A Neural Score-Based Particle Method for the Vlasov-Maxwell-Landau Equation

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IPAM Workshop: Learning Models from Data for Multi-Fidelity Fusion Plasma Physics
April 17, 2026

Joint work with Vasily Ilin (UW)

arXiv:2603.25832

Overview

1. Introduction
   - The Vlasov-Maxwell-Landau (VML) equation
   - Numerical challenge and goal of this work
2. A neural score-based particle method for the VML equation
   - Spatially homogeneous case
   - Spatially inhomogeneous case
   - Time discretization
3. Numerical results
4. Conclusion

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The Vlasov-Maxwell-Landau (VML) equation

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \nu \mathcal{Q}(f)$$

- A **kinetic model** often regarded as first-principles physics for weakly interacting plasmas
- $f = f(t, \mathbf{x}, \mathbf{v})$: electron **probability density function (PDF)**
  — $f \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v}$ gives the probability of finding a fixed electron at time t, position $\mathbf{x} \in \Omega_x \subset \mathbb{R}^3$ and velocity $\mathbf{v} \in \mathbb{R}^3$ in the phase space
- $\mathbf{E}$ and $\mathbf{B}$: **electric and magnetic fields**
  — given externally or determined self-consistently via the Maxwell's equations

$$\partial_t \mathbf{E} = \nabla_{\mathbf{x}} \times \mathbf{B} - \mathbf{J}, \quad \partial_t \mathbf{B} = -\nabla_{\mathbf{x}} \times \mathbf{E}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho - \rho_{\mathrm{i}}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0,$$

  where $\rho = \int_{\mathbb{R}^3} f \, \mathrm{d}\mathbf{v}$ is the charge density and $J = \int_{\mathbb{R}^3} f \mathbf{v} \, \mathrm{d}\mathbf{v}$ is the current density. ρ_{i} is the background ion density.
- $\mathcal{Q}(f)$: **Landau collision operator** (ν is the collision frequency)
  — a nonlinear integro-differential operator modeling Coulomb collisions between charged particles
  — can be derived from the Boltzmann collision operator when all collisions become grazing (i.e., the scattering angle $\to 0$)

The Landau collision operator[1]

$$\mathcal{Q}(f)(t,\mathbf{x},\mathbf{v}) = \nabla_{\mathbf{v}}\cdot \int_{\mathbb{R}^3} A(\mathbf{v}-\mathbf{v}_*)[f(t,\mathbf{x},\mathbf{v}_*)\nabla_{\mathbf{v}} f(t,\mathbf{x},\mathbf{v}) - f(t,\mathbf{x},\mathbf{v})\nabla_{\mathbf{v}_*} f(t,\mathbf{x},\mathbf{v}_*)]\,\mathrm{d}\mathbf{v}_*$$

where A is a semi-positive-definite matrix given by

$$A(\mathbf{v}-\mathbf{v}_*) = |\mathbf{v}-\mathbf{v}_*|^\gamma \left(|\mathbf{v}-\mathbf{v}_*|^2 I_3 - (\mathbf{v}-\mathbf{v}_*)\otimes(\mathbf{v}-\mathbf{v}_*)\right),$$

with $-3 \leq \gamma \leq 1$ ($\gamma = -3$: Coulomb collisions).

A nonlocal, quadratic, diffusive type operator **acting only in the velocity space**

Two equivalent formulations (neglecting $\mathbf{x}$ and t dependence):

- **Rosenbluth-Fokker-Planck form:**

$$\mathcal{Q}(f)(\mathbf{v}) = \nabla_{\mathbf{v}}\cdot(A_f\nabla_{\mathbf{v}} f - a_f f), \quad A_f = A * f, \; a_f = \nabla_{\mathbf{v}}\cdot A_f$$

- **"log" form**

$$\mathcal{Q}(f)(\mathbf{v}) = \nabla_{\mathbf{v}}\cdot \int_{\mathbb{R}^3} A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{v}) - \nabla_{\mathbf{v}_*}\log f(\mathbf{v}_*)] f(\mathbf{v}) f(\mathbf{v}_*)\,\mathrm{d}\mathbf{v}_*$$

[1] Landau, '36.

