

Implicit Brain Imaging

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F. Memoli, G. S., and P. Thompson,

“Implicit Brain Imaging”

Supported by NSF, ONR, PECASE, CAREER, NIH

Implicit Form Representation

S = level – set of $\Psi : R^n \rightarrow R = \{x : \Psi(x) = 0\}$

$$\Psi(x, y) = x^2 + y^2 - 1$$

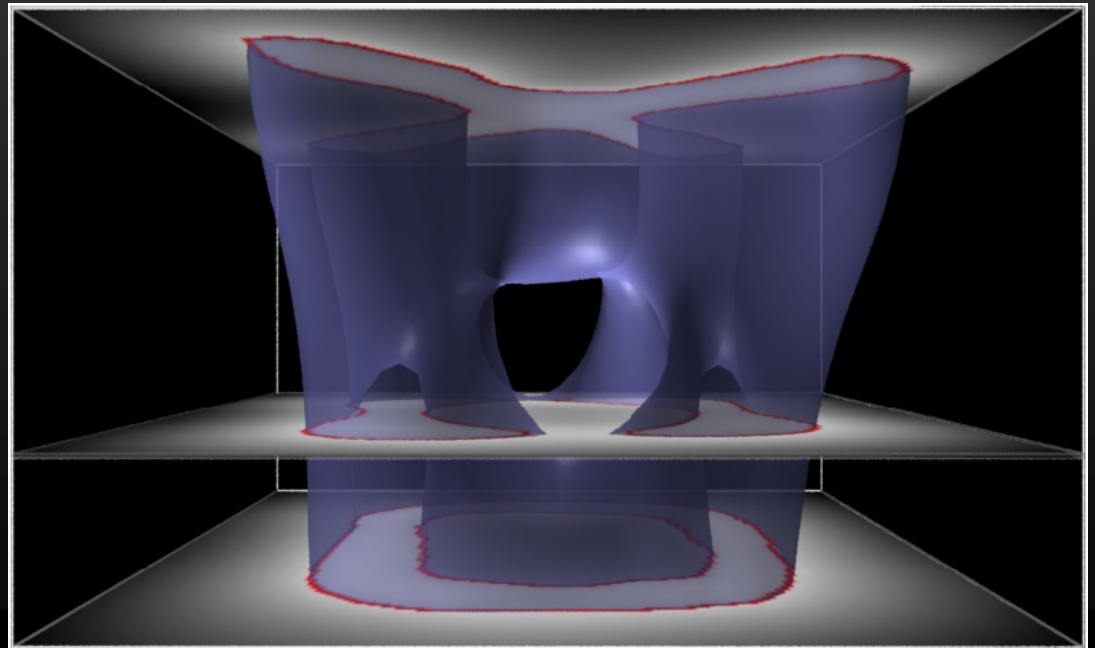


Figure from G. Turk

Overview

- **Motivation**
- **Geodesics on implicit brains**
- **Data regularization on implicit brains**
- **Mapping between implicit surfaces**
- **The future and concluding remarks**

Motivation:

Why implicit surfaces?

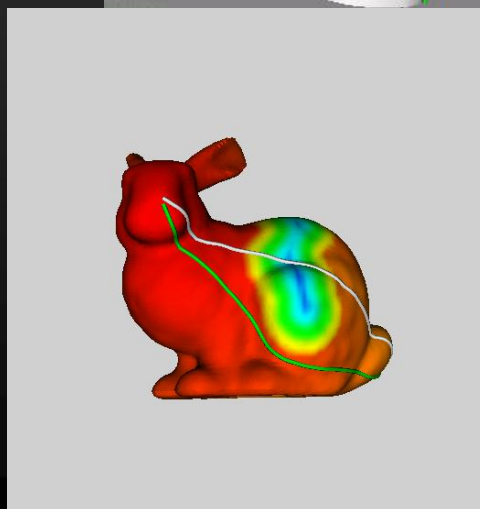
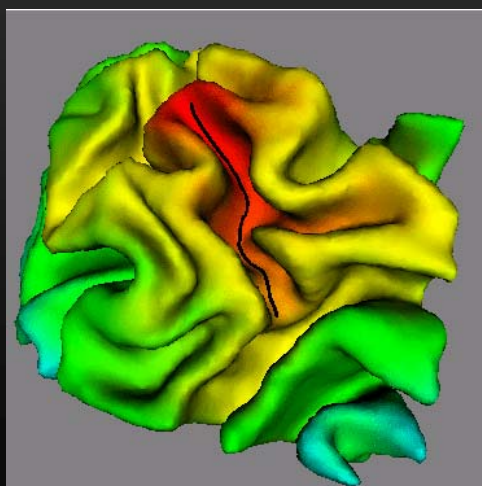
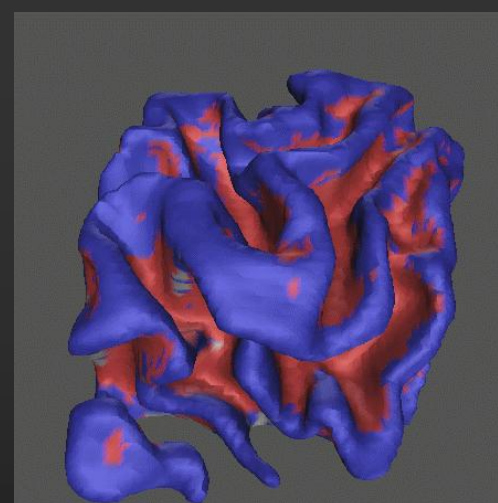
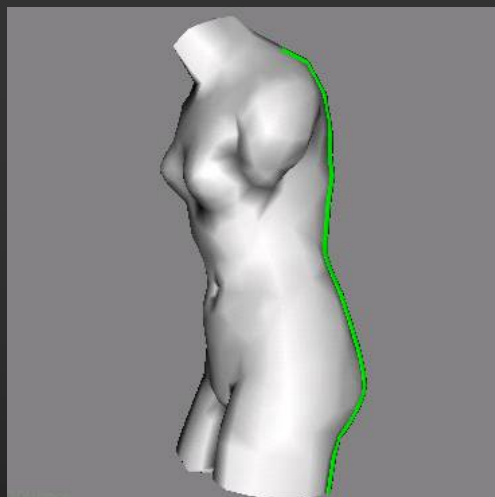
- **Natural representation for state-of-the-art segmentation algorithms**
- **Facilitate fundamental computations**
 - Simple differential geometry
 - Avoids intermediate projections/parameterizations
- **Dimensionality independent (17D brains 😊)**
- **Pure geometry (no artificial meshes, etc)**
- **Needed anyhow for projections, collision, etc**

Geodesics on Implicit Brains

Facundo Memoli and GS

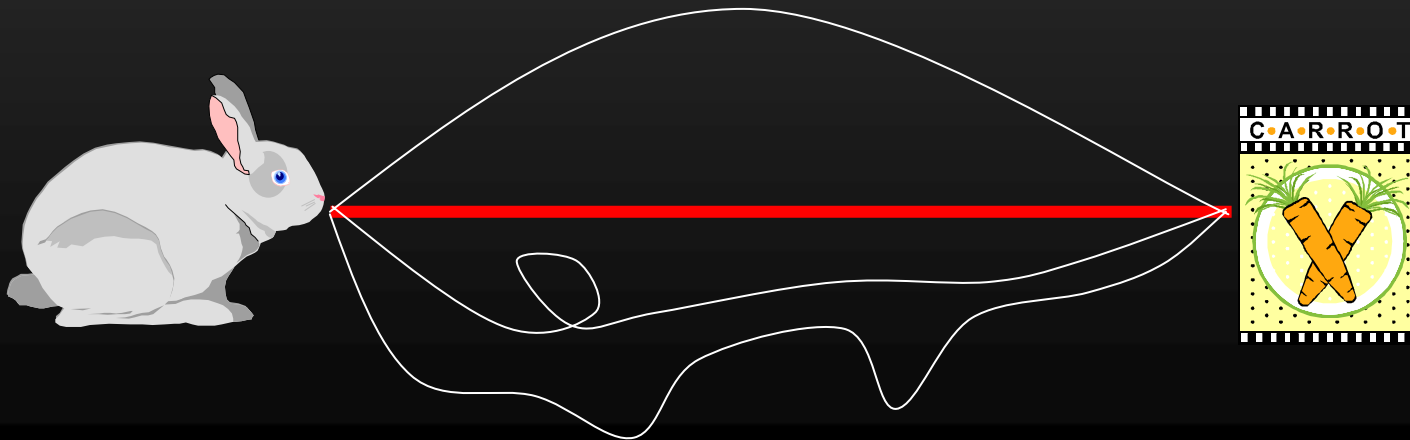
Journal of Computational Physics, 2002

Motivation: A Few Examples



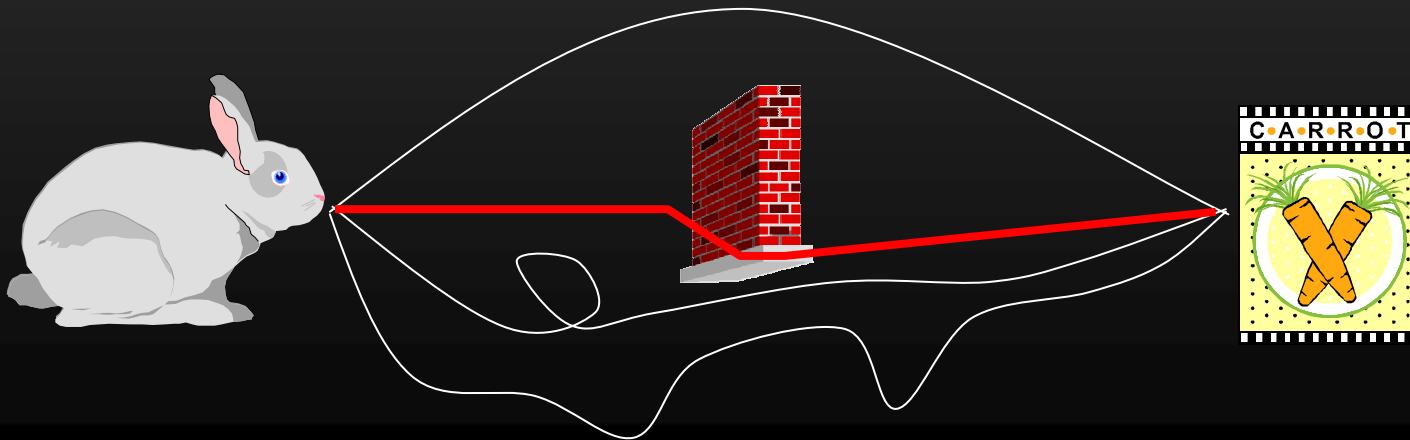
Motivation: What is a Geodesic?

