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Kinetic Limit for the Weakly Nonlinear Schrödinger Equation

(and other wave equations)

joint work with

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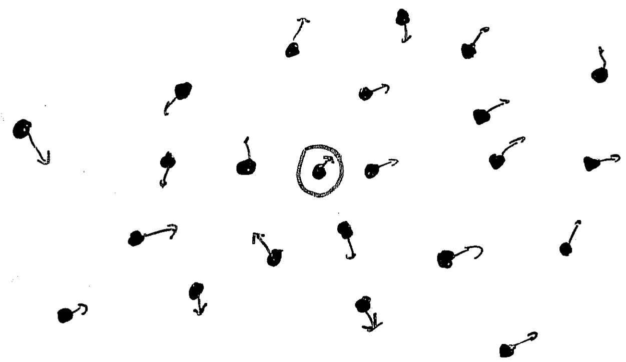
motion in random medium



particles

E. Frey

waves

$$i \frac{d}{dt} \psi = - \Delta \psi \quad + (\psi^* \psi)\, \psi$$

random potential $V(x,t)$

1.) Kinetic Limit

- classical particles interacting through short range potential

Boltzmann

kinetic limit   range a 3D

$$\| \, a \to 0, \; N \to \infty, \; a^2 N \text{ fixed} \, \|$$

Grad (1951), Lanford (1975)

random initial data!

- What about waves?

Peierls (1929)   obtained kinetic equation

wave turbulence $\lambda^2 = \varepsilon$

limit?   nonlinearity strength λ

time $\lambda^{-2} t$, space $\lambda^{-2} r$,

energy $\cong$ volume $= \lambda^{-6}$ $\lambda \to 0$

random initial data!

Boltzmann distribution function f

$\rightsquigarrow$ Wigner function of wave field $\psi^* \psi$

very rough picture

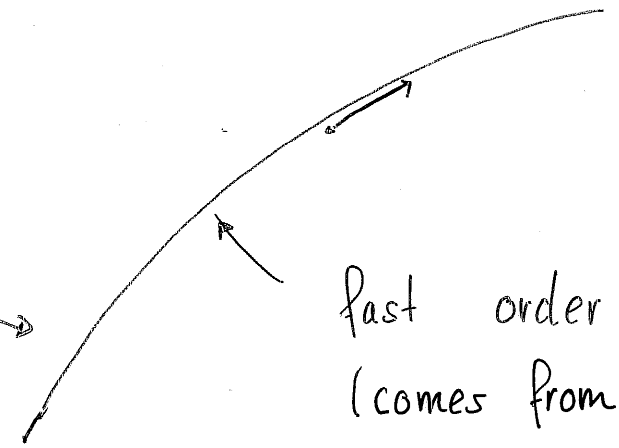
$$i \frac{d}{dt} \psi = - \Delta \psi + \lambda |\psi|^2 \psi$$

$\boxed{\text{space of measures on phase space}}$

order λ^2

spatially homogeneous

slow manifold



fast order 1
(comes from linear part)

slow manifold consists of "natural" measures invariant under the linear dynamics

particles : Poisson measures

waves : subclass of Gaussian measures

(gauge invariant)

Are there any proofs ?

- linear wave equation with weak random coefficients

⇒ Schrödinger with random potential

Erdös, Yau, ..., 2000 onwards

⇒ harmonic crystal with random masses

Lukkarinen, H.S (2007)

⇒ wave equation with random index of refraction

Bal, Komorowski, Ryzhik (2003)

(high frequency)

- nonlinear wave equation THIS TALK

nonlinear Schrödinger equation

symmetry between q, p

2.) NLS

wave field $\psi(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^D$

$$i \frac{\partial}{\partial t} \psi = -\Delta \psi + \lambda |\psi|^2 \psi$$

$$\lambda \geq 0$$

$\Rightarrow$ random initial data

singular at short distances

$$\mathbb{R}^D \rightsquigarrow \mathbb{Z}^D$$

$-\Delta \rightsquigarrow$ convolution with $\alpha(x)$,

|| α real , $\alpha(x) = \alpha(-x)$
compact support

$\rightsquigarrow$ dispersion relation

$$\omega(k) = \hat{\alpha}(k) \quad , \quad k \in \mathbb{T}^D = [0,1]^D$$

$\omega(k) = \omega(-k)$, real analytic

D-torus

$$\psi : \mathbb{Z}^D \to \mathbb{C}$$

$$i \frac{\partial}{\partial t} \psi(x,t) = \sum_y \alpha(x-y)\, \psi(y,t) + \lambda |\psi(x,t)|^2 \psi(x,t) \qquad (*)$$