Properties of the Landau operator

From the **"log" form** one can easily derive the **weak form**: for a test function $\phi(\mathbf{v})$

$$\int_{\mathbb{R}^3} \mathcal{Q}(f)(\mathbf{v})\phi(\mathbf{v})\,\mathrm{d}\mathbf{v}$$
$$= -\frac{1}{2}\iint_{\mathbb{R}^6} [\nabla_{\mathbf{v}}\phi(\mathbf{v}) - \nabla_{\mathbf{v}_*}\phi(\mathbf{v}_*)]^T A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}}\log f(\mathbf{v}) - \nabla_{\mathbf{v}_*}\log f(\mathbf{v}_*)] f(\mathbf{v}) f(\mathbf{v}_*)\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{v}_*$$

- **Conservation** of mass, momentum, and energy:

$$\int_{\mathbb{R}^3} \mathcal{Q}(f)\,\mathrm{d}\mathbf{v} = \int_{\mathbb{R}^3} \mathcal{Q}(f)\,\mathbf{v}\,\mathrm{d}\mathbf{v} = \int_{\mathbb{R}^3} \mathcal{Q}(f)\,|\mathbf{v}|^2\,\mathrm{d}\mathbf{v} = 0$$

- **Decay of entropy** (H-theorem):

$$\int_{\mathbb{R}^3} \mathcal{Q}(f)\,\log f\,\mathrm{d}\mathbf{v} \le 0$$

where "$=$" holds iff f is the **Maxwellian** $\mathcal{M}[f] := \frac{n}{(2\pi T)^{3/2}} e^{-\frac{|\mathbf{v}-\mathbf{u}|^2}{2T}}$

Overview

Particle-in-cell (PIC) method

Due to high dimensionality of the VML equation (6 dimension + time), the predominant numerical method is the **particle-in-cell method**[2]:

$$f^N(t,\mathbf{x},\mathbf{v}) = \sum_{p=1}^{N} w_p \delta(\mathbf{x}-\mathbf{x}_p(t))\delta(\mathbf{v}-\mathbf{v}_p(t))$$

where $\mathbf{x}_p$: particle position, $\mathbf{v}_p$: particle velocity, w_p: particle weight, N: number of particles.

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}_p}{\mathrm{d}t} = \mathbf{v}_p \\ \dfrac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = \mathbf{E}(t,\mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(t,\mathbf{x}_p) \end{cases}$$

— trace $\mathbf{x}_p$, $\mathbf{v}_p$ by following the characteristics and update $\mathbf{E}$ and $\mathbf{B}$ by solving the field equations on a mesh

- Natural to treat the Hamiltonian system (Vlasov equation without collisions)
- "Gold standard" in today's large scale plasma simulations
- Modern PIC can preserve the conservation properties[3]

[2] Birdsall and Langdon, '18. Hockney and Eastwood, '10.

[3] Chen, Chacon, and Barnes, '11. Lapenta, '17. Kraus, Kormann, Morrison, and Sonnendrucker, '17.

Numerical challenge when including collisions

- Coulomb collisions are an integral and important aspect of plasma dynamics but are often neglected due to their high complexity
- Some collisional algorithms:
  - **Monte Carlo Collisions (MCC)**[4]
  - **Direct Simulation Monte Carlo (DSMC)**[5]
  - **Langevin dynamics (SDE)**[6]

  Collisions are added as a separate step via time splitting. Stochastic nature, introducing statistical noise. First-order temporal accuracy at best. Nice properties preserved in the (collisionless) PIC are not guaranteed.

[4] Birdsall, '91.
[5] Takizuka and Abe, '77. Bobylev and Nanbu, '00. Dimarco, Caflisch, and Pareschi, '10.
[6] Manheimer, Lampe, and Joyce, '97.

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Goal of this work: Introduce a particle method to handle Coulomb collisions in plasmas, which can be naturally coupled with the classical PIC to create a structure-preserving particle method for the full Vlasov-Maxwell-Landau system.