$$d_s^g(p, x) = \inf_C \int_p^x g(C) ds$$

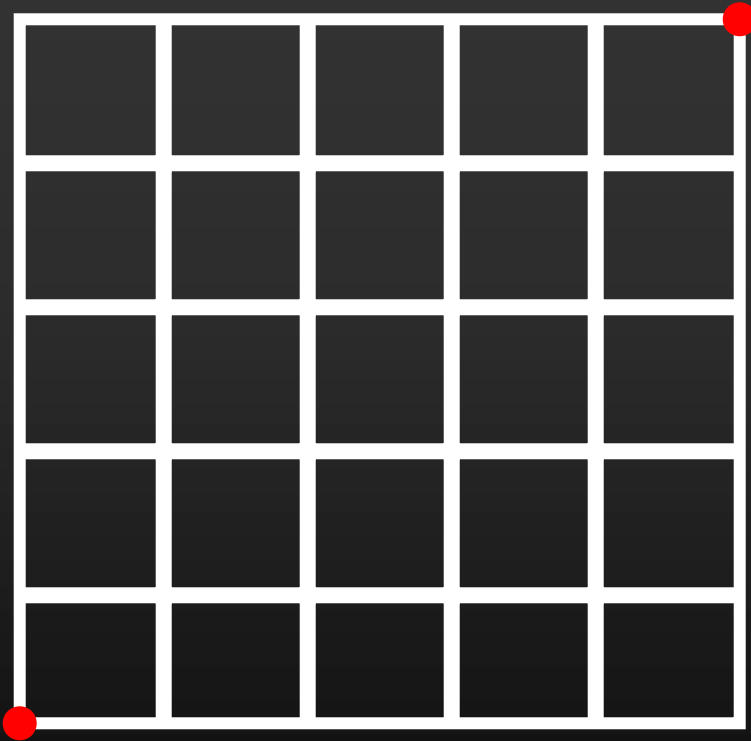


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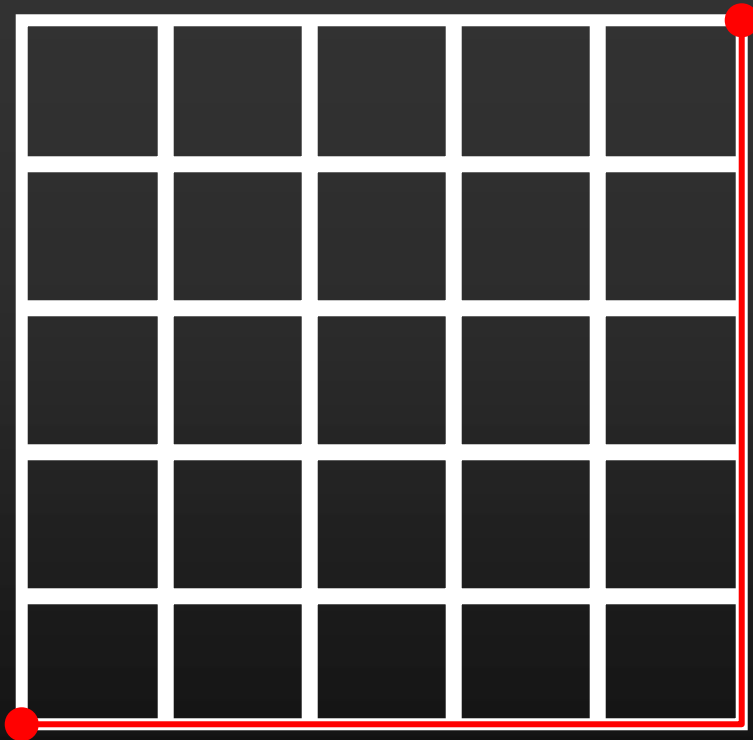


Background: Distance and Geodesic Computation via Dijkstra



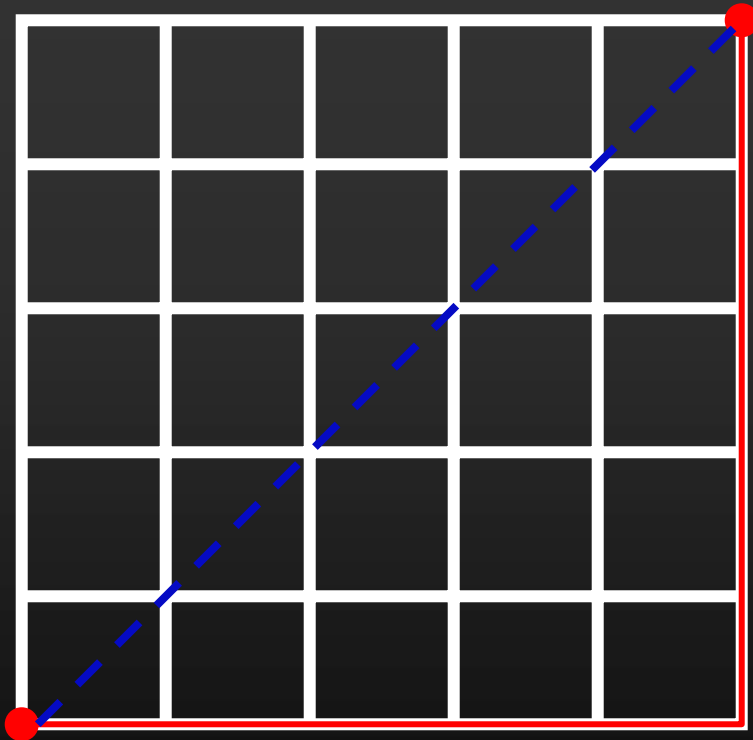
- **Complexity:** $O(n \log n)$
- **Advantage:** Works in any dimension and with any geometry (graphs)
- **Problems:**
 - Not consistent
 - Unorganized points?
 - Noise?
 - Implicit surfaces?

Background: Distance and Geodesic Computation via Dijkstra



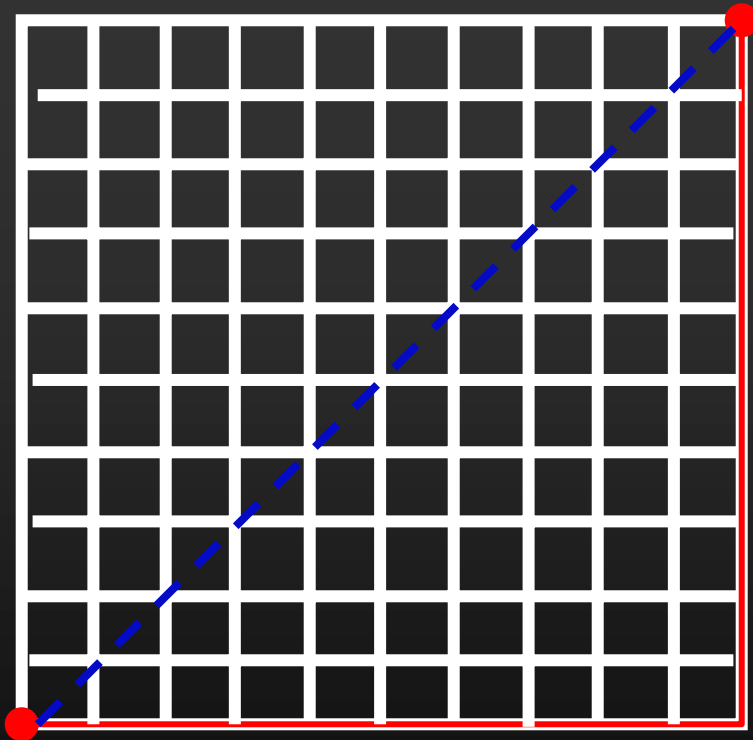
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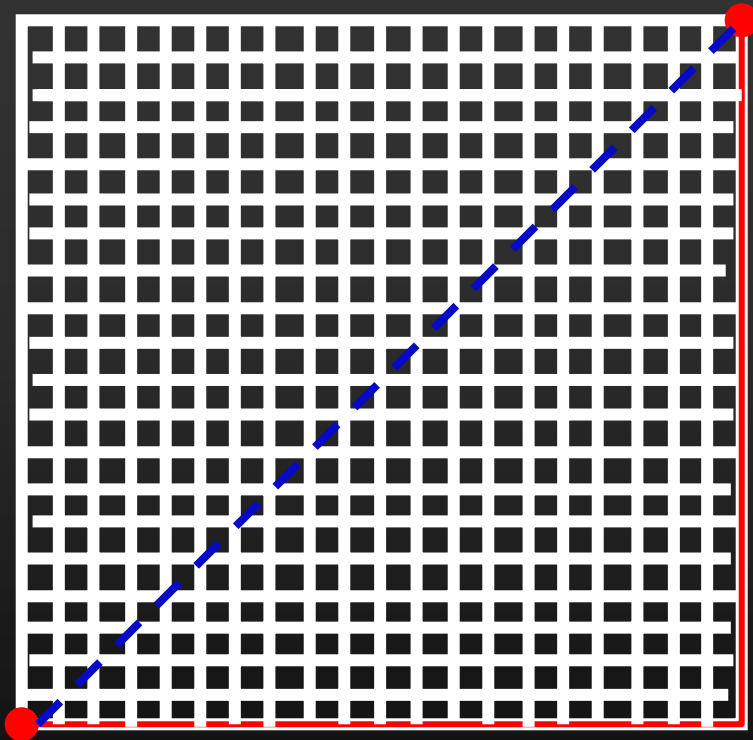
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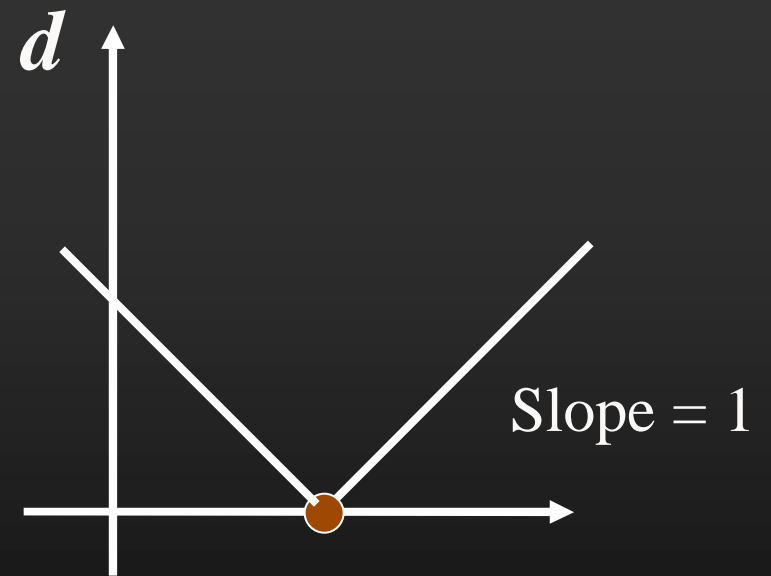
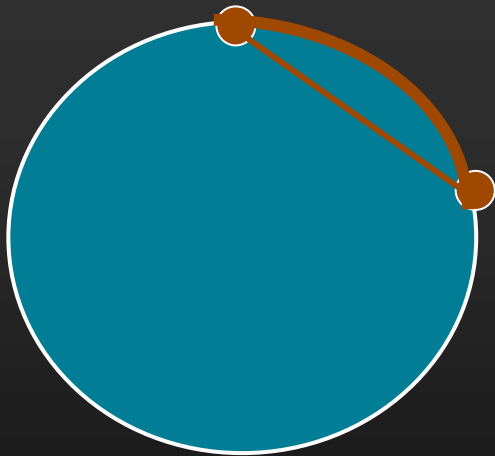
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Background: Distance Functions as Hamilton-Jacobi Equations

