STABILITY FOR RAYLEIGH-BENARD CONVECTIVE SOLUTIONS OF THE BOLTZMANN EQUATION

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The kinetic setting.

$$\frac{\partial F}{\partial t} + \frac{1}{\varepsilon}\, v_x \frac{\partial F}{\partial x} + \frac{1}{\varepsilon} v_z \frac{\partial F}{\partial z} - G \frac{\partial F}{\partial v_z} = \frac{1}{\varepsilon^2} Q(F,F),$$

$$F(0,x,z,v) = F_0(x,z,v), \quad (x,z) \in (-\mu\pi, \mu\pi) \times (-\pi,\pi), \ v \in \mathbb{R}^3$$

$$F(t,x,\mp\pi,v) \ = M_\mp(v) \int_{w_z \lessgtr 0} |w_z| F(t,x,\mp\pi,w) dw, t > 0, \ \ v_z \gtrless 0$$

for $x \in [-\mu\pi, \mu\pi]$, where

$$F_0 \geq 0, \quad M_- = \frac{1}{2\pi} e^{-\frac{v^2}{2}}, \quad M_+(v) = \frac{1}{2\pi(1-2\pi\varepsilon\lambda)^2} e^{-\frac{v^2}{2(1-2\pi\varepsilon\lambda)}},$$

$$\varepsilon = \frac{\ell_0}{d}, \quad G = \frac{1}{\varepsilon} \frac{dg}{2T_-}, \quad \lambda = \frac{1}{\varepsilon} \frac{T_- - T_+}{2\pi T_-}, \quad \mu = \frac{h}{d},$$

$$Q(f,g)(z,v,t) = \frac{1}{2} \int_{\mathbb{R}^3} dv_* \int_{S_2} d\omega B(\omega, v - v_*) \{ f'_* g' + f' g'_* - f_* g - g_* f \}.$$

The Rayleigh number $Ra = \dfrac{16G(2\pi\lambda)}{\pi}$ is independent of ε and chosen in $[Ra_c, (1+\delta)Ra_c]$, for δ small.
We construct a stationary solution $F_s = M + \varepsilon f_s + O(\varepsilon^2)$, with

$$M = \frac{1}{(2\pi)^{3/2}} e^{-\frac{v^2}{2}}, f_s = M\Big(\rho_s + u_s \cdot v + T_s \frac{|v|^2 - 3}{2}\Big) ,$$

where ρ_s, u_s, T_s are expressed in terms of the fluid solution

$$h_s = h_\ell + \delta\ h_{con} + O(\delta^2)$$

to the Oberbeck-Boussinesq system.
Moreover, we prove the kinetic non linear stability of F_s under suitable initial perturbations.

All solutions (stationary or evolutionary) to the Boltzmann equation will be weak L^1- solutions to the Boltzmann equation. This will be made possible by controlling the solutions in appropriate norms, in particular in the L^2_M norm in the v-variable of the L^∞ norm in the space variables.

We study the Boltzmann equation for the perturbation $\Phi = M^{-1}(F - F_s)$ with the initial datum

$$\Phi_0(x, z, v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(0, x, z, v) + \varepsilon^5 p_5,$$

where $\int dvdxdzMp_5 = 0$ and $F_s + M\Phi_0 \geq 0$. The time dependent solution is written

$$\Phi(t, x, z, v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(t, x, z, v) + \varepsilon R(t, x, z, v), \quad (x, z) \in \Omega_\mu.$$

The first term of the expansion in ε is

$$\Phi^{(1)} = \rho^1 + u^1 \cdot v + \theta^1 \frac{|v|^2 - 3}{2} \ ,$$

where the initial data for $\rho^1, u^1, \theta^1(t, x, z)$ are chosen small enough so that the solution $(u_s(x, z) + u^1(t, x, z), \theta_s(x, z) + \theta^1(t, x, z))$ of the initial boundary value problem for the O-B equations exists globally in time and converges to (u_s, θ_s) when $t \to \infty$.