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Spatially homogeneous Landau equation

$$\partial_t f = \mathcal{Q}(f) = \nabla_{\mathbf{v}} \cdot \int_{\mathbb{R}^3} A(\mathbf{v} - \mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{v}_*)] f(\mathbf{v}) f(\mathbf{v}_*) \, \mathrm{d}\mathbf{v}_*$$

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$$= \nabla_{\mathbf{v}} \cdot \{\mathbf{U}[f] f\},$$ a transport operator with "collision force" $\mathbf{U}[f]$

where

$$\mathbf{U}[f] := \int_{\mathbb{R}^3} A(\mathbf{v} - \mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{v}_*)] f(\mathbf{v}_*) \, \mathrm{d}\mathbf{v}_*$$

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The particle method seeks a (weak) solution of the form:

$$f^N(t,\mathbf{v}) = \sum_{p=1}^N w_p \delta(\mathbf{v}-\mathbf{v}_p(t)),$$

then $\mathbf{v}_p$ satisfies

$$\frac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = -\mathbf{U}[f^N](\mathbf{v}_p) = -\sum_{q=1}^N w_q A(\mathbf{v}_p - \mathbf{v}_q)[\nabla_{\mathbf{v}} \log f^N(\mathbf{v}_p) - \nabla_{\mathbf{v}} \log f^N(\mathbf{v}_q)]$$

Spatially homogeneous Landau equation

$$\partial_t f = Q(f) = \nabla_{\mathbf{v}} \cdot \int_{\mathbb{R}^3} A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{v}_*)] f(\mathbf{v}) f(\mathbf{v}_*)\, d\mathbf{v}_*$$

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Key observation: The collision force $\mathbf{U}[f^N]$ depends on the distribution function through the score function $\nabla_{\mathbf{v}} \log f^N$.

Score approximation

Therefore, if we know a good approximation $s(\mathbf{v})$ to $\nabla_{\mathbf{v}} \log f^N(\mathbf{v})$, we are done!

$$\frac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = -\mathbf{U}[f^N](\mathbf{v}_p) \approx -\sum_{q=1}^{N} w_q A(\mathbf{v}_p - \mathbf{v}_q)[s(\mathbf{v}_p) - s(\mathbf{v}_q)]$$

Q: Given particle samples $\{\mathbf{v}_p\}_{p=1}^N$, how can we approximate the score $\nabla_{\mathbf{v}} \log f^N$?

[7] Degond and Mustieles, '90.

[8] Carrillo, Craig, and Patacchini, '19.

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Q: Given particle samples $\{\mathbf{v}_p\}_{p=1}^N$, how can we approximate the score $\nabla_{\mathbf{v}} \log f^N$?

- **Traditional particle (blob) method**[7]:

$$s(\mathbf{v}) = \nabla_{\mathbf{v}} \log(\varphi^\varepsilon * f^N),$$

where φ^ε is a mollifier (typically Gaussian). Need to choose the bandwidth ε properly.

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- **Entropy-decaying (blob) particle method**[8]:

  Noting that $\nabla_{\mathbf{v}} \log f = \nabla_{\mathbf{v}} \frac{\delta H}{\delta f}$, where $H(f) = \int f \log f \, \mathrm{d}\mathbf{v}$

$$s(\mathbf{v}) = \nabla_{\mathbf{v}} \frac{\delta H^\varepsilon}{\delta f}[f^N], \quad H^\varepsilon = \int f \log(\varphi^\varepsilon * f) \, \mathrm{d}\mathbf{v}$$

[7] Degond and Mustieles, '90.

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Score approximation

- If we view the problem as **score matching**[9]:

$$\min_{s\in\mathcal{F}} \int |\nabla_{\mathbf{v}} \log f - s|^2 f \, \mathrm{d}\mathbf{v} \quad \Longrightarrow \quad \min_{s\in\mathcal{F}} \sum_{p=1}^{N} w_p [2\nabla_{\mathbf{v}} \cdot s(\mathbf{v}_p) + |s(\mathbf{v}_p)|^2]$$

$$\begin{aligned} \int |\nabla_{\mathbf{v}} \log f - s|^2 f \, \mathrm{d}\mathbf{v} &= \int (|\nabla_{\mathbf{v}} \log f|^2 - 2\nabla_{\mathbf{v}} \log f \cdot s + |s|^2) f \, \mathrm{d}\mathbf{v} \\ &= \int (2\nabla_{\mathbf{v}} \cdot s + |s|^2) f \, \mathrm{d}\mathbf{v} + C \end{aligned}$$

Choosing $\mathcal{F}$ as the neural network leads to the
score-based transport modeling (SBTM)[10]

Remark: The network is pretrained on the known analytic score of the initial condition. At each subsequent time step, perform a fixed number of steps of gradient descent.

[9] Hyvarinen, '05. Maoutsa, Reich, and Opper, '20.
[10] Boffi and Vanden-Eijnden, '23.

