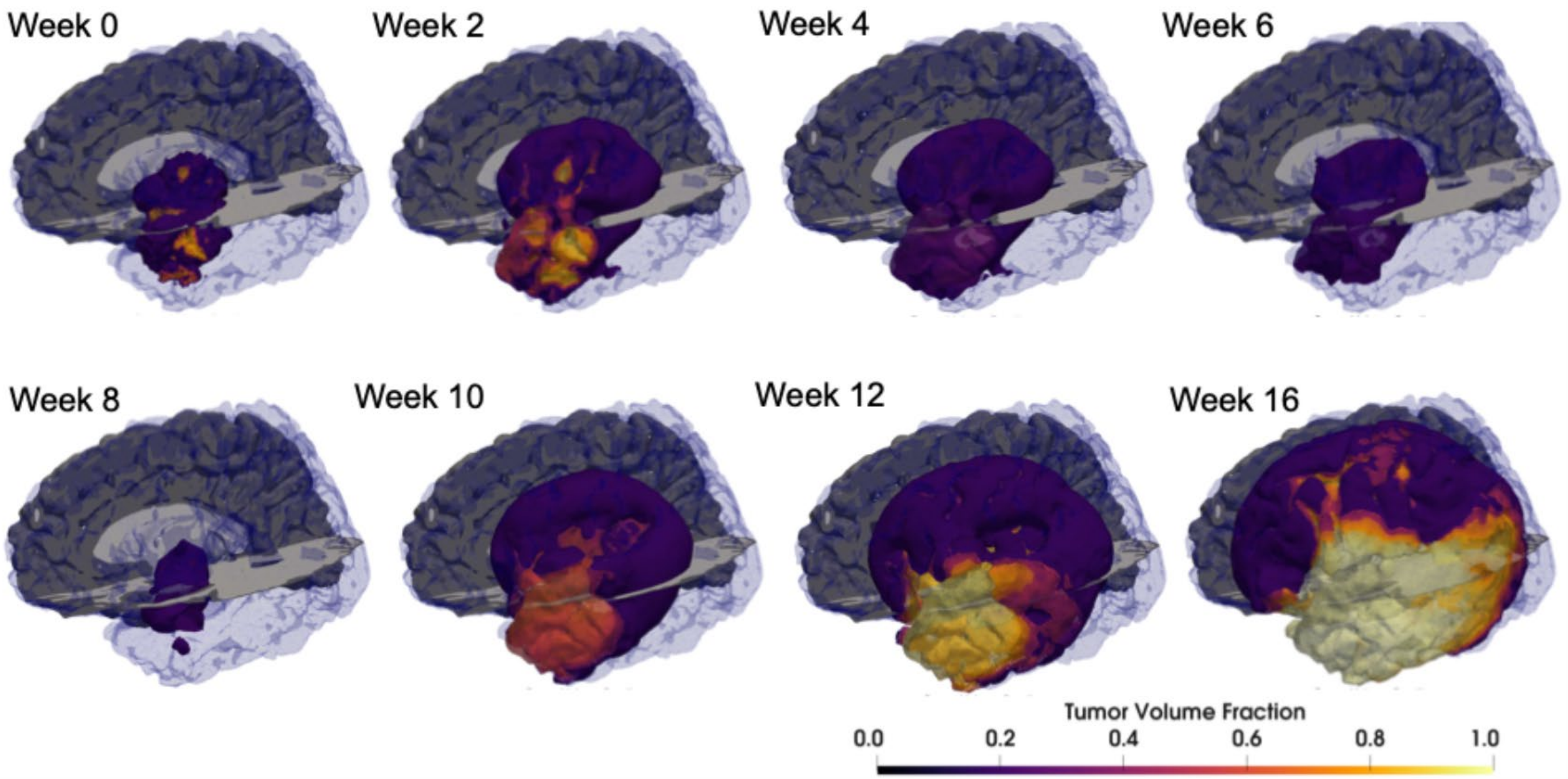
THE CENTER FOR COMPUTATIONAL ONCOLOGY

Collaboration with Center for Computational Oncology

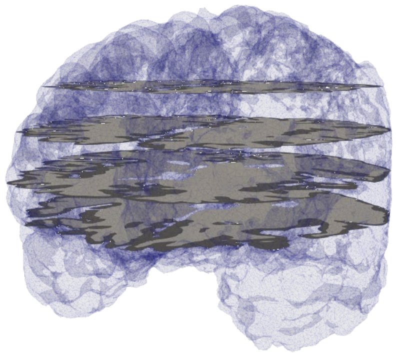


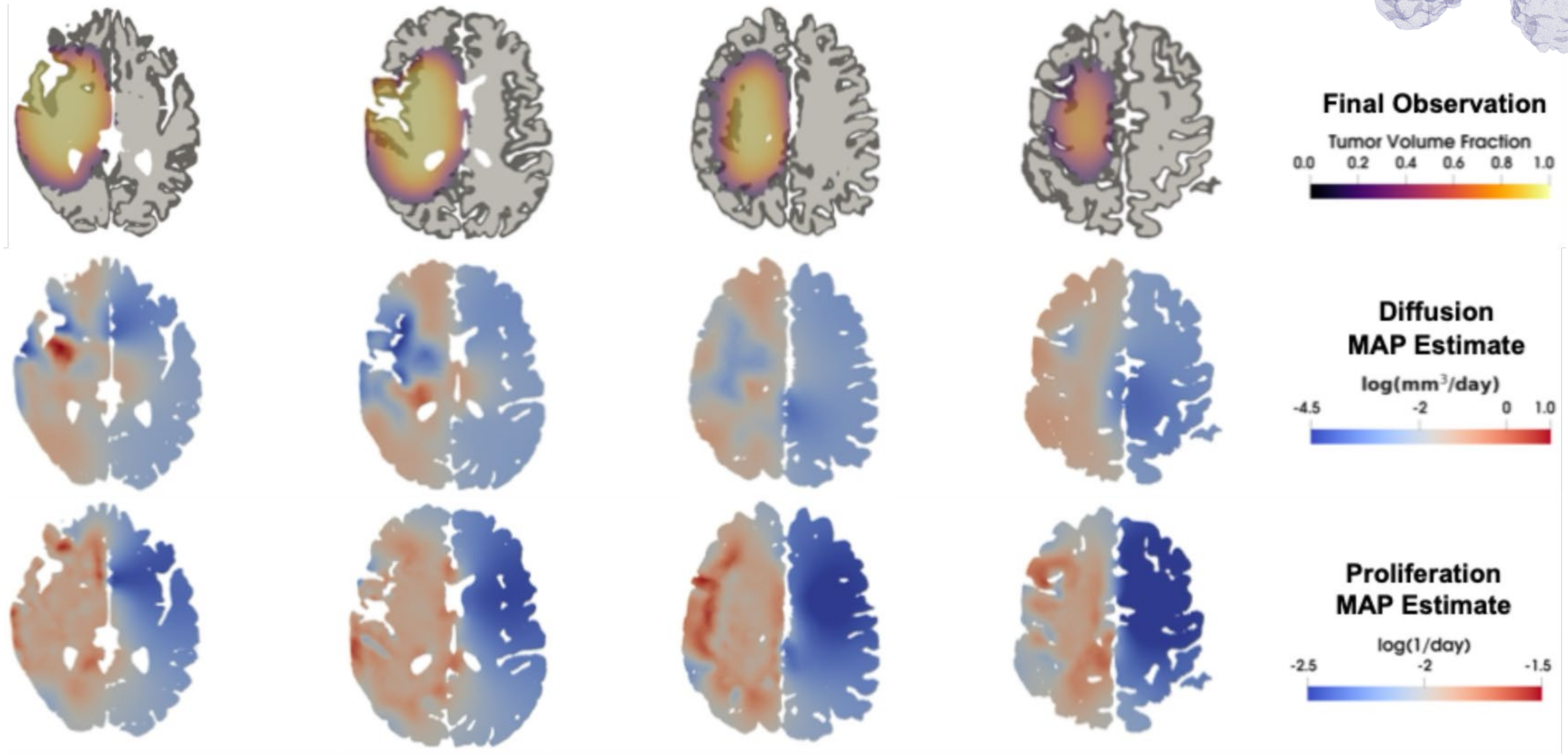
Simulated Disease Progression

Graham Pash
(PhD student)



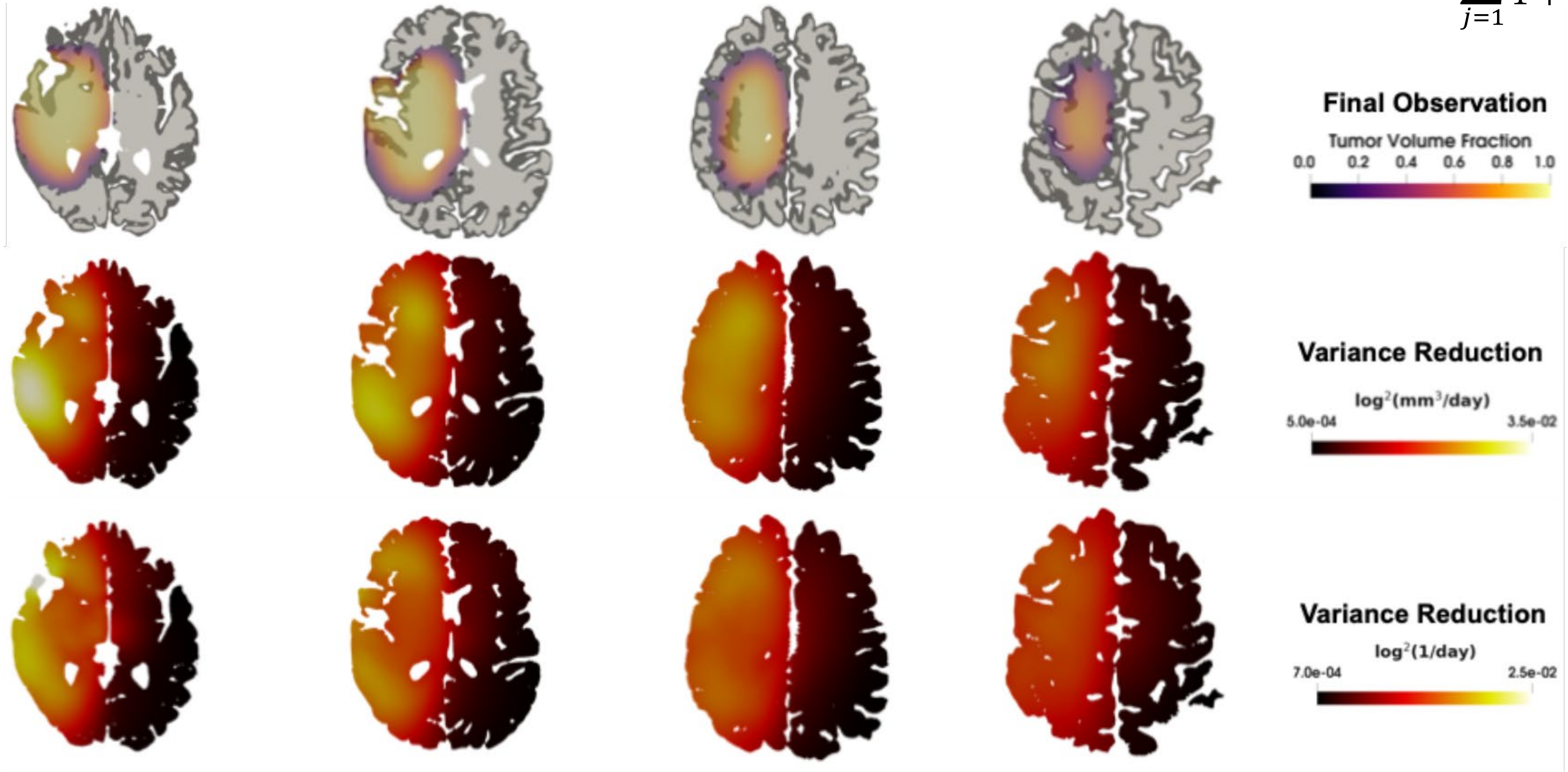
Model Parameter Reconstruction





Data-Informed Variance Reduction

$$\pi_{post} \approx \pi_{pr} - \sum_{j=1}^{r} \frac{\lambda_j}{1+\lambda_j} v_j v_j^{\top}$$