Stability : the remainder

We construct the rest term R, solution of

$$\frac{\partial R}{\partial t}+\frac{1}{\varepsilon}\mu\; v_x\frac{\partial R}{\partial x}+\frac{1}{\varepsilon}v_z\frac{\partial R}{\partial z}-GM^{-1}\frac{\partial(MR)}{\partial v_z}=\frac{1}{\varepsilon^2}LR+\frac{1}{\varepsilon}J(R,R)$$

$$+\frac{1}{\varepsilon}H(R)+A,$$

$$R(0,x,z,v)=R_0(x,z,v)=\varepsilon^4 p_5(x,z,v),$$

$$R(t,x,\mp\pi,v)=\frac{M_\mp}{M}\int_{w_z\lessgtr 0}\Big(R(t,x,\mp\pi,w)+\frac{\bar\psi}{\varepsilon}(t,x,\mp\pi,w)\Big)|w_z|M$$

$$-\frac{\bar\psi}{\varepsilon}(t,x,\mp\pi,v),\qquad x\in[-\pi,\pi],\quad t>0,\; v_z>0,$$

where

$$H(R)=\frac{1}{\varepsilon}J(R,\sum_{1}^{5}\Phi^{(j)}\varepsilon^j+\Phi_s).$$

The main result.

Theorem

There exists a solution R such that

$$\lim_{t\to\infty}\int_{[-\pi,\pi]^2\times\mathbb{R}^3} R^2(t,x,z,v)M(v)dxdzdv = 0.$$

Main lines of the proof.

$$\int_0^{+\infty}\int_{[-\pi,\pi]^2\times\mathbb{R}^3} R^2(t,x,z,v)M(v)dxdzdvdt < c\varepsilon^7,$$

$$\int R^2(t,x,z,v)M(v)dxdzdv < \frac{c}{\varepsilon^2}\Big(\int R^2(0,x,z,v)M(v)dxdzdv + \int_0^{+\infty} \parallel A(s) \parallel ds\Big).$$

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Three main problems.

- Avoid exponential growth of $R(t,.,.)$ when $t \to \infty$. Indeed, by

  $$-(R, LR) \geq C((1-P)R, \nu(1-P)R),$$

  and

  $$(R, J(\phi_H, PR)) \leq C\|\nu^{1/2}PR\| \, \|\nu^{1/2}(1-P)R\|.$$

  it holds that

  $$\frac{1}{2}\frac{d}{dt} \parallel R \parallel_{2,2}^2 \leq C \parallel R \parallel_{2,2}^2 + \int_{\Omega_\mu} |(B, R)|.$$

- Take care of the diffuse reflexion boundary conditions.
- Control the hydrodynamic moments.

Fix (x, z) and define

$$L_J R = LR + J(\sum_{n=1}^{5} \varepsilon^n \Phi^{(n)} + \Phi_s, PR).$$

Spectral gap property of L_J

Lemma

There is $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, there is c independent of ε and (x, z), for which the following inequalities hold :

$$-(L_J R, R) \geq c(\nu(I - P_J)R, (I - P_J)R),$$
$$-(L_J^* R, R) \geq c(\nu(I - P)R, (I - P)R).$$

The following norms are used,

$$\begin{aligned}
\parallel R \parallel_{2t,2} &= \Big(\int_0^t \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}^3} R^2(s,x,z,v)M(v)dsdxdzdv\Big)^{\frac{1}{2}},\\
\parallel R \parallel_{\infty,2} &= \sup_{t>0}\Big(\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}^3} R^2(t,x,z,v)M(v)dxdzdv\Big)^{\frac{1}{2}},\\
\parallel R \parallel_{\infty,\infty} &= \sup_{t>0}\Big(\int_{\mathbb{R}^3} \sup_{-\pi<x,z<\pi} R^2(t,x,z,v)M(v)dv\Big)^{\frac{1}{2}},\\
\parallel R \parallel_{2t,2,\sim} &= \Big(\int_0^t \int_{-\pi}^{\pi} \int_{v_z>0} v_z M(v) \mid R(s,x,-\pi,v) \mid^2 dvdxds\Big)^{\frac{1}{2}}\\
&+\Big(\int_0^t \int_{-\pi}^{\pi} \int_{v_z<0} \mid v_z \mid M(v) \mid R(s,x,\pi,v) \mid^2 dvdxds\Big)^{\frac{1}{2}},\\
\parallel R \parallel_{\infty,2,\sim} &= \Big(\sup_{t>0} \int_{-\pi}^{\pi} \int_{v_z>0} v_z M(v) \mid R(t,x,-\pi,v) \mid^2 dxdv\Big)^{\frac{1}{2}}\\
&+\Big(\sup_{t>0} \int_{-\pi}^{\pi} \int_{v_z<0} \mid v_z \mid M(v) \mid R(t,x,\pi,v) \mid^2 dxdv\Big)^{\frac{1}{2}}.
\end{aligned}$$

