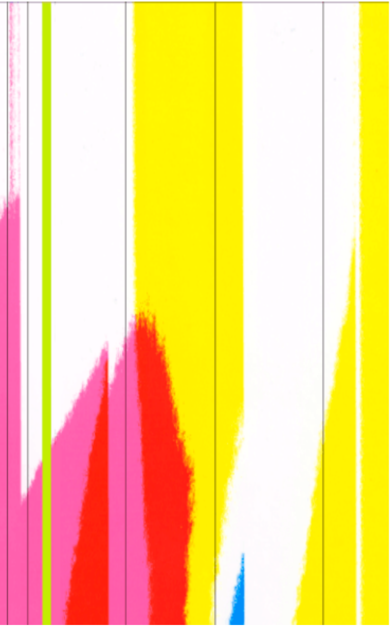

Tools for the Mathematical Analysis of Kinetic Equations

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I–Introduction to Kinetic Models

Statistical description

- $f(t,x,\xi)$ =particle distribution fct.=distribution of particles in phase space.
- t =time variable≥ 0, x =space variable$\in \mathbb{R}^N$, $\xi =$"state" variable$\in V$ (typically the velocity of the particle, $V = \mathbb{R}^N$)

Evolution PDE

$$\underbrace{\partial_t f + \nabla_x \cdot (v(\xi) f) + \nabla_\xi \cdot (Ff)}_{\text{Transport}} = \underbrace{\frac{1}{\tau} Q(f)}_{\text{Collisions}},$$

with $v : V \to \mathbb{R}^N$, the velocity (ex. $v(\xi) = \xi$), $F = F(t,x,\xi)$, $\tau > 0$.

Macroscopic quantities are Average wrt ξ (=observable quantities)

Ex. : $(\rho, \rho u, N\rho\theta)(t,x) = \displaystyle\int \big(1, v(\xi), |v(\xi) - u|^2\big) f \,\mathrm{d}\xi$

I.1–Colisionless eq., Particle viewpoint

Characteristics

Solve the ODE (Fundamental Principle of Mechanics) :

$$\frac{\mathrm{d}}{\mathrm{d}t}X_j = V(\Theta_j), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Theta_j = F(t, X_j, \Theta_j)$$

with initial data $X_j(0) = x_j$, $\Theta_j(0) = \xi_j$.
Then

$$f(t,x,\xi) = \sum_{j=1}^{J} \delta(x = X_j) \otimes \delta(\xi = \Theta_j)$$

solve

$$\partial_t f + \nabla_x \cdot (v(\xi) f) + \nabla_\xi \cdot (F f) = 0$$

as a consequence of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\varphi(t, X_j, \Theta_j)\Big] = \Big(\partial_t \varphi + v(\xi) \cdot \nabla_x \varphi + F(t,x,\xi) \cdot \nabla_\xi \varphi\Big)\Big|_{t, x = X_j, \xi = \Theta_j}$$

Colisionless eq., PDE viewpoint

Assume $F = F(t,x)$ and $Q = 0$

The PDE $\partial_t + v(\xi)\cdot\nabla_x f + F(t,x)\cdot\nabla_\xi f = 0$ recasts as $\frac{\mathrm{d}}{\mathrm{d}t}\Big[f(t,X,\Theta)\Big] = 0$ so that $f(t,x,\xi) = f_{\mathrm{Init}}\big(X(0;t,x,\xi),\Theta(0;t,x,\xi)\big)$.

Example and exercice

- $v(\xi)=\xi$, $F=0$ leads to $f(t,x,\xi) = f_{\mathrm{Init}}(x-t\xi,\xi)$ solution of $\partial_t f + \xi\cdot\nabla_x f = 0$.
- Generalize the formula when F depends on ξ. (Care of $\mathrm{div}_{x,\xi}\big(v(\xi),F(t,x,\xi)\big) = \nabla_\xi\cdot F(t,x,\xi)$)

So what ?

- L^p estimates are immediate
- But life is not so simple... since the def. of characteristics requires some regularity on $(x,\xi)\mapsto(a(\xi),F(t,x,\xi))$ (think of the Cauchy-Lipschitz theorem). Difficulties arise with **non-linear** models.

An example of non-linear model : Vlasov-Poisson eq.

The force field is defined by $F(t,x) = -\nabla_x \Phi(t,x)$ with Φ depending on the particle distribution through the Poisson eq.

$$-\Delta_x \Phi = \pm \int f(t,x,\xi)\,\mathrm{d}\xi = \pm\rho(t,x)$$

($+$=repulsive force, $-$=attractive force). Actually, the PDE could be misleading, it is better to define Φ by the convolution formula

$$\Phi(t,x) = \pm \int E_N(x-y)\rho(t,y)\,\mathrm{d}y$$

with $E_1(x) = -\dfrac{|x|}{2}$, $E_2(x) = -\dfrac{1}{2\pi}\ln(|x|)$, $E_N(x) = +\dfrac{C_N}{|x|^{N-2}}$.

Assuming L^p estimate on f what is the regularity of the potential Φ ?

A few useful comments

Source term, damping term

We solve $\partial_t f + \xi \cdot \nabla_x f + \nu f = q$ by the **Duhamel formula**

$$f(t,x,\xi) = \exp\Big(- \int_0^t \nu(s, x-(t-s)\xi, \xi)\,\mathrm{d}s \Big)\ f_{\mathrm{Init}}(x - t\xi, \xi)$$
$$+ \int_0^t \exp\Big(- \int_s^t \nu(\sigma, x-(t-\sigma)\xi, \xi)\,\mathrm{d}s \Big)\ q(s, x-(t-s)\xi, \xi)\,\mathrm{d}s$$

Stationary problems

For $\lambda > 0$, interpret $\lambda f + v \cdot \nabla_x f = q$ as
$\dfrac{\mathrm{d}}{\mathrm{d}t}\Big[e^{\lambda t} f(x + t\xi, \xi) \Big] = e^{\lambda t} q(x + t\xi, \xi)$. We get

$$f(x,\xi) = \int_0^\infty e^{-\lambda t} q(x - t\xi, \xi)\,\mathrm{d}t \quad \text{and thus} \quad \|f\|_{L^p} \le \frac{1}{\lambda} \|q\|_{L^p}.$$

I.2–Collisional models

Q has usually a specific structure

- Q usually acts only on ξ : integral or differential operator
- Q preserves the maximum principle ($f \geq 0$). For example : $Q(f) = Q^+(f) - \nu(f)\ f$ and think of an iterative process with the Duhamel formula : $\partial_t f_{n+1} + \xi \cdot \nabla_x f_{n+1} + \nu(f_n) f_{n+1} = Q^+(f_n)$.

and Q has some fundamental properties, crucial both on a physical and a mathematical viewpoints

- **Conservation :** There exists functions $m(\xi)$ such that
$$\int m(\xi) Q(f)\,\mathrm{d}\xi = 0$$
- **Equilibrium :** $Q(f) = 0$ iff f has a specific dependence wrt ξ : $f = M(\xi)$.
- **Dissipation :** There exists some function Ψ such that
$$\int \Psi(f) Q(f)\,\mathrm{d}\xi \leq 0$$

Examples : linear operators

Rexation operator

$$Q(f)(\xi) = M(\xi)\int f(\xi')\,\mathrm{d}\xi' - f(\xi) \text{ with } M(\xi) = (2\pi)^{-N/2}e^{-\xi^2/2}.$$

- Mass conservation : $\displaystyle\int Q(f)\,\mathrm{d}\xi = 0$,
- Equilibrium : $\mathrm{Ker}(Q) = \mathrm{Span}(M)$,
- Dissipation of any convex function : $\displaystyle\int Q(f)\Psi(f/M)\,\mathrm{d}\xi \leq 0$ for any non decreasing Ψ.

Fokker-Planck operator

$$Q(f)(\xi) = \nabla_\xi \cdot (\xi f + \nabla_\xi f) = \nabla_\xi \cdot \left[M\nabla_\xi\left(\frac{f}{M}\right)\right].$$

Examples : non-linear operators

BGK operator

$$Q(f) = M_{n,u,\theta} - f \text{ with } M_{n,u,\theta}(\xi) = \frac{n}{(2\pi\theta)^{N/2}} \exp\Big(-\frac{|\xi - u|^2}{2\theta}\Big).$$

Non-linear Fokker-Planck operator

$$Q(f)(\xi) = \nabla_\xi \cdot \big((\xi - u)f + \theta\nabla_\xi f\big).$$

Properties

- **Conservation of mass, momentum, energy** : $\displaystyle\int (1, \xi, \xi^2) Q(f)\,\mathrm{d}\xi = 0,$
- **Equilibrium** : $Q(f) = 0$ iff $f(\xi) = M_{n,u,\theta}(\xi),$
- **Entropy dissipation** $\displaystyle\int Q(f)\ln(f)\,\mathrm{d}\xi \le 0.$

Why could it be important ?

