Critical Thresholds in Hyperbolic Relaxation Systems

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It is generic phenomena that solutions of homogeneous systems of quasi-linear hyperbolic conservation laws break down after a finite elapse of time. A balance is often attained with the presence of various sources $S[U]$,

$$U_t + \nabla_x \cdot F(U) = S[U],$$

in which $F(U)$ is the flux and the source $S[U]$ may prevent finite time breakdown from happening if initial configuration is above certain threshold.

Consider a relaxation system of the form

$$v_t - u_x = 0, \tag{1}$$

$$u_t + p(v)_x = \frac{1}{\tau}(u_e(v) - u), \tag{2}$$

subject to bounded and differentiable initial data

$$(v, u)(x, 0) = (v_0, u_0)(x) \tag{3}$$

for all $x \in R$, where v and u are scalars, $u_e(v)$ is the equilibrium flux, $p(v)$ is the pressure satisfies $p'(v) < 0$ and $p''(v) > 0$, and $\tau > 0$ is a relaxation parameter.

When $\tau \to 0$, (1) (2) relax to the equilibrium equation

$$v_t - u_e(v)_x = 0 \tag{4}$$

with equilibrium characteristic

$$\lambda_*(v) = u_e'(v).$$

We calculate the eigenvalues of system (1) (2)

$$\lambda_1(v) = -\lambda_2(v) = -\sqrt{-p'(v)}. \tag{5}$$

We impose the following the stability conditions for (1) (2), i.e., the subcharacteristic conditions,

$$\lambda_1(v) \leq \lambda_*(v) \leq \lambda_2(v). \tag{6}$$

The system (1) (2) is dissipative if the subcharacteristic condition (6) is satisfied. It is known that the sub-characteristic type condition is necessary even for linear stability as evidenced by Whitham's work [Whit]. The sub-characteristic condition for a class of 2×2 relaxation systems is coined in [Tliu] for nonlinear stability of shock waves.

We are concerned with both global in time regularity and finite time singularity in solutions.

Under what conditions on the initial data,

we have existence of global smooth solutions?

or we have finite time singularity in solutions?

For a GNL scalar conservation law,

$$u_t + f(u)_x = 0$$

if $f''(u) > 0$, singularity develops in a finite time if the initial data has a negative slope. Shock formation. Indeed

$$u_x = \frac{u_0'(x)}{1 + u_0'(x) f''(u) t}.$$

Lax, 1964, Development of singularities, homogeneous, quasilinear, GNL, systems of two equations, finite time blow up if the Riemann invariant has a negative initial slope.

John, 1974, GNL, systems of n equations.

Liu, 1979, extended to include LDG field

Nishida, 1978, global smooth solutions to p-system with a damping exist if the initial Riemann invariant slopes are small, say, bounded by $C\alpha$.

$$v_t - u_x = 0$$

$$u_t - \sigma(v)_x = -\alpha u.$$

We calculate the Riemann invariants of system (1) (2)

$$R^{\pm} = u \mp m(v), \quad m(v) := \int_{v^*}^{v} \sqrt{-p'(s)}ds \tag{7}$$

where $v^* > 0$ is a fixed number. The above transformation maps (u, v) to (R^-, R^+), and vice versa by

$$u = \frac{1}{2}(R^- + R^+), v = m^{-1}\left(\frac{1}{2}(R^- - R^+)\right). \tag{8}$$

Riemann invariants thus satisfy

$$R_t^- + \lambda_1 R_x^- = \frac{1}{\tau}(u_e(v) - u) \tag{9}$$

and

$$R_t^+ + \lambda_2 R_x^+ = \frac{1}{\tau}(u_e(v) - u) \tag{10}$$

subject to the corresponding initial data

$$R^\pm(x, t) = R_0^\pm(x) = u_0(x) \mp m(v_0(x))$$

for all $x \in R$.

$p''(v) > 0$ implies that both characteristic fields are genuinely nonlinear.