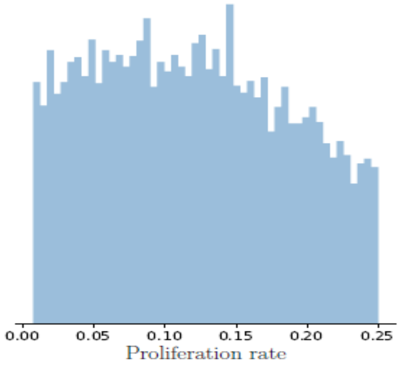
Uncertainty Quantification is critical for Digital Twins

Patient Prior

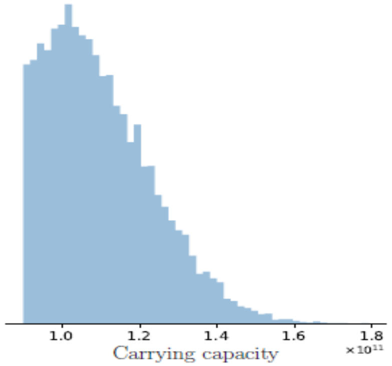
Estimates before infusing any patient data

prior

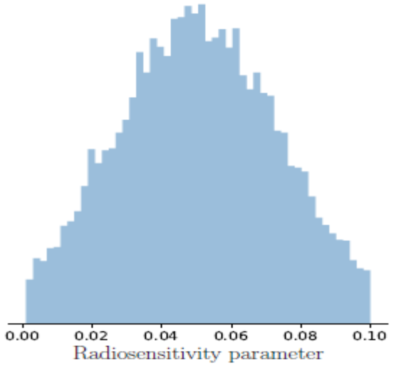
Proliferation Rate



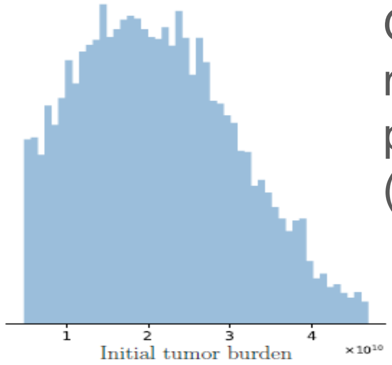
Carrying Capacity



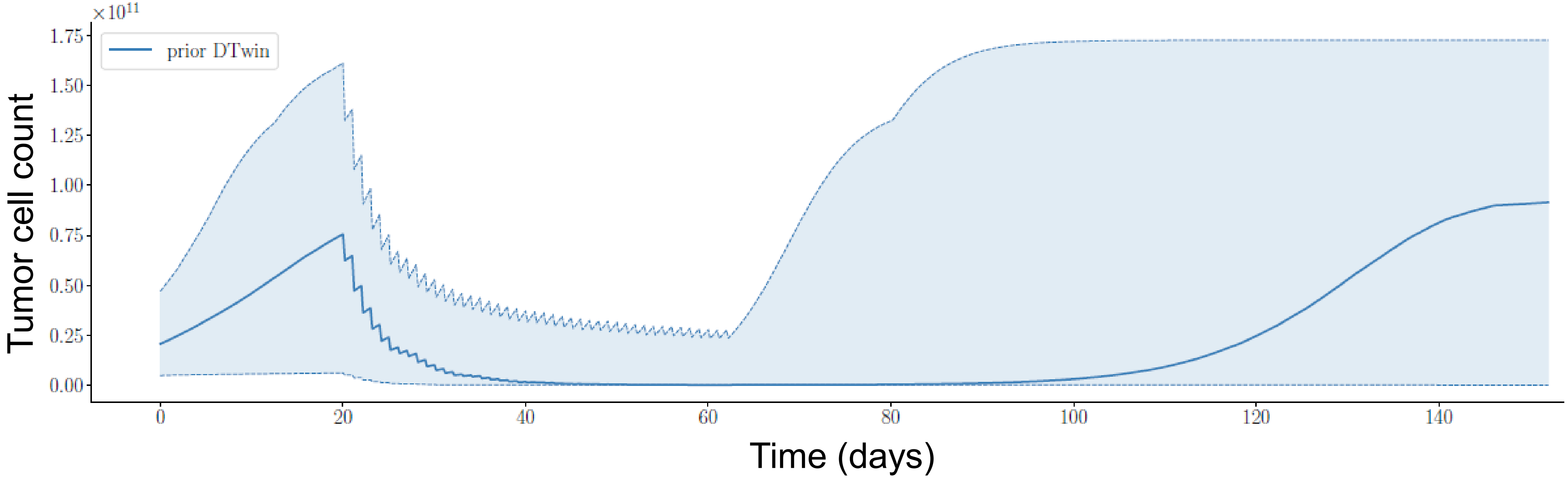
Radiosensitivity parameter



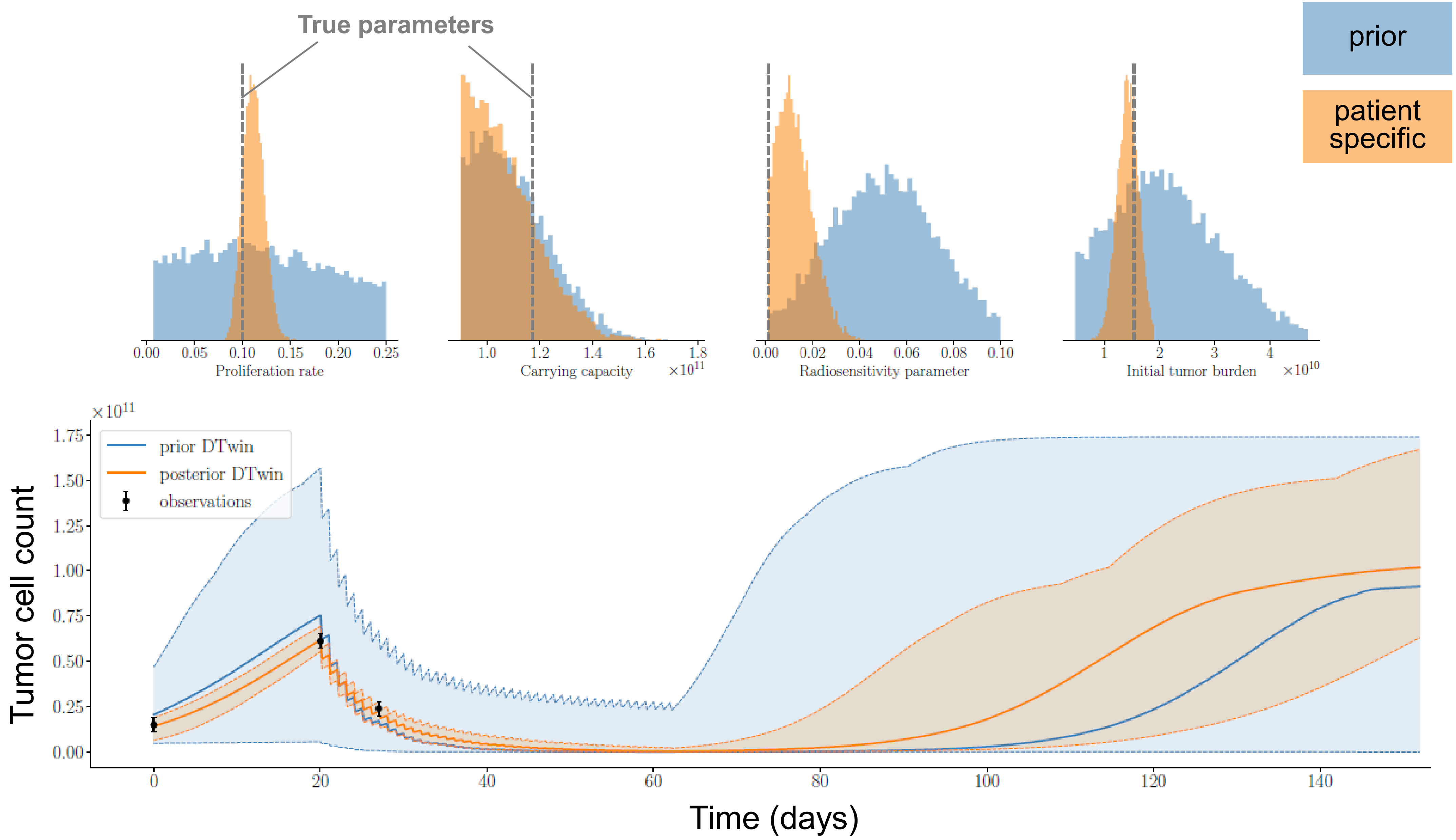
Initial Tumor Burden



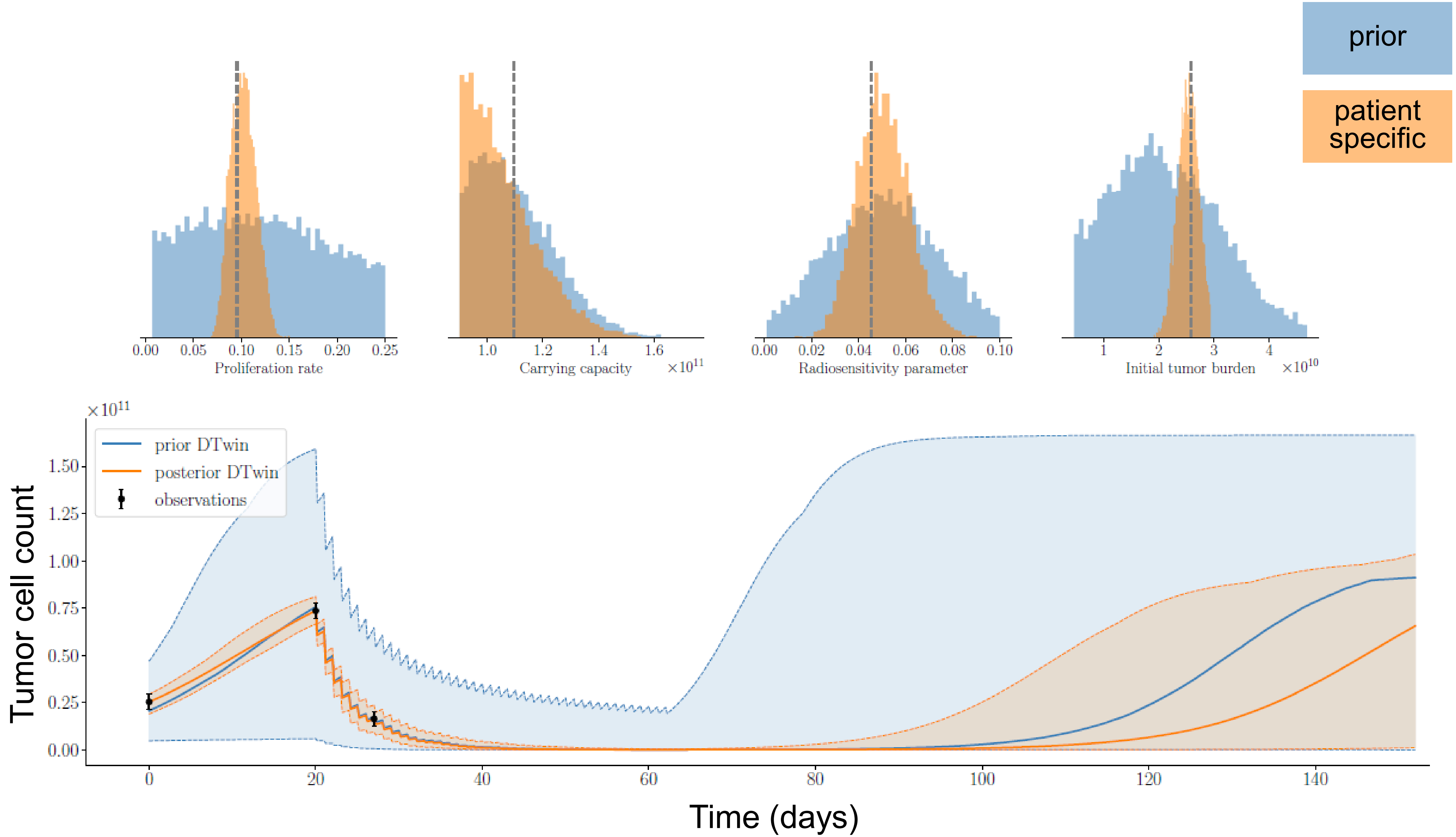
Optimal control results with lumped parameter model (no spatial variation)