- It corresponds to Physical Principles : collisions preserve mass, momentum, energy... and induces relaxation/dissipation effects (large time behavior, hydrodynamic limits...)
- Hence Numerical Schemes should preserve these properties
- It provides useful A priori Estimates : we deal with sequences of (approximate) solutions and stability properties of the eq. rely on compactness of these sequences.

I.3–Hydrodynamic limits : Compressible limits

Go back to BGK : $\partial_t f + \xi \cdot \nabla_x f = \frac{1}{\tau}\big(M_{n,u,\theta} - f\big)$

- Due to **conservation** we have

$$\partial_t \int \begin{pmatrix} 1 \\ \xi \\ \xi^2/2 \end{pmatrix} f \, \mathrm{d}\xi + \nabla_x \int \xi \begin{pmatrix} 1 \\ \xi \\ \xi^2/2 \end{pmatrix} f \, \mathrm{d}\xi = 0$$

- Due to **equlibrium and dissipation** we expect as $\tau \to 0$ that $f \simeq M_{n,u,\theta}$
- Then, replace f by $M_{n,u,\theta}$ in the conservation eq. and we get... the Euler system for (n, u, θ) with pressure law $p = n\theta$.

Hydrodynamic limits : Incompressible limits

Remark that $M = (2\pi)^{-N/2} e^{-\xi^2/2}$ solves the BGK eq

$$\mathrm{Ma}\ \partial_t f + \xi \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f)$$

Then

- look at a **perturbation** $f = M(1 + \delta g)$.
- Scale the eq. : $\mathrm{Ma} = \varepsilon$, $\mathrm{Kn} = \varepsilon^q$, $\delta = \varepsilon^r$.

$$\varepsilon \partial_t g_\varepsilon + \xi \cdot \nabla_x g_\varepsilon = \frac{1}{\varepsilon^q} L g_\varepsilon + \frac{\varepsilon^r}{\varepsilon^q} B(g_\varepsilon) + \dots$$

with $L = \frac{1}{M}\ DQ(M)$, $B = \frac{1}{M}\ D^2 Q(M)$...

Lemma.

L is a self-adjoint Fredholm operator, with

$$\mathrm{Ker}(L) = \mathrm{Span}\{1, \xi, \xi^2\},$$
$$\mathrm{Ran}(L) = \left\{\phi \in L^2(M\,\mathrm{d}\xi),\ \int (1, \xi, \xi^2)\phi\ M\,\mathrm{d}\xi = 0\right\}$$

Towards incompressible models

Step 1 : Suppose $g_\varepsilon \to g$

As $\varepsilon \to 0$: $Lg = 0$ so $g = \rho(t,x) + u(t,x)\cdot\xi + \theta(t,x)\xi^2$.

Step 2 : Moment eq. and incompressibility

By **conservation** $(\mu(\xi) = (1, \xi, \xi^2))$

$$\varepsilon\partial_t \int \mu(\xi) g_\varepsilon M(\xi)\,\mathrm{d}\xi + \nabla_x \cdot \int \xi\mu(\xi) g_\varepsilon M(\xi)\,\mathrm{d}\xi = 0.$$

so that $\nabla_x \cdot u = 0$ and $\nabla_x(\rho + \theta) = 0$.

Step 3 : Moment eq and evolution eq

With $\mathbb{P}$ =projector on $\mathrm{Ker}(L) = \mathrm{Span}\{1, \xi, \xi^2\}$, $\mathbb{Q} = (\mathbb{I} - \mathbb{P})$

$$\partial_t \int \mu(\xi) g_\varepsilon M(\xi)\,\mathrm{d}\xi + \frac{1}{\varepsilon}\nabla_x \cdot \int \mathbb{Q}\big(\xi\mu(\xi)\big) g_\varepsilon M(\xi)\,\mathrm{d}\xi$$
$$= -\frac{1}{\varepsilon}\nabla_x \cdot \int \mathbb{P}\big(\xi\mu(\xi)\big) g_\varepsilon M(\xi)\,\mathrm{d}\xi.$$

Towards incompressible models, continued

With $\mu(\xi) = \xi$, we get $\mathbb{P}(\xi \otimes \xi) = \frac{1}{N}\xi^2\mathbb{I}$ and $\mathbb{Q}(\xi \otimes \xi) = \xi \otimes \xi - \frac{1}{N}\xi^2\mathbb{I}$. Set $L\chi = \mathbb{Q}(\xi \otimes \xi)$, then with π_ε a scalar quantity

$$\partial_t \int \mu(\xi) g_\varepsilon M(\xi)\,\mathrm{d}\xi + \nabla_x \cdot \int \chi \, \frac{1}{\varepsilon} L g_\varepsilon \; M(\xi)\,\mathrm{d}\xi = -\nabla_x \pi_\varepsilon.$$

Then, use

$$\begin{aligned} \frac{1}{\varepsilon} L g_\varepsilon &= \frac{\varepsilon^q}{\varepsilon}\Big(\varepsilon \partial_t g_\varepsilon + \xi \cdot \nabla_x g_\varepsilon - \varepsilon^{r-q} B(g_\varepsilon) + \ldots\Big) \\ &\simeq \begin{cases} \xi \cdot \nabla_x g & \text{if } q = 1,\ r > 1 : \text{Stokes eq.,} \\ \xi \cdot \nabla_x g - B(g) & \text{if } q = 1,\ r = 1 : \text{Navier-Stokes eq.,} \\ -B(g) & \text{if } q > 1,\ r = 1 : \text{Euler eq.} \end{cases} \end{aligned}$$

Similarly, we obtain a convection $(q > 1)$ or convection-diffusion eq. $(q = 1)$ for θ...

I.4–A few mathematical tools, Compactness

We deal with sequences of (approximate) solutions and we wish to "pass to the limit"... possibly at the price of extracting subsequences. To this end :

Banach-Alaoglu(-Bourbaki) lemma.
If B is a Banach space, then the unit ball of the dual space B' is weakly-$\star$ compact.

It means that if $\big(x_n\big)_{n\in\mathbb{N}}$ is bounded in B' then, there exists a subsequence such that for any $y \in B$, $\lim_{k\to\infty}\langle x_{n_k}, y\rangle_{B',B} = \langle x, y\rangle_{B',B}$

In particular it applies for any Hilbert space or any reflexive Banach space ($(B')' \simeq B$). Thus it works with $B = L^p$, $1 < p \leq \infty$.

Weak Compactness in L^1

Go back to Kinetic Equations : L^1 is a natural framework...

but L^1 is not a dual space and bounded sequences in L^1 in general do not have subsequences that converge weakly in L^1 !

Dunford-Pettis Theorem. *A sequence* $\big(f_n\big)_{n\in\mathbb{N}}$ *is weakly compact in* $L^1(X, \mathrm{d}\mu)$ *iff it is equi-integrable.*

$$\text{A useful criterion :} \quad \sup_{n\in\mathbb{N}} \int \Big[|f_n|\big(1+\omega(x)\big) + \Phi(|f_n|)\Big]\,\mathrm{d}\mu$$

with $\displaystyle\lim_{|x|\to\infty} \omega(x) = \infty$ and $\displaystyle\lim_{z\to\infty} \frac{\Phi(z)}{z} = \infty$.

Application to BGK : Due to **conservation and dissipation** we show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f\big(1+\xi^2+\ln(f)\big)\,\mathrm{d}\xi\,\mathrm{d}x \le 0$$

A few mathematical tools, Average lemma

A specific tool of kinetic theory

- Goal : Having information on f and $\xi \cdot \nabla_x f$, can we improve the regularity (compactness) of $\rho_\psi(x) = \displaystyle\int \psi(\xi)\, f(x,\xi)\, \mathrm{d}x\xi$?
- Basic claim : *If both f and $\xi \cdot \nabla_x f$ belong to $L^2(\mathbb{R}^N \times \mathbb{R}^N)$, then ρ_ψ lies in $H^{1/2}(\mathbb{R}^N)$.*

Sketch of proof

- Fourier transform wrt to space
- Split into $|\xi \cdot k| \geq \delta |k|$ (Good) $|\xi \cdot k| \leq \delta |k|$ (Bad but with a small contribution)
- Optimize wrt δ : $|\widehat{\rho_\psi}(k)| \leq \dfrac{\big[F(k)G(k)\big]^{1/4}}{|k|^{1/2}}$ with $F, G \in L^1(\mathbb{R}^N)$.

Improvements and variants

- Replacing $\xi \cdot \nabla_x$ by $\partial_t + \xi \cdot \nabla_x$ is not a big deal...
- Crucial : "having enough velocity" that is for any k$\in \mathbb{S}^{N-1}$, $\left|\left\{\xi \in B(0,R),\ \xi \cdot k = 0\right\}\right| = 0$
- Dealing with L^p spaces (ok at least for $p > 1$)
- Dealing with derivatives in the rhs

Theorem [Bouchut, Perthame-Souganidis].

