Nonlinear Waves in Mathematical Models of Traffic Flow

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Introduction

- Question: how to combine, either at discrete or at continuum level, enough instability, experimentally observed in stop and go waves ... and enough stability, in order to prevent paradoxes, e.g. negative velocities, or unpleasant effects (crashes)?
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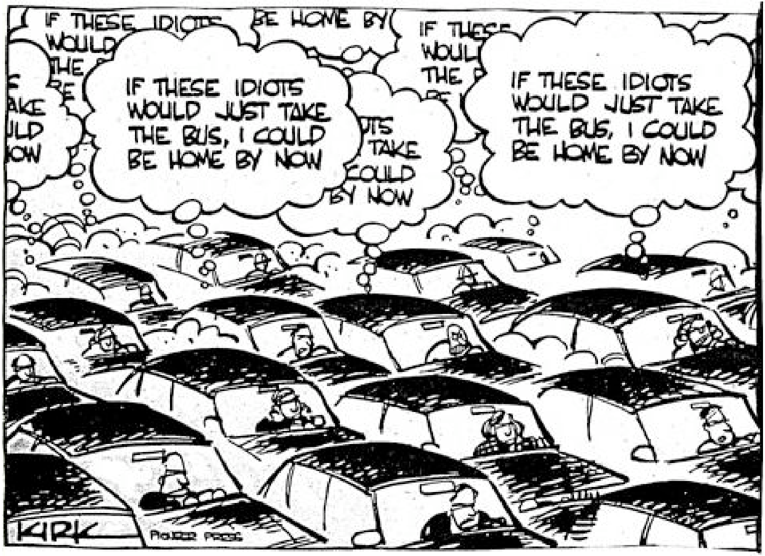
Outline

- Introduction
- Discrete / Fluid Models
- The Fluid Model (Without Relaxation)
    - The Eulerian System
    - Riemann Problem. Waves
    - Motivations. Lagrangian version
    - Link with Microscopic Models (FLM)
    - Lagrangian Godunov Scheme
    - Passing to the limit(s)
- With Relaxation: Traveling Waves and Oscillations
    - motivations
    - Remark: Whitham Subcharacteristic condition
    - Smooth "simple waves" are are generically Traveling Waves
    - J. Greenberg's work periodic solutions. Extensions...
    - An Example: the Intelligent Driver Model
- Additional Remarks. Conclusion


IF THESE IDIOTS WOULD JUST TAKE THE BUS, I COULD BE HOME BY NOW
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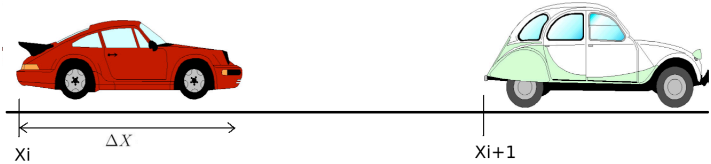
KIRK
PIONEER PRESS


ΔX
Xi
Xi+1

Discrete / Fluid Models

- Discrete Models: Follow the Leader Models...
  Car length: $l = \Delta X$. Spacing:
  $\tau_j := x_{j+1} - x_j \, ; s_j = 1/\rho_j \; = \tau_j / l$
  specific volume, density.

$$\begin{cases} \dot{x}_j = v_j \Longrightarrow \dot{s}_j = \frac{v_{j+1}-v_j}{l} \\ \dot{v}_j = F(x_j, x_{j+1}, v_j, v_{j+1}) \\ (e.g.) = \alpha v_j^m \, V'(\frac{x_{j+1}-x_j}{l}) \, \frac{v_{j+1}-v_j}{l} + \beta(V_e(\frac{x_{j+1}-x_j}{l}) - v_j) \end{cases} \tag{2.1}$$

Convective part (fast reaction) + (slow) relaxation part ...
Examples, see also Gazis-Herman-Rothery:

- $\alpha = 0, \beta > 0$: Bando's Optimal Velocity Model
- $\alpha > 0, \beta = m = 0$: Aw-Klar-Materne-Rascle, SIAP 2002
- $\alpha > 0, \beta > 0, m = 0$: J. Greenberg and/or Aw-Rascle, SIAP 2000-2004
- Intelligent Driver Model (IDM): Helbing-Treiber, ∼ 2000
  $\dot{v}_j = a \, [1 - v_j^m - (\frac{s_b(v_j) - v_j(v_{j+1}-v_j)}{s_j})^2] \; ; s_b(v) := s_0 + s_1\sqrt{v} + s_2(v)$

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  - First Order: Lighthill-Whitham-Richards (LWR) [↔ Hamilton-Jacobi]

    $$\partial_t\rho + \partial_x(\rho v) = 0,\ v = V(\rho), V'(\rho) < 0, (\rho V)" < 0,$$

    Fundamental diagram: flux $q = \rho V(\rho)$.
    Riemann Pb: $\rho(x,0) = \rho_\pm$ for $\pm x > 0$:
    - centered rarefaction waves (acceleration) if $v_- < v_+$,
    - shock waves (braking) if $v_- > v_+$. Very robust, (too) stable. Figures.