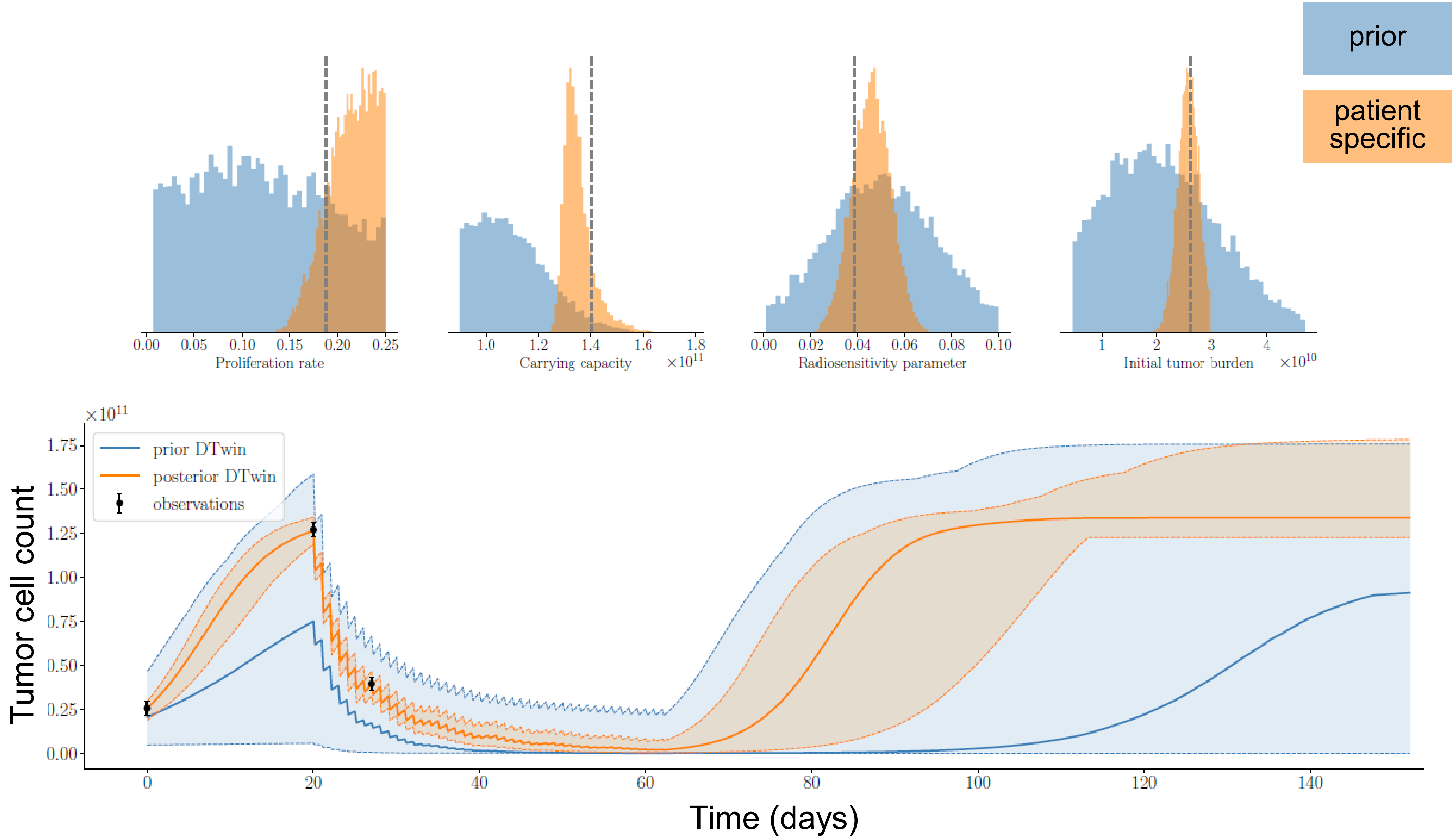
Patient 1: Patient-specific Digital Twin



Patient 2: Patient-specific Digital Twin

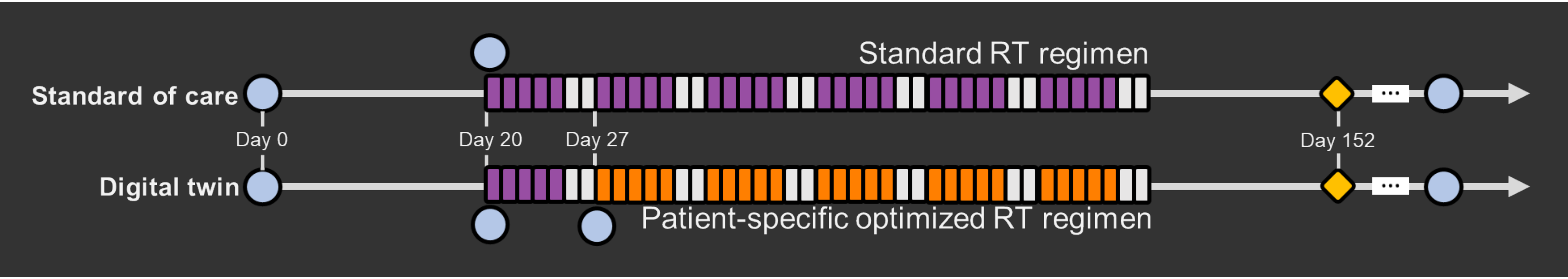


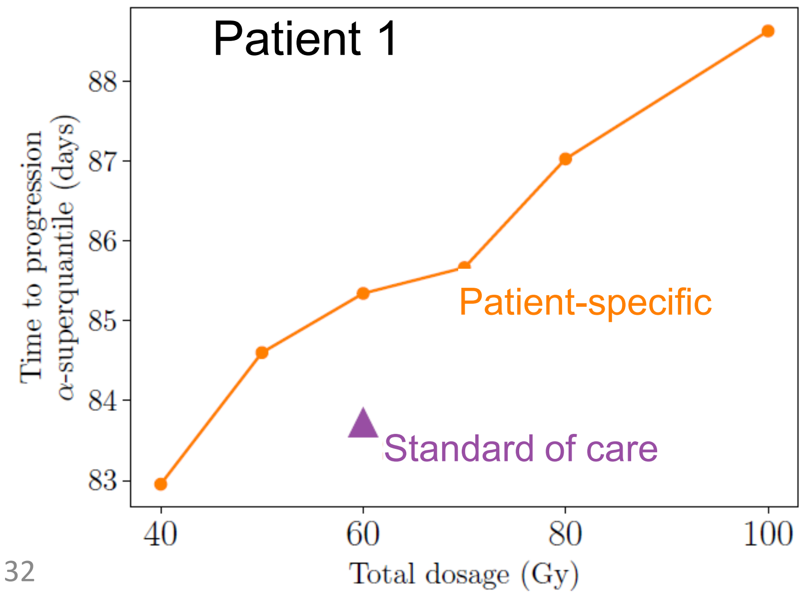
Patient 3: Patient-specific Digital Twin

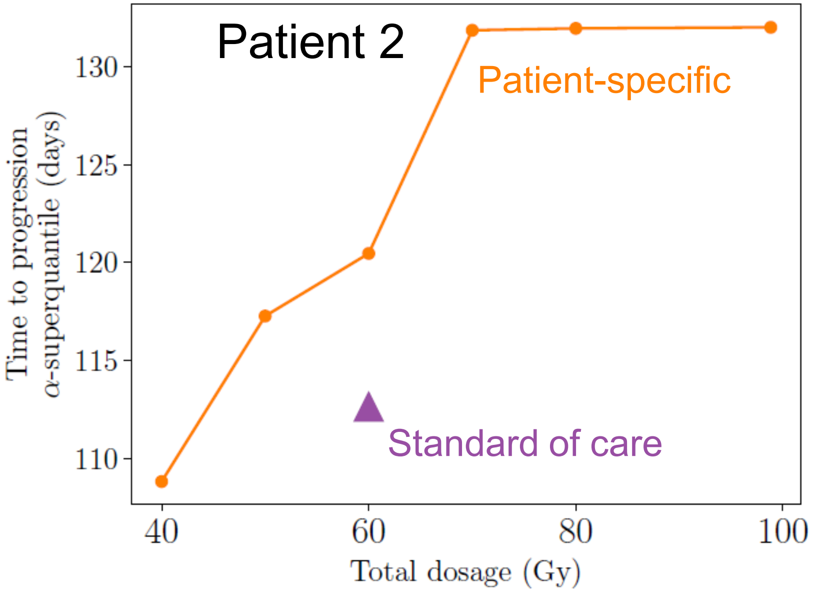


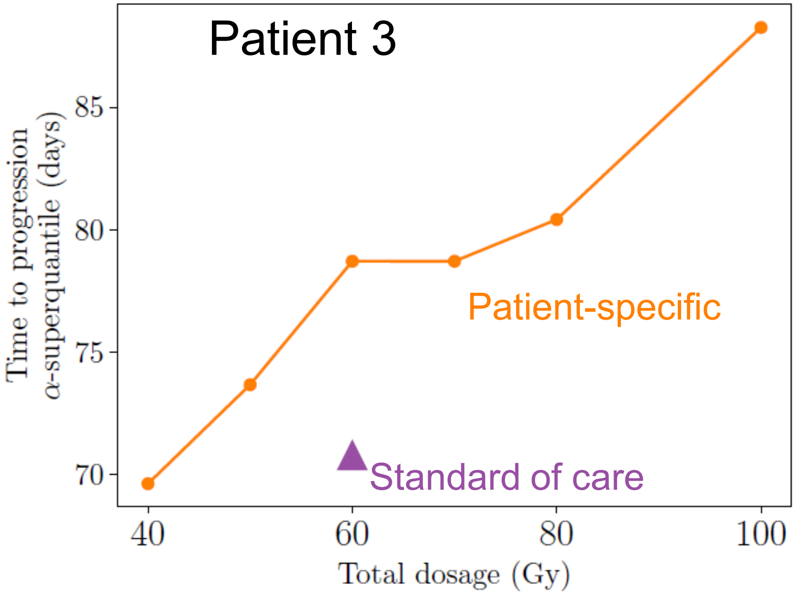
Patient-specific treatment plan
Increases the tumor time to progression

Optimal control results with lumped parameter model (no spatial variation)



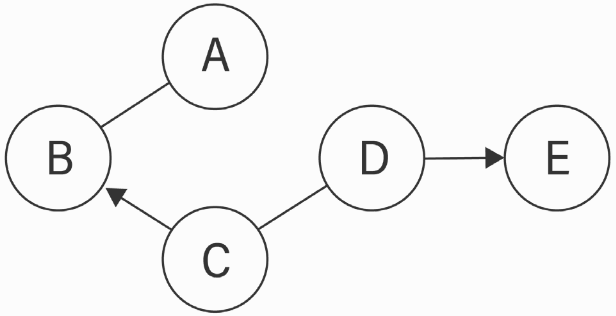






Graphical Representations of Digital Twins

GRAPHICAL REPRESENTATION
of a digital twin



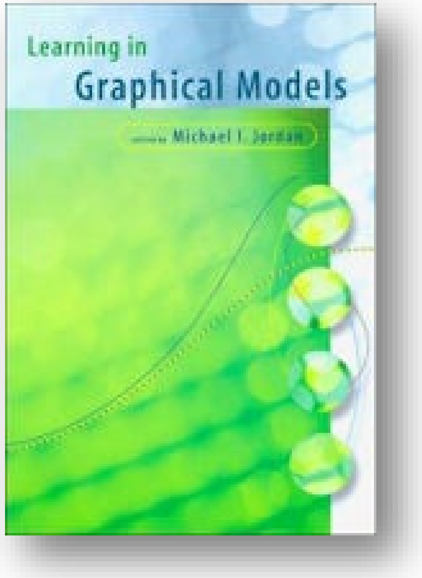
$G = (V, E)$

$V = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$

$E = \{\mathrm{A} \leftrightarrow \mathrm{B}, \mathrm{C} \rightarrow \mathrm{B}, \mathrm{C} \leftrightarrow \mathrm{D}, \mathrm{D} \rightarrow \mathrm{E}\}$

"Graphical models, a marriage between probability theory and graph theory, provide a natural tool for dealing with two problems that occur throughout applied mathematics and engineering — uncertainty and complexity."