Overview

Spatially inhomogeneous Landau operator

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \nu \mathcal{Q}(f)$$

$$\begin{aligned}\mathcal{Q}(f)(\mathbf{x},\mathbf{v}) &= \nabla_{\mathbf{v}} \cdot \int_{\mathbb{R}^3} A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{x},\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{x},\mathbf{v}_*)] f(\mathbf{x},\mathbf{v}) f(\mathbf{x},\mathbf{v}_*)\, \mathrm{d}\mathbf{v}_* \\ &= \nabla_{\mathbf{v}} \cdot \{\mathbf{U}[f] f\}\end{aligned}$$

where

$$\mathbf{U}[f](\mathbf{x},\mathbf{v}) := \int_{\mathbb{R}^3} A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{x},\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{x},\mathbf{v}_*)] f(\mathbf{x},\mathbf{v}_*)\, \mathrm{d}\mathbf{v}_*$$

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For a particle solution of the form

$$f^N(t,\mathbf{v}) = \sum_{p=1}^{N} w_p \delta(\mathbf{x} - \mathbf{x}_p(t)) \delta(\mathbf{v} - \mathbf{v}_p(t)),$$

$\mathbf{v}_p$ satisfies

$$\frac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = \mathbf{E}(t,\mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(t,\mathbf{x}_p) - \nu \underbrace{\mathbf{U}[f^N](\mathbf{x}_p,\mathbf{v}_p)}_{??}$$

Regularized Landau operator

$$\mathbf{U}[f](\mathbf{x},\mathbf{v}) = \int_{\mathbb{R}^3} A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{x},\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{x},\mathbf{v}_*)] f(\mathbf{x},\mathbf{v}_*) \, d\mathbf{v}_*$$

We propose to regularize $\mathbf{U}[f]$ as[11]

$$\mathbf{U}^h[f](\mathbf{x},\mathbf{v}) = \iint_{\Omega_x \times \mathbb{R}^3} \psi_h(\mathbf{x}-\mathbf{x}_*) A(\mathbf{v}-\mathbf{v}_*)[\nabla_{\mathbf{v}} \log f(\mathbf{x},\mathbf{v}) - \nabla_{\mathbf{v}_*} \log f(\mathbf{x}_*,\mathbf{v}_*)] \times f(\mathbf{x}_*,\mathbf{v}_*) \, d\mathbf{v}_* \, d\mathbf{x}_*$$

- When considering collisions at a spatial point $\mathbf{x}$, we first *localize* particles to the neighborhood of $\mathbf{x}$; then within this neighborhood, we *delocalize* to associate each particle with a distinct position.
- We choose $\psi_h(\mathbf{x})$ as the B-spline (h is the cell size)

[11] Bailo, Carrillo, and H., '24.

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Now if we plug in f^N, we obtain

$$\mathbf{U}^h[f^N](\mathbf{x}_p,\mathbf{v}_p) = \sum_{q=1}^{N} w_q \psi_h(\mathbf{x}_p-\mathbf{x}_q) A(\mathbf{v}_p-\mathbf{v}_q)[\nabla_{\mathbf{v}} \log f^N(\mathbf{x}_p,\mathbf{v}_p) - \nabla_{\mathbf{v}} \log f^N(\mathbf{x}_q,\mathbf{v}_q)]$$

[11] Bailo, Carrillo, and H., '24.

Replacing $\nabla_{\mathbf{v}} \log f^N$ by the score approximation...

We have a simple neural score-based particle method for the VML equation[12]:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}_p}{\mathrm{d}t} = \mathbf{v}_p \\ \dfrac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = \mathbf{E}(t, \mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(t, \mathbf{x}_p) - \nu \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) \end{cases}$$

where

$$\mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) = \sum_{q=1}^{N} w_q \psi_h(\mathbf{x}_p - \mathbf{x}_q) A(\mathbf{v}_p - \mathbf{v}_q)[s(\mathbf{x}_p, \mathbf{v}_p) - s(\mathbf{x}_q, \mathbf{v}_q)]$$

- Learn $s(\mathbf{x}_p, \mathbf{v}_p)$ from $\{\mathbf{x}_p, \mathbf{v}_p\}_{p=1}^N$

$$\min_{s \in \mathcal{F}} \int_{\Omega_x \times \mathbb{R}^3} |\nabla_{\mathbf{v}} \log f - s|^2 f \, \mathrm{d}\mathbf{v} \, \mathrm{d}\mathbf{x} \implies \min_{s \in \mathcal{F}} \sum_{p=1}^{N} w_p [2 \nabla_{\mathbf{v}} \cdot s(\mathbf{x}_p, \mathbf{v}_p) + |s(\mathbf{x}_p, \mathbf{v}_p)|^2]$$