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  - Second Order: Payne-Whitham (cf Gas Dynamics)

    $$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + v\,\partial_x v = -\rho^{-1} p'(\rho)\partial_x \rho + ... := -\tilde{p}'(\rho)\partial_x \rho + ... \end{cases}$$

  - Daganzo (Requiem, 95) PW is a terrible model!! [Diffusion still worse !]
    Paradoxes: 1: $v < 0$ **and 2:** $\lambda_2 = v + c > v$ **!!**
  - Aw-Rascle (Resurrection ?, 2000), Zhang(2002). Fixing:
    $\partial_x p \rightarrow \partial_t p + v\partial_x p$
  - Second equation in (PW) becomes:

    $$\partial_t v + v\,\partial_x v = -\tilde{p}'(\rho)(\partial_t + v\partial_x)(\rho)$$

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2 The Fluid Model

- The Eulerian System
- Motivations. Lagrangian version
- Link with microscopic models (FLM)
- Lagrangian Godunov Scheme
- Passing to the limit(s)

3 With Relaxation. Traveling Waves and Oscillations

- Motivations
- Remark: Whitham Subcharacteristic Condition
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4 Additional Remarks. Conclusion

The Fluid Model. Eulerian System

- Therefore, setting (new) $p(\rho) := \tilde{p}(\rho)$,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t w + v\, \partial_x w = 0. \end{cases}$$

Here, w : Lagrangian marker ("color") **defines the fundamental diagram**, e.g. $w := v + p(\rho) := v + v_{max} - V(\rho)$ or (better) $w := v - V(\rho)$**,** could be much more general

- In conservative form, the system becomes (E):

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0 \end{cases} \tag{3.1}$$

- Here $V(\rho)$ is a known function, with $V'(\rho) < 0$ and (strict concavity, again can be extended !), λ_1 is GNL: either shocks or rarefactions

Riemann Problem (RP). (Very) quick version

- **Riemann Problem: IVP with** $U(x,0) = U_\pm$ **for** $\pm x > 0$
- Strictly hyperbolic system, (except for $\rho = 0$ ...)
- Eigenvalues of 2x2 matrix : $\lambda_1(U) = v + \rho V'(\rho) < \lambda_2(U) = v$
- λ_1 : genuinely nonlinear shock (braking) or rarefaction (acceleration),whose curves coincide here, since $[\rho w(v-\sigma)] = ((\rho(v-\sigma)_\pm) \cdot [w] = 0$
- λ_2 is linearly degenerate : 2-contact discontinuity.
- Diagonalization: Riemann invariants (say on road i) :

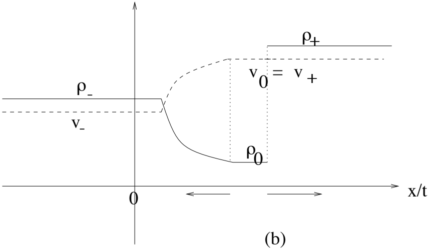
$$w(U) := w_i(U) = v - V_i(\rho) \text{ and } v(U) = v$$
$$\partial_t w + v\,\partial_x w = 0, \quad \partial_t v + \lambda_{1,i}(U)\partial_x v \approx 0$$

- **Solution of Riemann Pb** with initial data U_- and U_+: **first find U_0 with same w as U_- and same v as U_+.** Then, see Figure, construct:
  - a $1-$ wave connecting U_- and U_0 by a shock or rarefaction as for first order model, with fundamental diagram $v = V(\rho) + w(U_-)$:
    - ★ a rarefaction: $w(U_0) := v_0 - V(\rho_0) = w(U_-)$, if $v_0 = v_+ > v_-$,
    - ★ or a shock: $w(U_0) := v_0 - V(\rho_0) = w(U_-)$, if $v_+ > v_-$ (coinciding),
  - followed by a $2-$ wave between U_0 and U_+: contact discontinuity: $v_0 = v_+$
- **In all cases**,if $d(U^1, U^2)) := |v_1 - v_2| + |w_1 - w_2|$, then (BV estimates) (no wild oscillation)