- g = weight on the hyper-surface
- The g -weighted distance function between two points p and x on the hyper-surface S is:

$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$

$$\|\nabla_S d_S^g(p, x)\| = g$$

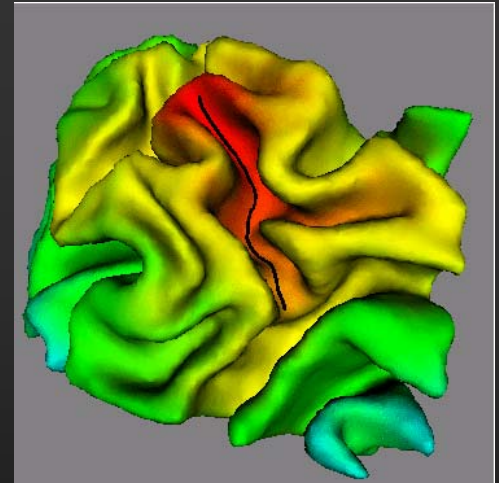
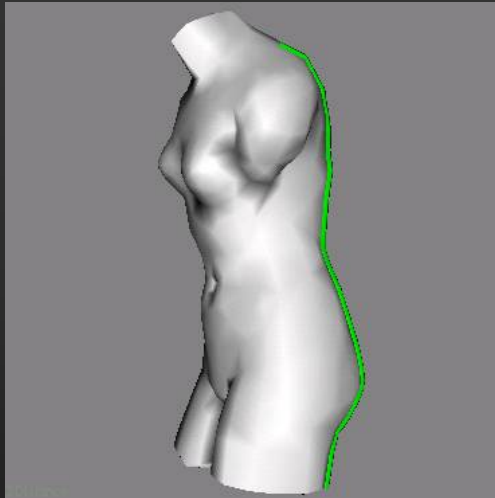


Background: Computing Distance Functions as Hamilton-Jacobi Equations

- Solved in $O(n \log n)$ by Tsitsiklis, by Sethian, and by Helmsen, **only** for Euclidean spaces and Cartesian grids

$$\left\| \nabla d^g(p, x) \right\| = g$$

- Solved **only** for acute 3D triangulations by Kimmel and Sethian



A real time example

The Problem

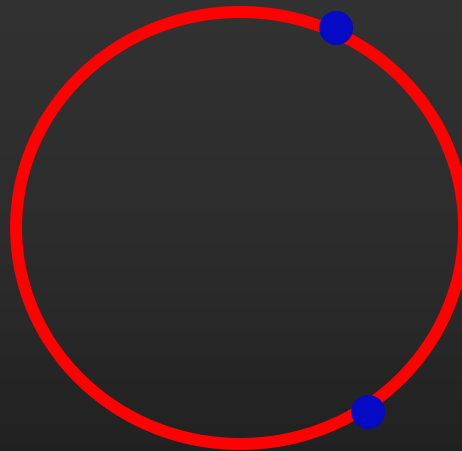
- **How to compute intrinsic distances and geodesics for**
 - General dimensions
 - Implicit surfaces
 - Unorganized noisy points (hyper-surfaces just given by examples)

Our Approach

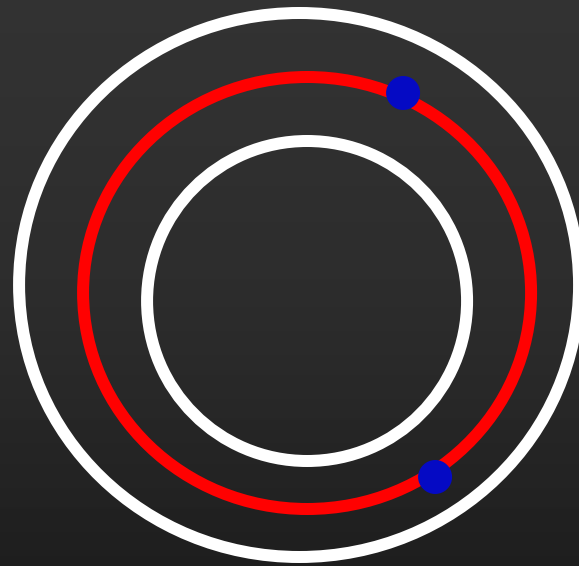
- We have to solve

$$\left\| \nabla_s d_s^g(p, x) \right\| = g$$

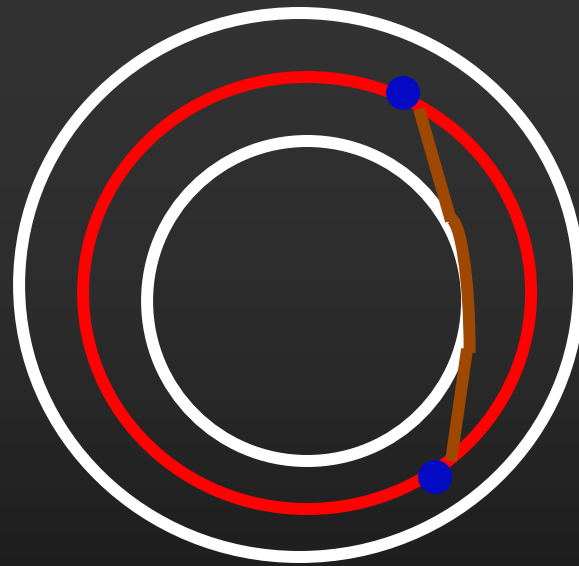
Basic Idea



Basic Idea



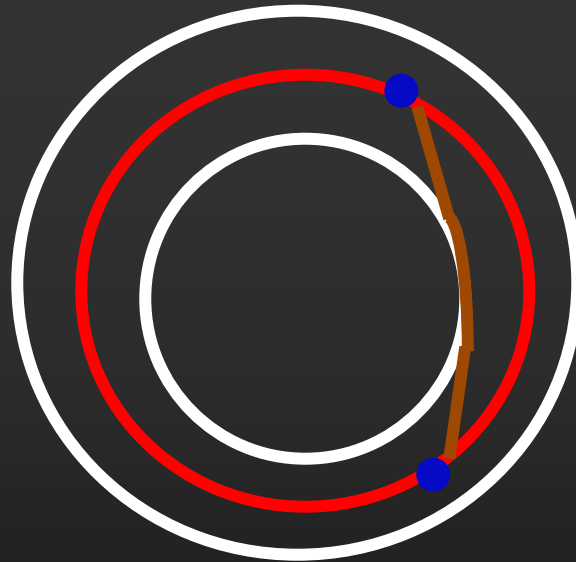
Basic Idea



Theorem (Memoli-S):

