ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE VLASOV–POISSON & VLASOV–MAXWELL EQUATIONS IN LOW DIMENSION

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THE VLASOV–POISSON system (VP):

$$\frac{\partial f}{\partial t} + v f_x + E f_v = 0$$

$$\frac{\partial g}{\partial t} + v g_x - E g_v = 0$$

$$\rho(t,x) = \int_{\mathbb{R}^1} (f(t,x,v) - g(t,x,v))\, dv.$$

$$E(t,x) = \frac{1}{2}\left(\int_{-\infty}^{x} \rho(t,y)\ dy - \int_{x}^{\infty} \rho(t,y)\ dy\right).$$

Here $t \geq 0$ is time, $x \in \mathbb{R}$ is position, $v \in \mathbb{R}$ is velocity. Also consider the relativistic version (RVP) with the same ρ and E,

$$\frac{\partial f}{\partial t} + \hat{v} f_x + E f_v = 0$$

$$\frac{\partial g}{\partial t} + \hat{v} g_x - E g_v = 0$$

$$\hat{v} = \frac{v}{v_0}, \quad v_0 = \sqrt{1+v^2}.$$

The RELATIVISTIC Vlasov–Maxwell system (RVM) is with $x,\ v \in \mathbb{R}^3$

$$\frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0$$

$$\frac{\partial g}{\partial t} + \hat{v} \cdot \nabla_x g - (E + \hat{v} \times B) \cdot \nabla_v g = 0$$

$$\frac{\partial E}{\partial t} = \nabla \times B - j$$

$$\frac{\partial B}{\partial t} = -\nabla \times E$$

$$\nabla \cdot E = \rho$$

$$\nabla \cdot B = 0$$

$$\rho(t,x) \equiv \int_{\mathbb{R}^3} (f(t,x,v) - g(t,x,v))\, dv$$

$$j(t,x) \equiv \int_{\mathbb{R}^3} \hat{v}(f(t,x,v) - g(t,x,v))\, dv.$$

The 1.5D Version: $x \in \mathbb{R}^1, \ v \in \mathbb{R}^2$

$E(t,x) = (E_1(t,x), E_2(t,x), 0), \ B(t,x)$ scalar

$$\frac{\partial f}{\partial t} + \hat{v}_1 f_x + (E_1 + \hat{v}_2 B) f_{v_1} + (E_2 - \hat{v}_1 B) f_{v_2} = 0$$

$$\frac{\partial g}{\partial t} + \hat{v}_1 g_x - (E_1 + \hat{v}_2 B) g_{v_1} - (E_2 - \hat{v}_1 B) g_{v_2} = 0$$

$$\partial_x E_1 = \rho = \int (f - g)\, dv,$$

$$\partial_t E_1 = -j_1 = -\int \hat{v}_1 (f - g)\, dv$$

$$\partial_t E_2 = -\partial_x B - j_2, \quad \partial_t B = -\partial_x E_2$$

$$j_2 = \int \hat{v}_2 (f - g)\, dv$$

Consider 1D VP/RVP with C_0^∞ data ≥ 0. Let

$$F(t,x) = \int f(t,x,v)\,dv,$$

$$G(t,x) = \int g(t,x,v)\,dv.$$

Then $\rho = F - G$. Assume **Neutrality:**

$$\int \rho(t,x)\,dx = 0.$$

We will show for (VP)&(RVP) that

$$\int_0^\infty \int E^2(F+G)\,dx\,dt < \infty.$$

Also for (VP)

$$\int_0^\infty \int (F^4 + G^4)\,dx\,dt < \infty$$

while for (RVP)

$$\int_0^\infty \left(\int \left(F(t,x)^{\frac{7}{4}} + G(t,x)^{\frac{7}{4}} \right) \, dx \right)^4 dt < \infty.$$