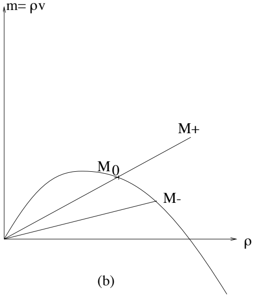
$$d(U^-, U^+) = d(U^-, U^0) + d(U^0, U^+),$$

and **(bounded) rectangles in (v, w) plane are invariant regions: L^∞ estimates. No more paradox 1 ($v < 0$) or 2 ($\lambda_2 > v$). No crash if no crazy driver (invariant region) ... Compare with PW, or Bando!**

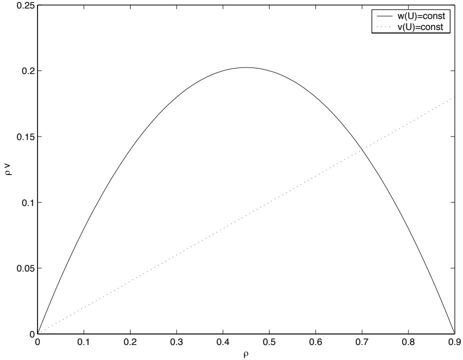
Figures

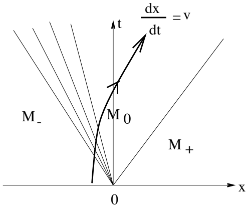

ρ+
v0 = v+
ρ-
v-
ρ0
0
x/t

(b)


m= ρv
M+
M0
M-
ρ

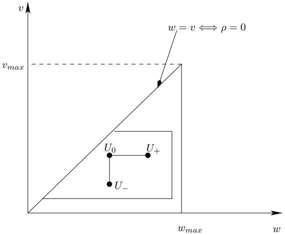
(b)

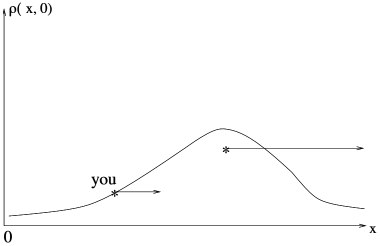

w(U)=const
v(U)=const
ρ v
ρ
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


dx/dt = v
t
M-
M0
M+
x
0

(b)

Riemann Pb in (v, w) plane BV estimate:
$d(U^-, U^+) = d(U^-, U^0) + d(U^0, U^+)$. No oscillation ...

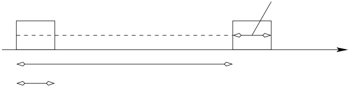




First motivation: x or t dependence? Do we react to flow variations in x or t : if the "wave" is faster than you, should you brake (cf gas dynamics), or accelerate (cf our model)?? Compare:
$\partial_t v + v\,\partial_x v = -\,\partial_x\,\tilde{p}(\rho)$ or $= -(\partial_t + v\,\partial_x)(\tilde{p}(\rho))$

Motivations. Lagrangian version

- Lagrangian mass coordinates (Courant-Friedrichs)



- From mass conservation,

$$\partial_t \partial_x X = \partial_x \partial_t X \,; \quad X(x,t) = \int_{-\infty}^{x} \rho(y,t)dy$$

  Essentially, $X = -N$, N : cumulated flow. Discrete X_j = position of car j if parked nose to tail. Also, s is additive (on a single lane), not ρ !! Important for homogenization.

- s and ρ are adimensional (occupancy), therefore invariant in a **hyperbolic scaling:** let a zoom parameter $\varepsilon \to 0$ and $(x', t', X', \Delta t', \Delta X') := \varepsilon\,(x, t, X, \Delta t, \Delta X)$

Link with microscopic Models (FLM)

- Follow The Leader Model (FLM) . We set $w = v - V(\rho)$ or $v - V(\tau)$

$$\begin{cases} \dot{x}_j = v_j \Longrightarrow \dot{s}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{v}_j = V'\left(\frac{x_j+1-x_j}{\Delta X}\right)\frac{v_j+1-v_j}{\Delta X} = V'(s_j)\dot{s}_j \end{cases} \tag{3.2}$$

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- Equivalent form (FLM'):

$$\begin{cases} \dot{s}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{w}_j = 0 \quad ; \quad w_j := v_j - V(s_j) \end{cases} \tag{3.3}$$

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- When $\Delta X \to 0$, (FLM') formally (in fact, rigorously) CV to Lagrangian System (L):

$$\begin{cases} \partial_t s - \partial_X v = 0\,,\ s := \rho^{-1}\,, \\ \partial_t w = 0\,,\ w = v - V(s) := v - V(\rho). \end{cases} \tag{3.4}$$

- First consider the first order Euler explicit discretization of (FLM') :

$$\begin{cases} s_j^{n+1} = s_j^n + \frac{\Delta t}{\Delta X}\left(v_{j+1}^n - v_j^n\right) \\ w_j^{n+1} = w_j^n = ... = w_j \; ... \end{cases} \tag{3.5}$$

- and its numerical solution CV to the solution of (FLM') when $\Delta t \to 0$, with ΔX fixed.