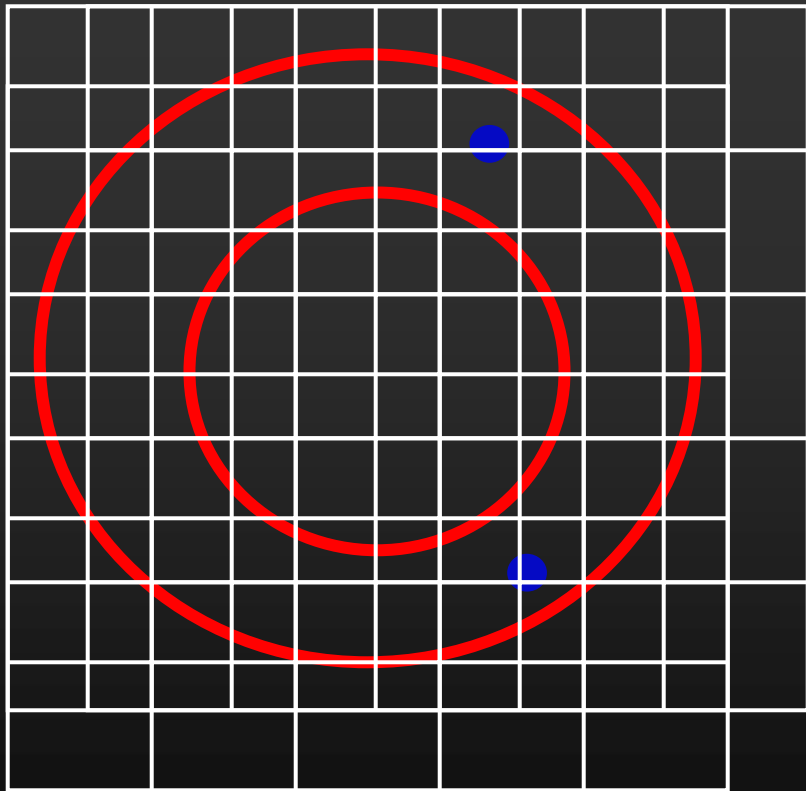
$$\left| d^g - d_S^g \right| \rightarrow \mathbf{0}$$

Basic idea



$$\left| d^g - d_S^g \right| \rightarrow \begin{cases} h^{1/2} & \text{general} \\ h & \text{local analytic} \\ h^\gamma, \gamma > 1 & \text{"smart" metric} \end{cases}$$

Why is this good?



$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$



$$\left\| \nabla d^g(p, x) \right\| = g$$

Implicit Form Representation

S = level – set of $\Psi : R^n \rightarrow R = \{x : \Psi(x) = 0\}$

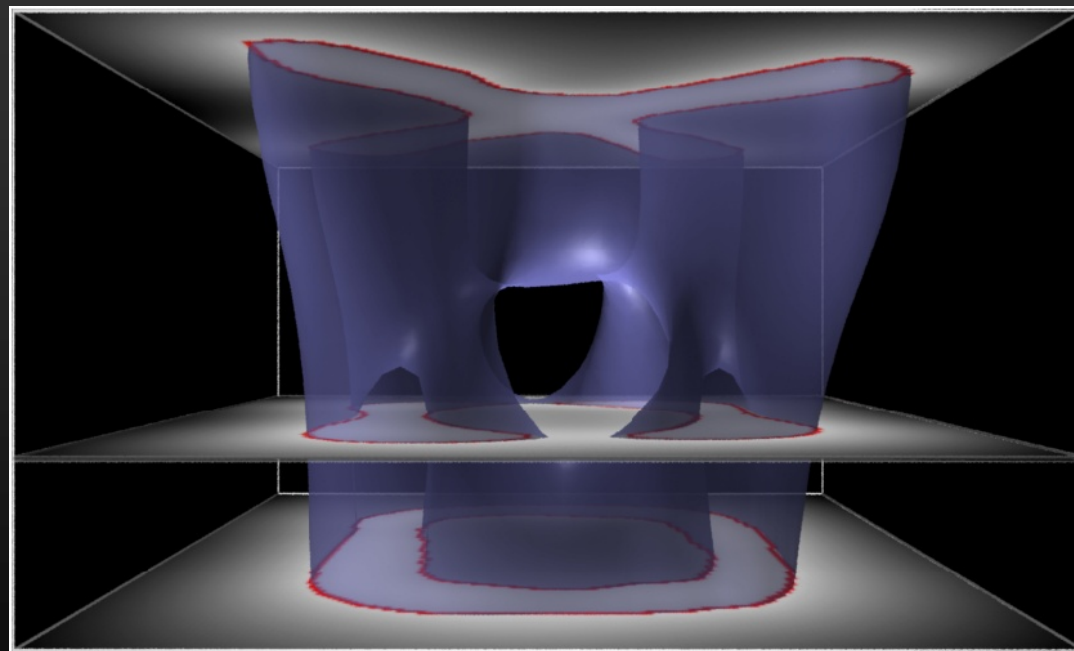


Figure from G. Turk

Data extension

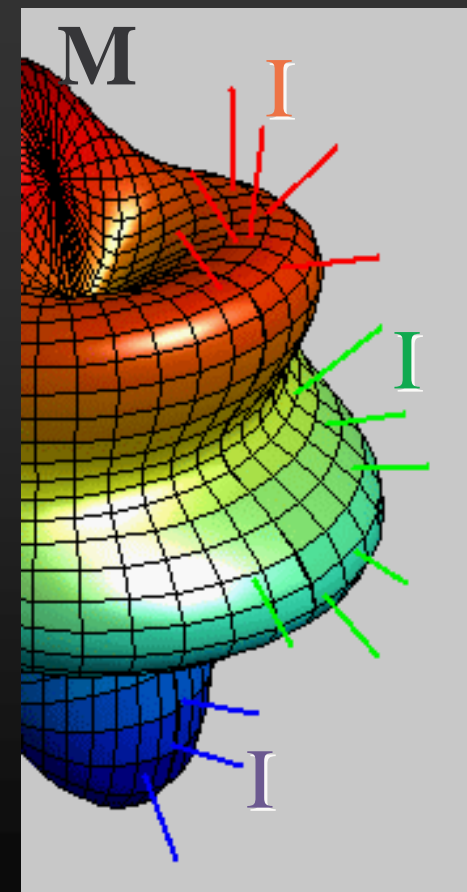
$$I: M \rightarrow \mathbb{R}$$

- Embed M:

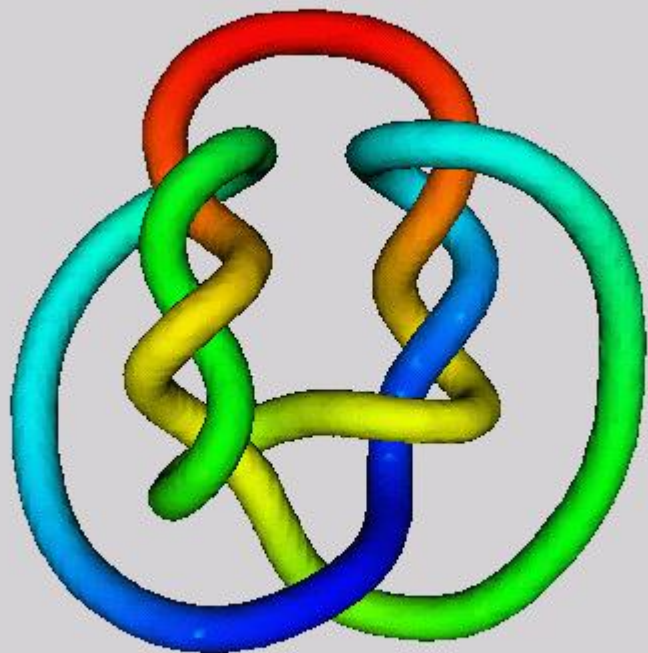
$$M = \{x : \Psi(x) = 0\}$$

- Extend I outside M:

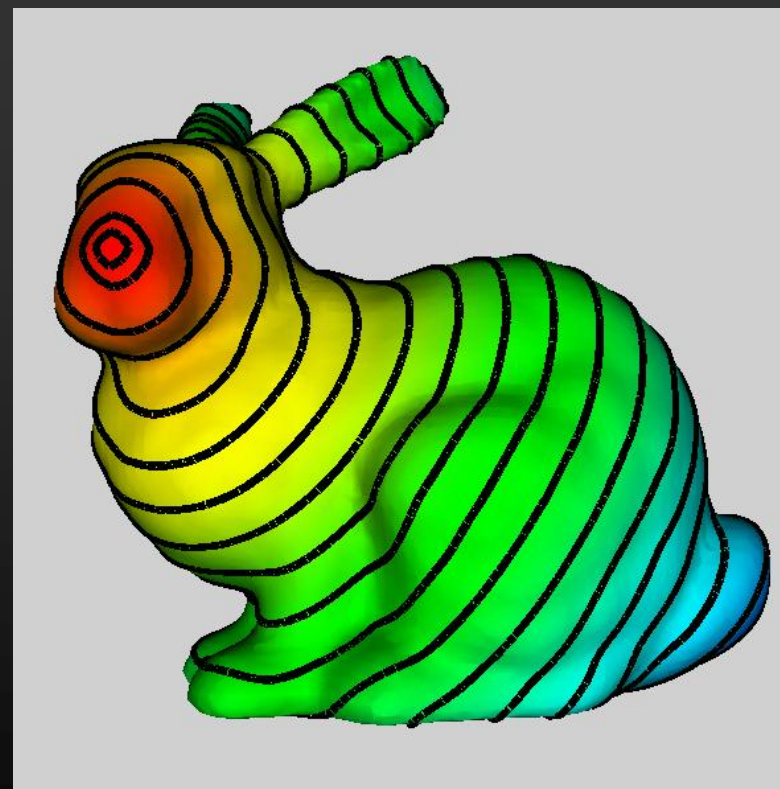
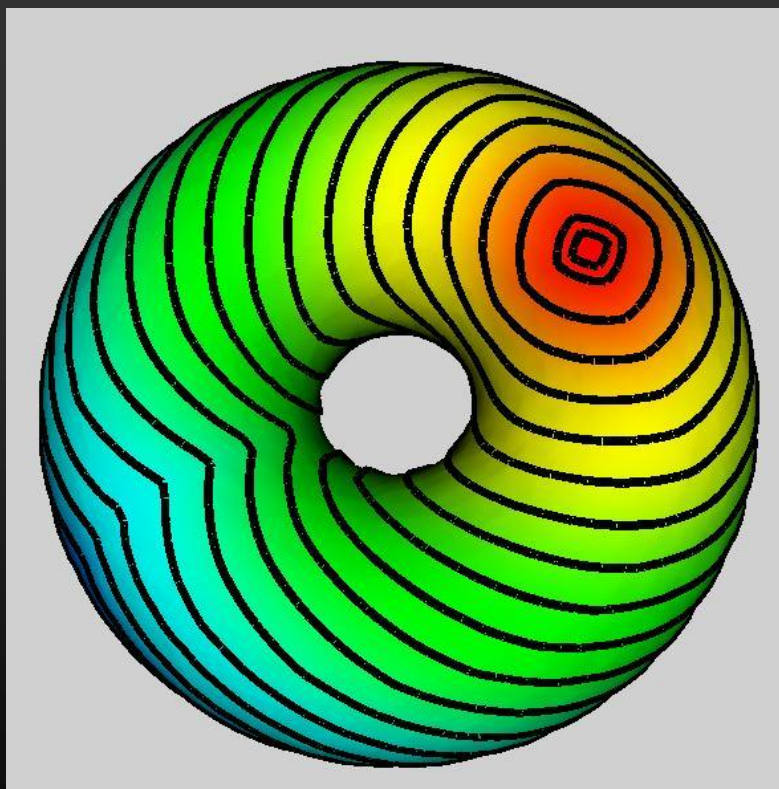
$$\frac{\partial I}{\partial t} + \text{sign}(\psi) (\nabla I \cdot \nabla \psi) = 0$$