"Fundamental to the idea of a graphical model is the notion of modularity: a complex system is built by combining simpler parts. Probability theory serves as the glue whereby the parts are combined, ensuring that the system as a whole is consistent and providing ways to interface models to data."



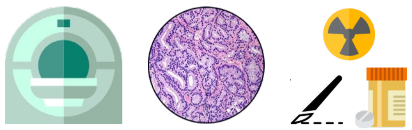
Graphical models emphasize relationships
→ key to a digital twin as "more than just simulation & modeling"

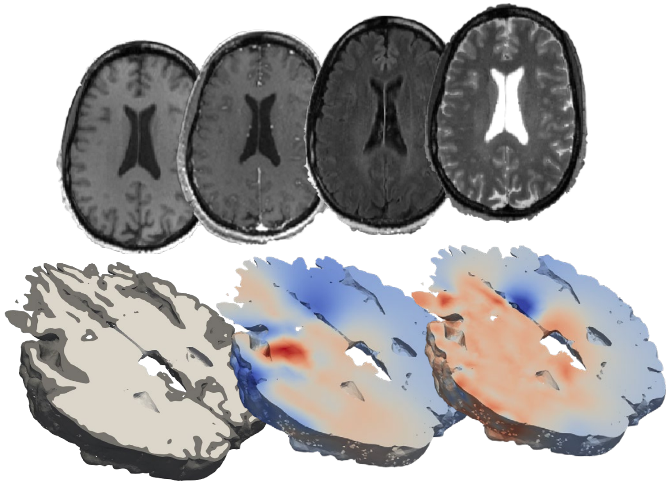
GRAPHICAL REPRESENTATION
of a patient-twin system

Collaboration with Center for Computational Oncology

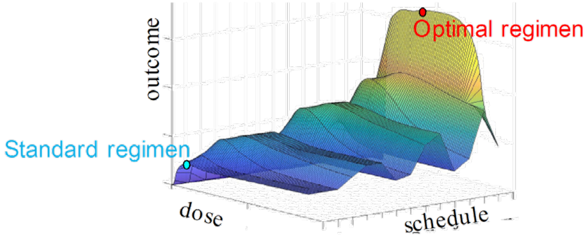
D_t Digital Twin State
Tumor dynamics, anatomical & mechanical properties

U_t Control inputs
MRI studies, biopsies, treatment regimens



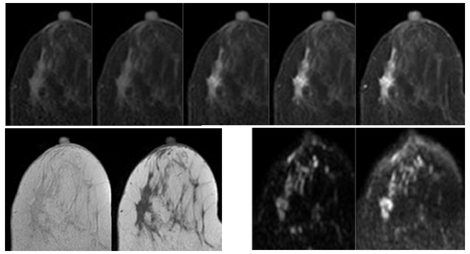


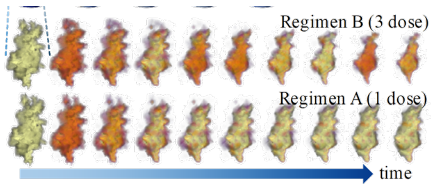
R_t Reward
Patient outcomes: treatment efficacy, toxicity



S_t Physical State
Anatomy & morphology, mechanical & physiological state

O_t Observational data
Anatomy, perfusion, permeability, cell density, metabolism





Q_t Quantities of Interest
Distribution of therapies, tumor shape, cell density, time to progression

Chaudhuri et al., *Frontiers in AI*, 2023

GRAPHICAL REPRESENTATION

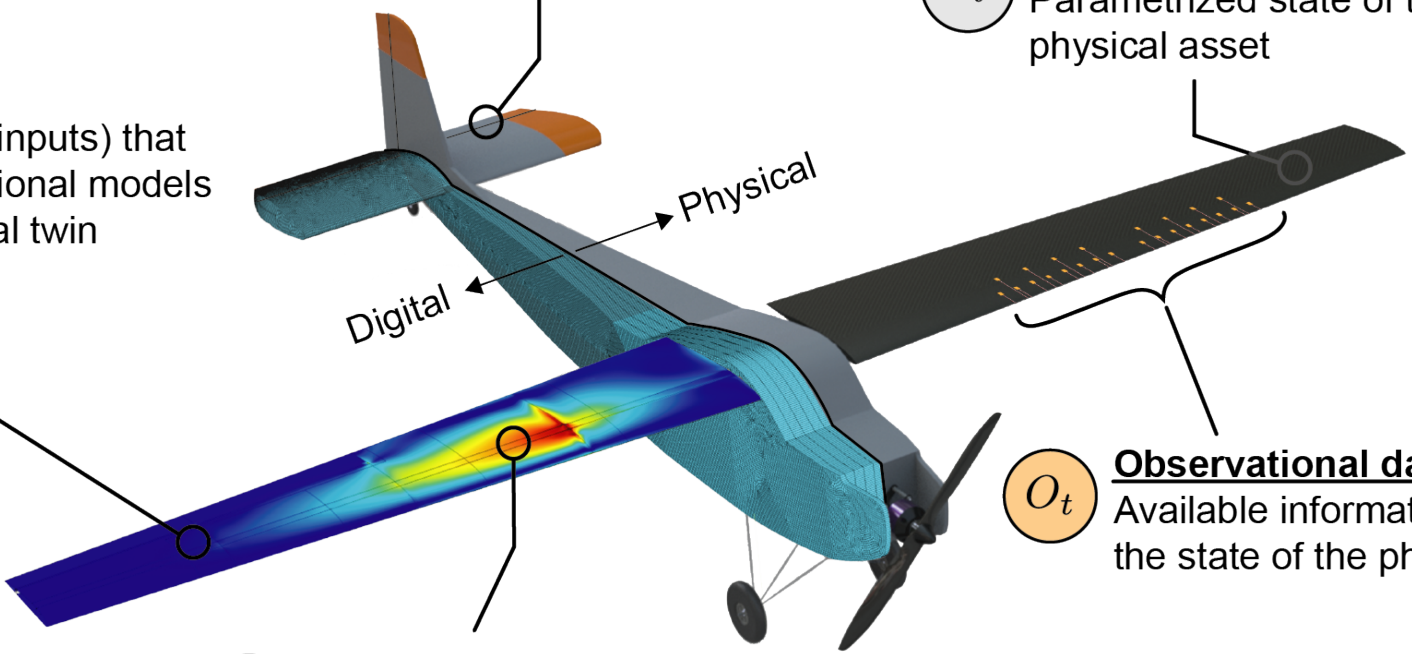
of an asset-twin system

Michael Kapteyn

U_t **Control inputs:**
Actions or decisions that influence the physical asset

S_t **Physical State:**
Parametrized state of the physical asset

D_t **Digital State:**
Parameters (model inputs) that define the computational models comprising the digital twin

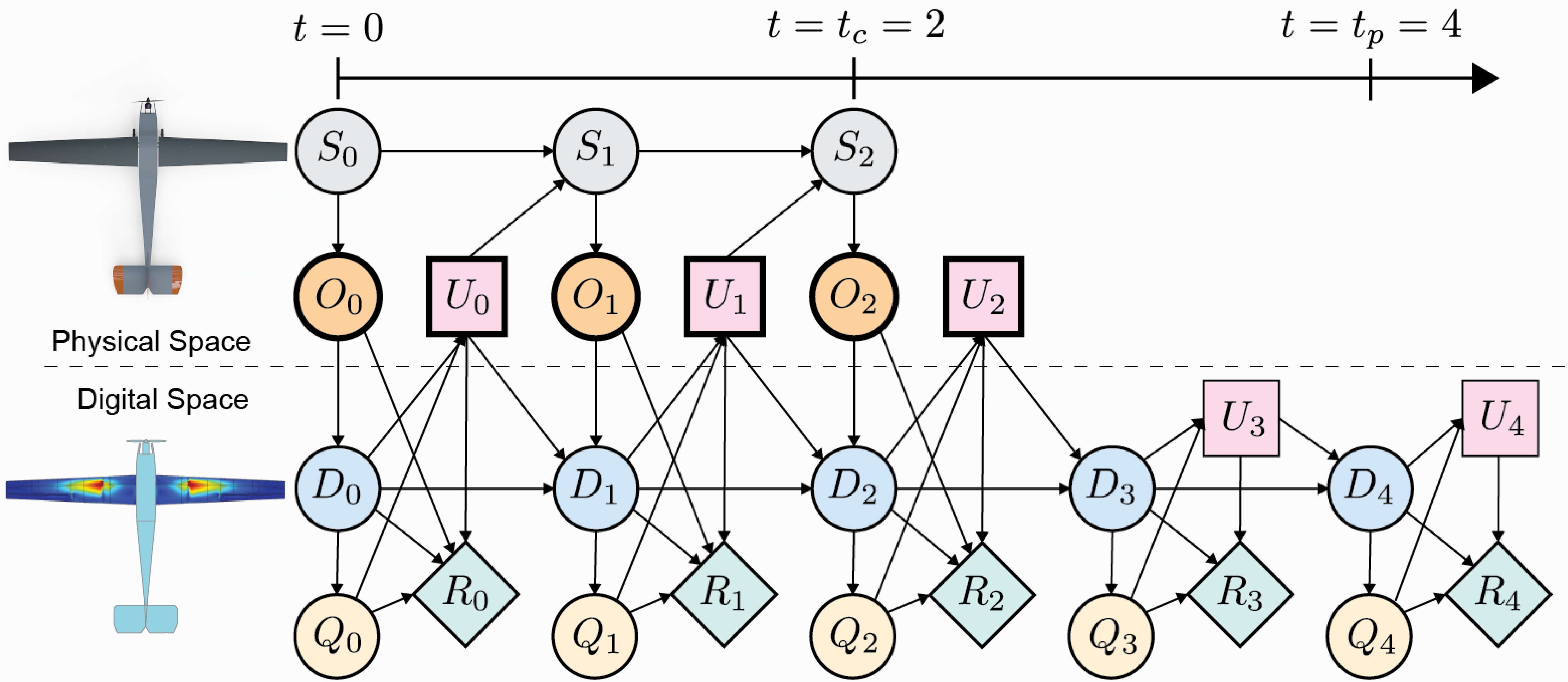


O_t **Observational data:**
Available information describing the state of the physical asset

R_t **Reward:**
Quantifies overall performance of the asset-twin system

Q_t **Quantities of Interest:**
Quantities describing the asset, estimated via model outputs

Kapteyn, Pretorius & W.,
Nature Comp. Sci., Vol. 1, No. 5, 2021.