- Compared with the traditional particle (blob) method, it does not require regularization/mollification in velocity space

$$s(\mathbf{x}_p, \mathbf{v}_p) = \nabla_{\mathbf{v}} \log \left(\sum_{q=1}^{N} w_q \varphi_\varepsilon(\mathbf{v}_p - \mathbf{v}_q) \psi_h(\mathbf{x}_p - \mathbf{x}_q) \right)$$

[12] Ilin and H., 2026.

Preserving the properties

For the regularized "collision force" $\mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p)$, one can show that for a test function $\phi(\mathbf{x}, \mathbf{v})$:

$$\sum_p w_p \, \phi(\mathbf{x}_p, \mathbf{v}_p) \cdot \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) = \frac{1}{2} \sum_{p,q} w_p w_q \psi_h(\mathbf{x}_p - \mathbf{x}_q) [\phi(\mathbf{x}_p, \mathbf{v}_p) - \phi(\mathbf{x}_q, \mathbf{v}_q)]^T \times A(\mathbf{v}_p - \mathbf{v}_q) \, [s(\mathbf{x}_p, \mathbf{v}_p) - s(\mathbf{x}_q, \mathbf{v}_q)]$$

— This is the discrete weak form of the regularized Landau operator.

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— This is the discrete weak form of the regularized Landau operator.

- **Conservation** of momentum and energy:

$$\sum_p w_p \, \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) = 0, \quad \sum_p w_p \, \mathbf{v}_p \cdot \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) = 0$$

- **Estimated entropy decay**:

$$-\sum_p w_p \, s(\mathbf{x}_p, \mathbf{v}_p) \cdot \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) \leq 0$$

Overview

How about the time discretization?

We have obtained a (semi-discrete) particle method for the VML equation:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}_p}{\mathrm{d}t} = \mathbf{v}_p \\ \dfrac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = \mathbf{E}(t, \mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(t, \mathbf{x}_p) - \nu \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) \end{cases}$$

Provided the Maxwell's equations are discretized properly (e.g., Yee's lattice), this scheme can conserve the total (semi-discrete) energy of the system:

$$E_{\text{total}} := \sum_{p=1}^{N} w_p \frac{1}{2}|\mathbf{v}_p|^2 + \sum_h \left(\frac{1}{2}|\mathbf{E}(t, \mathbf{x}_h)|^2 + \frac{1}{2}|\mathbf{B}(t, \mathbf{x}_h)|^2 \right) h^3 = \text{const}$$

How about the time discretization?

We have obtained a (semi-discrete) particle method for the VML equation:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}_p}{\mathrm{d}t} = \mathbf{v}_p \\ \dfrac{\mathrm{d}\mathbf{v}_p}{\mathrm{d}t} = \mathbf{E}(t, \mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(t, \mathbf{x}_p) - \nu \mathbf{U}^h[f^N](\mathbf{x}_p, \mathbf{v}_p) \end{cases}$$

Provided the Maxwell's equations are discretized properly (e.g., Yee's lattice), this scheme can conserve the total (semi-discrete) energy of the system:

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Q: Can we design a time discretization such that the fully discrete scheme is energy-conserving?

— An implicit scheme can achieve so (e.g., implicit midpoint); however, collisions are the most computationally expensive part of the simulation and are better treated explicitly.

An explicit energy-conserving time discretization[13]

$$
\begin{aligned}
\mathbf{x}_p^* &= \mathbf{x}_p^n + \frac{\Delta t}{2}\mathbf{v}_p^n,\\
\mathbf{v}_p^* &= \mathbf{v}_p^n + \frac{\Delta t}{2}\mathbf{E}_p^{n,*} - \frac{\Delta t}{2}\nu \mathbf{U}^h[f^N](\mathbf{x}_p^*, \mathbf{v}_p^n),\\
\mathbf{x}_p^{n+1} &= \mathbf{x}_p^n + \Delta t\, \mathbf{v}_p^*,\\
\mathbf{E}_h^{n+1} &= \mathbf{E}_h^n - \Delta t\, \mathbf{J}_h^{*,*},\\
\mathbf{v}_p^\dagger &= \mathbf{v}_p^n + \Delta t \mathbf{E}_p^{n+\frac{1}{2},*} - \Delta t \nu \mathbf{U}^h[f^N](\mathbf{x}_p^*, \mathbf{v}_p^*),\\
\mathbf{v}_p^{n+1} &= \Gamma_p^n \mathbf{v}_p^\dagger, \quad \Gamma_p^n = \sqrt{1 + 2(\mathbf{v}_p^* - \frac{\mathbf{v}_p^\dagger + \mathbf{v}_p^n}{2}) \cdot \left(\mathbf{v}_p^\dagger - \mathbf{v}_p^n\right) / |\mathbf{v}_p^\dagger|^2}.
\end{aligned}
$$