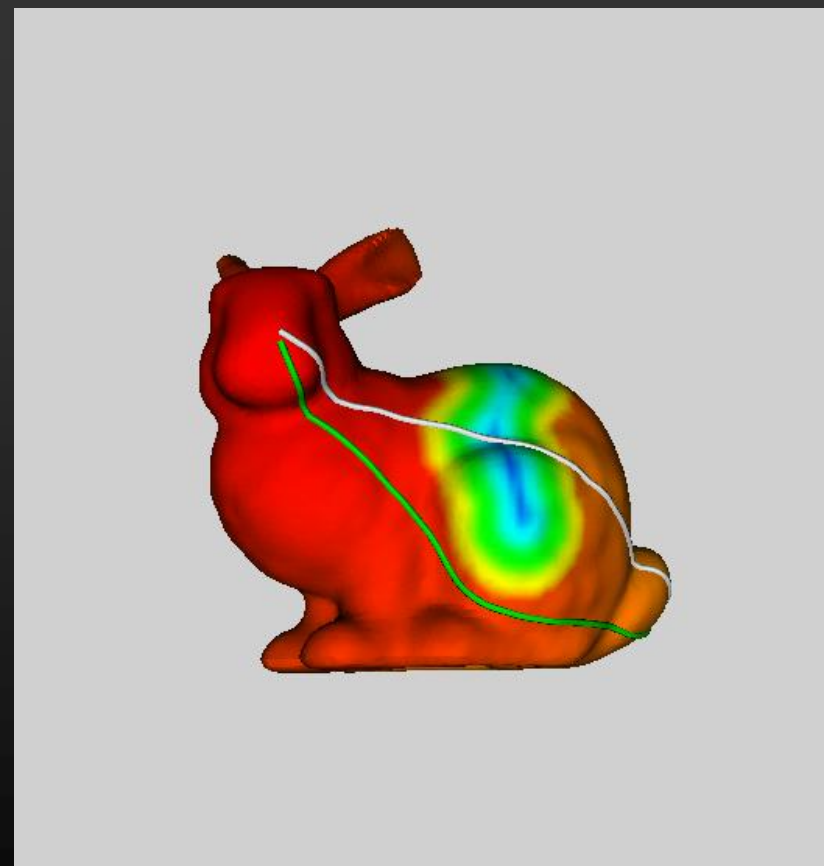
Examples



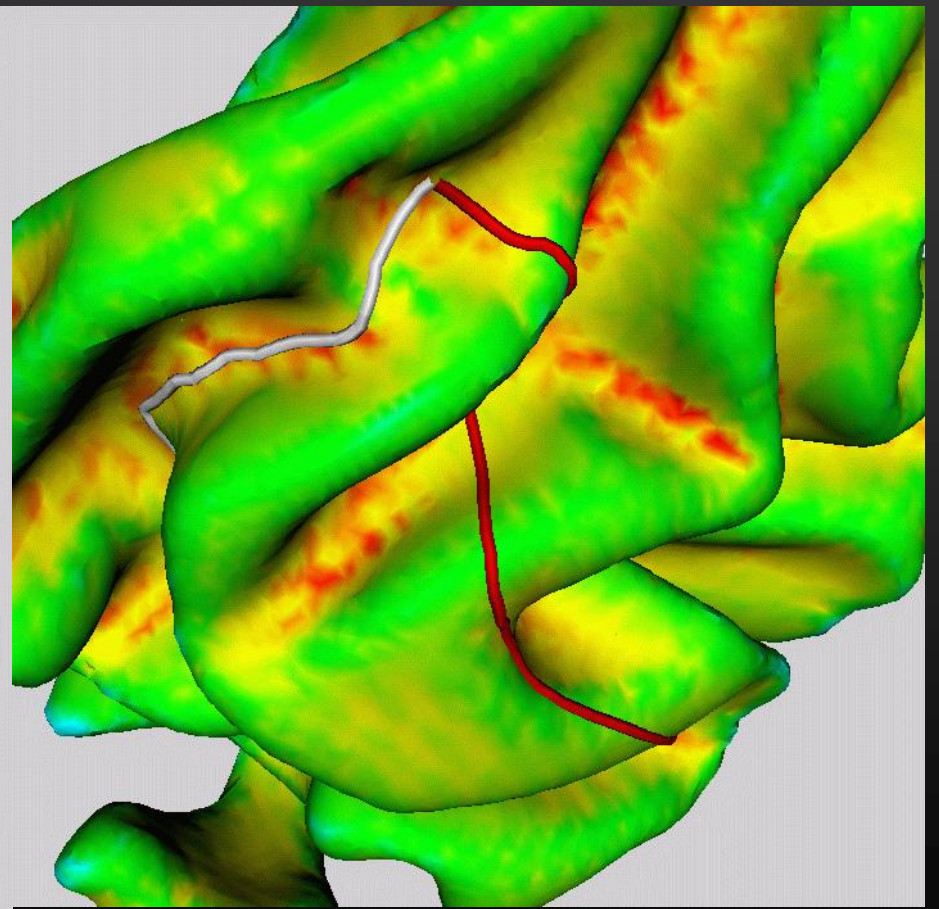
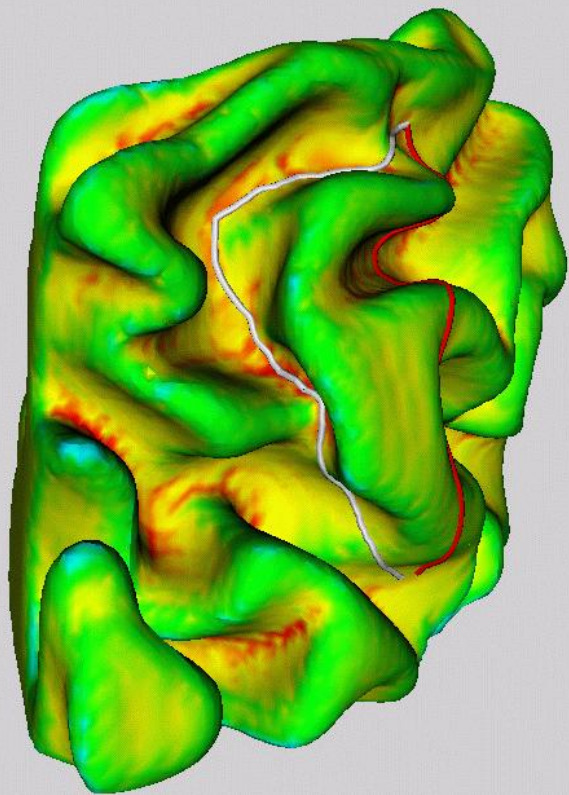
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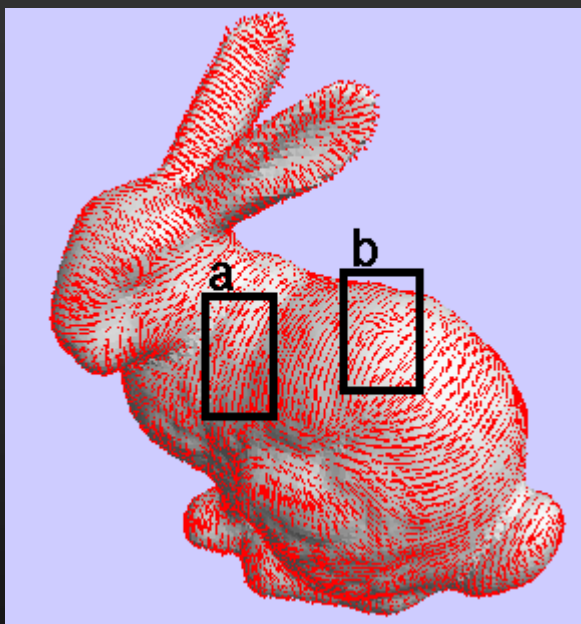
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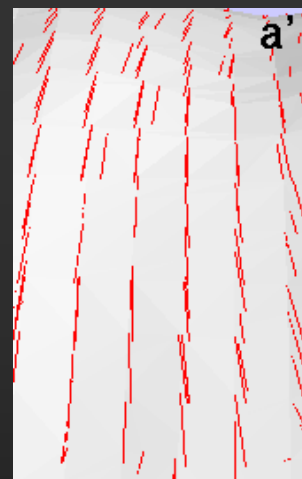
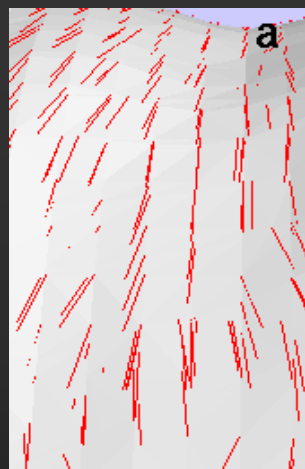
Examples