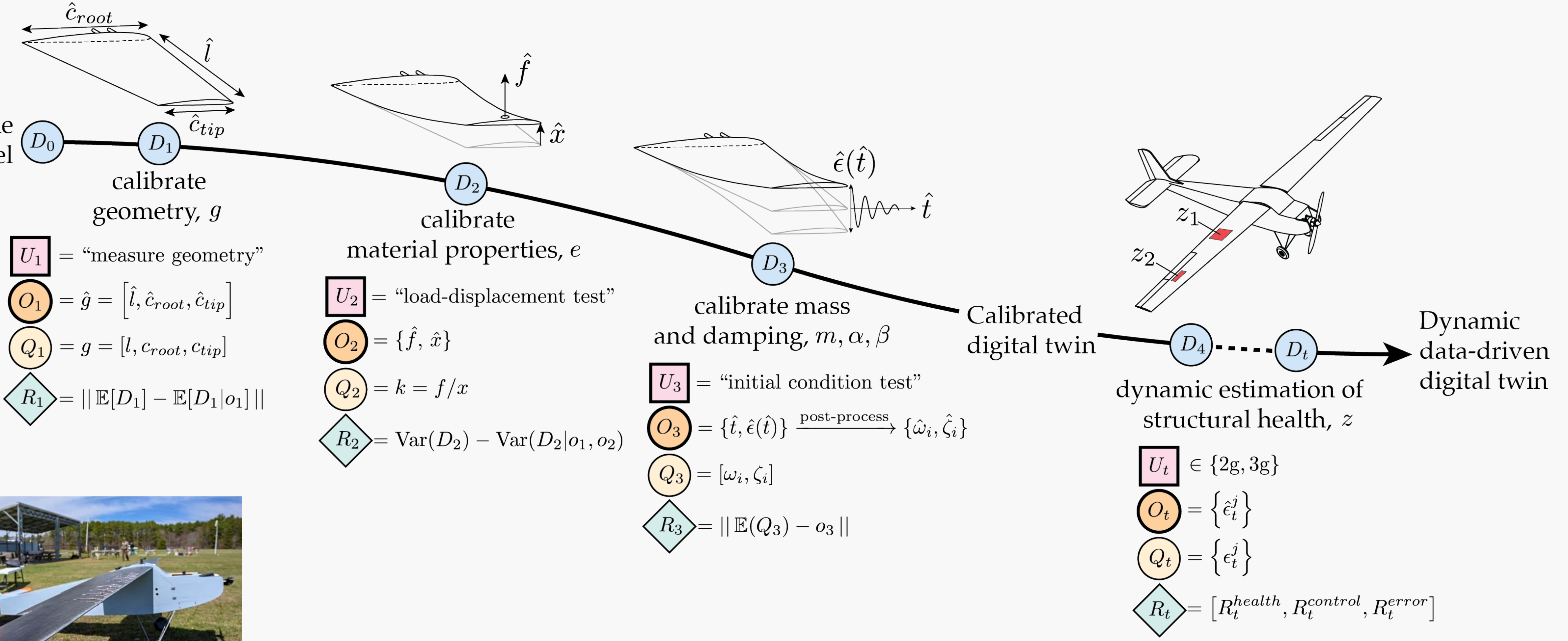
Graph represents joint probability distribution: $p\left(D_0, \ldots, D_{t_p}, Q_0, \ldots, Q_{t_p}, R_0, \ldots, R_{t_p}, U_{t_c+1}, \ldots, U_{t_p} \;\middle|\; o_0, \ldots, o_{t_c}, u_0, \ldots, u_{t_c}\right)$

PROBABILISTIC GRAPHICAL MODEL: A FOUNDATION FOR BAYESIAN CALIBRATION & DECISION UNDER UNCERTAINTY

Predictive Digital Twin Use-case		Mathematical Formulation via Probabilistic Graphical Model
Virtual inspections, asset/patient-specific calibration, simulation-based certification	**Inverse problem / data assimilation:**	$p(D_{t_c}, Q_{t_c}, R_{t_c} \mid u_0, \dots u_{t_c}, o_0, \dots o_{t_c})$ $\frac{d\pi_{\text{post}}}{d\pi_{\text{pr}}} \propto \pi_{like}(\mathbf{o} \mid m, u)$
Forecasting, predictive maintenance, prognosis, treatment planning	**Prediction:**	$p(D_{t_p}, Q_{t_p}, R_{t_p} \mid u_0, \dots u_{t_c}, o_0, \dots o_{t_c})$ $\mathcal{Q}(\cdot, u)_{\#}\pi_{\text{post}}$
Operational decision-making for system control, treatment, data acquisition	**Optimal control:**	$\min_{u \in \mathcal{U}_{\text{ad}}} \rho_{m \sim \pi_{\text{post}}} [\mathcal{Q}(m,u)] + \mathcal{P}(u)$
Learn from historical data / across assets	**Learning:**	$\phi_t^{\text{dynamics}} = p(D_t \mid D_{t-1}, U_t)$ $\phi_t^{\text{assimilation}} = p(O_t \mid D_t)$

Creating & evolving a structural digital twin
for an unmanned aerial vehicle

Michael Kapteyn



What about problems where there are no governing differential equations?

EDUCATIONAL DIGITAL TWIN

S_t **Physical State**

Student enrollments, engagement, outcomes, demographics;
course enrollments, success rates;
institutional resources, faculty capacity

U_t **Control inputs**

Curriculum changes, student advising, tutoring interventions, added resources

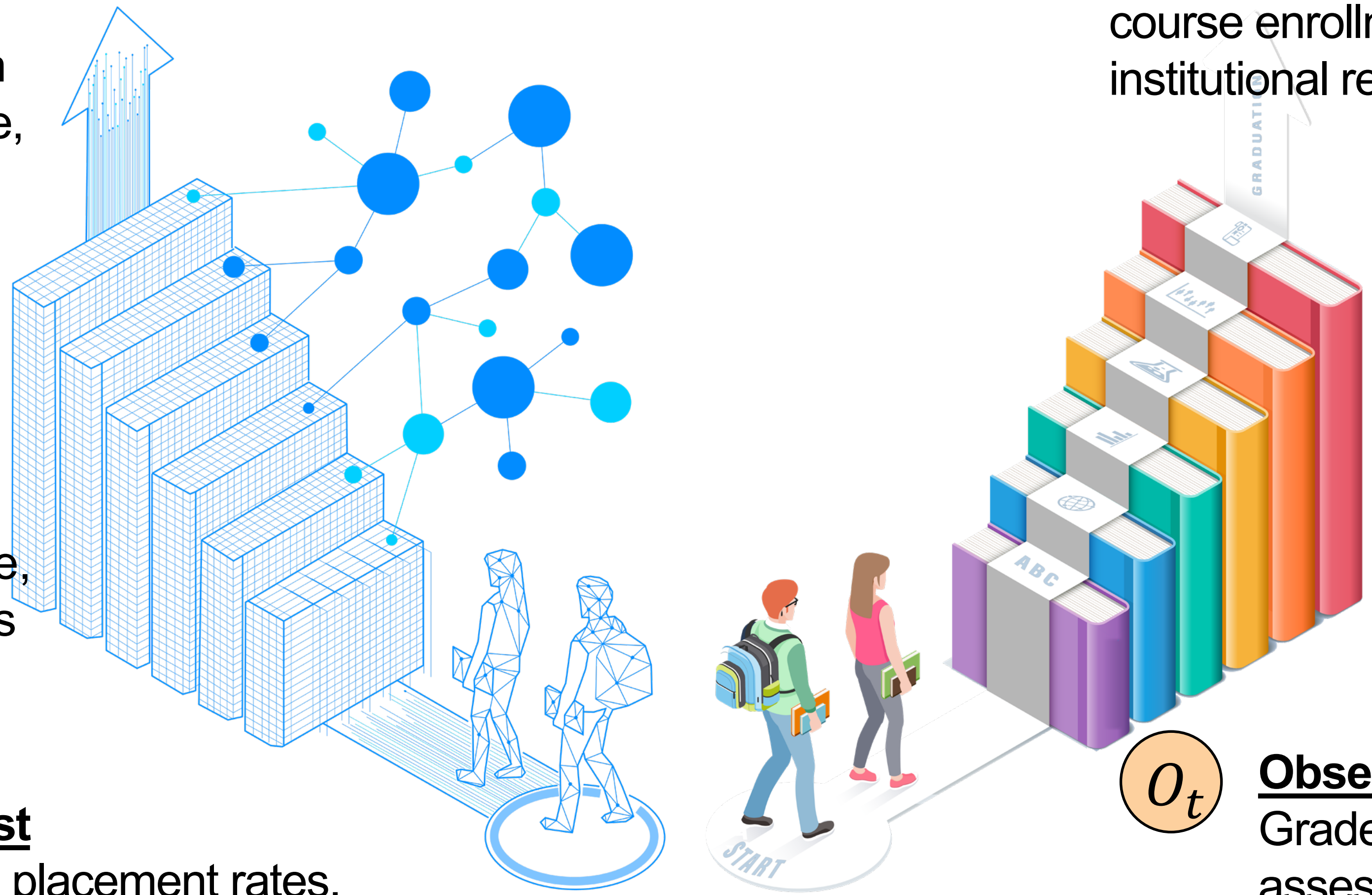
O_t **Observational data**

Grades, attendance records, assessment results, institutional records, engagement analytics

EDUCATIONAL DIGITAL TWIN

S_t **Physical State**
Student enrollments, engagement, outcomes, demographics; course enrollments, success rates; institutional resources, faculty capacity

D_t **Digital Twin State**
Virtual representation at the student, course, program, institution, and state levels



U_t **Control inputs**
Curriculum changes, student advising, tutoring interventions, added resources

R_t **Reward**
Student outcomes, student performance, drop-out rates, costs

Q_t **Quantities of Interest**
Graduation rates, job placement rates, teacher-student ratios, advising time

O_t **Observational data**
Grades, attendance records, assessment results, institutional records, engagement analytics

Educational Pathways

Academic trajectories are nonlinear, multi-faceted and time-dependent

Fall 2015

Jennifer enrolls in community college for the first time.

She meets the Texas requirements in Reading and Writing, but not in math.

She enrolls in College Algebra I, Drama I, Biology I and English I.

Spring 2015

She fails College Algebra I at the end of term.

She does not enroll the following term.

Fall 2016

Spring 2016

She re-enrolls.

She takes Pre-Algebra II, Govt I, and Engl II.

Part of recommended sequence of courses for the Biology pathway.

She passes all classes.

She earns a Core Completer certificate.

Fall 2017

She transfers to a four-year institution.

All data shown are notional.

The Scale & Complexity of Educational Pathways in Texas

All data shown are notional

Volume: millions of students, hundreds of public postsecondary institutions

Variety: Grade reports, HTML, excel spreadsheets, etc.