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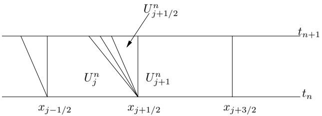
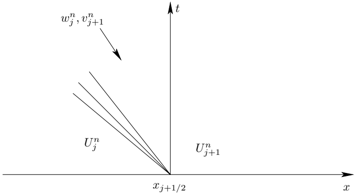
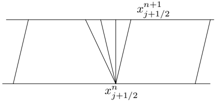
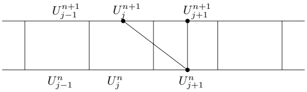
- On the other hand, (3.5) is exactly the Godunov approximation of Lagrangian system and (**exceptional**) has same BV stability as the Riemann Pb (in each Lagrangian cell, v is monotonous, since $w = C$)
- Even for weak solutions (Wagner, 87) (L) is equivalent to system (E).
- Eigenvalues become: $\lambda_1 = -V'(s) < 0$ (GNL), and $\lambda_2 = 0$ (LD), with same Riemann Invariants v, w and same structure (coinciding ...)

Lagrangian Godunov Scheme









Indeed, in Eulerian moving coordinates, $x^{n+1}_{j+1/2} = x^n_{j+1/2} + \Delta t\, v^n_{j+1}$.

Therefore $s_j^{n+1} = \frac{x^{n+1}_{j+1/2} - x^{n+1}_{j-1/2}}{\Delta X} = s_j^n + \Delta t\, \frac{v^n_{j+1} - v^n_j}{\Delta X}$, and $w_j^{n+1} = w_j^n$: the Godunov scheme for (3.4) is **exactly** (3.5) and defines numerically the trajectories. Moreover, w remains constant in each cell. Finally, we refresh v by $v_j^{n+1} = w_j^{n+1} + V(s_j^{n+1})$.

Passing to the limit(s)

- Thanks to uniform BV estimates, we can either let $\Delta t \to 0$, with ΔX fixed: (GOD) $\equiv$ the explicit Euler scheme CV to (FLM), which inherits same BV and L^∞ estimates (not obvious directly!)
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- By-product: (FLM) CV to (L): that can be used directly
- With no relaxation term, this procedure combines nicely with a hyperbolic scaling, with a zoom parameter $\epsilon \to 0$: and: $(x', t', X', \Delta t', \Delta X') := \epsilon\,(x, t, X, \Delta t, \Delta X)$.
- ρ, s, v, system (L) and (God) are unchanged in this scaling, but not the **initial data**

$$U_0(X, \epsilon X) := U_0(\frac{X'}{\epsilon}, X')$$

- Therefore, **if there is no small scale $\frac{X'}{\epsilon}$** in the initial data the solution of (God) converges to the **(unique)** solution of (L) when $\epsilon \longrightarrow 0$: withAw-Klar-Materne-Rascle, SIAP 2002)
- Independent, formal M. Zhang (2002)
- First ∃ result (no scaling): J. Greenberg (SIAP 2001), with Relax, (sub)"characteristic" case; Aw, PhD
- If ∃ small scales in initial data (oscillations in w and s), **homogenize** : with P. Bagnerini, SIMA 2003, cf also Hamilton-Jacobi approach...
- Oscillations in w (mixture) on outgoing roads in junctions: with Herty, Moutari, see further

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- Summary: start from (FLM'). Between two cars $j, j+1$, make $\rho = \rho_j(t)$ in Eulerian coord or $\tau = \tau_j(t)$ in Lagrangian coordinates, then choose between ODE an PDE approach, or both, e.g. for hybrid schemes.

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Motivations

- Stop and go waves: experimental observations show appearance and persistence of large oscillations, with almost constant negative propagation speed of fronts, both uspstream and downstream ⇒ no rarefaction
- Shocks are OK, but they tend to kill oscillations
- Contact discontinuities would be OK, but they only propagate oscillations.
- Here we want to see the effect of adding a relaxation term in the second equation

Whitham Subcharacteristic Condition

- If we use the **same** hyperbolic scaling, relaxation term becomes $\frac{1}{\varepsilon}(V_e(\rho) - v)$, with $\varepsilon \to 0$: **zero-relaxation limit** problem.
- Whitham **Subcharacteristic Condition** is then necessary for stability: **(SC)** : on the equilibrium curve: $v = V_e(\rho)$, characteristic speed of the formal equilibrium system, here

$$\partial_t \rho + \partial_x(\rho V_e(\rho)) = 0$$

must be between the two eigenvalues of the non-equilibrium system:

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \ q = \rho v \,, \\ \partial_t w + v\, \partial_x w = R(\rho, v) := \frac{1}{\varepsilon}(V_e(\rho) - v) \ : \end{cases} \tag{4.1}$$

"Convection must dominate relaxation". Pb: if so, our previous model is too stable (TVD), many others, e.g. Bando, too unstable ...