The local charges for solutions to both systems will satisfy for any fixed $R > 0$

$$\lim_{t\to\infty} \int_{|x|<R} F(t,x)\, dx = 0$$

$$\lim_{t\to\infty} \int_{|x|<R} G(t,x)\, dx = 0.$$

Finally, for solutions to (VP) or (RVP) we have that

$$\lim_{t\to\infty} \|E(t,\cdot)\|_\infty = 0.$$

Sketch of the proof: A new identity. Take (VP):

$$F_t = -\int (v f_x + E f_v)\, dv = -\partial_x \int v f \, dv$$

and thus

$$\partial_t \int_{-\infty}^{x} F(t,y)\,dy = -\int vf(t,x,v)\,dv$$

with a similar result for g. Multiply the f equation in (VP) by $v \cdot \int_{-\infty}^{x} F(t,y)\,dy$ and integrate over v and x:

$$\frac{d}{dt}\int \left[\int vf \int_{-\infty}^{x} F(t,y)\,dy\,dv\right] dx$$

$$+\int \left(\int vf\,dv\right)^2 dx - \int F(t,x) \int v^2 f\,dv\,dx$$

$$+\frac{1}{2}\int \rho(t,x)\left[\int_{-\infty}^{x} F(t,y)\,dy\right]^2 dx = 0.$$

Now repeat this calculation with f replaced by g and add:

$$0=\frac{d}{dt}\int\Big[\int vf\int_{-\infty}^{x}F(t,y)\,dy\,dv$$

$$+\int vg\int_{-\infty}^{x}G(t,y)\,dy\,dv\Big]\,dx$$

$$+\int\left(\int vf\,dv\right)^2dx-\int F(t,x)\int v^2f\,dv\,dx$$

$$+\int\left(\int vg\,dv\right)^2dx-\int G(t,x)\int v^2g\,dv\,dx$$

$$+\frac{1}{2}\int\rho\left(\left[\int_{-\infty}^{x}F\,dy\right]^2-\left[\int_{-\infty}^{x}G\,dy\right]^2\right)dx$$

The first line is bounded when integrated in time. The second and third lines are nonpositive. Call L the last term above. Then because $\rho=\int(f-g)\,dv=F-G$ and $E=\int_{-\infty}^{x}\rho(t,y)\,dy=\int_{-\infty}^{x}(F-G)\,dy$ we get

$$L=-\frac{1}{4}\int E^2(F+G)\,dx.$$

Thus in particular

$$\int_0^\infty \int E^2(F+G)\,dx\,dt < \infty$$

and

$$\int_0^\infty \int \left[F \int v^2 f\,dv - \left(\int vf\,dv \right)^2 \right] dx\,dt < \infty$$

$$\int_0^\infty \int \left[G \int v^2 g\,dv - \left(\int vg\,dv \right)^2 \right] dx\,dt < \infty.$$

We can use these inequalities directly to establish the L^4 estimate. We can write

$$F(t,x) \int v^2 f\,dv - \left(\int vf\,dv \right)^2$$

as

$$\frac{1}{2} \int \int (w-v)^2 f(v) f(w)\,dv\,dw \equiv \frac{1}{2} k.$$

$$F(t,x)^2 = \int\int f(v)f(w)\,dv\,dw$$

$$= \int_{|v-w|<R} + \int_{|v-w|>R} \equiv I_1 + I_2.$$

So $I_2 \le R^{-2}k(t,x)$ and in I_1

$$\int_{|v-w|<R} f(w)\,dw = \int_{v-R}^{v+R} f(w)\,dw \le cR.$$

Thus

$$I_1 \le c \cdot R \cdot F.$$

Optimize on R: $F^4 \le ck$

Uniform decay of E: from $E_x = \rho = F - G$,

$$\frac{\partial}{\partial x} E^3 = 3E^2\rho = 3E^2(F-G).$$

$$Q(t) := \int_{-\infty}^{\infty} E^2(t,x)\Big[F(t,x) + G(t,x)\Big]\,dx$$

is integrable in time and it's easy to show that $|\dot{Q}|$ is bounded. So $\|E\|_\infty \to 0$

REMARKS

1. Proof is very 1 dimensional

2. Seems to work for only 2 species

3. No known rate of decay for $\|E\|_\infty$

4. Does $\|E\|_2 \to 0$? Known proofs in 3D of Rein, Illner, Perthame fail.

In the following, we always assume C_0^∞ data ≥ 0 for f, g etc.