Through this reformulated system, the existence of a uniform invariant region for the relaxation system (1) (2) is ensured by the sub-characteristic condition (6).

We now estimate the derivatives of the solution through

$$r^{\pm} = (-p'(v))^{1/4} R_x^{\pm}$$

$$= (-p'(v))^{1/4}(u_x \mp \sqrt{-p'(v)}v_x).$$

It is clear that the boundedness of (u_x, v_x) is equivalent to the boundedness of $r^{\pm}$ for $v \in I$.

Lemma *The dynamic systems for $r^{\pm}$ are*

$$(\partial_t + \lambda_1 \partial_x)(r^- - g^-) + a(r^-)^2 + b^- r^- = 0 \quad (11)$$

and

$$(\partial_t + \lambda_2 \partial_x)(r^+ - g^+) + a(r^+)^2 + b^+ r^+ = 0. \quad (12)$$

Proof. Differentiate (9) and (10) w.r.t. x to have

$$s_t^- + \lambda_1 s_x^- + \lambda_{1v} v_x s^- = (\frac{1}{\tau}(u_e(v) - u))_x$$

and

$$s_t^+ + \lambda_2 s_x^+ + \lambda_{2v} v_x s^+ = (\frac{1}{\tau}(u_e(v) - u))_x$$

where

$$s^\pm = R_x^\pm .$$

Differentiate v along the first characteristic curve $x_1'(t) = \lambda_1$ to have

$$(\partial_t + \lambda_1 \partial_x) v =: v' = \frac{(R^-)' - (R^+)'}{2m'(v)}$$

$$= \frac{(\lambda_2 - \lambda_1) R_x^+}{2\lambda_2} = s^+. \tag{13}$$

Similarly, we have $(\partial_t + \lambda_2 \partial_x) v = s^-$. Let

$$h = \frac{1}{2}\ln(\lambda_2) = \frac{1}{2}\ln(-\lambda_1).$$

Using again $m(v) = (R^- - R^+)/2$, we obtain

$$\lambda_{1v} v_x = \frac{\lambda_{1v}}{2m'(v)}(s^- - s^+) = h_v (s^+ - s^-).$$

This and (13) lead to

$$\lambda_{1v} v_x = h_v v' - h_v s^- = h' - h_v s^-.$$

Let

$$a = \frac{1}{4} p''(v)(-p'(v))^{-5/4} > 0$$

and

$$w = u_e'(v)/\lambda_2(v).$$

Then we have

$$(\frac{1}{\tau}(u_e(v) - u))_x = \frac{-1+w}{2\tau} s^- - \frac{1+w}{2\tau} s^+.$$

Substitution of these into the above equation for s^- yields

$$(s^-)' + h' s^- - h_v (s^-)^2 = \frac{-1+w}{2\tau} s^- - \frac{1+w}{2\tau} s^+.$$

Note $r^- = s^- e^h$. This together with (13) gives

$$(r^-)' - h_v e^{-h} (r^-)^2 = -b^- r^- - \frac{1+w}{2\tau} e^h v'$$

where

$$b^{\pm} = \frac{1 \pm w}{2\tau}.$$

Let

$$g^{\pm} = -\frac{1}{2\tau} \int_{v^*}^{v} (1 \mp w)(-p'(s))^{1/4} ds.$$

Thus we derive (11). Similarly, (12) can be derived. The lemma is proved.
Applying a key lemma to (11) and (12), we obtain the following critical threshold results.