- From now on, we assume that there are more than one small parameter, (e.g. two in the IDM,) **and a relaxation time small, but nonzero**. Nevertheless, some weak form of (SC) must be enforced to exclude crashes (Bando) or negative velocities (PW) (invariant regions), see below.
- With a suitable scaling, the RHS is a Lipschitz function of the solution. Therefore, the classical results (existence, uniqueness, continuous dependence in L^1 ...) apply. Of course, we lose the TVD property. We want to study Traveling wave solutions

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- With a suitable scaling, the RHS is a Lipschitz function of the solution. Therefore, the classical results (existence, uniqueness, continuous dependence in L^1 ...) apply. Of course, we lose the TVD property. We want to study Traveling wave solutions
- **Thm (Le Roux)** For a large class of systems, including (4.2) below , traveling waves are "generic" in the sense: **any smooth "simple wave"**, i.e. any smooth solution whose all components are functions of one of them, (e.g. of ρ) **must be a traveling wave.**

Smooth "simple waves" are generically Traveling Waves

- Of course, discontinuous solutions (shocks or contacts) persist, since they can't "see" the relaxation term. In contrast, in some sense, **"T-waves replace rarefaction waves when there is a RHS".**
- **Proof:**
$$\begin{cases}\partial_t \rho + \partial_x q = 0, \; q = \rho v \,, \\ \partial_t w + v \, \partial_x w = R(\rho, v) := (V_e(\rho) - v) \;\; : \end{cases} \tag{4.2}$$

  Assume that v, q and $w = v - V(\rho)$ are (unknown) functions of ρ. Then by (4.2,i), we have

$$\partial_t \rho = -q'(\rho) \partial_{x\rho}. \tag{4.3}$$

  Now divide (4.2,ii) by $R(\rho, v)$ and use (4.3) to obtain, for some function F,

$$F'(\rho) \partial_x \rho = 1.$$

- Therefore $F(\rho(x,t)) = x - A(t)$. Now, multiply (4.2,i) by $F'(\rho)$, so that, for some function A

  $$F'(\rho)\partial_t\rho = -A'(t) = -F'(\rho).q'(\rho)\partial_x\rho = -q'(\rho).$$

  Differentiating this relation in x or in t shows first that $q"(\rho)\partial_x\,\rho \equiv 0$ and then, using (4.3), that
  $A"(t) = q"(\rho)\,\partial_t\rho = -q'(\rho)q"(\rho)\,\partial_x\rho \equiv 0$.
  Therefore, the solution is a function of $x - A(t) = x - ct$.

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  $A''(t) = q''(\rho)\,\partial_t \rho = -q'(\rho)q''(\rho)\,\partial_x \rho \equiv 0.$
  Therefore, the solution is a function of $x - A(t) = x - ct$.
- Of course, this is only true locally ...

- Therefore $F(\rho(x,t)) = x - A(t)$. Now, multiply (4.2,i) by $F'(\rho)$, so that, for some function A
$$F'(\rho)\partial_t \rho = -A'(t) = -F'(\rho).q'(\rho)\partial_x \rho = -q'(\rho).$$
Differentiating this relation in x or in t shows first that $q''(\rho)\partial_x \rho \equiv 0$ and then, using (4.3), that
$A''(t) = q''(\rho)\,\partial_t \rho = -q'(\rho)q''(\rho)\,\partial_x \rho \equiv 0$.
Therefore, the solution is a function of $x - A(t) = x - ct$.
- Of course, this is only true locally ...
- In the sequel, we will work in Lagrangian coordinates. We consider
$$\begin{cases} \partial_t s - \partial_X v = 0, \\ \partial_t w = R(s,v) := V_e(s) - v)\ , \end{cases} \tag{4.4}$$
assuming there are given reasonable functions such that the relations
$$v = V(s,w) \Leftrightarrow w = W(s,v) \Leftrightarrow s = S(v,w)$$
are equivalent and that, e.g. for $v = V(s) + w$, we have:
$$v = V(s) + w = V(s) + W_e(s) = V_e(s)) \Leftrightarrow w = W_e(s) = V_e(s) - V(s)$$

J. Greenberg's periodic solutions. Extensions

- Here, we show how to construct periodic solutions of (4.4) with one T-wave $U_- U_+$ and one adjacent shock $U_+ U_-$.
- First, we seek a T-wave $U(\xi) := U(X + ct)$, $c > 0$ (U travels backwards) in Lagrangian coordinates, with e.g. $w = v - V(s)$