Lemma

Let $\varphi(\bar\tau, x, z, v)$ *be solution to*

$$\frac{\partial\varphi}{\partial\bar\tau} + \; v_x\frac{\partial\varphi}{\partial x} + v_z\frac{\partial\varphi}{\partial z} - \varepsilon G M^{-1}\frac{\partial(M\varphi)}{\partial v_z} = \frac{1}{\varepsilon}L_J^*\varphi + g, \tag{1}$$

periodic in x *of period* 2π*, with zero initial and ingoing boundary values at* $z = -\pi, \pi$*, and* g x*-periodic of period* 2π*. Set* $\tilde\varphi = \varphi - <\varphi> = \varphi - (2\pi)^{-2}\int \varphi dxdz$*.*
Then, if $\varepsilon \le \varepsilon_0$*,* $\delta \le \delta_0$*, for* ε_0, δ_0 *small enough, there exists* η *small such that,*

$$\begin{aligned} \parallel \varphi \parallel_{\infty,2} \quad & \le c\Big(\varepsilon^{\frac{1}{2}} \parallel \nu^{-\frac{1}{2}}(I-P)g \parallel_{2,2} + \varepsilon^{-\frac{1}{2}} \parallel Pg \parallel_{2,2} \\ & + \eta\varepsilon^{\frac{1}{2}} \parallel < P\varphi > \parallel_{2,2} \Big), \\ \parallel \nu^{\frac{1}{2}}(I-P)\varphi \parallel_{2,2} \quad & \le c\Big(\varepsilon \parallel \nu^{-\frac{1}{2}}(I-P)g \parallel_{2,2} + \parallel Pg \parallel_{2,2} \\ & + \eta\varepsilon \parallel < P\varphi > \parallel_{2} \Big), \end{aligned}$$

Proof of the Lemma.

Denote by $\hat{\varphi}(\bar{\tau}, \xi, v)$, $\xi = (\xi_x, \xi_z) \in \mathbb{Z}^2$ the Fourier transform of φ with respect to space.
Then for $\xi \neq (0, 0)$,

$$\frac{\partial \hat{\varphi}}{\partial \bar{\tau}} = \frac{1}{\varepsilon} \widehat{L_J^* \varphi} - i\xi \cdot v \ \hat{\varphi} + \varepsilon G M^{-1} \frac{\partial (M\hat{\varphi})}{\partial v_z} + \hat{g} - |v_z| r (-1)^{\xi_z}.$$

Here $r = \mathcal{F}_x \varphi(\bar{\tau}, \xi_x, \pm\pi, v)$ for $v_z \gtrless 0$. Then,

$$\int_0^\infty d\bar{\tau} \int (P\hat{\varphi})^2(\bar{\tau}, \xi, v) M dv \leq C\Big(\frac{1}{\varepsilon^2} \int_0^\infty d\bar{\tau} \Big(\parallel \zeta_{-s}(v) \widehat{L_J^* \varphi})(\bar{\tau}, \xi, \cdot$$

$$+ \parallel (I - P)\hat{\varphi}(\bar{\tau}, \xi, \cdot) \parallel^2 \Big) + \int_0^\infty d\bar{\tau} \int \nu^{-1} \hat{g}^2(\bar{\tau}, \xi, v) M dv$$

$$+ \int_0^\infty d\bar{\tau} \frac{\parallel \sqrt{|v_z|} r \parallel^2}{\delta_1 |\xi|^2} \Big).$$