Let f_n and g_n^α satisfy

$$(\partial_t + v \cdot \nabla_x) f_n = \sum_{|\alpha| \leq m} \partial_v^\alpha g_n^\alpha$$

for some $m \in \mathbb{N}$. Let Q be a open set in $\mathbb{R} \times \mathbb{R}^N$. We suppose that $\left(f_n\right)_{n\in\mathbb{N}}$ is bounded in $L^p(Q \times \mathbb{R}^N)$ for some $p > 1$ and the $\left(g_n^\alpha\right)_{n\in\mathbb{N}}$'s are bounded in $L^1(Q \times \mathbb{R}^N)$. Then, for any $\phi \in C_c^\infty(\mathbb{R}^N)$, the sequence defined by $\rho_n(t,x) = \int_{\mathbb{R}^N} f_n \phi \, \mathrm{d}v$ is relatively compact in $L^p(Q)$.

What about L^1 ?

Theorem [Golse-StRaymond].

Let f_n be weakly compact in L^1 and let g_n be bounded in L^1 such that

$$\xi \cdot \nabla_x f_n = g_n$$

Then, for any $\phi \in C_c^\infty(\mathbb{R}^N)$, the sequence defined by $\rho_n(t,x) = \int_{\mathbb{R}^N} f_n \phi \, \mathrm{d}v$ is relatively compact in $L^1(Q)$.

Sketch of proof

- Assume first that $g_n = g_n(\mathbb{1}_{|g_n| \leq M} + \mathbb{1}_{|g_n| \geq M})$ is also weakly compact in L^1
- Write $f_n = \lambda R_\lambda f_n + R_\lambda \xi \cdot \nabla_x f_n$ with $R_\lambda = (\lambda + \xi \cdot \nabla_x)^{-1}$ and $\lambda \gg 1$.

II. Colisionless Models : Analysis of the Vlasov-Poisson system

The VP system

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = 0$$

where the potential is defined by

$$-\Delta_x \Phi = n(t,x) = \pm \int_{\mathbb{R}^N} f(t,x,v)\,\mathrm{d}v.$$

The latter equation should actually be understood as the convolution formula

$$\Phi(t,x) = \int_{\mathbb{R}^N} E_N(x-y) n(t,y)\,\mathrm{d}y = \int_{\mathbb{R}^N} E_N(x-y) f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v$$

where E_N stands for the elementary solution of $-\Delta$ $\left(E_3(x) = \frac{1}{4\pi|x|}\right)$

Difficulties and strategy

Nonlinearity makes the problem difficult

- Determine a functional framework so that the quadratic term $f\nabla_x\Phi$ makes sense
- We will approach the pb by regularizing the kernel : justify the necessary compactness to pass to the limit

Blow-up can occur : Attractive case in dimension 4

Consider $N = 4$ and the attractive potential. Then there exists initial data $f^{\mathrm{Init}} \in C^2_c(\mathbb{R}^4 \times \mathbb{R}^4)$ such that the solution does not remain of class C^2 for any positive time.

Indeed $\displaystyle\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{x^2}{2} f \,\mathrm{d}v\,\mathrm{d}x = \mathscr{E}_0 t^2 + I_1 t + I_0$ with $\mathscr{E}_0$ possibly < 0.

The one dimension case

The one-dimension case is interesting since it allows to perform a proof with **elementary tools**.

Further "nice" estimates in 1D

We write

$$F(t,x) = \partial_x \Phi(t,x) = \mp\frac{1}{2}\Big(\int_{-\infty}^{x} n(t,y)\,\mathrm{d}y - \int_{x}^{+\infty} n(t,y)\,\mathrm{d}y\Big).$$

The L^1 bound on n implies that F is continuous and bounded. If n belongs to L^∞, F is uniformly lipschitizian.

Then the proof uses the a priori estimates, sharp estimates on the characteristics and the preservation of support properties (which provides the L^∞ estimate on n on finite time interval...).

A priori estimates

Mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f \,\mathrm{d}v\,\mathrm{d}x = 0,$$

Maximum principle

Reasoning (formally) with characteristics we get

$$\|f(t)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \leq \|f^{\mathrm{Init}}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}.$$

Energy conservation

$$E_c(t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2}{2}\, f \,\mathrm{d}v\,\mathrm{d}x$$

$$E_p(t) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi f \,\mathrm{d}v\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} E_N(x-y)n(t,y)n(t,x)\,\mathrm{d}y\,\mathrm{d}x.$$

$$\text{verify} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(E_c(t)+E_p(t)) = 0.$$

HLS

Lemma.

Let $1 < \alpha < \infty$ and

$$T : f \mapsto \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N/\alpha}} \,\mathrm{d}y.$$

Then, T is a bounded operator from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for

$$1 < p < \alpha' = \frac{\alpha}{\alpha - 1}, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{\alpha} - 1.$$

Remark

Obtaining $\nabla_x \Phi \in L^2$ is not affordable in dimension 1 and 2 ; in dimension 3 it requires $p = 6/5$ ($\alpha = N/(N-1)$).

Existence of solutions : repulsive case, $N = 3$

Step 1. Regularization of the convolution kernel

- $U_\varepsilon(x) = \zeta_\varepsilon \star E_3(x)$,
- $E_3 \in L^q(B(0,R))$ for $1 \le q < 3$, $\nabla_x E_3(x) = -\dfrac{1}{4\pi}\dfrac{x}{|x|^3} \in L^q(B(0,R))$ for $1 \le q < 3/2$
- $U_\varepsilon \to E_3$, $\nabla_x U_\varepsilon \to \nabla_x E_3$ in these spaces.

Step 2. Fixed point argument and basic estimates

- Existence-uniqueness of a solution $(f_\varepsilon, \Phi_\varepsilon)$,
- $f_\varepsilon \ge 0$ and $\|f_\varepsilon(t)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \|f^{\mathrm{Init}}\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}$, $1 \le p \le \infty$,
- Energy conservation (Repulsive case : $E_p \ge 0$)

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2}{2} f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_\varepsilon f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x = E_c(0) + E_p(0) < \infty.$$

An interpolation trick

Lemma.

Let $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that, for some $m > 0$,

$$f \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N) \qquad \text{and} \qquad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v|^m \, |f(x,v)| \, \mathrm{d}v \, \mathrm{d}x < \infty.$$

Then $n(x) = \int_{\mathbb{R}^N} f(x,v)\,\mathrm{d}v$ belongs to $L^p(\mathbb{R}^N)$ for $p = (N+m)/N$.

Corollary

$\big(n_\varepsilon\big)_{\varepsilon>0}$ is bounded in $L^\infty(0,\infty; L^{5/3}(\mathbb{R}^3))$. The sequence $\big(\nabla_x \Phi_\varepsilon\big)_{\varepsilon>0}$ is bounded in $L^\infty(0,\infty; L^q(\mathbb{R}^3))$, for $3/2 < q < 15/4$.

Conclusion

- $f_\varepsilon \nabla_x \Phi_\varepsilon$ is bounded in $L^\infty(0,\infty; L^q(\mathbb{R}^3))$, for $3/2 < q < 15/4$
- Compactness uses either the smoothing effect of the Poisson eq. or the average lemma to pass to the limit in the delicate term

$$\int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_\varepsilon \nabla_x \Phi \cdot \nabla_v \psi \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\infty \int_{\mathbb{R}^N} \nabla_x \Phi_\varepsilon(t,x) \cdot \Big(\int_{\mathbb{R}^N} f_\varepsilon \nabla_v \psi(t,x,v)\,\mathrm{d}v \Big)\,\mathrm{d}x\,\mathrm{d}t$$

Theorem.

Let $N = 3$ and let $f^{\mathrm{Init}} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, $f^{\mathrm{Init}} \geq 0$, such that $(E_c + E_P)(0) < \infty$. Then, there exists $f \in C^0([0,\infty); L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) - \mathrm{weak} - \star)$, solution of the Vlasov-Poisson system with initial data f^{Init}. Furthermore, we have

$$\|f(t)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} \leq \|f^{\mathrm{Init}}\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}, \qquad (E_c + E_p)(t) \leq (E_c + E_p)(0).$$

The attractive case ($N = 3$)

The potential energy is ≤ 0... but similar estimates can be obtained. Hence the same statement applies to the attractive case as well.

An interpolation lemma.