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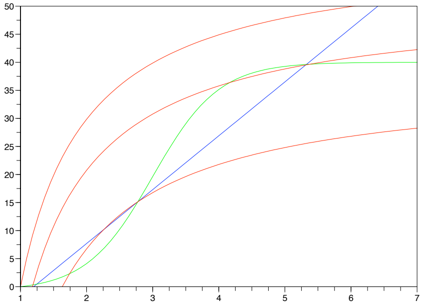
- Subcharacteristic Condition: $-V'(s) < -V_e'(s) < 0$ **only** satisfied on eq. curve for small or large s: invariant regions $v \geq 0$, no crash ...
- A T-wave connecting U_- to U_+ must satisfy: $c\dot{s} - \dot{v} = 0$ and (therefore) on the straight line $U_- U_+$, we must have:

  $$\dot{s}(\xi) = \frac{R(s, v = cs + C)}{c(c - V'(s)} := \frac{N}{D}\text{: } N \text{ and } D \text{ must vanish together, i.e.}$$

  D must vanish at the intersection point$U_0 := U_- U_+ \cap \{v = V_e(s)\}$: since $D = c\frac{d}{ds}(w(s, v = cs + C))$, the level curve $\{w(U) = w(U_0)\}$ must be **tangent** at U_0 to the straight line $U_- U_+$.

- **Existence** of such a solution (heteroclinic orbit) by intermediate value Theorem, when $U_\pm \in$ "(SC) stable" region of $\{v = V_e(\tau)\}$. Figure. Uniqueness??
- Similar solutions exist with nearby endpoints $U_\pm$ not at rest (thus reached in finite time), with $w(U_-) = w(U_+)$. Then the T-wave can be interrupted (before reaching equilibrium) by an adjacent shock wave U_+U_- with **same** speed (Rankine-Hugoniot): $\exists$ periodic solutions, typically on a ring road
- Stability of such waves? How relevant is linear stability analysis ??

Ve(s) V(s) v(s)



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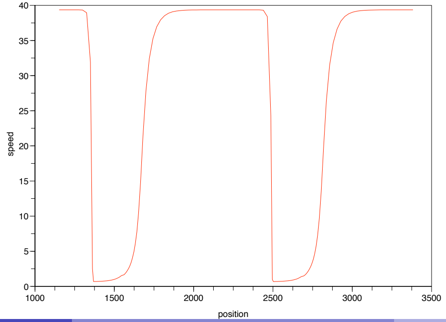

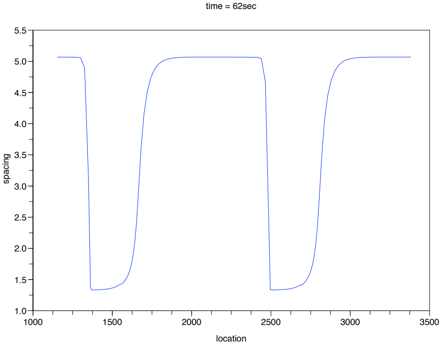
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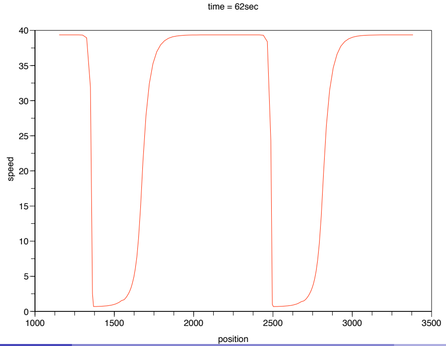
2000
2500
3000
3500
location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30
25
20
15
10
5
0
1000
1500
2000
2500
3000
3500
position

An example: the Intelligent Driver Model

With some modifications (e.g. on the length l) , this model writes

$$\begin{cases} \dot{x}_j = v_j, \\ \dot{v}_j = a\,[1 - (\frac{v_j}{v_0})^m - (\frac{s_b(v_j) - v_j \frac{v_{j+1} - v_j}{2ab}}{x_{j+1} - x_j - l})^p], \end{cases} \tag{4.5}$$

where

$$s_b(v) := s_0 + s_1\,\frac{v}{v_0} + Tv,$$

with $m = 1, 2$ or 4, $p = 1$ or 2, and

$$a = b = 1m/s^2;\; v_0 = 33m/s:\; l = 1m;\; s_0 = 1m;\; s_1 = 10m;\; T = 1sec.$$

We introduce reference quantities: x_r, t_r, v_r. Assume that $v_r = x_r/t_r = v_0$.
Two (small) dimensionless parameters appear $\varepsilon := \frac{l}{x_r}$, $\mu := \frac{a\,l}{v_0^2}$.
Typically, we choose: $x_r = 1000m$; $t_r = 30sec$, $v_r = v_0 = 33m/sec$ ...