Can show that $E_{\text{total}}^{n+1} = E_{\text{total}}^n$ – a key property used to achieve energy conservation

$$\sum_p w_p\, \mathbf{v}_p^* \cdot \mathbf{U}^h[f^N](\mathbf{x}_p^*, \mathbf{v}_p^*) = 0 \Longleftarrow \text{ exactly satisfied by our particle method!}$$

Remark: Second order accuracy; can be extended to the electromagnetic case.

[13] Yoo, H., and Ricketson, '25 (arXiv:2510.03960).

Overview

Numerical results – Landau damping

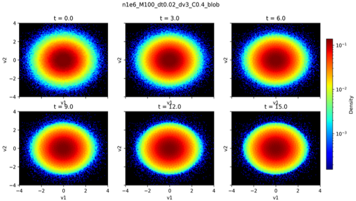
We initialize the plasma as

$$f_0(x, \mathbf{v}) = \frac{1 + \alpha \cos(kx)}{(2\pi)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2}\right)$$

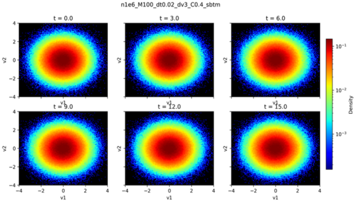
with wavenumber $k = 0.5$ and domain $L = 2\pi/k$, $\alpha = 0.1$, $M = 100$ spatial cells, $\Delta t = 0.2$, $\nu = 0.4$, $t_{\text{final}} = 15$, $K = 100$ gradient steps, sweeping over particle numbers $N = 5 \times 10^5, 10^6, 3 \times 10^6$.

We compare SBTM and the blob method. Forward Euler is used for simplicity.

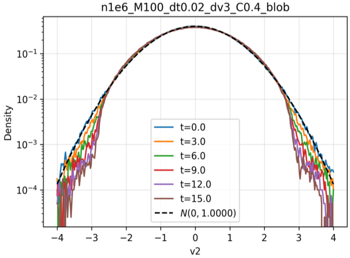
Numerical results – Landau damping



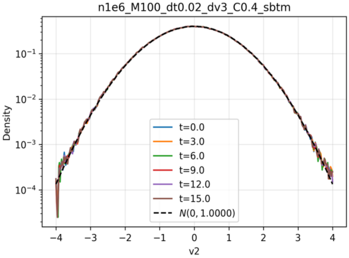
(a) Blob, $\nu = 0.4$



(b) SBTM, $\nu = 0.4$



(c) Blob, v_2-marginal (log), $\nu = 0.4$



(d) SBTM, v_2-marginal (log), $\nu = 0.4$

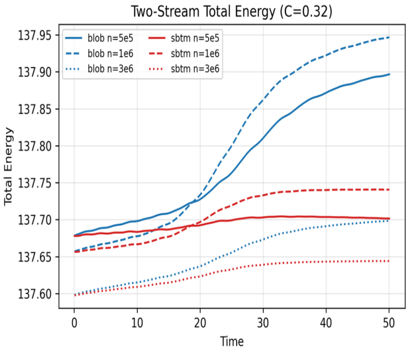
Numerical results – Two stream instability