By the Parseval inequality,

$$\int_0^\infty \int (\widetilde{P\varphi})^2(\bar\tau, x, z, v) M dv dx dz d\bar\tau$$
$$\leq c\Big(\frac{1}{\varepsilon^2} \int_0^\infty \int \nu((I-P)\varphi)^2(\bar\tau, x, z, v) M dv dx dz d\bar\tau$$
$$+ \int_0^\infty \int \nu^{-1} g^2(\bar\tau, x, z, v) M dv dx dz d\bar\tau + \parallel \gamma^-\varphi \parallel^2_{2\infty,2,\sim} + \eta \parallel \varphi \parallel^2_{2,2}$$

By Green's formula,

$$\parallel \gamma^-\varphi \parallel^2_{2\bar T,2,\sim} + \parallel \varphi \parallel^2_{2\bar T,2} + \frac{1}{\varepsilon} \parallel \nu^{\frac12}(I-P)\varphi \parallel^2_{2\bar T,2}$$
$$\leq c(\varepsilon \parallel \nu^{-\frac12}(I-P)g \parallel^2_{2\bar T,2} + \eta_1 \parallel P\varphi \parallel^2_{2\bar T,2} + \frac{1}{\eta_1} \parallel Pg \parallel^2_{2\bar T,2}).$$

Write R as the sum $R_1 + R_2$, where R_1 and R_2 are solutions of two different problems. R_1 solves

$$\begin{aligned}
&\frac{\partial R_1}{\partial t} + \frac{1}{\varepsilon} v_x \cdot \frac{\partial R_1}{\partial x} + \frac{1}{\varepsilon} v_z \cdot \frac{\partial R_1}{\partial z} - \frac{G}{M}\frac{\partial (MR_1)}{\partial v_z} = \frac{1}{\varepsilon^2} L_J R_1 \\
&\qquad\qquad\qquad + \frac{1}{\varepsilon} H_1(R_1) + \frac{1}{\varepsilon} g, \\
&R_1(0,x,z,v) = R_0(x,z,v), \\
&R_1(t,x,\mp\pi,v) = -\frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,v), \quad t>0,\ v_z \gtrless 0.
\end{aligned}$$

The non-hydrodynamic part of R_1 is estimated by Green's formula : for every $\eta_1 > 0$,

$$\begin{aligned}
\| \gamma^- R_1 \|^2_{2\bar{T},2,\sim} \quad & + \| R_1(\bar{T}) \|^2_{2,2} + \frac{1}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P_J)R_1 \|^2_{2\bar{T},2} \\
& \le c\Big(\| R_0 \|^2_{2,2} + \varepsilon \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2\bar{T},2} \\
& + \frac{\eta_1}{2} \| P_J R_1 \|^2_{2\bar{T},2} + \frac{1}{2\eta_1} \| P_J g \|^2_{2\bar{T},2} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{2\bar{T},2,}
\end{aligned}$$

An a priori bound for $P_J R_1$ is obtained in the following lemma based on dual techniques.

Lemma

Set $h := P_J R_1$. *Then*

$$\begin{aligned} \| h \|_{2,2}^2 &\le c(\| R_0 \|_2^2 + \| \nu^{-\frac{1}{2}}(I - P_J)g \|_{2,2}^2 \\ &+ \frac{1}{\varepsilon^2} \| P_J g \|_{2,2}^2 + \frac{1}{\varepsilon^3} \| \bar{\psi} \|_{2,2,\sim}^2). \end{aligned}$$