Let $n : \mathbb{R}^3 \to \mathbb{R}$. Then, we have

$$\int_{\mathbb{R}^3} |n|^{6/5} \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^3} |n| \,\mathrm{d}x \right)^{7/10} \left(\int_{\mathbb{R}^3} |n|^{5/3} \,\mathrm{d}x \right)^{3/10} .$$

Consequences on the a priori estimates

- $\|\nabla_x \Phi\|_{L^2(\mathbb{R}^3)} \leq C \|n\|_{L^{6/5}(\mathbb{R}^3)} \leq C \|n\|^{5/12}_{L^{5/3}(\mathbb{R}^3)} \leq C \big(E_c\big)^{1/4}$,
- $\mathscr{E}_0 = E_c + E_p \geq E_c - C \big(E_c\big)^{1/2}$

The Vlasov-Maxwell equation

The method (regularization, compactness by average lemma) applies to treat the Vlasov-Maxwell system as well

$$
\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\big((E+v\wedge B)f\big) = 0,\\
&\partial_t E - \mathrm{curl}_x B = -\int vf\,\mathrm{d}v, \quad \mathrm{div}_x E = \int f\,\mathrm{d}v,\\
&\partial_t B + \mathrm{curl}_x E = 0, \qquad \mathrm{div}_x B = 0,
\end{aligned}
$$

Indeed, we still have the a priori estimates $\|f(t)\|_{L^1} = \|f_0\|_{L^1}$, and $\|f(t)\|_{L^\infty} = \|f_0\|_{L^\infty}$ and the Energy Conservation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int\int \frac{v^2}{2} f\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int (E^2+B^2)\,\mathrm{d}x = 0.\Big)
$$

Dispersive effects : VP, $N = 3$, repulsive case

Goal

Spread of the particles under the effect of repulsive forces : time decay of L^p norms.

A tricky computation [Illner-Rein, Perthame]

- $\frac{\mathrm{d}}{\mathrm{d}t}\int |x-tv|^2 \; f \,\mathrm{d}v\,\mathrm{d}x = -\Big(t^2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x\Phi\|^2_{L^2} + t\|\nabla_x\Phi\|^2_{L^2}\Big)$
- Set $\Gamma(t) = \frac{t^2}{2}\|\nabla_x\Phi\|^2_{L^2}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int |x-tv|^2 \; f\,\mathrm{d}v\,\mathrm{d}x + 2\Gamma\Big) = t\|\nabla_x\Phi\|^2_{L^2} = \frac{2\Gamma(t)}{t}.$$

- Deduce $\Gamma(t) \le Ct$, $\|\nabla_x\Phi\|^2_{L^2} \le C/t$ and

$$\int |x-tv|^2 \; f\,\mathrm{d}v\,\mathrm{d}x \le C(1+t), \qquad \|n\|_{L^{5/3}} \le \frac{C}{(1+t)^{3/5}}.$$

Classical solutions

Can we construct C^1 solutions based on characteristics formulae ?

A lemma on transport eq.

If $a \in C^1$ with $\nabla_x a \in L^\infty$ and $\partial_t F + a \cdot \nabla_x F = 0$, then

$$\|\nabla_x F(t)\|_\infty \leq \|\nabla_x F_0\|_\infty \exp\left(\int_0^t \|\nabla_x a(\tau)\|_\infty \,\mathrm{d}\tau\right).$$

A lemma from harmonic analysis ($N = 3$)

If $\Phi = E_3 \star \rho$, then

$$|\partial^2_{x_i x_j}\Phi(x)| \leq C\Big(1 + \|\rho\|_1 + \|\rho\|_\infty \big(1 + \ln(1 + \|\nabla_x \rho\|_\infty)\big)\Big).$$

Role of the support (wrt v)

If $f \in L^1 \cap L^\infty$ solves VP with $\mathrm{supp}(f(t,x,\cdot)) \subset B(0, R(t))$, then $\|\rho(t)\|_\infty \leq C\|f_0\|_\infty R^3(t)$. Thus $x \mapsto D^2\Phi(t,x)$ bounded in $L^\infty(\mathbb{R}^3)$.

Control of the support

A local in time estimate

We have $f(t,x,v) = f_0\big(X(0;t,x,v),\Theta(0,t,x,v)\big)$ so that

$\mathrm{supp}(f_0(x,\cdot)) \subset B(0,R_0)$ implies $|v| \leq \left| \int_0^t \nabla_x\Phi(s,X(s;t,x,v))\,\mathrm{d}s \right| + R_0$

for $v \in \mathrm{supp}(f(t,x,\cdot))$.
Using the $L^{5/3}(\mathbb{R}^3)$ estimate on $\rho(t)$, we get

$$\begin{aligned} P(t) &= \sup\big\{|v| s.t.\ \exists (t,x)\ f(t,x,v) \neq 0\big\} \\ &\leq R_0 + C\int_0^t \|\rho(s)\|_\infty^{4/9}\,\mathrm{d}s \leq R_0 + C\int_0^t P(s)^{4/3}\,\mathrm{d}s \end{aligned}$$

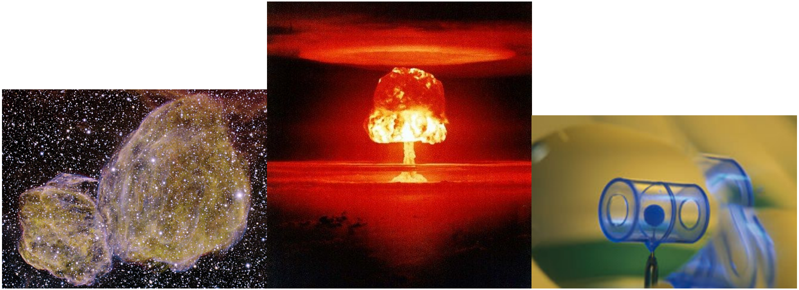
Towards a global estimate

To obtain better

- Either a finer estimate on $\nabla_x\Phi(s,X(s;t,x,v))$ [Schaeffer '91]
- or a higher moment estimate [Lions-Perthame '92]

III. Hydrodynamic limits : diffusion approximation

Application to Radiative Transfer : why could it be interesting ?



Applications in Astrophysics : Description of stars atmospheres, supernovae explosion

Many people are interested in Radiative Hydrodynamics at CEA and the Research Centre of Los Alamos... and, more recently, for Fusion by Inertial Confinement

Many other industrial and scientific applications...

Nuclear engineering
Electricity distribution network
Fires and Furnaces
Atmospheric measurements (estimate of chemical fluxes, inverse problems), Radiotherapy...

To learn more on radiative transfer models...

See Martin Frank !

The basic problem in radiative transfer

Radiation field= photon gas

- Let $F(t,x,p)$=distribution of photons in phase space, with momentum variable $p = \frac{h\nu}{c}\ v$ where c=light speed, v=direction of flight $\in \mathbb{S}^{N-1}$, ν=frequency variable
- $F\,\mathrm{d}p\,\mathrm{d}x = \frac{1}{c}\ f\,\mathrm{d}\nu\,\mathrm{d}v\,\mathrm{d}x$=energy ; $f(t,x,v,\nu)$=specific intensity of radiation
- Grey models : we get rid of ν.

Energy exchanges

$$\frac{1}{c}\partial_t f + v\cdot\nabla_x f = \frac{\sigma(T)}{\ell}(aT^4 - f),$$
$$C\partial_t T = \frac{\sigma(T)}{\ell}\int_{\mathbb{S}^{N-1}}(f - aT^4)\,\mathrm{d}v$$

(with $\int\,\mathrm{d}v = 1$).

The Rosseland approximation

Optically thick medium

- When $\ell \ll 1$, we expect $f \simeq aT^4$... and a diffusion eq. for T
- Interest : we get rid of the direction variable, and we have a simpler eq.

Simplified model

$$\begin{aligned}
&\varepsilon\partial_t f + v\cdot\nabla_x f = \frac{\sigma(\rho)}{\varepsilon}(\rho - f),\\
&t \geq 0, \qquad x \in \mathbb{T}^N, \qquad v \in \mathbb{S}^{N-1},\\
&\int \mathrm{d}v = 1, \qquad \int v\,\mathrm{d}v = 0, \qquad \rho(t,x) = \int f\,\mathrm{d}v,\\
&0 < \sigma_\star \leq \sigma(\rho) \leq \sigma^\star < \infty \quad \text{continuous.}
\end{aligned}$$

Remarkable properties of the collision operator

Conservation

$$\int Q(f)\,\mathrm{d}v = \sigma(\rho)\int(\rho - f)\,\mathrm{d}v = 0.$$

Equilibrium

$$Q(f) = 0 \text{ iff } f(v) = \rho\,\mathbb{1}(v).$$

Dissipation

For any convex function Ψ

$$\int \Psi(f)\;Q(f)\,\mathrm{d}v = \int \sigma(\rho)(\rho - f)\;\big(\Psi'(\rho) - \Psi'(f)\big)\,\mathrm{d}v \le 0.$$

Existence of solution

Fixed point method

- Set $\mathscr{T}(\widetilde{\rho}) = \rho = \int f \, \mathrm{d}v$ where

$$\varepsilon \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \sigma(\widetilde{\rho})(\rho - f).$$

- We have $0 \le f(t, x, v) \le \|f_0\|_\infty$,
- $\mathscr{T}$ is continuous (L^1 norm)
- $\mathscr{T}$ is compact by virtue of the average lemma.