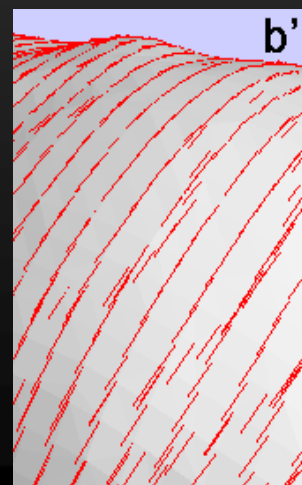
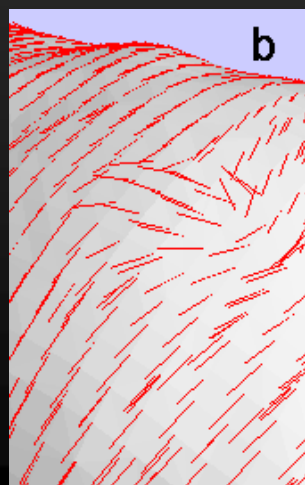
Is this a geodesic?



noisy



cleaned



Generalized geodesics: Harmonic maps

- Find a smooth map from two manifolds (M, g) and (N, h) such that

$$\min_{C: M \rightarrow N} \int_{\Omega} \|\nabla_M C\|^p d\text{vol}_M$$

$$\left(\frac{\partial C}{\partial t} = \right) \quad \Delta_M C + A_N(C) \langle \nabla_M C, \nabla_M C \rangle = 0$$

Examples

- M is an Euclidean space and N the real line

$$\Delta C = 0$$

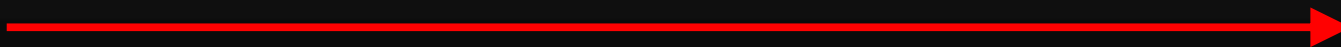
- M = [0,1], **geodesics!**

$$\frac{\partial^2 C}{\partial t^2} + A_N(C) < \nabla_M C, \nabla_M C > = 0$$

- **How we implement this for general manifolds?**
 - Consider M and N defined in implicit form.

Heat flow on the plane

$$E = \int \|\nabla I\|^2 \Rightarrow \frac{\partial I}{\partial t} = \Delta I$$



t

Embedding the domain surface

(Bertalmio-Cheng-Osher-S, JCP 2002)

- **Example: $I:M \rightarrow \mathbb{R}$**

- A map from a generic domain surface onto the real line

$$\min_{I:M \rightarrow \mathbb{R}} \int_M \|\nabla_M I\|^2 dvol_M$$

$$\frac{\partial I}{\partial t} = \Delta_M I$$

Embedding the domain surface (cont.)

$$\begin{aligned} \int_M \|\nabla_M I\|^2 d\text{vol}_M &= \int_M \|P_{\nabla\Psi} \nabla I\|^2 d\text{vol}_M \\ &= \int_{R^3} \|P_{\nabla\Psi} \nabla I\|^2 \delta(\Psi) \|\nabla\Psi\| d\mathbf{x} \end{aligned}$$

Framework

- **Gradient descent flow:**

$$\frac{\partial I}{\partial t} = \operatorname{div}(P_{\nabla \Psi} \nabla I)$$

- **Compare with planar case:**

$$\frac{\partial I}{\partial t} = \operatorname{div}(\nabla I) = \Delta I$$

Example: intrinsic heat flow



L1 denoising on implicit surfaces

$$\frac{\partial I}{\partial t} = \operatorname{div} \left(\frac{P_{\nabla \Psi} \nabla I}{\|P_{\nabla \Psi} \nabla I\|} \right)$$

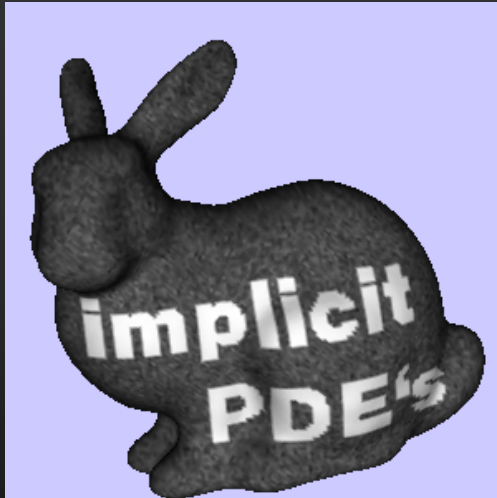
intrinsic

$$\frac{\partial I}{\partial t} = \operatorname{div} \left(\frac{\nabla I}{\|\nabla I\|} \right)$$

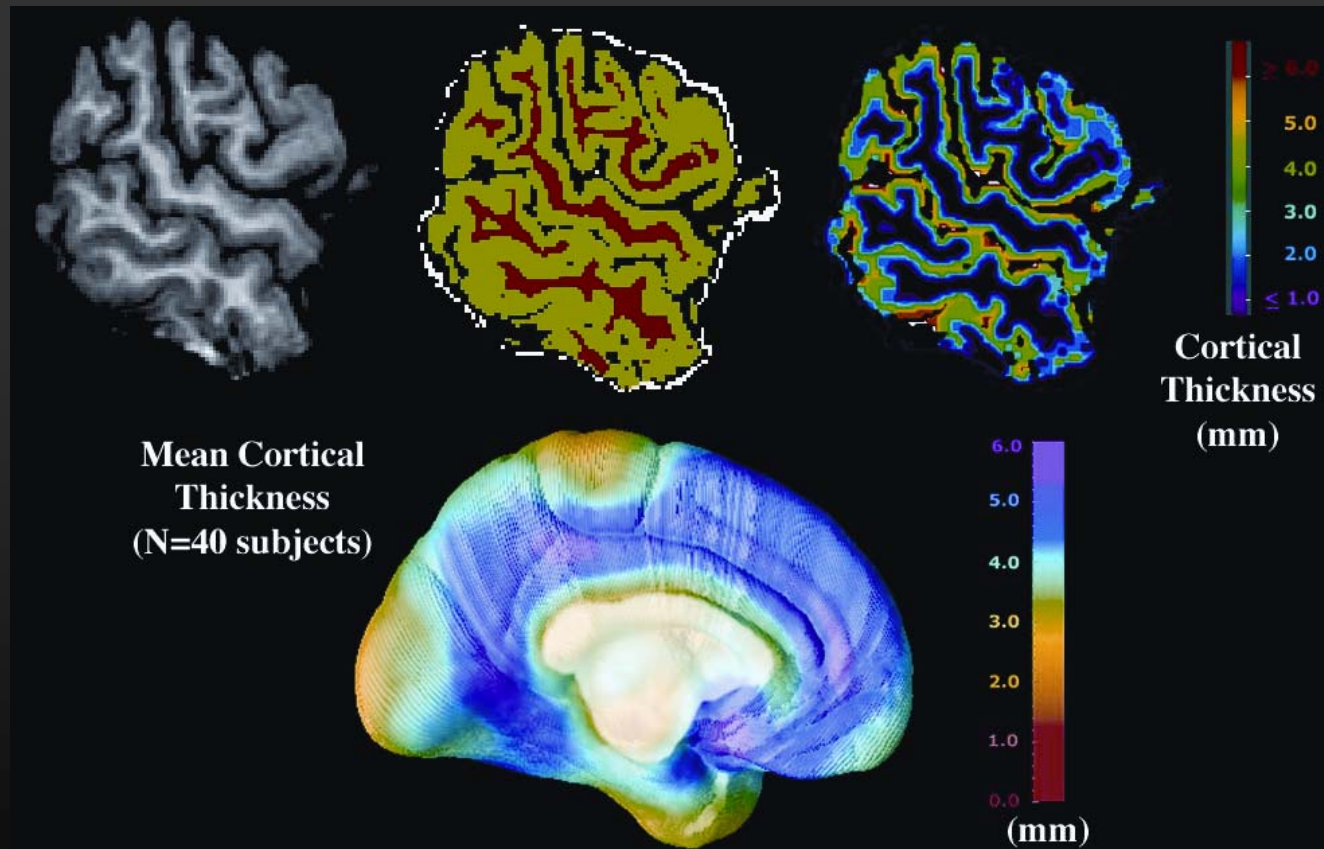
planar

Example:

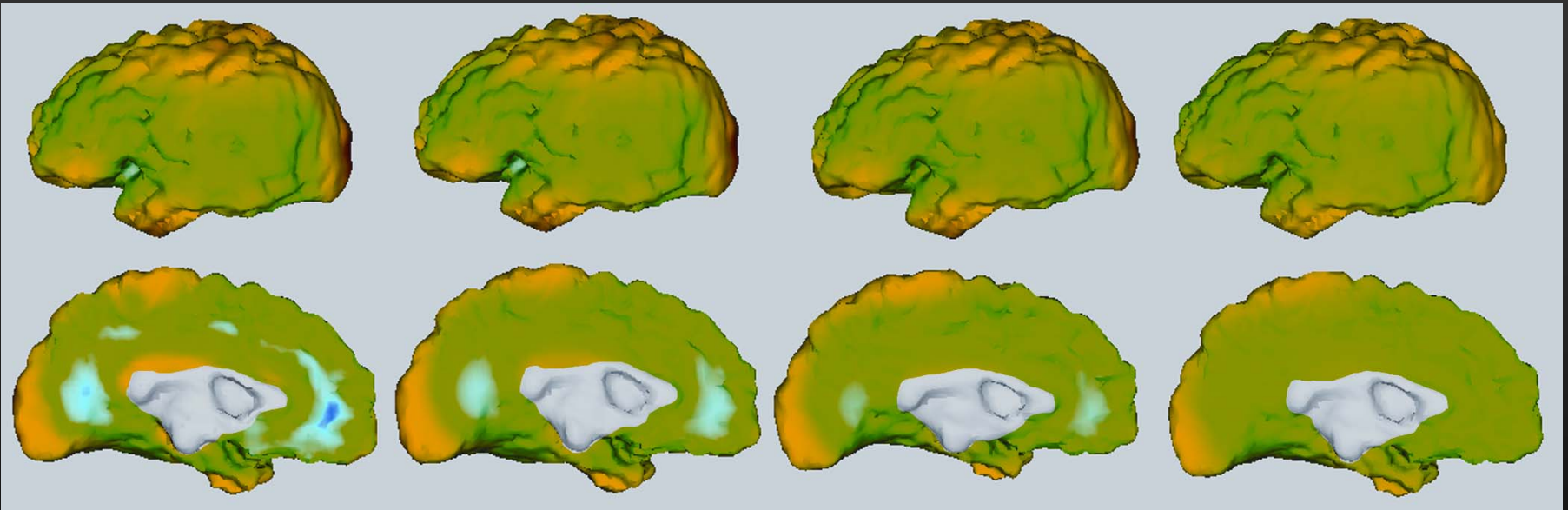
L1 denoising with constraints



Cortical Thickness Regularization



Regularization on Implicit Brains (Memoli-S-Thompson)