Other Asymptotics

Consider the relativistic, monocharged case ($g = 0$) in (RVP).

Theorem: Consider (RVP) and assume $f_0 \geq 0 \in C_0^1(\mathbb{R}^2)$ is $\not\equiv 0$. Then $\exists\ C > 0$ s.t.

$$\|\rho(t)\|_p \geq C$$

for all $t \geq 0, p \in [1, \infty]$.

[Contrast to Batt, Kruse, Rein: the charge density decays in sup–norm like t^{-1} for the classical, monocharged system (VP)]. This remains true for the 1.5D relativistic (VP):

$$\partial_t f + \hat{v}_1 \partial_x f + E_1 \partial_{v_1} f = 0$$

$$\rho(t, x) = \int f(t, x, v)\ dv$$

$$E_1(t, x) = \frac{1}{2}\left(\int_{-\infty}^{x} \rho(t, y) dy - \int_{x}^{\infty} \rho(t, y) dy\right)$$

Here $f = f(t, x, v_1, v_2)$ depends on two components of momentum. Why is this of onterest? Consider 1.5D (RVM). Assume that $E_2 = B = 0$ initially and that f is initially even in v_2. Then these properties persist for all $t > 0$, and the system reduces to the "one and one-half" dimensional (RVP) system. Hence ρ does not decay in any L^p norm for certain solutions of 1.5D RVM.

Growth of the momentum support

Theorem: Let $f(t,x,v)$ be a solution of monocharged 1.5D (RVP) and define

$Q_1(t) = \sup\{|v_1| : \exists x, v_2 \in \mathbb{R} s.t. f(t,x,v) \neq 0\}$.

Then for large t

$$C_1 t \leq Q_1(t) \leq C_2 t.$$

True for classical (VP) as well.

Theorem: Let f, g satisfy neutral 1.5D (RVP). Define

$Q_1(t) = \sup\{|v_1| : \exists\, x, v_2 \in \mathbb{R} s.t. f + g \neq 0\}$.

Then, $\exists\ C > 0$ such that for any $t \geq 0$ we have

$$Q_1(t) \leq C\sqrt{1+t}.$$

True for classical neutral (VP) as well.

Growth of momentum supp 1.5D RVM

The Cauchy problem: specify compactly supported data for $E_2(0,x)$, $B(0,x)$, neutral

$$\{f,g\}(0,x,v) = \{f^0(x,v),\, g^0(x,v)\} \geq 0$$

Theorem On supp$(f+g)$ as $t \to \infty$,

$$|v_2| = O(t^{\frac{1}{2}}), \quad |v_1| = O(t^{\frac{3}{4}})$$

Conservation Laws:

$$\rho_t + \partial_x j_1 = 0 \quad \textbf{Mass}$$

$$\partial_t e + \partial_x m = 0 \quad \textbf{Energy}$$

where with $v_0 = \sqrt{1+|v|^2}$

$$e = \frac{1}{2}\Big[|E|^2 + B^2\Big] + \int v_0(f+g)\,dv$$

$$m = -\int v_1(f+g)\,dv - BE_2.$$

hence

$$\int \rho(t,x)\,dx = \text{const.}; \int e(t,x)\,dx = \text{const.}$$

Assume **neutrality:**

$$\int \rho(0,x)\,dx = 0.$$

$$E_1(t,x) = \int_{-\infty}^{x} \rho(t,y)\,dy$$

Computing $\partial_t(E_2 \pm B)$ we find that E_2, B are given (mod data terms) by integrals like

$$\int_0^t j_2(\tau, x \pm t \mp \tau)\, d\tau$$

To get bounds on the fields E_2, B, integrate the energy identity over a cone to find