Velocity: Semester-based grade reports, multiyear policy changes

Veracity: Accurate course records, self-reported student intentions

Value: Insights support both student interventions and policy making across Texas

Variability: Multiple use cases across multiple scales e.g., advising tool for counselors vs. student nightly registration status vs. institutional resource decisions

Degree Plans –

ASSOCIATE DEGREE

Biology

TSI Liable

RECOMMENDED COURSE SEQUENCI

First Year	
Fall	
ENGL 1301	Composition I
Creative Arts Elective - Core Curriculum	
Mathematics Elective - Core Curriculum	
Select one of the following:	
MATH 1414	College Algebra
MATH 2412	Pre-Calculus Math

(a) Selected columns from Report CMB001 ("Student")

student	fice	gender	ethnic	ecodis	...
15	1	1	2	1	
16	2	0	7	1	
17	2	0	7	1	

(b) Selected columns from Report CMB00S ("Student Schedule")

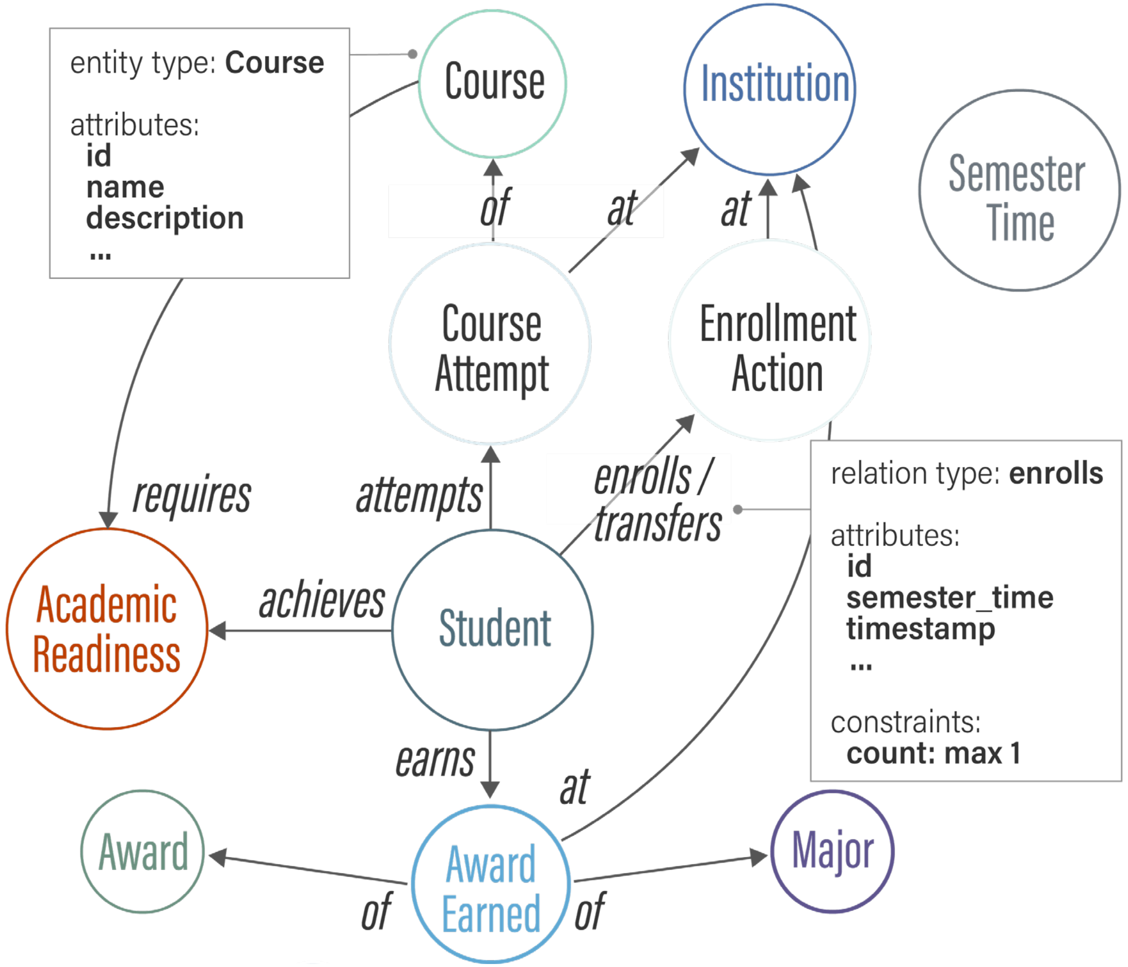
student	course	grade	credit	fcl	...
15	Math 1	A	TRUE	1	
15	Eng 1	B	TRUE	0	
16	Math 1	C	TRUE	3	
17	Eng 1	A	FALSE	4	
17	Econ 2	A	FALSE	4	

(c) Selected columns from Report CMB009 ("Graduation"):

student	fice	degree	level	major	type	...
9	1	AA	1	012	Academic	
13	1	ATC	2	156	Technical	
25	2	CCC	5	213	Tech-Prep	

Educational Digital Twin Knowledge Graph

EDT-KG ontology

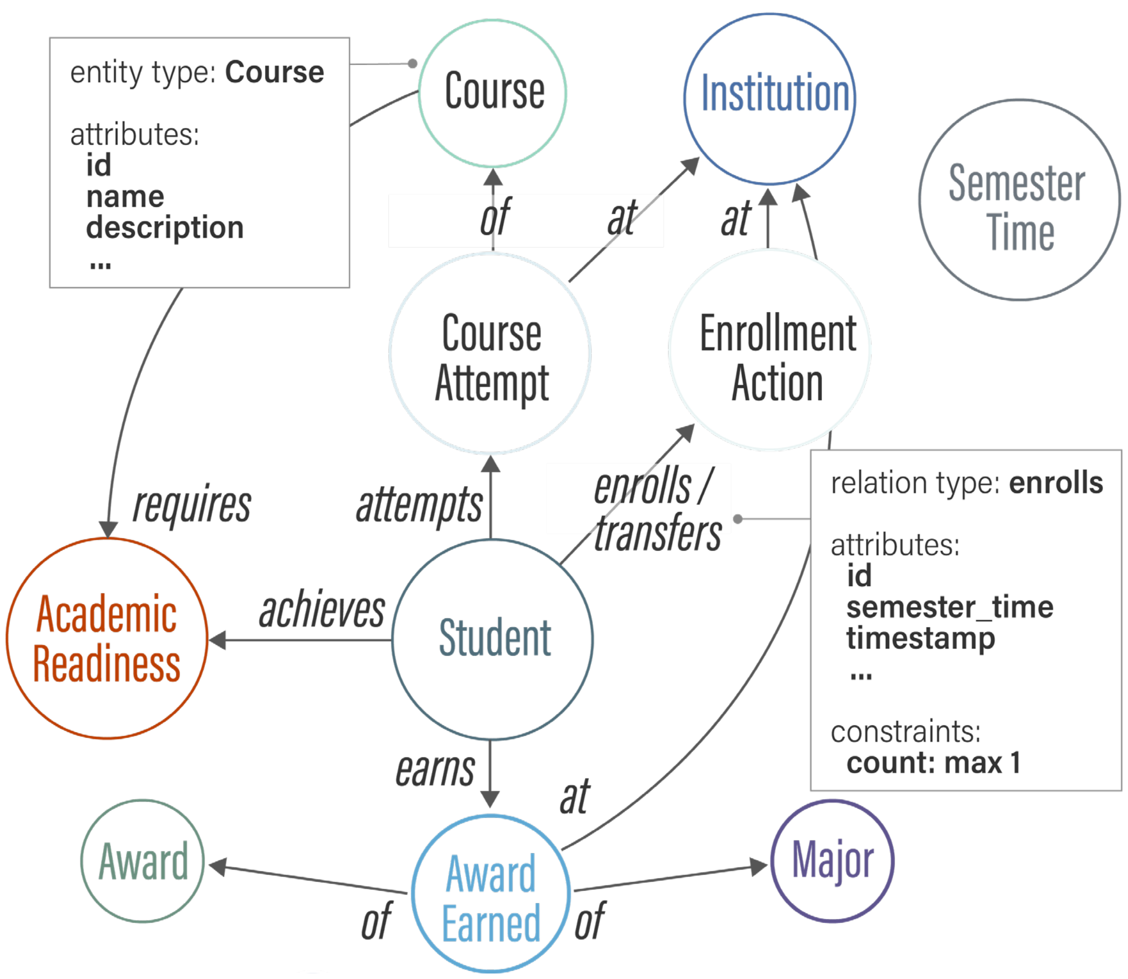


Luwen Huang
Research Associate

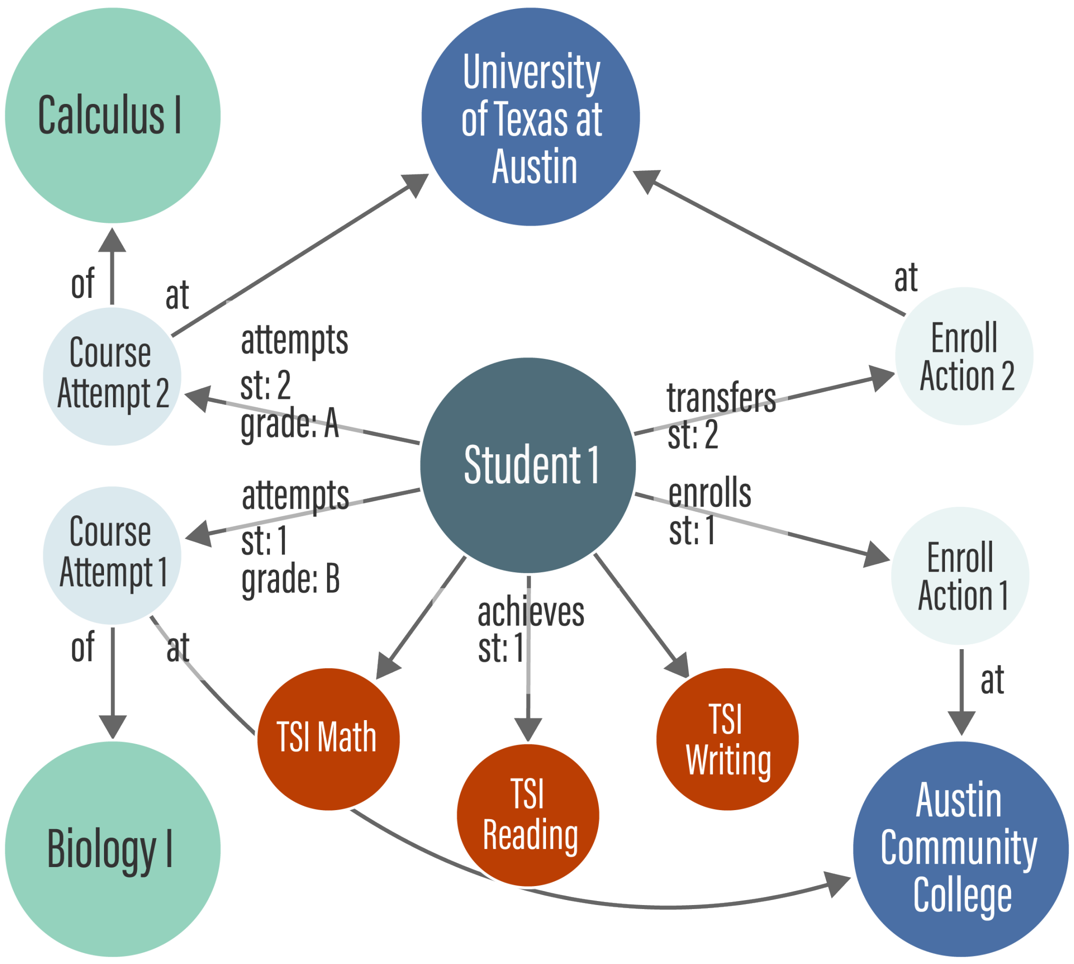
Educational Digital Twin Knowledge Graph