The remaining part R_2 of R satisfies the equation

$$\varepsilon\frac{\partial R_2}{\partial t} + v_x\frac{\partial R_2}{\partial x} + v_z\frac{\partial R_2}{\partial z} - \varepsilon\frac{G}{M}\frac{\partial(MR_2)}{\partial v_z} = \frac{1}{\varepsilon}L_J R_2 + H_1(R_2),$$

$$R_2(0,x,z,v) = 0,$$

$$R_2(t,x,\mp\pi,v) = \frac{M_\mp(v)}{M(v)}\int_{w_z \lesseqgtr 0}\Big(R_1(t,x,\mp\pi,w) + R_2(t,x,\mp\pi,w) + \frac{1}{\varepsilon}\bar{\psi}(t,x,\mp\pi,w)\Big)|w_z|Mdw, \quad t>0,\ v_z \gtrless 0.$$

By Green's formula, and noting that $H_1(R_2)$ only depends on $(I-P)R_2$, we get

$$\varepsilon\|R_2(t)\|_2^2 + \| \gamma^- R_2 \|_{2t,2,\sim}^2 + \frac{c}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P_J)R_2 \|_{2t,2}^2 \leq \| \gamma^+ R_2 \|_{2t,2,\sim}^2 \ .$$

Treatment of the diffuse reflexion boundary conditions :

$$\begin{gathered}\varepsilon\|R_2\|_{2,2}^2(t) + \frac{c}{\varepsilon} \| \nu^{\frac{1}{2}}(I-P_J)R_2 \|_{2t,2}^2 \\ \leq \frac{1}{\varepsilon\eta} \| f^- \|_{2t,2,\sim}^2 + C\varepsilon\eta \| P_J R_2 \|_{2t,2}^2, \\ \| \gamma^- R_2 \|_{2t,2,\sim}^2 \leq \frac{1}{\varepsilon^2} \| f^- \|_{2t,2,\sim}^2 + C \| P_J R_2 \|_{2t,2}^2 \ .\end{gathered}$$

Hydrodynamic estimates for R_2.

Lemma

$$\| P_J R_2 \|_{2,2}^2 \leq \frac{c_1}{\varepsilon^2} \| f^- \|_{2,2\sim}^2 + c_2 \| P_J R_1 \|_{2,2}^2 .$$

Lemma

Any solution R_2 satisfies the a priori estimates

$$\begin{aligned}
\| \nu^{\frac{1}{2}}(I-P_J)R_2 \|^2_{2,2} &\leq c\Big(\varepsilon \| R_0 \|^2_2 + \varepsilon \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2} \\
&\quad + \frac{1}{\varepsilon} \| P_J g \|^2_{2,2} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{2,2,\sim} \Big), \\
\| P_J R_2 \|^2_{2,2} &\leq c\Big(\frac{1}{\varepsilon}(\| R_0 \|^2_2 + \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2}) \\
&\quad + \frac{1}{\varepsilon^3} \| P_J g \|^2_{2,2} + \frac{1}{\varepsilon^4} \| \bar{\psi} \|^2_{2,2,\sim} \Big), \\
\| \nu^{\frac{1}{2}} R_2 \|^2_{\infty,\infty} &\leq c(\frac{1}{\varepsilon^2} \| R_2 \|^2_{\infty,2} + \| \gamma^- R_1 \|^2_{\infty,2,\sim} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{\infty} \\
&\leq c\Big(\frac{1}{\varepsilon^3} \| R_0 \|^2_2 + \frac{1}{\varepsilon^3} \| \nu^{-\frac{1}{2}}(I-P_J)g \|^2_{2,2} \\
&\quad + \frac{1}{\varepsilon^5} \| P_J g \|^2_{2,2} + \frac{1}{\varepsilon^6} \| \bar{\psi} \|^2_{2,2,\sim} + \| R_0 \|^2_{\infty,2} \\
&\quad + \varepsilon^2 \| \nu^{-\frac{1}{2}} g \|^2_{\infty,\infty} + \frac{1}{\varepsilon^2} \| \bar{\psi} \|^2_{\infty,2,\sim} \Big).
\end{aligned}$$

Theorem

There exists a solution R to the rest term problem such that

$$\int_0^{+\infty}\int_{[-\pi,\pi]^2\times\mathbb{R}^3} |R(t,x,z,v)|^2 M(v)\,dtdxdzdv < c\varepsilon^7.$$

Proof. R is obtained as the limit of an approximating sequence.