$$\int_0^t (e \mp m)(\tau, x \pm t \mp \tau)\, d\tau \leq \text{const.}$$

Now

$$e \pm m = \frac{1}{2}(E_1^2 + (E_2 \mp B)^2) + \int (v_0 \mp v_1)(f+g)\, dv$$

and

$$v_0 \mp v_1 \geq \frac{1 + v_2^2}{2v_0} \geq \frac{|v_2|}{v_0} \equiv |\hat{v}_2|$$

so the fields E_2, B are bounded and

$$\int_0^t \left[E_1^2 + (E_2 \mp B)^2\right](\tau, x \pm t \mp \tau)\, d\tau \leq \text{const.}$$

The characteristic ODE's for f are solutions $X(s,t,x,v)$, $V(s,t,x,v)$ to

$$\dot{X} = \widehat{V}_1; \quad X(t,t,x,v) = x$$

$$\dot{V}_1 = E_1 + \widehat{V}_2 B; \quad V_1(t,t,x,v) = v_1$$

$$\dot{V}_2 = E_2 - \widehat{V}_1 B; \quad V_2(t,t,x,v) = v_2$$

f is constant on these:

$$f(s, X(s,t,x,v), V(s,t,x,v)) = \text{const.}$$

so that

$$f(t,x,v) = f^0(X(0,t,x,v), V(0,t,x,v)).$$

Support Property: assume $f^0(x,v) = 0$ for $|x| \geq k$, $|v| \geq k$. Then $f(t,x,v) = 0$ for $|x| \geq k + t,\ |v| \geq k + \beta t$. Same for g.

Asymptotics: introduce a potential $A(t,x)$ s.t. $E_2 = -A_t$, $B = A_x$. The last Maxwell equation holds and

$$\frac{d}{ds}(V_2 + A) = E_2 - \widehat{V}_1 B + A_t + A_x \dot{X} = 0$$

$V_2 + A =$ const. along characteristics:

$$|v_2 + A(t,x)| = |V_2(0,t,x,v) + A(0, X(0,t,x,v))|$$

is bounded by assumption. Now A satisfies

$$A_{tt} - A_{xx} = j_2$$

Represent A as an integral over a backward cone:

$$A(t,x) = \text{data} + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} j_2(\tau, y)\, dy\, d\tau.$$

then $A = O(t)$: too weak. Rewrite as

$$\partial_t A_t - \partial_x A_x = j_2$$

integrate over the cone and use the divergence theorem. Then

$$\int_0^t \int_{x-t+\tau}^{x+t-\tau} j_2(\tau, y)\, dy\, d\tau$$

is bounded by integrals like

$$\int_0^t (E_2 \mp B)(\tau, x \pm t \mp \tau)\, d\tau$$

which are $O(\sqrt{t})$. Thus A and therefore v_2 are $O(\sqrt{t})$. Using the support property we can strengthen this: on $\mathrm{supp}(f+g)$,

$$|v_2| \leq c_1 + c_2\sqrt{t - |x| + 2k}.$$

Estimate on the growth of $V_1(t)$

We'll show that $V_1(t) \le ct^{3/4}$. From the energy we have

$$\int f\,dv \le 3(\sigma k)^{1/2} \quad \text{where}$$

$$k = \int v_0 f\,dv;\ \sigma = \int (v_0 - v_1) f\,dv.$$

Take characteristic $X(t), V(t)$ with $V_1(t) > 0$ and $f(t, X(t), V(t)) \ne 0$. WLOG $V_1(t) \ge 1$. Define

$$\Delta = \sup\Big\{\tau \in [0,t] : V_1(s) \ge \frac{1}{2} V_1(t)$$

$$\forall s \in [t-\tau, t]\Big\}.$$

Define $X_C(s) = X(t) - t + s$. Then for $s \in [t-\Delta, t]$

$$\left|\frac{d}{ds}(X(s) - X_C(s))\right| = 1 - \frac{V_1(s)}{V_0(s)} \quad \text{so}$$