Theorem 1 *To appear on JDE.*
i) If at least at one point $x \in R$, either

$$r_1(0,x) < g_1(v_0(x)) + \inf_{v \in I} \left(-g_1(v) - \frac{b_1(v)}{a(v)} \right)$$

or

$$r_2(0,x) < g_2(v_0(x)) + \inf_{v \in I} \left(-g_2(v) - \frac{b_2(v)}{a(v)} \right)$$

holds, the solution of system (11) (12) must develop singularity at a finite time.

ii) If

$$\inf_{v\in I}\left(\frac{b_i}{a}\right)(v) \geq \sup_{v\in I} g_i(v) - \inf_{v\in I} g_i(v), \quad (14)$$

for $i = 1,\ \ 2$, *then the solution of system (11) (12) remains smooth for all time, provided for all* $x \in R$

$$r_i(0,x) \geq g_i(v_0(x)) + \sup_{v\in I}\left(-g_i(v) - \frac{b_i(v)}{a(v)}\right)$$

for $i = 1,\ \ 2$.

The condition (14) actually requires that the sub-characteristic condition (6) is satisfied strictly, i.e., $\lambda_1(v) < u_e'(v) < \lambda_2(v)$.

Under condition (14), the lower thresholds on the right hand side are nonpositive.

Indeed,

$$g^{\pm}(v_0(x)) + \sup_{v\in I}\left(-g^{\pm}(v) - \frac{b^{\pm}(v)}{a(v)}\right)$$

$$\leq g^{\pm}(v_0(x)) + \sup_{v\in I}(-g^{\pm}(v)) + \sup_{v\in I}\left(-\frac{b^{\pm}(v)}{a(v)}\right)$$

$$\leq \sup_{v\in I} g^{\pm}(v) - \inf_{v\in I} g^{\pm}(v) - \inf_{v\in I} \frac{b^{\pm}(v)}{a(v)} \leq 0$$

where the last inequality holds due to condition (14).

Thus the set of initial data leading to global regularty is rich. In particular, it allows initial Riemann invariant slopes to be negative. Furthermore, the magnitudes of the negative slopes are proportional to $\frac{1}{\tau}$ which are not necessarily small. This is in sharp contrast to the generic breakdown in the homogeneous hyperbolic systems, Lax.

Along each characteristic field, the two equations (11) (12) for $r^{\pm}$ are ordinary differential equations of the same form

$$\frac{d}{dt}(r-g) + ar^2 + br = 0, \quad r(0) = r_0$$

which can be written as

$$\frac{d}{dt}A + a(t)(A - b_1(t))(A - b_2(t)) = 0 \quad (15)$$

and

$$A(0) = A_0 \tag{16}$$

where

$$A = r^{\pm} - g^{\pm}$$

and

$$b_2 = -g^{\pm}, \quad b_1 = -g^{\pm} - \frac{b^{\pm}}{a}.$$

A key lemma *Consider equation (15) for A with $\inf a > 0$, $b_1 \le b_2$ and that a, b_1, b_2 are uniformly bounded. We have*
(i) If $A_0 < \min b_1$, then solution to (15) (16) will experience a finite time blow up at $t_* \le t^*$

$$\lim_{t \to t_*} A(t) = -\infty,$$

where t^* satisfies

$$\int_0^{t^*} a(s)ds$$

$$= \frac{1}{\min b_2 - \min b_1} \ln\left(1 + \frac{\min b_2 - \min b_1}{\min b_1 - A_0}\right).$$

(ii) If there exists a constant $\bar{b}$ *such that*

$$b_1(t) \leq \bar{b} \leq b_2(t),$$

then (15) (16) admits a unique global bounded solution satisfying

$$\bar{b} \leq A(t) \leq \max\{A_0, \max b_2\}$$

provided that $A_0 \geq \bar{b}$*.*

Proof. Set $\tau = \int_0^t a(s)ds$, which maps $t \in [0, \infty)$ to $\tau \in [0, \infty)$ with $\infty = \int_0^\infty a(s)ds$. Then equation (15) reduces to

$$\frac{d}{d\tau}A + (A - b_1)(A - b_2) = 0, \quad A(0) = A_0.$$

(i) In order to prove the blow up result, we consider the following auxiliary problem