Hilbert expansion

Plug the ansatz $f_\varepsilon = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots$ into

$$\varepsilon \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \sigma(\rho)(\rho - f)$$

We get

$$f^{(0)} = \rho(t,x), \qquad f^{(1)} = -\frac{v}{\sigma(\rho)} \cdot \nabla_x \rho(t,x) \ (+q(t,x)),$$

and

$$\partial_t \int f^{(0)} \, \mathrm{d}v + \nabla_x \cdot \int v f^{(1)} \, \mathrm{d}v = \partial_t \rho - \nabla_x \cdot \Big(\frac{1}{N\sigma(\rho)} \nabla_x \rho \Big) = 0.$$

Non linear diffusion equation : Rosseland Approximation.

Possible proof

Expand $f_\varepsilon = \rho - \varepsilon v \partial_x \rho + \varepsilon^2 f_2 + \varepsilon g_\varepsilon$ and estimate the remainder g_ε...

Entropy and Moment Equations

Entropy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}|f_\varepsilon|^2\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\sigma(\rho_\varepsilon)\,|f_\varepsilon-\rho_\varepsilon|^2\,\mathrm{d}v\,\mathrm{d}x=0$$

so that $f_\varepsilon=\rho_\varepsilon+\varepsilon r_\varepsilon$ with r_ε bounded in L^2.

Moment equations

Set $(\rho_\varepsilon,J_\varepsilon,\mathbb{P}_\varepsilon)(t,x)=\displaystyle\int_{\mathbb{S}^{N-1}}(1,\frac{v}{\varepsilon},v\otimes v)\ f_\varepsilon(t,x,v)\,\mathrm{d}v$.

$$\text{We get}\qquad\left\{\begin{array}{l}\partial_t\rho_\varepsilon+\mathrm{div}_x J_\varepsilon=0,\\ \quad\text{and}\quad\varepsilon^2\partial_t J_\varepsilon+\mathrm{Div}_x\mathbb{P}_\varepsilon=-\sigma(\rho_\varepsilon)J_\varepsilon.\end{array}\right.$$

As $\varepsilon\to 0$, $\mathbb{P}_\varepsilon=\displaystyle\int_{\mathbb{S}^{N-1}}v\otimes v\,\mathrm{d}v\ \rho_\varepsilon+\mathcal{O}(\varepsilon)$ yields the Limit System

$$\mathrm{Div}_x\mathbb{P}=\nabla_x\rho=-\sigma(\rho)J\qquad\text{and}\qquad\partial_t\rho+\mathrm{div}_x J=0.$$

Proof by using the average lemma

Estimate

- f_ε is bounded in $L^\infty(0,T;L^2(\mathbb{T}^N\times\mathbb{R}^N))$,
- ρ_ε is bounded in $L^\infty(0,T;L^2(\mathbb{T}^N))$,
- $r_\varepsilon = \frac{1}{\varepsilon}(f_\varepsilon - \rho_\varepsilon)$ is bounded in $L^2((0,T)\times\mathbb{T}^N\times\mathbb{R}^N)$,
- $J_\varepsilon = \dfrac{1}{\varepsilon}\displaystyle\int v f_\varepsilon\,\mathrm{d}v = \int v r_\varepsilon\,\mathrm{d}v$ is bounded in $L^2((0,T)\times\mathbb{T}^N)$.

Compactness

- $\varepsilon\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon = -\sigma(\rho_\varepsilon) r_\varepsilon$ is bounded in $L^2((0,T)\times\mathbb{T}^N)$,
- The average lemma provides compactness of ρ_ε "with respect to the space variable",
- but also $\partial_t\rho_\varepsilon = -\nabla_x\cdot J_\varepsilon$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}^N))$, which provides compactness "with respect to the time variable".

An alternative approach by compensated compactness

Div-Curl Lemma [Tartar, Tartar-Murat]

Let $U_n = (u_n^1, \ldots, u_n^N) \rightharpoonup U$, $V_n \rightharpoonup V$ in $L^2(\Omega)$ with furthermore $\mathrm{div}\, U_n = \sum \partial_i u_n^i$ and $\mathrm{curl}\, V_n = \left[\partial_j v_n^i - \partial_i v_n^j\right]_{ij}$ compact in H^{-1} then

$$U_n \cdot V_n = \sum_{i=1}^{N} u_n^i v_n^i \rightharpoonup U \cdot V \text{ in } \mathcal{D}'.$$

Crucial Assumption

$\forall \xi \in \mathbb{S}^{N-1}$, $\left|\{v \in \mathcal{V} \text{ such that } v \cdot \xi \neq 0\}\right| > 0$, while the average lemma needs $\left|\{v \in \mathcal{V} \text{ such that } v \cdot \xi = 0\}\right| = 0$. Thus it works for discrete velocity models $v \in \{v^1, \ldots, v^M\}$, $\mathrm{d}v = \sum_{i=1}^{M} \omega_i \delta_{v = v^i}$. (cf. $\langle v \otimes v \rangle > 0$).

How does it work ?

Using $f_\varepsilon = \rho_\varepsilon + \varepsilon r_\varepsilon$ we rewrite the Moment equations as

$$\begin{cases} \mathrm{div}_{t,x}(\rho_\varepsilon, J_\varepsilon) = 0, \\ \left(\int_{\mathbb{S}^{N-1}} v \otimes v \,\mathrm{d}v\right) \nabla_x \rho_\varepsilon = -\varepsilon^2 \partial_t J_\varepsilon - \sigma(\rho_\varepsilon) J_\varepsilon - \varepsilon \mathrm{Div}_x(R_\varepsilon) \end{cases}$$

so that

$$\mathrm{div}_{t,x}(\rho_\varepsilon, J_\varepsilon) = 0$$
$$\mathrm{curl}_{t,x}(\rho_\varepsilon, 0, \ldots, 0) = \begin{pmatrix} 0 & -\nabla_x \rho_\varepsilon^T \\ \nabla_x \rho_\varepsilon & 0 \end{pmatrix}$$

belong to a compact set of $H^{-1}_{\mathrm{loc}}((0, T) \times \mathbb{R}^N)$.

Coupling to Homogeneization

We seek reduced models for routine computations that take into account heterogeneities of the medium

$$\varepsilon\partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}\Big(\int \sigma(x, x/\varepsilon, v, v_\star) f(v_\star)\,\mathrm{d}v_\star - \int \sigma(x, x/\varepsilon, v_\star, v)\,\mathrm{d}v_\star\ f(v)\Big)$$

Set $T = v\cdot\nabla_y - Q$, $y = x/\varepsilon$ and expand $f_\varepsilon = \sum \varepsilon^j f^{(j)}(t, x, x/\varepsilon, v)$

- $Tf^{(0)} = 0$ is solved by $\rho(t,x)M(x,y,v)$
- $Tf^{(1)} = v\cdot\nabla_x f_0 = vM\cdot\nabla_x\rho + v\cdot\nabla_x M\rho$. If $\int vM\,\mathrm{d}v\,\mathrm{d}y = 0$ then $f^{(1)} = -\chi\cdot\nabla_x\rho + \lambda\rho$ where $T\chi = -vM$, and $T\lambda = v\cdot\nabla_x M$

Effective equations

$$\partial_t\rho - \nabla_x\cdot(D(x)\nabla_x\rho - U(x)\rho) = 0,$$

$$D(x) = \int\int v\otimes\chi(x,y,v)\,\mathrm{d}v\,\mathrm{d}y, \qquad U(x) = \int\int v\lambda(x,y,v)\,\mathrm{d}v\,\mathrm{d}y$$

Intermediate Models

Assuming $\mathcal{V} \subset (-1, +1)$, $\int_{\mathcal{V}} \mathrm{d}v = 1$, $\int_{\mathcal{V}} v \,\mathrm{d}v = 0$, $\int_{\mathcal{V}} v^2 \,\mathrm{d}v = d > 0$, solutions of

$$\varepsilon\partial_t f_\varepsilon + v\partial_x f_\varepsilon = \frac{1}{\varepsilon}\Big(\int_{\mathcal{V}} f_\varepsilon \,\mathrm{d}v - f_\varepsilon\Big)$$

converge to $\rho(t, x)$ solution of

$$\partial_t\rho - d\partial^2_{xx}\rho = 0.$$

One seeks intermediate models for $0 < \varepsilon \ll 1$:

⋆ heat eq. propagates at infinite speed instead of $\mathcal{O}(1/\varepsilon)$,

⋆ $\rho - \varepsilon v\partial_x\rho$ does not preserve non-negativeness, nor the flux limited condition

$$\Big|\int_{\mathcal{V}} \frac{v}{\varepsilon} f_\varepsilon \,\mathrm{d}v\Big| \leq \frac{1}{\varepsilon}\int_{\mathcal{V}} f_\varepsilon \,\mathrm{d}v.$$