All data shown are notional

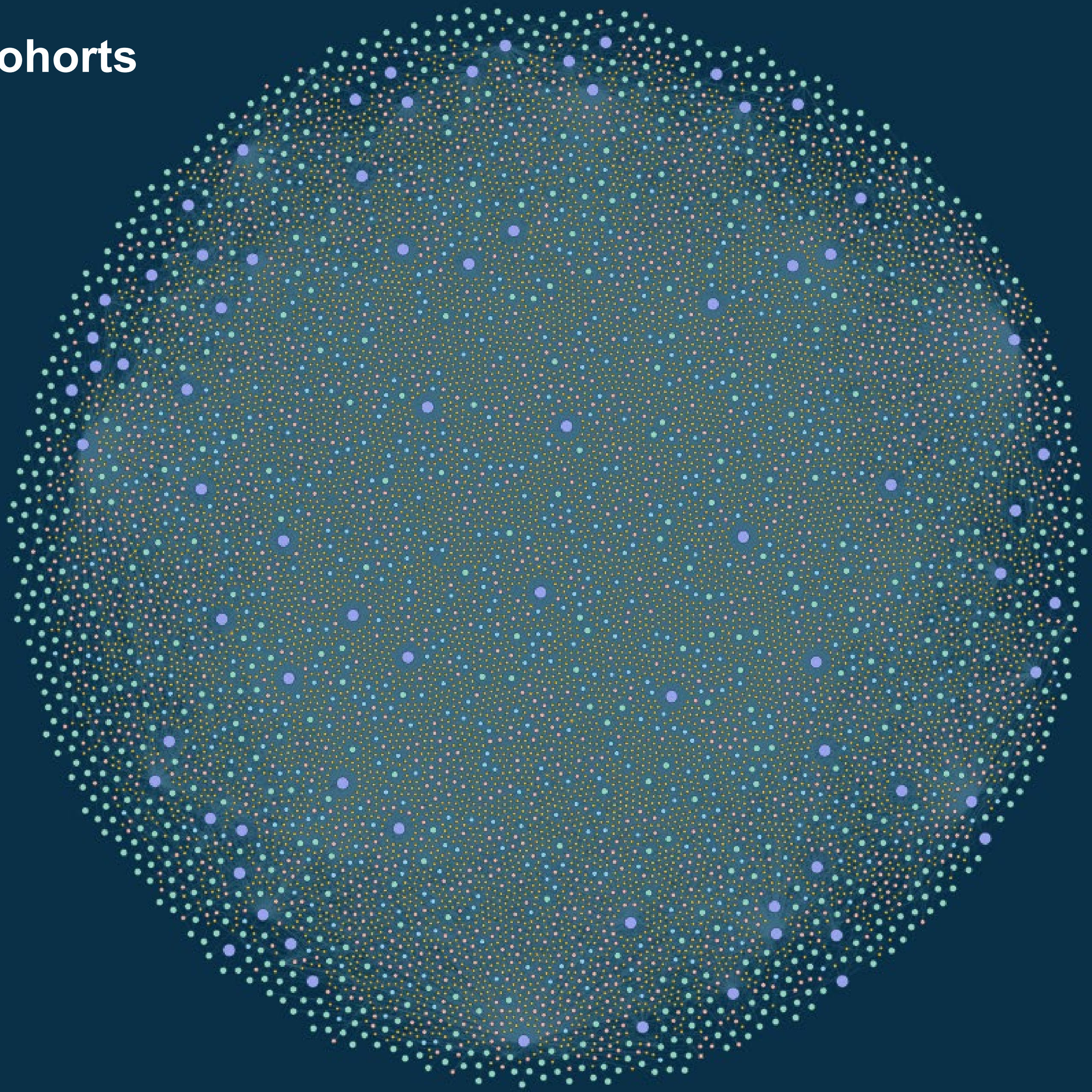
EDT-KG ontology



EDT-KG instantiation (illustrative)



7 majors spanning 6 cohorts from 2012 – 2020
15,901,092 nodes
39,130,256 edges

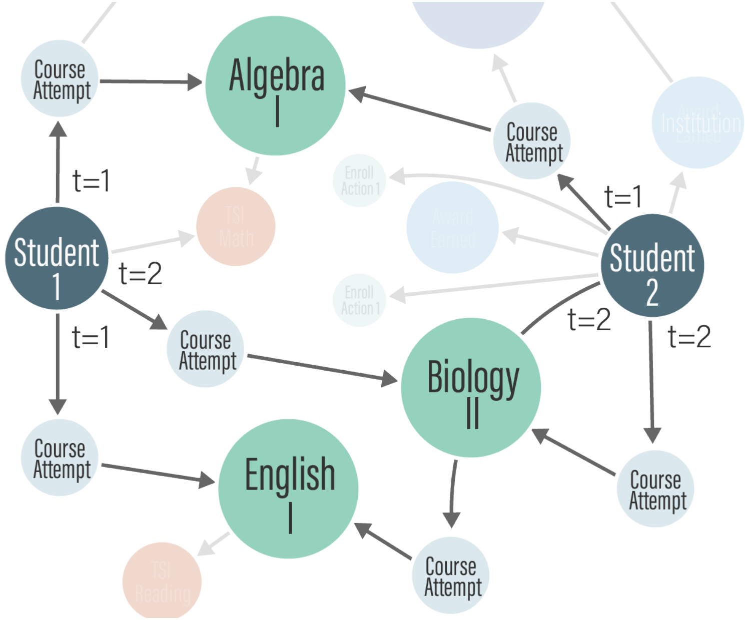


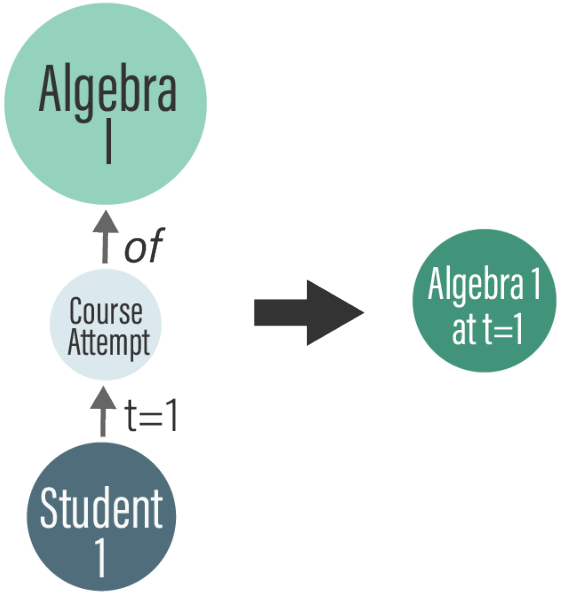
Authors: Luwen Huang and Karen E. Willcox

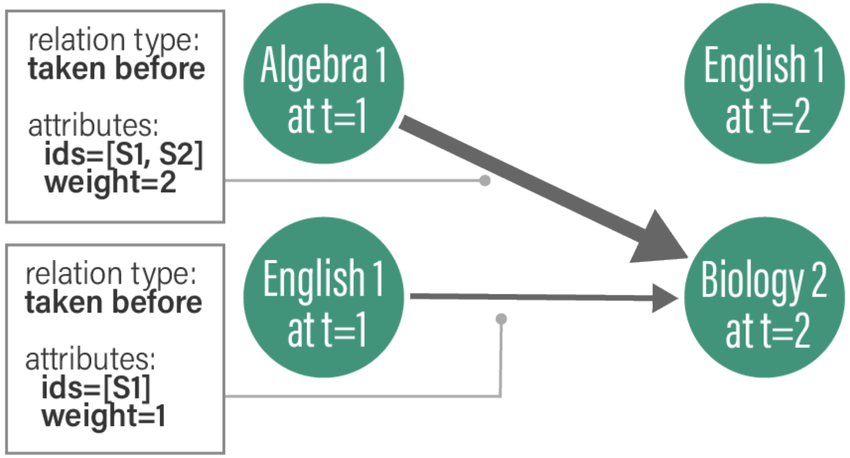
Multiscale Queries: Graph Transformations

All data shown are notional

Enable **scalable** bidirectional flow between physical and digital







Selection

Defines selection of vertices and edges with logical predicates

Projection

Create new vertices and edges based on selection

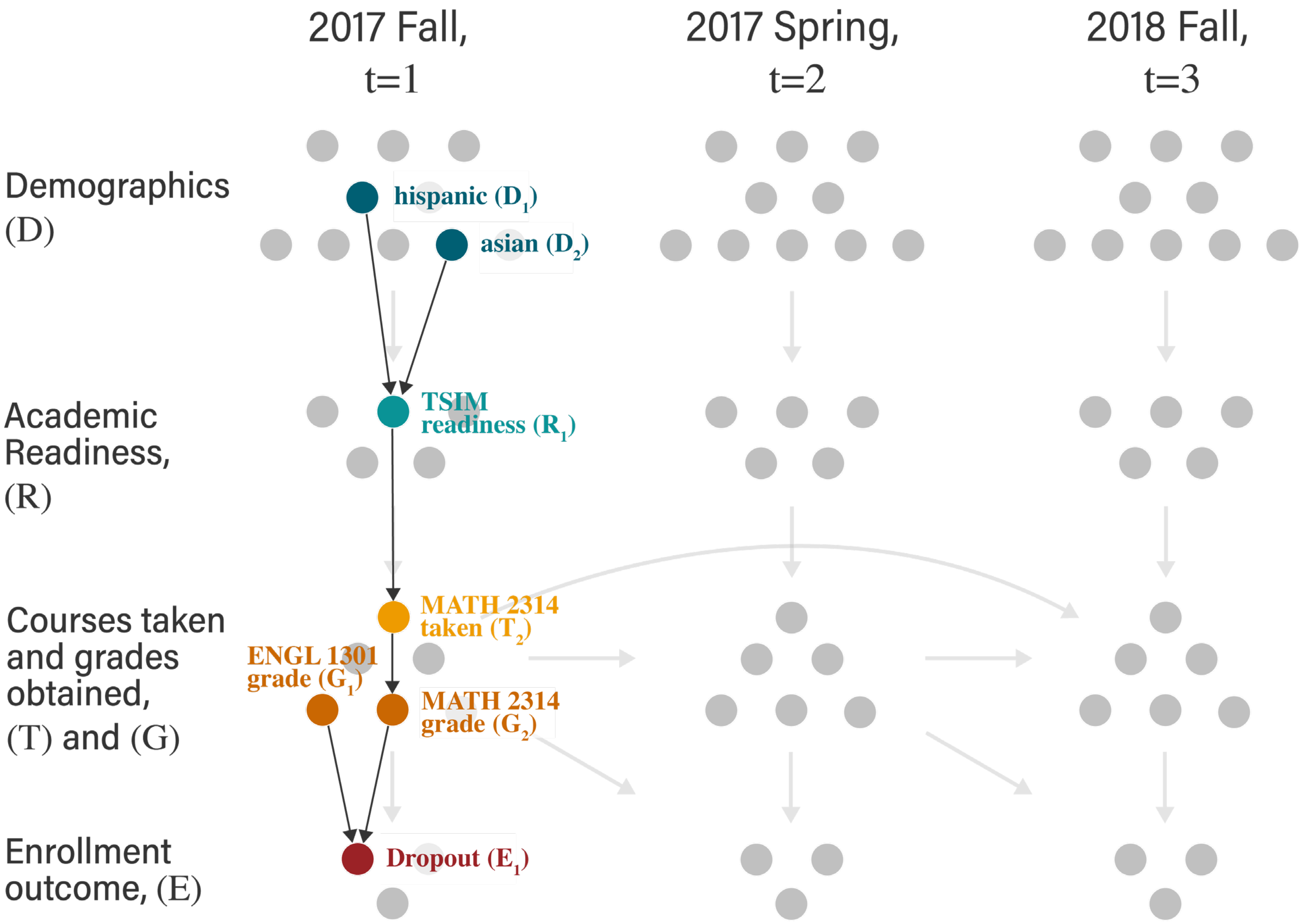
Aggregation

Traverse over selection and projection to attach attributes

The Educational Digital Twin Probabilistic Graphical Model

All quantities are represented as random variables. The PGM represents the joint probability distribution over all variables.

We construct the EDT PGM using domain knowledge to impose structure on the graph (e.g., `MATH 2314` *requires* `MATH 2311`) combined with data-driven learning of the posterior distribution.