$$\frac{d}{d\tau}A^* + (A^* - \min b_1)(A^* - \min b_2) = 0,$$

$$A^*(0) = A_0$$

which has only local solution up to τ^* if $A_0 < \min b_1$, with τ^* defined as

$$\frac{1}{\min b_2 - \min b_1} \ln \left(1 + \frac{\min b_2 - \min b_1}{\min b_1 - A_0} \right),$$

which equals to $\frac{1}{\min b_2 - A_0}$ if $\min b_2 = \min b_1$. Let $B = A - A^*$, then it solves the following equation

$$\frac{d}{d\tau} B + B(B + 2A^* - b_1 - b_2) + C = 0,$$

where

$$C =$$

$$(A^* - b_1)(A^* - b_2) - (A^* - \min b_1)(A^* - \min b_2)$$

$$= (A^* - \min b_1)(\min b_1 + \min b_2 - b_1 - b_2)$$

$$+ (b_1 - \min b_1)(b_2 - \min b_1) \geq 0,$$

where $b_2 \geq \min b_1$ has been used. These together lead to

$$\frac{d}{d\tau} B + B(B + 2A^* - b_1 - b_2) \leq 0, \quad B(0) = 0.$$

Therefore

$$B(\tau) \leq 0$$

for as long as the solution B exists.

Thus the blow up time of A is less than or equal to τ^* since

$$A(\tau) \leq A^*(\tau)$$

for as long as both solutions A and A^* exist.

ii) We now consider an auxiliary problem

$$\frac{d}{d\tau}\bar{A} + (\bar{A} - b_1)(\bar{A} - \bar{b}) = 0, \quad \bar{A}(0) = A_0,$$

which has a global bounded solution if $\bar{A}_0 \geq \bar{b}$ and

$$\bar{A}(\tau) \geq \bar{b} \geq b_1, \quad \forall \tau > 0.$$

Indeed, $\bar{A}$ satisfies

$$\frac{d}{d\tau}(\bar{A} - \bar{b})e^{\int_0^\tau (\bar{A}(s) - b_1(s))ds} = 0$$

which implies that

$$\bar{A} = \bar{b} + (A_0 - \bar{b})e^{-\int_0^\tau (\bar{A}(s) - b_1(s))ds} \geq \bar{b} > -\infty$$

provided that $\bar{A}_0 \geq \bar{b}$.

Let $B = A - \bar{A}$, then it solves the following equation

$$\frac{d}{d\tau}B + B(B + 2\bar{A} - b_1 - b_2) + C = 0,$$

where

$$C = (\bar{A} - b_1)(\bar{b} - b_2) \leq 0.$$

These together lead to

$$\frac{d}{d\tau}B + B(B + 2\bar{A} - b_1 - b_2) \geq 0, \quad B(0) = 0.$$

Therefore

$$B(\tau) \geq 0, \quad \forall \tau > 0.$$

Hence if $A_0 \geq \bar{b}$ we have

$$A \geq \bar{A} \geq \bar{b},$$

where $\bar{b}$ serves as a lower threshold. This and the upper bound $A \leq \max\{A_0, \max b_2\}$ lead to the desired estimate.

Applications

1. Global smoothness in semi-linear relaxation system

Consider the semi-linear case [jinz], $p = -\alpha^2 v$, $\alpha > 0$, in the relaxation system (1) (2)

$$v_t - u_x = 0 \tag{17}$$

$$u_t - \alpha^2 v_x = \frac{1}{\tau}(u_e(v) - u). \tag{18}$$

The sub-characteristic condition (6) thus reduces to

$$-\alpha \leq u_e'(v) \leq \alpha \tag{19}$$

for v in question.