A zeroth order closure

- A modified Hilbert expnasion $f_\varepsilon = \exp(a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots)$
- $a_0 = a_0(t,x)$ and $a_1(t,x,v) = -v\partial_x a_0$
- Then impose that $\widetilde{f}_\varepsilon = \exp(a_0 + \varepsilon a_1)$ satisfies mass conservation.
- $\widetilde{\rho}_\varepsilon = \displaystyle\int \widetilde{f}_\varepsilon \,\mathrm{d}v$ verifies $\partial_t \widetilde{\rho}_\varepsilon - \partial_x\Big(\dfrac{\widetilde{\rho}_\varepsilon}{\varepsilon}\mathbb{G}(\varepsilon\partial_x a_0)\Big) = 0,$
- $\partial_x \ln(\widetilde{\rho}_\varepsilon) \simeq \partial_x a_0$ leads to (Lebesgue measure)

$$\partial_t \widetilde{\rho}_\varepsilon - \partial_x\Big(\frac{\widetilde{\rho}_\varepsilon}{\varepsilon}\big(\coth\big(\frac{\varepsilon\partial_x\widetilde{\rho}_\varepsilon}{\widetilde{\rho}_\varepsilon}\big) - \frac{\widetilde{\rho}_\varepsilon}{\varepsilon\partial_x\widetilde{\rho}_\varepsilon}\big)\Big) = 0.$$

Minimum Entropy Principle Closure

The Moment System

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x J_\varepsilon = 0 \\ \varepsilon^2 \partial_t J_\varepsilon + \mathrm{Div}_x \mathbb{P}_\varepsilon = -J_\varepsilon \end{cases}$$

is closed by imposing [Levermore'97, Dubroca-Feugeas'99, Fort'97]

$$\mathbb{P}_\varepsilon = \int_{\mathcal{V}} v^2 f_\varepsilon^\star \,\mathrm{d}v$$

where $f_\varepsilon^\star$ minimizes

$$\int_{\mathcal{V}} f \ln f \,\mathrm{d}v, \ \text{with the constraints} \ \int_{\mathcal{V}} (1,\ v/\varepsilon) f \,\mathrm{d}v = (\rho_\varepsilon,\ J_\varepsilon)$$

Remarkable facts

One obtains $\mathbb{P}_\varepsilon = \rho_\varepsilon \psi(\varepsilon J_\varepsilon / \rho_\varepsilon)$, a (strictly) hyperbolic system which is globally well-posed for small enough initial data, and consistent to the diffusion eq. as ε goes to 0 ($\psi(0) = d$).

Remarks

- The model is exact for the 2-velocities model
- For the Gaussian measure, the model is nothing but the isothermal Euler system.

On the Entropy-Based Model

Theorem. [Coulombel-Golse-G.'06]

Let $\overline{\rho} > 0$. There exist $\delta > 0$, $C > 0$ such that, for any $\varepsilon \in]0,1]$, and for any (ρ_0, J_0) with $\|\rho_0 - \overline{\rho}\|_{H^2(\mathbb{R})} \leq \delta$ and $\|\varepsilon J_0\|_{H^2(\mathbb{R})} \leq \delta$, there exists a unique global solution $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon)$ to the "Levermore System" with initial data (ρ_0, J_0), and that satisfies $(\widehat{\rho}_\varepsilon - \overline{\rho}, \widehat{J}_\varepsilon) \in \mathcal{C}(\mathbb{R}^+; H^2(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; H^1(\mathbb{R}))$. For r solution to the heat equation with initial data ρ_0, we have

$$\|\widehat{\rho}_\varepsilon - r\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \leq C\,\varepsilon, \qquad \|f_\varepsilon^\star - f_\varepsilon\|_{L^2((0,T)\times\mathbb{R}\times\mathbb{R})} \leq C\,\varepsilon.$$

Arguments

- Use Relaxation (it looks like $y' = y^2 - \lambda y$)
- Strong Coupling : "Kawashima-Shizuta Condition"
- Adapt Hanouzet-Natalini'03 analysis... and make it uniform wrt ε !
- Junca-Rascle'02's trick for the convergence to the heat eq.

Towards more realistic Radiative Transfer Problems

Euler System

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2 + p) = -S_m, \\ \partial_t(nE) + \partial_x((nE + p)u) = -S_e \end{cases}$$

Radiation transport eq.

$$\varepsilon \partial_t f + v \partial_x f = \frac{1}{\varepsilon} Q_s + \varepsilon Q_a$$

$$Q_s = \sigma_s \Big(\frac{1}{\Lambda^3} \langle \Lambda^2 f \rangle - \Lambda f \Big), \qquad Q_a = \sigma_a \Big(\frac{1}{\Lambda^3} \frac{1}{\pi} \theta^4 - \Lambda f \Big).$$

with $\Lambda = (1 - \varepsilon uv)/\sqrt{1 - \varepsilon^2 u^2}$ and $S_m = \frac{1}{\varepsilon} \langle v Q_s \rangle + \varepsilon \langle v Q_a \rangle$, $S_e = \frac{1}{\varepsilon^2} \langle Q_s \rangle + \langle Q_a \rangle$.

As $\varepsilon \to 0$ f_ε becomes proportional to Λ^{-4}, which has a $\mathcal{O}(\varepsilon)$ flux.

Non Equilibrium Diffusion Regime

Scattering dominates : relaxation to an isotropic distribution but final model with TWO temperatures $\theta \neq \theta_{rad} (\simeq \rho^{1/4})$.
Ref. : Lowrie-Morel-Hittinger'99, Buet-Després'04
Full Model :

$$\begin{cases} \partial_t n + \partial_x (nu) = 0, \\ \partial_t (nu) + \partial_x (nu^2 + p) = -\mathcal{P} \dfrac{\partial_x \rho}{3}, \\ \partial_t (nE) + \partial_x (nEu + pu) = -\mathcal{P} \dfrac{1}{3} u \partial_x \rho + \mathcal{P} \sigma_a (\rho - \theta^4), \\ \partial_t \rho - \dfrac{1}{3\sigma_s} \partial^2_{xx} \rho + \dfrac{4}{3} \partial_x (\rho u) - \dfrac{1}{3} u \partial_x \rho = \sigma_a (\theta^4 - \rho). \end{cases}$$

Doppler corrections make non conservative $p_{rad} \partial_x u$ terms appear

Non Equilibrium Diffusion Regime

Questions are related to the effects of the Energy Exchanges on the features of the usual Euler system :
- Well posedness of the kinetic/hyperbolic system [Lin'06, Zhong-Jiang'06]
- Asymptotic problems : diffusion regime [G.-Lafitte '06]
- (Smoothing ?) effects on the shock profile [Lin-Coulombel-G. '06]
- Stability questions (of constants, of shocks profiles...) [Lin-Coulombel-G. '06, Coulombel-Mascia'0 ?]
- Numerical Experiments

Radiative Shock Profiles

$$\text{Simplified Model :}\quad \begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2 + p) = 0, \\ \partial_t(nE) + \partial_x(nEu + pu) = \rho - \theta^4, \\ -\partial^2_{xx}\rho = \theta^4 - \rho, \end{cases}$$

The last eq. recasts as

$$\rho(t,x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}\theta^4(t,y)\,\mathrm{d}y, \qquad q = -\partial_x\rho, \qquad \partial_x q = -(\rho - \theta^4)$$

System version of the toy model [Kawashima-Nishibata'99]

$$\partial_t u + \partial_x \frac{u^2}{2} = -\partial_x q = Ku - u, \qquad Ku(t,x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}u(t,y)\,\mathrm{d}y$$

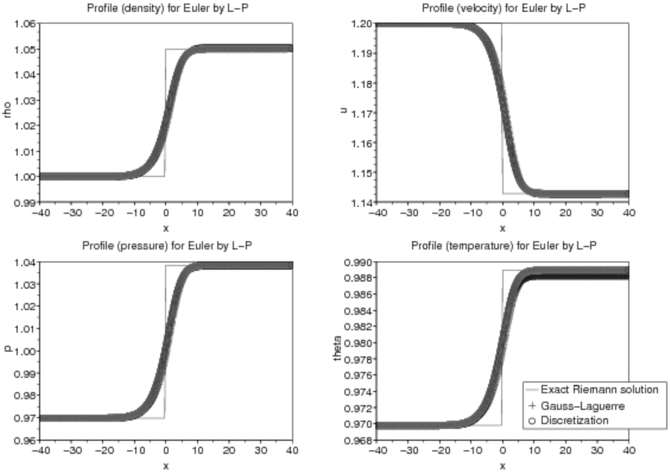
Radiative (Small) Shock Profiles

Theorem. [Lin, Coulombel, G.'07] Let γ satisfy $1 < \gamma < \dfrac{\sqrt{7}+1}{\sqrt{7}-1} \simeq 2.215$ and let (ρ_-, u_-, e_-) be fixed. Then there exists a positive constant δ (that depends on (ρ_-, u_-, e_-), and γ) such that, for all state (ρ_+, u_+, e_+) verifying :

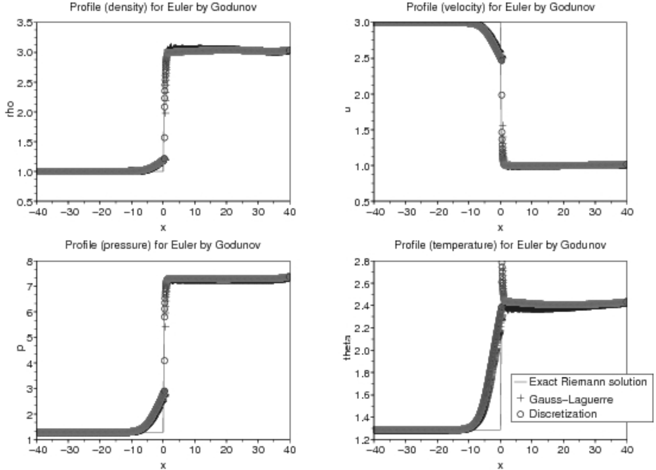
$$\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta$$

and $(\rho_\pm, u_\pm, e_\pm)$ is a shock wave, with speed σ, for the (standard) Euler equations, then there exists a C^2 traveling wave $(\rho, u, e)(x - \sigma t)$ solution of the Radiative Euler eq. Furthermore, there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ if $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta_n$, then the profile is C^{n+2}. The profile can be shown to be asymptotically stable wrt zero mass perturbation.