Embedding the target manifold

(Memoli-S-Osher, IMA 2002, JCP 2004)

- $I: M \rightarrow N$

$N = \text{level set of } \Phi = \{x : \Phi(x) = 0\}$

$$\min_{I: M \rightarrow \{\Phi=0\}} \int_M \|\nabla_M I\|^p d\text{vol}_M$$

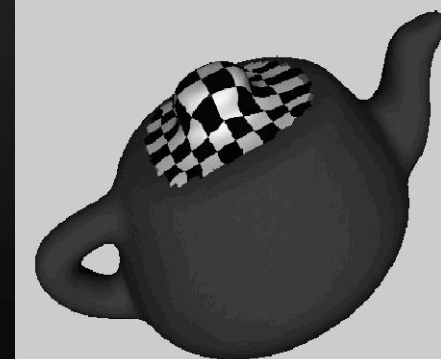
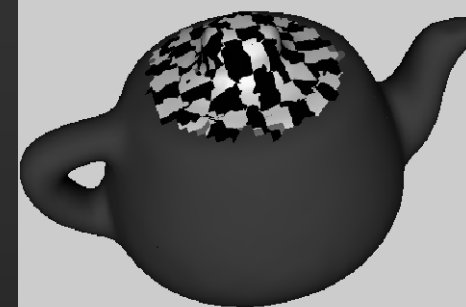
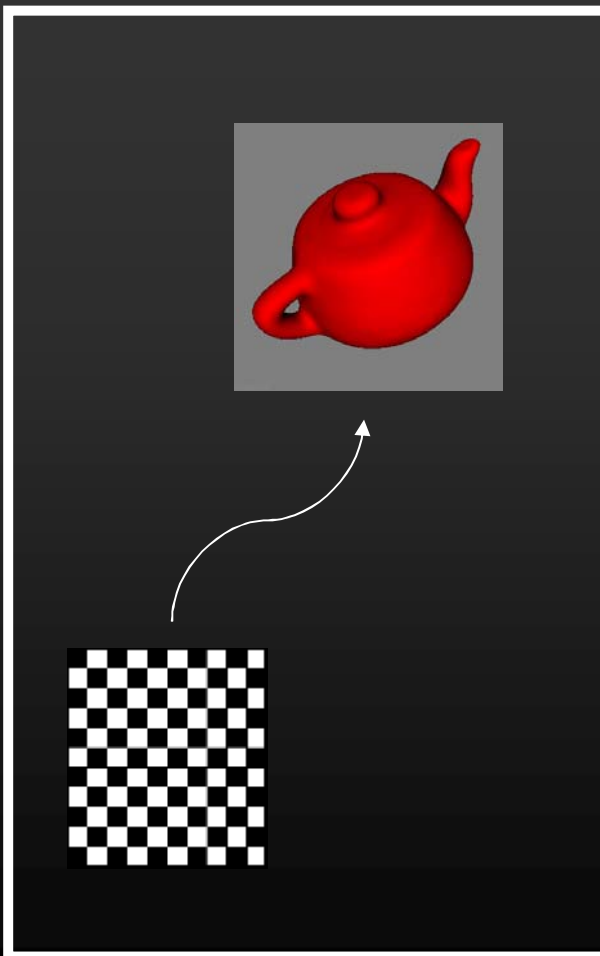
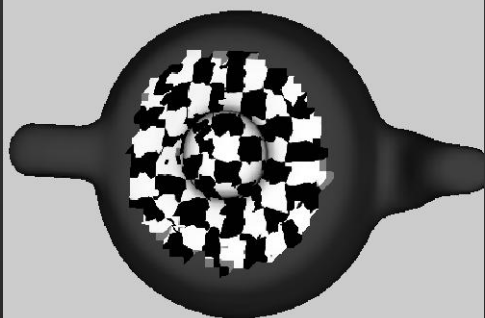
$$\frac{\partial I}{\partial t} = \Delta_M I + A_{\{\Phi=0\}}(I) \langle \nabla_M I, \nabla_M I \rangle$$

Embedding the target manifold (cont.)

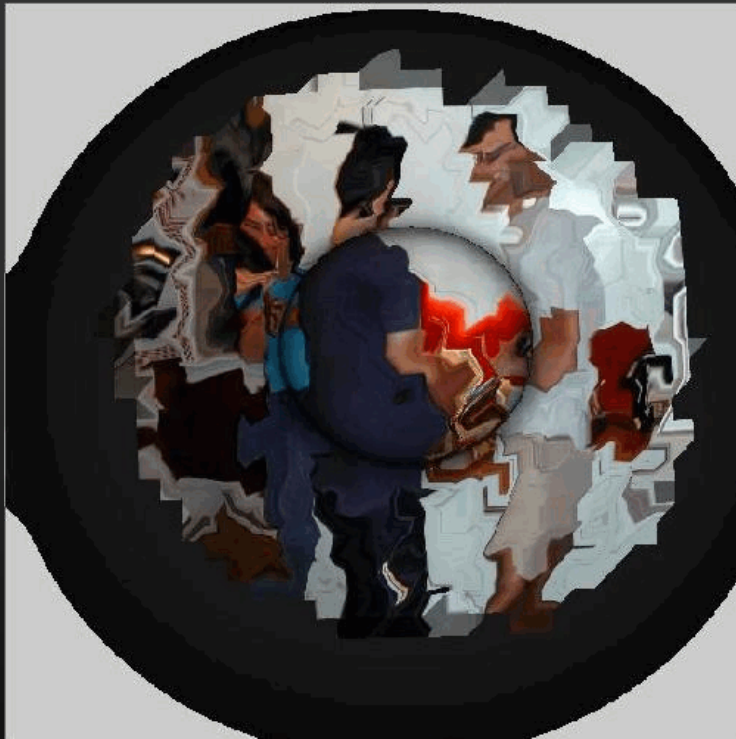
$$\min_{\mathbf{l}: R^k \rightarrow \{\Phi=0\}} \int_{R^k} \|\nabla_M \mathbf{l}\|^2 d\mathbf{x}$$

$$\frac{\partial \mathbf{l}}{\partial \mathbf{t}} = \Delta \mathbf{l} + \left(\sum_{\mathbf{k}} \mathbf{H}_{\Phi} \left\langle \frac{\partial \mathbf{l}}{\partial \mathbf{x}_{\mathbf{k}}}, \frac{\partial \mathbf{l}}{\partial \mathbf{x}_{\mathbf{k}}} \right\rangle \right) \|\nabla \Phi\|$$