We initialize the plasma as

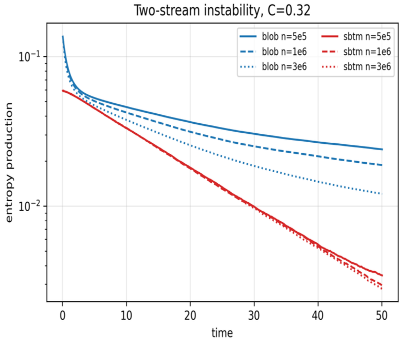
$$f_0(x, \mathbf{v}) = \frac{1 + \alpha \cos(kx)}{2} [\mathcal{N}(v_1; c, 1) + \mathcal{N}(v_1; -c, 1)] \prod_{j=2,3} \mathcal{N}(v_j; 0, 1)$$

with $c = 2.4$, wavenumber $k = 1/5$ and domain $L = 2\pi/k$, $\alpha = 1/200$, $M = 100$ spatial cells, $\Delta t = 0.05$, $\nu = 0.32$, $t_{\text{final}} = 50$, $K = 100$ gradient steps, sweeping over particle numbers $N = 5 \times 10^5, 10^6, 3 \times 10^6$.

Numerical results – Two stream instability

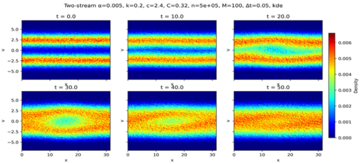


(c) Total energy

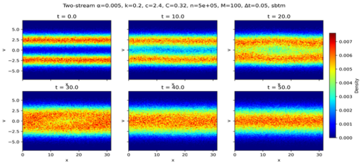


(d) Estimated entropy production

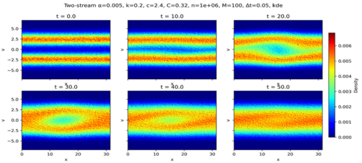
Numerical results – Two stream instability



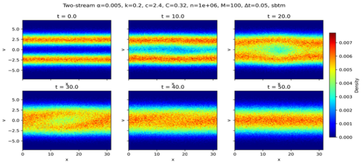
(a) Blob, $n = 5 \times 10^5$



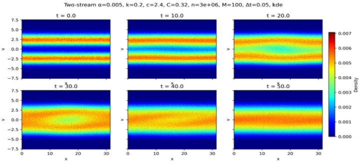
(b) SBTM, $n = 5 \times 10^5$



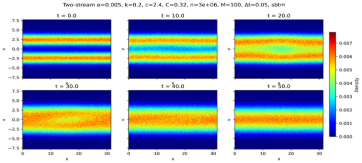
(c) Blob, $n = 10^6$



(d) SBTM, $n = 10^6$



(e) Blob, $n = 3 \times 10^6$



(f) SBTM, $n = 3 \times 10^6$

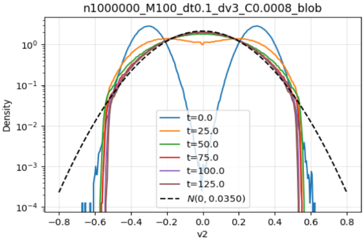
Numerical results – Weibel instability

We initialize the plasma as

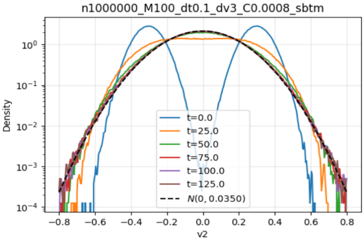
$$f_0(x,\mathbf{v}) \propto \exp\left(-\frac{v_1^2}{\beta}\right)\left[\exp\left(-\frac{(v_2-c)^2}{\beta}\right)+\exp\left(-\frac{(v_2+c)^2}{\beta}\right)\right]\exp\left(-\frac{v_3^2}{\beta}\right)$$

with $c = 0.3$, $\beta = 0.01$, $B_3(0,x) = \alpha_B \sin(kx)$, $\alpha_B = 10^{-3}$, wavenumber $k = 1/5$ and domain $L = 2\pi/k$, $M = 100$ spatial cells, $\Delta t = 0.1$, $N = 10^6$, $t_{\text{final}} = 125$, $K = 100$ gradient steps, sweeping over collision frequencies $\nu = 10^{-4}, 2\times 10^{-4}, 4\times 10^{-4}, 8\times 10^{-4}$.