A smooth profile : $\delta \simeq 0.2$ [Coulombel-Lafitte]



A non-smooth profile (Zeldovich spike) : $\delta \simeq 2$ [Coulombel-Lafitte]



As a conclusion

- Highly nonlinear, strongly coupled models
- Multiscale features
- Many asymptotic problems
- A large variety of relevant models (with maybe different behavior...)
- Many challenging questions both for analysis and simulations

Hydrodynamic limits : from kinetic equation to drift-diffusion models

VPFP

$$\varepsilon\partial_t f_\varepsilon + v\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_v f = \frac{1}{\varepsilon}\nabla_v\cdot(vf+\nabla_v f),$$
$$-\Delta\Phi = \pm\int f\,\mathrm{d}v = \pm\rho, \qquad \Phi = \pm E_N \star \rho.$$

Formal asymptotics

- $f_\varepsilon(t,x,v) \simeq \rho(t,x)\ \dfrac{\mathrm{e}^{-v^2/2}}{(2\pi)^{N/2}}$
- $\partial_t\rho_\varepsilon + \nabla_x\cdot J_\varepsilon = 0$, and $\varepsilon^2\partial_t J_\varepsilon + \mathrm{Div}_x\mathbb{P}_\varepsilon = -\rho_\varepsilon\nabla_x\Phi_\varepsilon - J_\varepsilon$
- $\mathbb{P}_\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^N} v\otimes v\ f_\varepsilon\,\mathrm{d}v \simeq \rho\mathbb{I}$

Smoluchowski (Keller-Segel) equation

Limit system

$$\begin{aligned} &\partial_t \rho - \nabla_x \cdot (\rho \nabla_x \Phi + \nabla_x \rho), \\ &\Phi = \pm E_N \star \rho. \end{aligned}$$

Attractive case

Blow up in finite time can occur !

Under a threshold condition we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int x^2 \rho \, \mathrm{d}x = -C < 0$$

Energy-entropy dissipation

$$D_\varepsilon = \frac{4}{\varepsilon^2}\int \left|\nabla_v \sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x = \frac{1}{\varepsilon^2}\int \left|v\sqrt{f_\varepsilon}+2\nabla_v\sqrt{f_\varepsilon}\right|^2 \mathrm{d}v\,\mathrm{d}x.$$

Dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon \ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{v^2}{2} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon \Phi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\right\} = -D_\varepsilon \leq 0.$$

Control of the tails

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x| f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x \leq \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2} \sqrt{D_\varepsilon}.$$

Jensen inequality

$$\int_{\mathbb{R}^N} \rho \ln(\rho_\varepsilon)\,\mathrm{d}x \leq \int_{\mathbb{R}^N\times\mathbb{R}^N} \left(f_\varepsilon \ln(f_\varepsilon) + \frac{N}{2}\ln(2\pi) + \frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x.$$

Estimate of the potential energy : $N = 2$

Repulsive case

$$\begin{aligned}\int_{\mathbb{R}^2} \rho\Phi \,\mathrm{d}x &= -\frac{1}{2\pi}\iint \ln(|x-y|)\rho(y)\rho(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \iint_{|x-y|\le k} \dots + \iint_{|x-y|\ge k} \dots\\ &\ge -\frac{\ln(k)}{2\pi}\left(\int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x\right)^2 - \frac{\ln(k)}{\pi k}\int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x\int_{\mathbb{R}^2}|x|\rho(x)\,\mathrm{d}x.\end{aligned}$$

Attractive case

Use the Beckner, Carlen-Loss inequality : Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that $\rho \ge 0$ and $\int_{\mathbb{R}^2}\rho\,\mathrm{d}x = 1$. Then, there exists a constant $C_* > 0$ such that

$$-4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x)\rho(y)\ln(|x-y|)\mathrm{d}y\,\mathrm{d}x \le C_* + 2\int_{\mathbb{R}^2}\rho\ln(\rho)\,\mathrm{d}x.$$

Conclusion

Bounds and compactness

- We get uniform estimates on $\rho_\varepsilon(1+|x|+\ln(\rho_\varepsilon))$ in $L^\infty(0,T);L^1(\mathbb{R}^N))$, and on $\frac{1}{\varepsilon}|\nabla_v(f_\varepsilon/M)|\sqrt{M}$ in $L^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$.

  … under a threshold condition for the attractive case
- $\mathbb{P}_\varepsilon = \rho_\varepsilon\mathbb{I} + \mathcal{O}(\varepsilon)$ and J_ε is bounded in $L^1((0,T)\times\mathbb{R}^N)$.

Passage to the limit $N=2$

Use a symmetry argument

$$\int \rho\nabla_x\Phi\ \psi\,\mathrm{d}x = \pm\frac{1}{2\pi}\int \rho(x)\rho(y)\quad \underbrace{\frac{x-y}{|x-y|^2}\ (\psi(x)-\psi(y))}_{\text{belongs to } L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\ \mathrm{d}x$$

IV. Hydrodynamic limits and hyperbolic problems

Start with the BGK equation

$$\partial_t f + \xi \cdot \nabla_x f = \frac{1}{\tau}\big(M_{n,u,\theta} - f\big)$$

As $\tau \longrightarrow 0$, we guess that

$$f \simeq M_{n,u,\theta}$$

and due to the **Conservation properties** we obtain (at least formally...) the **Euler system**

$$\begin{aligned}
&\partial_t n + \mathrm{div}_x(nu) = 0,\\
&\partial_t(nu) + \mathrm{Div}_x(nu \otimes u + n\Theta\mathbb{I}) = 0,\\
&\partial_t\Big(\frac{nu^2}{2} + N\frac{n\theta}{2}\Big) + \mathrm{div}_x\Big(\Big(\frac{nu^2}{2} + N\frac{n\theta}{2} + n\theta\Big)u\Big) = 0.
\end{aligned}$$

IV-1. A BGK-like toy model [Perthame-Tadmor '91]

Let $a : \mathbb{R} \to \mathbb{R}$ and consider

$$\partial_t f + a(v)\partial_x f = \frac{1}{\epsilon}(\mathbb{1}_{0\leq v\leq\rho} - f), \qquad \rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v,$$

Here, the "Maxwellian" is $\mathbb{1}_{0\leq v\leq\rho}$

Mass conservation

$$\int_{\mathbb{R}} (\mathbb{1}_{0\leq v\leq\rho} - f)\,\mathrm{d}v = \int_0^\rho \mathrm{d}v - \int_{\mathbb{R}} f\,\mathrm{d}v = 0.$$

which leads to the local conservation law

$$\partial_t \int_{\mathbb{R}} f\,\mathrm{d}v + \partial_x \int_{\mathbb{R}} a(v) f\,\mathrm{d}v = 0$$

Can we guess the limit $\epsilon \to 0$? (Yes, we can !)