- on-site

ALSO $\sum_y |\psi(y,t)|^2 V(y-x)\, \psi(x,t)$

- Hamiltonian system

$$\psi(x) = \frac{1}{\sqrt{2}} (p_x + i q_x)$$

energy

$$H = \sum_{x,y} \alpha(x-y)\, \psi(x)^* \psi(y) + \frac{1}{2} \lambda \sum_{x,y} |\psi(x)|^2 V(x-y) |\psi(y)|^2$$

(*) has solutions global in time of
infinite energy

Lanford, Lieb, Lebowitz 1974

Butta, Caglioti, Di Ruzza, Marchioro 2007

(propagation estimates)

number $\|\psi\|^2$

energy $H(\psi)$

are locally conserved

Fourier (momentum) space

$$i\frac{\partial}{\partial t}\hat{\psi}(k,t) = \omega(k)\hat{\psi}(k,t)$$

$$+\lambda\int_{(T^D)^3} dk_2\, dk_3\, dk_4\; \delta(k+k_2-k_3-k_4) \times \hat{V}(k_2-k_3)\,\hat{\psi}(k_2,t)^*\,\hat{\psi}(k_3,t)\,\hat{\psi}(k_4,t)$$

3. The Boltzmann-Peierls equation

random initial data Gaussian, translation invariant

$$\langle \psi \rangle = 0 \quad , \quad \langle \psi \psi \rangle = 0$$

$$\langle \psi(x)^* \psi(y) \rangle = \int dk \, e^{i 2\pi (y-x) k} \, W(k)$$

Wigner function

$t > 0$ $$\langle \psi(x,t)^* \psi(x,t) \rangle = \int dk \, e^{i 2\pi (y-x) k} \, W_\lambda(k,t)$$

Conjecture

$$\lim_{\lambda \to 0} W_\lambda(k, \lambda^{-2} t) = W(k,t)$$

and

W is solution of

(1D)

$$\frac{d}{dt} W(t) = \mathcal{C} W(t)$$

$$\mathcal{C} W(k_1) = 4\pi \int dk_2 \, dk_3 \, dk_4 \, \underbrace{\delta(k_1 + k_2 - k_3 - k_4)}_{\text{momentum}} \, \underbrace{\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)}_{\text{energy}}$$

$$[W_1 W_3 W_4 + W_2 W_3 W_4 - W_1 W_2 W_3 - W_1 W_2 W_4]$$

formal derivation

$$\frac{d}{dt}\langle\psi\psi\rangle_t = \langle\psi\psi\psi\psi\rangle_t$$

$$= \langle\psi\psi\psi\psi\rangle_0 + \int_0^t ds \, \underbrace{\langle\psi\psi\psi\psi\psi\psi\rangle_s}_{\text{Gaussian truncation}}$$

|| problem remains OPEN ||

slow spatial variation

$\Rightarrow$ transport $\frac{1}{2\pi} \nabla_k \omega \cdot \nabla_r$

law of large numbers

B-P holds almost surely

4. Equilibrium time correlations

$\langle \cdot \rangle_\beta$ equilibrium measure small λ

$$\frac{1}{Z} e^{-\beta H(\psi) + \beta\mu \|\psi\|^2} \prod_{x\in\Lambda} d\psi(x)\, d\psi(x)^*$$

$\| \min_k \omega(k) > \mu \|$ $\rightsquigarrow$ exponential clustering

$\|\,\psi(x,t)$ is stationary random field $\|$

$$\langle \psi(x,t)\, \psi(y)^* \rangle_\beta = \int_{\mathbb{T}^D} dk\, C_\lambda(k,t)\, e^{ik(x-y)}$$