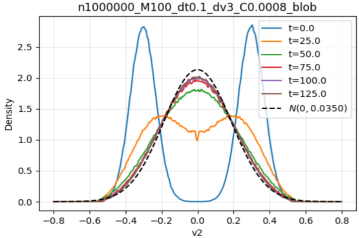
Numerical results – Weibel instability



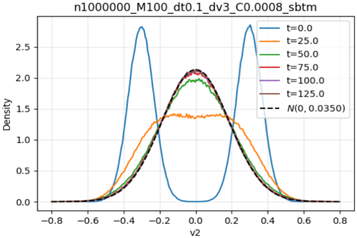
(a) Blob, log scale



(b) SBTM, log scale

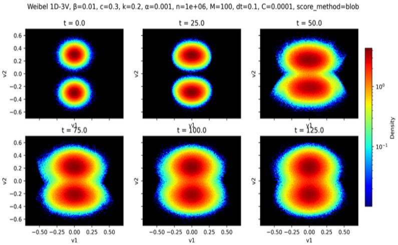


(c) Blob, linear scale

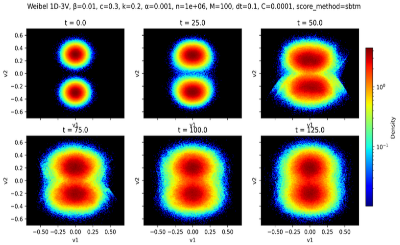


(d) SBTM, linear scale

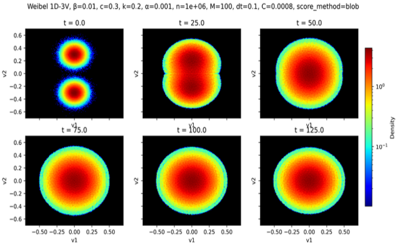
Numerical results – Weibel instability



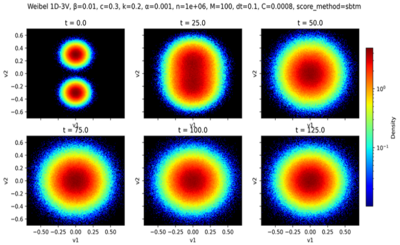
(a) Blob, $\nu = 10^{-4}$



(b) SBTM, $\nu = 10^{-4}$

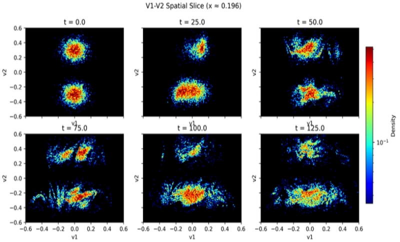


(c) Blob, $\nu = 8 \times 10^{-4}$

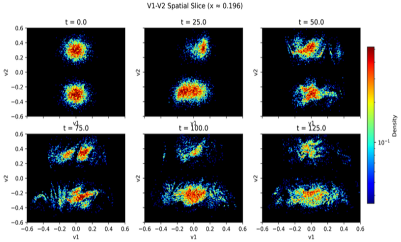


(d) SBTM, $\nu = 8 \times 10^{-4}$

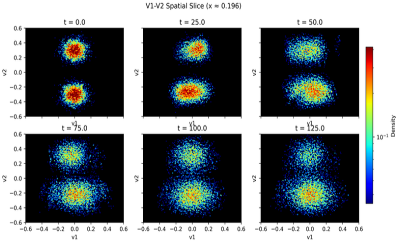
Numerical results – Weibel instability



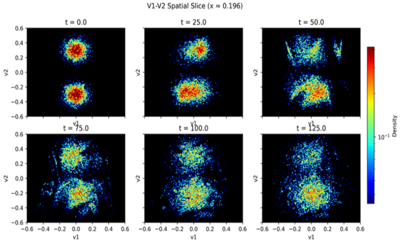
(e) Blob, $\nu = 0$



(f) SBTM, $\nu = 0$



(g) Blob, $\nu = 10^{-4}$



(h) SBTM, $\nu = 10^{-4}$

Numerical results – computational efficiency

Method	**Runtime** ($n = 10^6$) Time	Speedup	**Peak Memory (GB)** $n = 10^6$	$n = 3 \times 10^6$
Blob (KDE)	11h 27m	–	16.4	65.1
SBTM	7h 17m	1.57×	4.4	17.2

Figure: This table compares runtime and peak GPU memory for the two-stream instability test on the same architecture. SBTM achieves a 1.57×speedup at $N = 10^6$ and uses $2 - 4\times$less memory across all particle counts, with the memory advantage growing at larger N.

Overview

Conclusion

A particle method is introduced for the Vlasov-Maxwell-Landau equation:

- offers a unified approach to handle transport, field effects, and collisions in a plasma system
- replaces the traditional particle/blob method by the neural score matching
- preserves the conservation and entropy decay structure of the Landau operator

arXiv:2603.25832

Thank you!