$$\left|\frac{d}{ds}(X(s)-X_C(s))\right| \le \frac{(1+V_2(s)^2)}{V_1^2(s)}$$

$$\left|\frac{d}{ds}(X(s)-X_C(s))\right| \le c\frac{1+s}{(V_1(t))^2}.$$

$$|X(s)-X_C(s)| \le \frac{c(1+t)\Delta}{(V_1(t))^2}.$$

Now write the characteristic integral for $V_1(t)$:

$$V_1(t) = V_1(t-\Delta) + \int_{t-\Delta}^{t} E_1(s, X(s))\, ds$$

$$+ \int_{t-\Delta}^{t} \widehat{V}_2(s) B(s, X(s))\, ds.$$

For the E_1 term

$$I \equiv \int_{t-\Delta}^{t} E_1(s, X(s))\, ds =$$

$$\int_{t-\Delta}^{t} E_1(s, X_C(s))\, ds$$

$$+ \int_{t-\Delta}^{t} \Big(E_1(s, X(s)) - E_1(s, X_C(s)) \Big)\, ds.$$

The first term is $O(\Delta^{1/2})$ by the cone integral and Schwarz. The "f-part" of the second term is

$$\int_{t-\Delta}^{t} \int_{X_C(s)}^{X(s)} \int f\, dv\, dx\, ds.$$

Use here

$$\int f\, dv \leq 3(\sigma k)^{1/2}$$

The result is

$$I \le c\Delta^{1/2} + c\Delta(1+t)^{1/2}(V_1(t))^{-1}$$

How we should treat the magnetic term?

$$J \equiv \int_{t-\Delta}^{t} \widehat{V}_2(s) B(s, X(s))\, ds.$$

For this recall $|B(t,x)| \le c$ everywhere. Also $V_2^2(s) \le c(1+s)$ and on $[t-\Delta, t]$, $V(s) \ge \frac{1}{2}V(t)$ by the definition of Δ. Thus on this interval

$$|\widehat{V}_2(s)| \equiv \frac{|V_2(s)|}{\sqrt{1+V_1(s)^2+V_2(s)^2}}$$

$$\le \frac{c(1+s)^{1/2}}{V_1(s)} \le \frac{c(1+s)^{1/2}}{V_1(t)}.$$

So we have

$$J \le \frac{c}{V_1(t)} \int_{t-\Delta}^{t} (1+s)^{1/2}\, ds \le \frac{c(1+t)^{1/2}\Delta}{V_1(t)}.$$

(*)

$$V_1(t) \le V_1(t-\Delta) + c\Delta^{1/2} + c\frac{\Delta(1+t)^{1/2}}{(V_1(t))}.$$

The last term in this estimate is dominant so since $V_1(t) \ge 1$

$$V_1(t) \le V_1(t-\Delta) + \frac{c\Delta(1+t)^{1/2}}{V_1(t)}$$

$$(2) \qquad \le V_1(t-\Delta) + \frac{c\Delta(1+t)^{1/2}}{1+V_1(t)}.$$

$$\text{Let} \quad Q(t) = V_1^+(t) + 1.$$

Then (2) yields

$$Q(t) \le Q(t-\Delta) + \frac{c\Delta(1+t)^{1/2}}{Q(t)}$$

or

$$\frac{Q(t) - Q(t-\Delta)}{\Delta} \le c\frac{(1+t)^{1/2}}{Q(t)}$$

A continuous analogue of this is

$$\dot{Q} \leq c\frac{(1+t)^{1/2}}{Q(t)}$$

or

$$Q\dot{Q} \leq c(1+t)^{\frac{1}{2}}.$$

So for large t, $Q^2(t) \leq ct^{3/2}$ and it follows that

$$V_1(t) \leq ct^{3/4}.$$

Use of the support property gives in fact

$$v_1(t) \leq ct^{\frac{1}{2}}(t-|x|+2k)^{\frac{1}{4}}$$

on the support of $f+g$.