$t=0$ $\quad \lim_{\lambda\to 0} C_\lambda(k,0) = \dfrac{1}{\beta(\omega(k)-\mu)}$

$\lambda=0$ $\quad C_0(k,t) = e^{-i\omega(k)t} \dfrac{1}{\beta(\omega(k)-\mu)}$

RESULT $\quad t \geq 0$

$$\lim_{\lambda \to 0} e^{i\omega^\lambda(k)t/\lambda^2} C_\lambda(k, \lambda^{-2} t) = \frac{1}{\beta(\omega - \mu)} e^{-\nu(k)t}$$

- $\mathrm{Re}\, \nu(k) > 0$ DECAY
- renormalized dispersion

$$\omega^\lambda(k) = \omega(k) + \lambda R_0(k) + \lambda^2 R_1(k) + O(\lambda^3)$$

- ν, R_0, R_1 are explicit

dangerous

There must be some conditions!

conditions

A) ℓ_1-clustering (static)

$n \geq 4$

fully truncated

$$\sum_{x \in \mathbb{Z}^{nD}} \delta_{x_1 0} \left| \left\langle \prod_{j=1}^{n} \psi(x_j)^{\#} \right\rangle_\beta^T \right| \leq \lambda (c_0)^n \, n!$$

Abdessalam, Proccacia, Scoppola (2009)

B) ℓ_3-bound

$$p_t(x) = \int_{\mathbb{T}^D} dk \; e^{-i\omega(k)t} \, e^{ikx}$$

$$\sum_x |p_t(x)|^3 \leq c(1+|t|)^{-1-\delta}$$

$\delta > 0$

- stationary phase

  $D \geq 3$

c) constructive interference

bad set $\mathcal{M}$

$$\left| \int_{\mathbb{T}^D} dk \; e^{-it(\omega(k) \pm \omega(k-k_0))} \right| \leq \frac{c}{1+|t|} \; \frac{1}{d(k_0, \mathcal{M})}$$

distance

- $0 \in \mathcal{M}$
- $\mathcal{M}$ discrete ? not clear ?

requires $D \geq 3$ from other parts of proof

- $\mathcal{M}$ finite collection of smooth curves

o.k. for $-\Delta$ (n.n.)

requires $D \geq 4$

$D \geq 4$

D] crossing bound

o.k. for $D \geqslant 3$, nearest neighbor

in addition

finite kinetic time

$|t| < t_0$

5. Strategy

- Dohamel expansion

$$\frac{d}{dt} a_t = \lambda (a_t)^3 \qquad a_t \in \mathbb{R}$$

$$\frac{d}{dt} (a_t)^n = \lambda n (a_t)^{n+2}$$

$$a_t = a_0 + \lambda \int_0^t ds\, (a_s)^3$$

iterate

$$\Rightarrow \quad a_t = \sum_{n=0}^{\infty} \lambda^n c_n(t) (a_0)^{2n+1}$$

- convergence

$$c_n(t) = n! \frac{1}{n!} t^n$$

times Gaussian pairings for $\psi(t=0)$

$n!$ zero radius

- truncate series

$$\lambda^{\alpha} N! = 1$$

$$\psi(t) = \psi_{main}(t) + \underbrace{\psi_{error}(t)}_{\text{full time evolution}}$$

Duhamel expanded at order N

only $\psi(t=0)$

- stationarity

$$|\langle \psi_{error}(t)^{*} \psi(0) \rangle|^{2}$$

$$\leq \langle | \int_0^t ds \, \mathcal{A}(t-s) \underbrace{\psi(s)}_{\downarrow \, \psi(0)}^{\otimes(2N+1)} |^2 \rangle \langle |\psi(0)|^2 \rangle$$

// large, but finite N analysis //

$\cup$

Main task

high dimensional oscillatory integrals

with constraints

interaction: $\int \delta(k_1+k_2-k_3-k_4)\, \hat{\varphi}^* \hat{\varphi} \hat{\varphi}$

initial conditions:

$$\langle (\hat{\varphi}^{\#})^{2n} \rangle = \sum \text{pairings}$$

$$\langle \hat{\varphi}(k)^* \hat{\varphi}(k') \rangle = \delta(k-k') \frac{1}{\omega(k)-\mu}$$