Within the same framework we proved

Theorem *Consider the semi-linear relaxation system (17) (18) subject to* C^1 *bounded initial data* $(v_0, u_0)(x)$*. Under subcharacteristic assumption (19), the Cauchy problem has a unique* C^1 *solution for all time* $t > 0$*.*

2. Damping system

Taking $u_e(v) = 0$ in the relaxation system (1) (2), we obtain the isentropic Euler system with damping:

$$v_t - u_x = 0 \tag{20}$$

$$u_t + p(v)_x = -\frac{1}{\tau}u. \tag{21}$$

Notice that the subcharacteristic condition (6) becomes

$$-\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)}.$$

Let $p(v) = e^{-v}$. In this quasilinear case, condition (14) becomes

$$0 < v_{\max} - v_{\min} \leq 4 \ln 2.$$

Corollary *Consider the damping system (20) (21) with $p(v) = e^{-v}$, subject to C^1 bounded initial data.*

i)If for at least one point $x \in R$ either of

$$u_{0x} \mp e^{-v_0/2} v_{0x} < \frac{2}{\tau}\left(1 - 2e^{\frac{v_0(x)-v_{min}}{4}}\right)$$

holds, then the solution must develop a finite time singularity when either u_x *or* v_x *becomes unbounded.*
ii) Assume that $0 < v_{max} - v_{min} \leq 4\ln 2$*, the solution remains smooth for all time, provided for all* $x \in R$ *it holds*

$$u_{0x} \mp e^{-v_0/2} v_{0x} \geq \frac{2}{\tau}\left(1 - 2e^{\frac{v_0(x)-v_{max}}{4}}\right).$$

3. In relaxation system (1) (2), let $p(v) = \frac{1}{\gamma}v^{-\gamma}$ and $u_e(v) = \frac{1}{1-\gamma}v^{(\gamma-1)/2}$ where $\gamma > 3$. Condition (14) becomes

$$1 < \frac{v_{max}}{v_{min}} \leq \left(\frac{4\gamma}{3(\gamma+1)}\right)^{4/(\gamma-3)}$$

where the right hand side is greater than one when $\gamma > 3$. In the case $1 \leq \gamma \leq 3$ similar but different condition can be derived.

Corollary *Consider relaxation system (1) (2) with* $p(v) = \frac{1}{\gamma}v^{-\gamma}$ *and* $u_e(v) = \frac{1}{1-\gamma}v^{(\gamma-1)/2}$ *for* $\gamma > 3$*, subject to* C^1 *bounded initial data.*

i) If for at least one point $x \in R$ either

$$u_{0x} - v_0^{-(\gamma+1)/2} v_{0x}$$

$$< \frac{v_0(x)}{\tau(\gamma-3)} \left(1 - \frac{4(\gamma-2)}{\gamma+1} \left(\frac{v_{min}}{v_0(x)}\right)^{(3-\gamma)/4}\right)$$

or

$$u_{0x} + v_0^{-(\gamma+1)/2} v_{0x}$$

$$< \frac{v_0(x)}{\tau(\gamma-3)} \left(1 - \frac{4\gamma}{\gamma+1} \left(\frac{v_{min}}{v_0(x)}\right)^{(3-\gamma)/4}\right)$$

holds, then the solution must develop a finite time singularity when either u_x or v_x becomes unbounded.

ii) Assume that

$$1 < \frac{v_{max}}{v_{min}} \le \left(\frac{4\gamma}{3(\gamma+1)}\right)^{4/(\gamma-3)},$$

the solution remains smooth for all time, provided for all $x \in R$ it holds

$$u_{0x} - v_0^{-(\gamma+1)/2} v_{0x}$$

$$\geq \frac{v_0(x)}{\tau(\gamma-3)}\left(1-\frac{4(\gamma-2)}{\gamma+1}\left(\frac{v_{max}}{v_0(x)}\right)^{(3-\gamma)/4}\right)$$

and

$$u_{0x}+v_0^{-(\gamma+1)/2}v_{0x}$$

$$\geq \frac{v_0(x)}{\tau(\gamma-3)}\left(1-\frac{4\gamma}{\gamma+1}\left(\frac{v_{max}}{v_0(x)}\right)^{(3-\gamma)/4}\right).$$