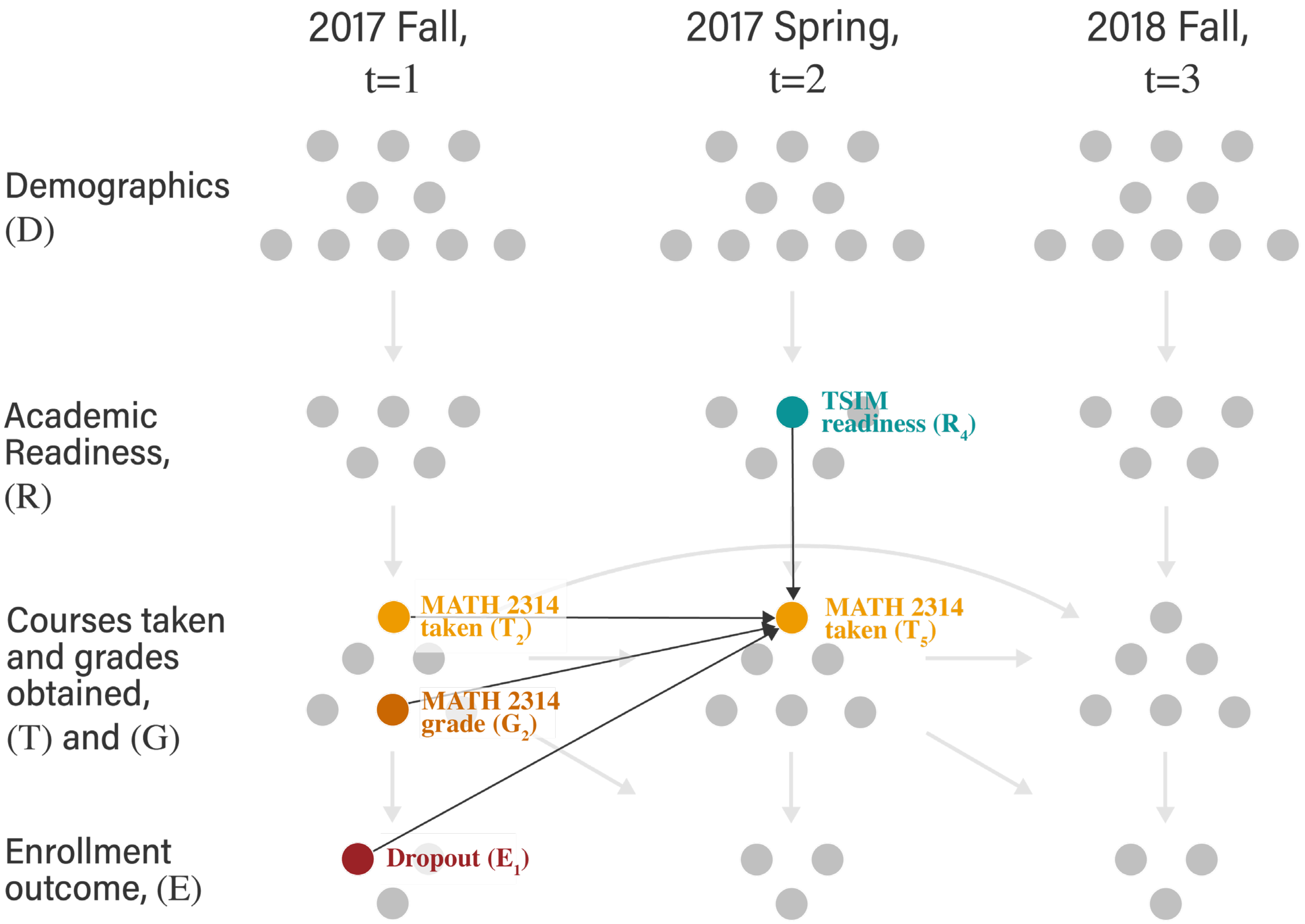


The Educational Digital Twin Probabilistic Graphical Model

Unroll PGM to continue forward in time.

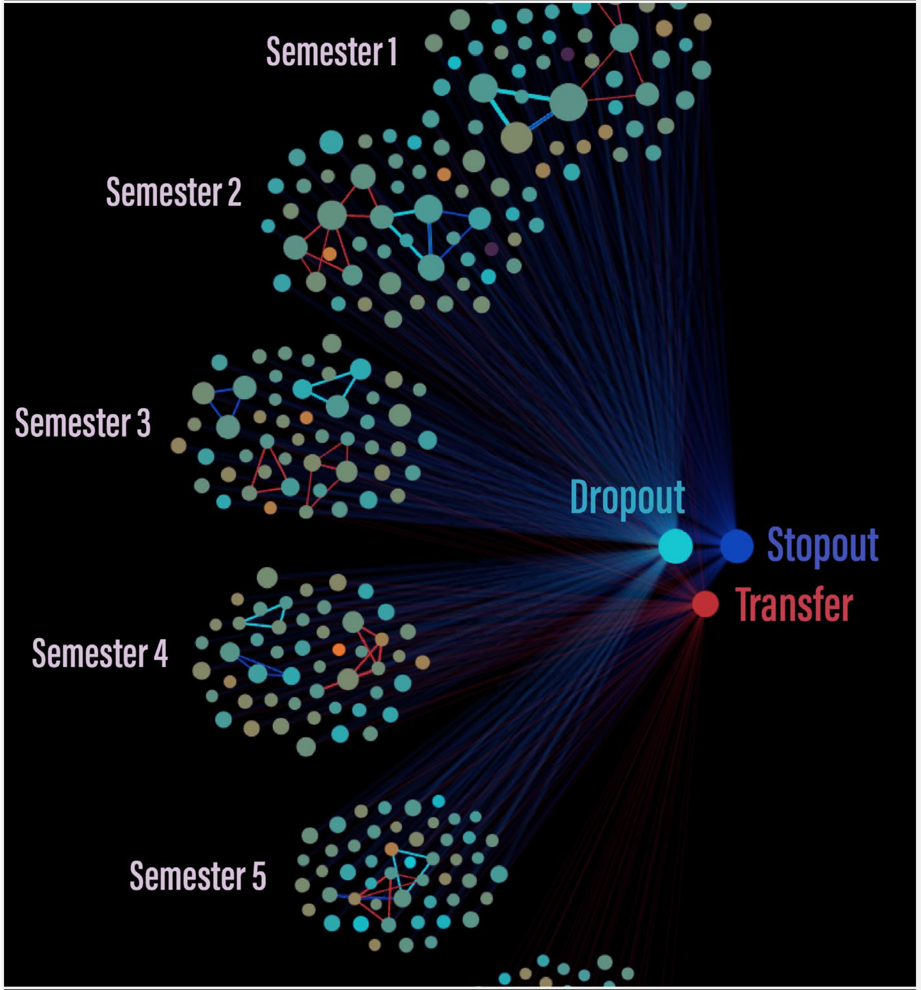
As we unroll more timesteps, use feature engineering to create new summary statistic for courses taken e.g., “max difficulty passed.”

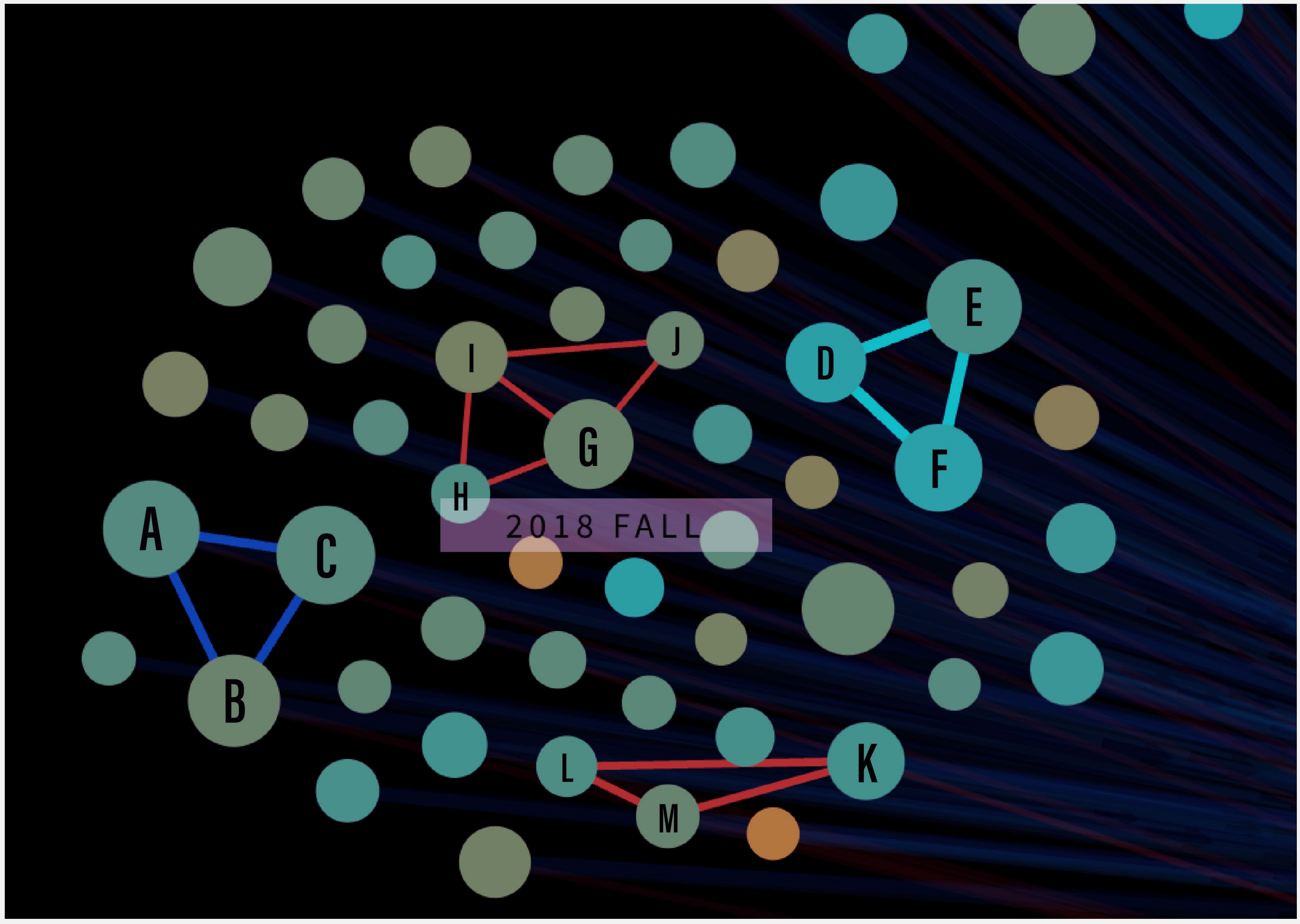
The resulting PGM can also be used to create synthetic student pathways, enabling broader research, tool development, and reproducibility without exposing protected records.



The University of Texas at Austin
Oden Institute for Computational Engineering and Sciences

TRANSFORMATION: COURSE OUTCOMES





DOMAIN KNOWLEDGE

PREDICTIVE PHYSICS-BASED MODELING & SIMULATION

UNCERTAINTY QUANTIFICATION

OPTIMIZATION & CONTROL

HIGH-PERFORMANCE COMPUTING

EDGE COMPUTING

SURROGATE MODELING

INVERSE PROBLEMS

DATA ASSIMILATION

ARTIFICIAL INTELLIGENCE

SCIENTIFIC MACHINE LEARNING

Digital Twins

A scientific grand challenge building on next-generation **mathematical modeling** and **computational science**

Interdisciplinary research & education in computational engineering & sciences

advancing computational science to address society's grand challenges

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ODEN INSTITUTE
FOR COMPUTATIONAL ENGINEERING & SCIENCES

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