so that $\varepsilon := \frac{5}{1000} = \frac{1}{200} = \mu := \frac{5}{33^2}$.
The term: $\frac{x_{j+1}-x_j-l}{x_r}$ becomes: $\frac{x_{j+1}-x_j}{\varepsilon} - 1 := s_j$ in rescaled coordinates, and the system rewrites

$$\begin{cases} \dot{s}_j = \frac{v_{j+1}-v_j}{\varepsilon}, \\ \dot{v}_j = \frac{\mu}{\varepsilon}\,[1 - v_j^m - (\frac{s_b(v_j) - v_j \, \frac{\varepsilon}{\mu} \, \frac{v_{j+1}-v_j}{\varepsilon}}{s_j})^p], \end{cases} \tag{4.6}$$

with now $s_b(v) := a_0 + a_1 v + a_2 v$ and $a_0 = \frac{s_0}{l}$, $a_1 = \frac{s_1}{l}$, $a_2 = \frac{T v_0}{l}$.
Now, first $\mu = \varepsilon$ and next, say for $p = 1$, (4.6,ii) rewrites:

$$\dot{v}_j = A(v) - (s_j)^{-1}\,[s_b(v_j) - v_j\,\dot{s}_j], \text{ with } A(v) := 1 - v_j^m. \tag{4.7}$$

Now, divide both members by v_j and define $w = W(s, v) := ln(s/v)$. The second equation rewrites:

$$\dot{w} = \frac{1}{v}\,[A(v) - \frac{s_b(v)}{s}] = \frac{A(v)}{v}\,[1 - \frac{s_e(v)}{s}], \tag{4.8}$$

with $s_e(v) := \frac{s_b(v)}{A(v)}$. Finally, we recognize here a system quite similar to the Follow the Leader Model (3.2), namely:

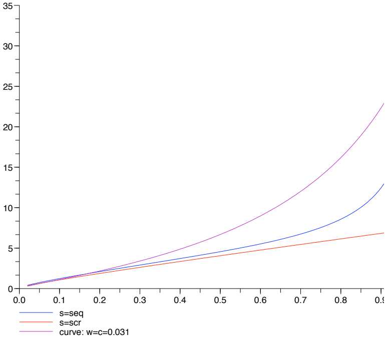
$$\begin{cases} \dot{s}_j = \frac{v_{j+1} - v_j}{\varepsilon}, \\ \dot{w}_j = \frac{A(v_j)}{v_j}\,[1 - \ \frac{s_e(v_j)}{s_j}]\,; w = ln(s/v), \end{cases}$$

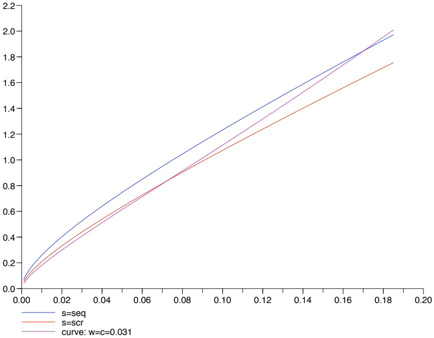
whose natural macroscopic version is thus:

$$\begin{cases} \partial_t s = \partial_X v, \\ \partial_t w = \frac{A(v)}{v}\,[1 - \ \frac{s_e(v)}{s}]\,; w = ln(s/v), \end{cases}$$

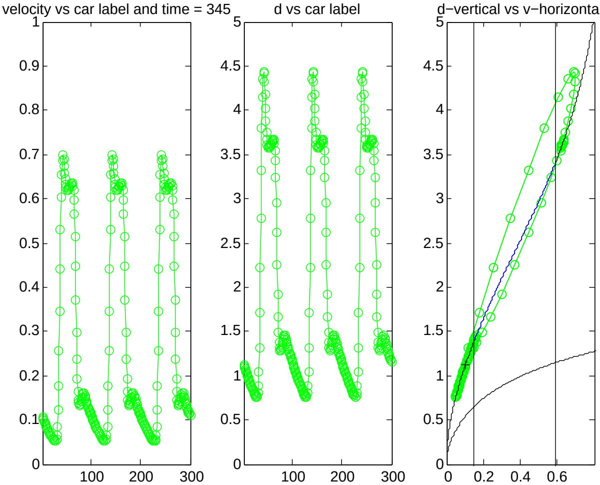
which is quite similar to (4.4). Natural question: can we exhibit Traveling Waves previous style for this system? Answer: NO for this one precisely, for hidden geometric reasons. Good hope for variants of this system. The Helbing-Treiber approach is thus much nearer by ours than we thought ... see the discussions on Payne-Whitham type of model. Work in progress, collaboration or contact with J. Greenberg, I. Gasser, C. Schmeiser ...