Mapping denoising

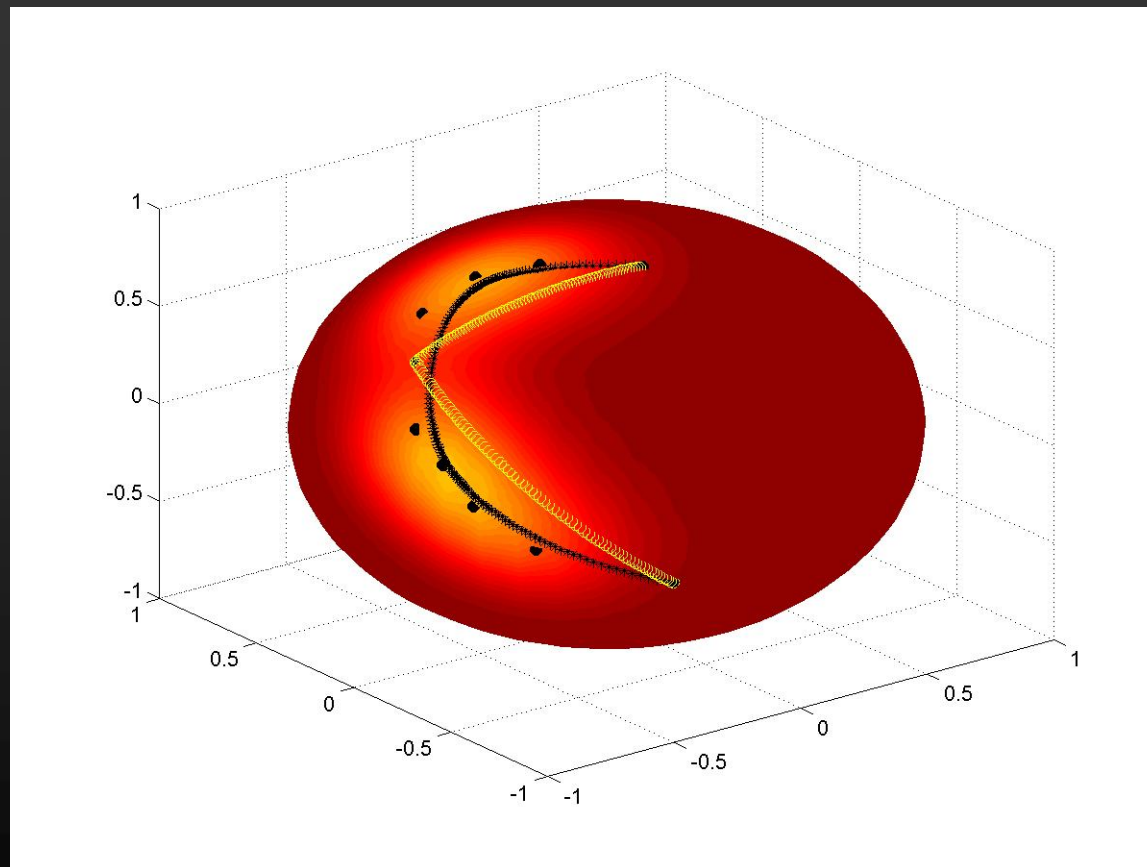


Texture mapping denoising

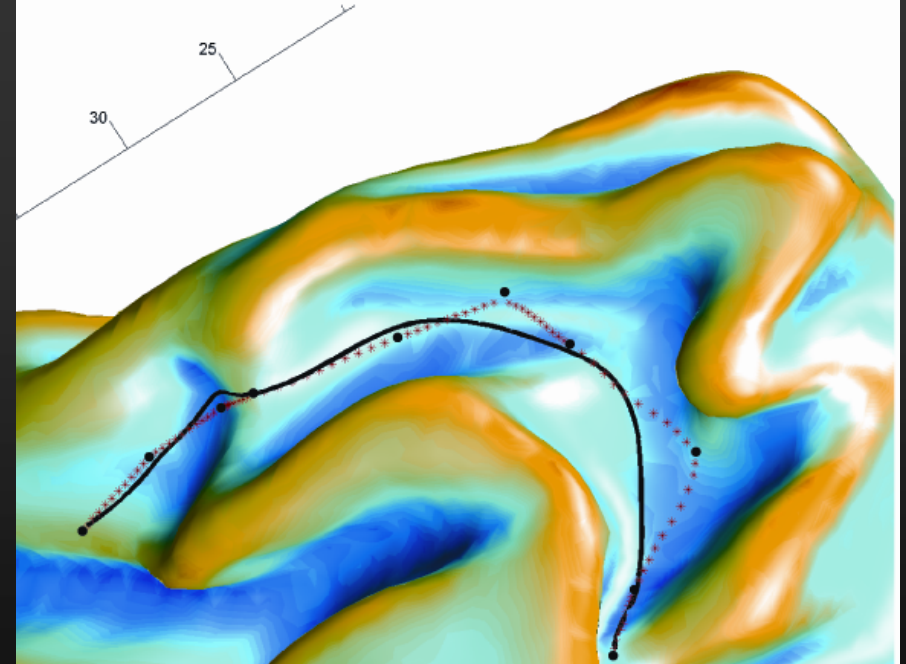
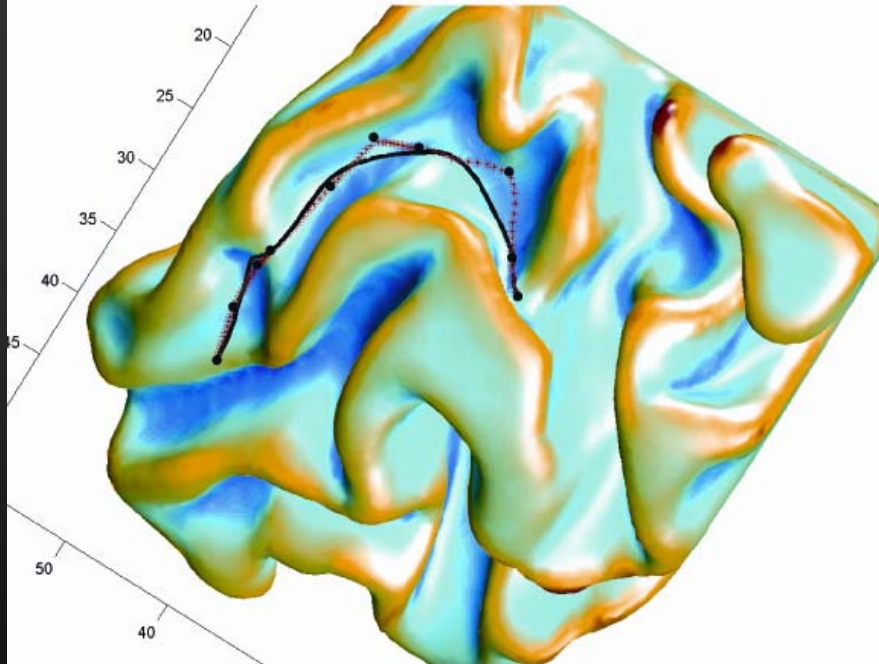


Constrained Curves on Surfaces

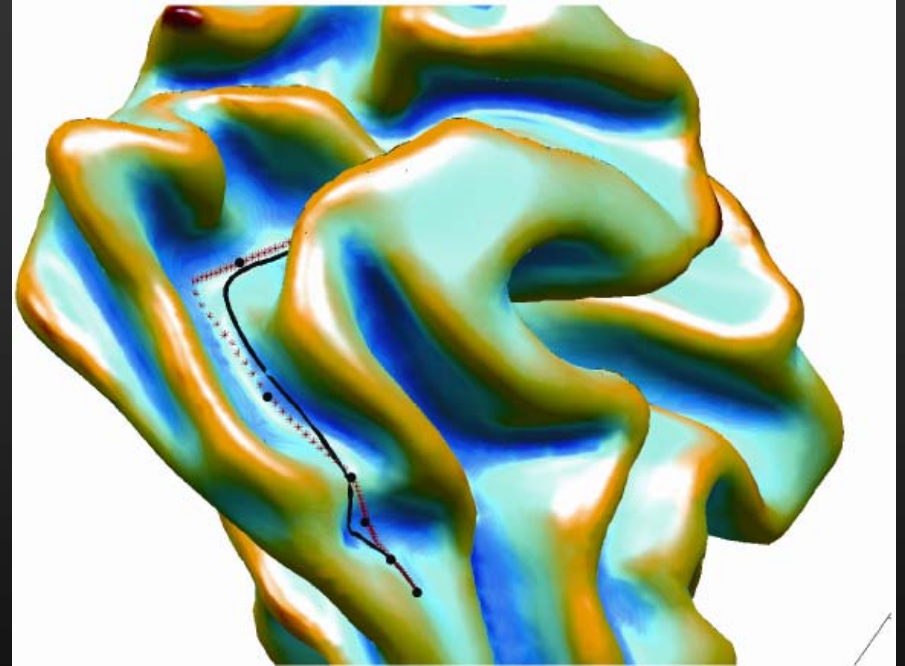
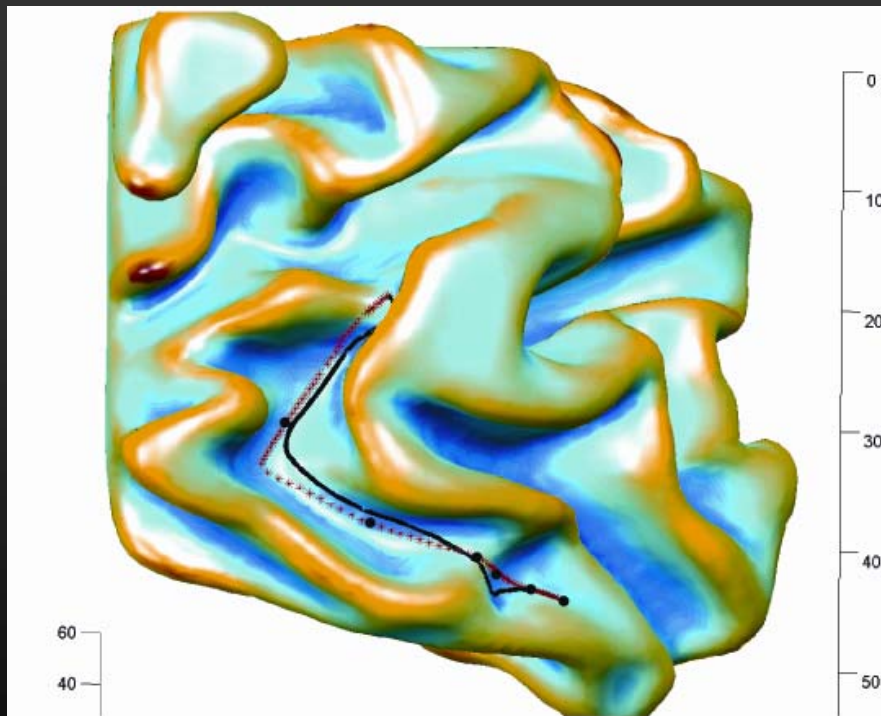
(Memoli-S-Thompson)



Constrained Curves on Surfaces



Constrained Curves on Surfaces



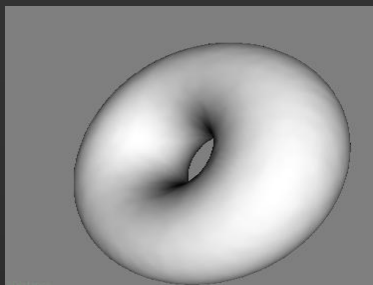
Complete Implicit Surfaces Representation

- Domain and target are implicitly represented: Simple Cartesian numerics

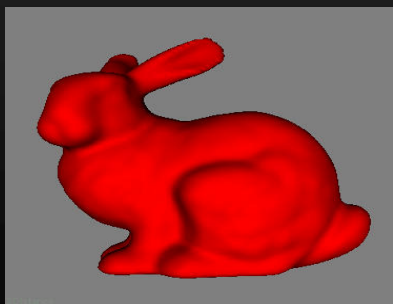


$$\frac{\partial I}{\partial t} = \mathbf{div}(\mathbf{P}_{\nabla\psi} \nabla I) + \left(\sum_k \mathbf{H}_{\Phi} \left\langle \frac{\partial I}{\partial \mathbf{x}_k}, \frac{\partial I}{\partial \mathbf{x}_k} \right\rangle \right) \|\nabla \Phi\|$$

Implicit Torus to Implicit Bunny



I



Concluding remarks

- A general computational framework for distance functions, geodesics, and generalized geodesics
- **Next:** Brain warping with landmark constraints

Thanks