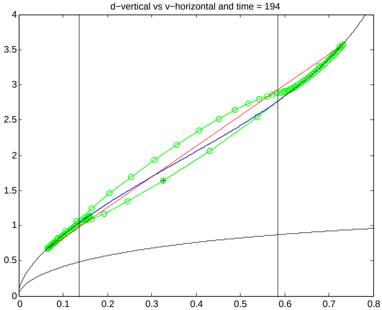
Analyzing the nature and the number of intersection points between these curves is not easy ... here for the case $p = 2$: the (non ?)-existence of T-waves previous style depends on these details ..





Examples of numerical results for a variant: J.Greenberg. Here, the waves look like for Bando's model:No shock, periodic orbits, at the expenses of adding a diffusion term *in the first equation*: philosophy of "equivalent equation to higher order"...




d−vertical vs v−horizontal and time = 194
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0.5
1
1.5
2
2.5
3
3.5
4
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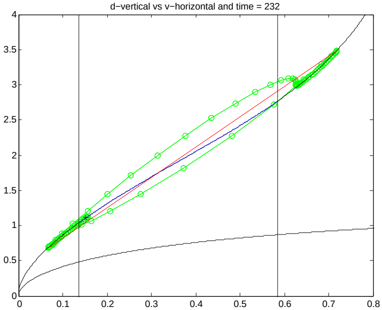
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8


d−vertical vs v−horizontal and time = 232
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1.5
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3.5
4
0
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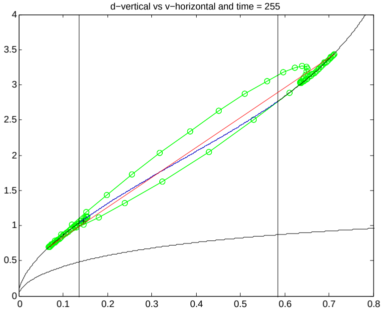
0.2
0.3
0.4
0.5
0.6
0.7
0.8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0
0.5
1
1.5
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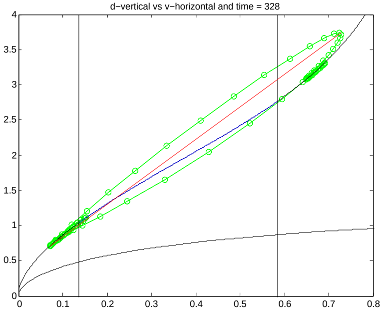
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8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4
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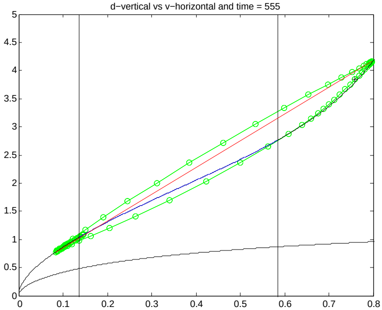
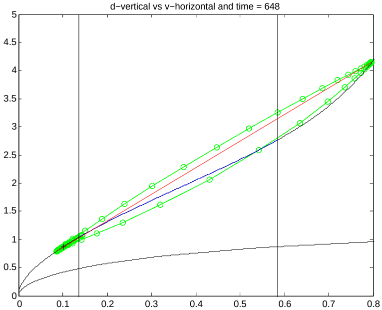
0.4
0.5
0.6
0.7
0.8


d−vertical vs v−horizontal and time = 555
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8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4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

1. Discrete / Fuid Models
2. The Fluid Model
   - The Eulerian System
   - Motivations. Lagrangian version
   - Link with microscopic models (FLM)
   - Lagrangian Godunov Scheme
   - Passing to the limit(s)
3. With Relaxation. Traveling Waves and Oscillations
   - Motivations
   - Remark: Whitham Subcharacteristic Condition
   - Smooth "simple waves" are generically Traveling Waves
   - J. Greenberg's periodic solutions. Extensions
   - An example: the Intelligent Driver Model
4. **Additional Remarks. Conclusion**

Additional Remarks. Conclusion

- Adding a relaxation term which slightly violates the subcharacteristic (only at intermediate densities) can start giving nice qualitative results, still avoiding any crash or negative speed (invariant regions). Compare with Bando or PW.
- If we took the zero-relaxation limit the limit solution would be a non-entropic solution of the limit First order model.
- The ARG philosophy (crucial role of Jim!) and the IDM philosophies are much nearer than we thought!
- Also, don't forget the junctions, in which an excitation-relaxation model can produce and sustain instabilities upstream, and therefore strongly modify the flow on a whole network.