Set $\displaystyle\int_{\mathbb{R}} a(v) \mathbb{1}_{0\leq v\leq \rho} \, \mathrm{d}v = \int_0^\rho a(v) \, \mathrm{d}v = A(\rho)$

Write $f = \mathbb{1}_{0\leq v\leq\rho} - \epsilon(\partial_t f + a(v)\partial_x f)$ so that

$$\partial_t \rho + \partial_x \int_{\mathbb{R}} a(v)\mathbb{1}_{0\leq v\leq\rho} \, \mathrm{d}v - \epsilon\partial_x \left(\int_{\mathbb{R}} a(v)\big(\partial_t f + a(v)\partial_x f\big) \, \mathrm{d}v \right) = 0$$

$$= \partial_t\rho + \partial_x A(\rho) - \epsilon \left(\partial^2_{tx} \int_{\mathbb{R}} a(v) \; f \, \mathrm{d}v + \partial^2_{xx} \int_{\mathbb{R}} a(v)^2 \; f \, \mathrm{d}v \right) = 0.$$

Then, we suspect that $f \simeq \mathbb{1}_{0\leq v\leq\rho}$ with ρ solution of the scalar conservation law

$$\partial_t\rho + \partial_x A(\rho) = 0.$$

Step 1. Existence of solutions and a priori estimates

Initial data

Let $f_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ verify $0 \leq f_0 \leq 1$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x < \infty, \quad f_0(x,v) = 0 \text{ for } v \leq 0 \text{ or } v \geq V_0.$$

Duhamel formula

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[e^{s/\epsilon}\ f(t+s, x+sa(v), v)\right] = \frac{1}{\epsilon}\ e^{s/\epsilon}\ \mathbb{1}_{0 \leq v \leq \rho(t, x+sa(v))}.$$

Iterative scheme

- $f^{(0)} = 0$,
- $f^{(n)}$ being given, set $\rho^{(n)}(t,x) = \int_{\mathbb{R}} f^{(n)}(t,x,v)\,\mathrm{d}v$ and

$$f^{(n+1)}(t,x,v) = e^{-t/\epsilon} f_0(x - ta(v), v) + \frac{1}{\epsilon}\int_0^t e^{-(t-s)/\epsilon}\ \mathbb{1}_{0 \leq v \leq \rho^{(n)}(s, x+(s-t)a(v))}\,\mathrm{d}s$$

Iterative properties

- $0 \le f^{(n)} \le 1$ implies $0 \le f^{(n+1)} \le 1$ since $\dfrac{1}{\epsilon}\displaystyle\int_0^t e^{-(t-s)/\epsilon}\,\mathrm{d}s = 1 - e^{-t/\epsilon}$.
- $\mathrm{supp}\big(f^{(n)}(t,x,\cdot)\big) \subset [0, V_0]$ implies $\mathbb{1}_{0 \le v \le \rho^{(n)}} = 0$ on $\complement[0, V_0]$. Thus $\mathrm{supp}\big(f^{(n+1)}(t,x,\cdot)\big) \subset [0, V_0]$.
- If $0 \le \rho_1 \le \rho_2$ then $\mathbb{1}_{0 \le v \le \rho_1} \le \mathbb{1}_{0 \le v \le \rho_2}$ so that $0 = f^{(0)} \le f^{(1)} \le \dots f^{(n)} \le f^{(n+1)} \le \dots \le 1$.
- $\displaystyle\int_{\mathbb{R}} \big(\mathbb{1}_{0 \le v \le \rho_2} - \mathbb{1}_{0 \le v \le \rho_1}\big)\,\mathrm{d}v = \rho_2 - \rho_1$ yields $\displaystyle\int_{\mathbb{R}} |\mathbb{1}_{0 \le v \le \rho_2} - \mathbb{1}_{0 \le v \le \rho_1}|\,\mathrm{d}v \le |\rho_2 - \rho_1|$.

Contractive scheme

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f^{(n+1)} - f^{(n)}|(t)\,\mathrm{d}v\,\mathrm{d}x \le \frac{1}{\epsilon}\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{t-s}{\epsilon}} |f^{(n)} - f^{(n-1)}|(s)\,\mathrm{d}x\,\mathrm{d}s.$$

Step 2. Dissipation Properties

Lemma.

Let $f \in L^1(\mathbb{R})$ *verify* $0 \leq f \leq 1$. *Let* H *be a non decreasing function :*

$$\int_{\mathbb{R}} (\mathbb{1}_{0 \leq v \leq \rho} - f) \ H(v) \, \mathrm{d}v \leq 0.$$

Rewrite the collision term

- Set $m_\epsilon(t,x,v) = \dfrac{1}{\epsilon} \displaystyle\int_0^v \big(\mathbb{1}_{0 \leq w \leq \rho_\epsilon} - f_\epsilon\big)(t,x,w) \, \mathrm{d}w$
- m_ϵ is a sequence of non negative measures on $(0,T) \times \mathbb{R} \times \mathbb{R}$. since for $h \geq 0$,

$$\int_{\mathbb{R}} m_\epsilon h \, \mathrm{d}v = - \int_{\mathbb{R}} \partial_v m_\epsilon \int_0^v h(w) \, \mathrm{d}w \, \mathrm{d}v \geq 0.$$

- The sequence $\big(m_\epsilon\big)_{\epsilon > 0}$ is bounded in $\mathcal{M}^1((0,T) \times \mathbb{R} \times \mathbb{R})$.

Step 3. Compactness

We have

$$\partial_t f_\epsilon + a(v)\partial_x f_\epsilon = \partial_v m_\epsilon$$

with

- f_ϵ bouded in L^∞,
- m_ϵ bounded in $\mathcal{M}^1((0,T)\times\mathbb{R}\times\mathbb{R})$.
- Suppose $a(v) = A'(v) \neq 0$ for a. e. v.

Average lemma applies and ρ_ϵ converges strongly in $L^p((0,T)\times\mathbb{R})$, $1 \leq p < \infty$.

Suppose further $f_{0,\epsilon} \rightharpoonup \mathbb{1}_{0\leq v\leq \rho_0(x)}$ (preparation of data) : it guarantees

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int |\rho(t,x)-\rho_0(x)|\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Step 4. Conclusion

Conservation law

We have $\partial_t \rho + \partial_x A(\rho) = 0$... but this is not enough

Entropies

$$\int_0^t \int_{\mathbb{R}} \left(\eta(\rho)\partial_t \psi + q(\rho)\partial_x \psi\right) \,\mathrm{d}x\,\mathrm{d}t \geq 0$$

for any positive $\psi \in C_c^\infty((0,\infty) \times \mathbb{R} \times \mathbb{R})$ and any pair entropy/entropy flux (η, q), with η convex and $\eta' A' = q'$. Indeed

$$\partial_t \eta(\rho_\epsilon) + \partial_x q(\rho_\epsilon) = \underbrace{-\int_{\mathbb{R}} \eta''(v)\ m_\epsilon \,\mathrm{d}v}_{\text{which is } \leq 0} + \underbrace{\text{remainder}}_{\text{which is small}}$$

IV-2. The relative entropy approach

We start from the Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_v \cdot \big((v - u) f + \nabla_v f \big),$$

$$\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v, \qquad \rho u(t,x) = J(t,x) = \int v \ f(t,x,v)\,\mathrm{d}v$$

Conservation of Mass and Momentum

$$\int Q(f)\,\mathrm{d}v = 0, \qquad \int v \ Q(f)\,\mathrm{d}v = 0$$

Equilibrium $f(t,x,v) = \frac{\rho(t,x)}{(2\pi)^{N/2}} \exp\Big(- \frac{|v - u(t,x)|^2}{2} \Big)$

Dissipation (periodic BC)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \Big(\frac{v^2}{2} + \ln(f) \Big)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\epsilon} \int \Big| (v - u)\sqrt{f} + 2\nabla_v \sqrt{f} \Big|^2 u dv\,\mathrm{d}x \leq 0.$$

Guess the limit

Moment conservation

$$\partial_t \rho + \mathrm{div}_x J = 0,$$
$$\partial_t J + \mathrm{Div}_x \left(\int v \otimes v f \, \mathrm{d}v \right) = 0,$$

Equilibrium and dissipation

We suspect that $f \simeq \frac{\rho(t,x)}{(2\pi)^{N/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2} \right)$ which imposes the dependence of the particle distribution function wrt v

Isothermal Euler system

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \mathrm{Div}_x\big(\rho u \otimes u + \rho \mathbb{I}\big) = 0.$$

Strategy of proof

Difficulty related to non linear terms $(\rho_\epsilon u_\epsilon,\ \rho_\epsilon u_\epsilon \otimes u_\epsilon)$ and a lack of compactness : we only know

A priori estimates

- $f_\epsilon(1 + v^2/2 + |\ln(f_\epsilon)|)$ is bounded in $L^\infty(0, T; L^1(\mathbb{T}^N \times \mathbb{R}^N))$,
- $\dfrac{1}{\sqrt{\epsilon}}\big((v - u_\epsilon)\sqrt{f_\epsilon} + 2\nabla_v \sqrt{f_\epsilon}\big)$ is bounded in $L^2((0, T) \times \mathbb{T}^N \times \mathbb{R}^N)$.

Idea

Evaluate how far $f_\epsilon, \rho_\epsilon, J_\epsilon$ are from the expected limit $M_{\rho,u,1}, \rho, \rho u$... but with a more clever idea than using a mere L^p norm.

- Use the entropy of the limit system
- Use compatibility conditions between the kinetic and the fluid eq.

Entropy and Relative Entropy (Let us assume $N = 1$)

Flux, Entropy, Entropy Flux

Set $U = (\rho, J)$, $A(U) = \left(J, \dfrac{J^2}{\rho} + \rho\right)$ so that $\partial_t U + \partial_x A(U) = 0$. At least for smooth solution of the Euler eq. $\partial_t \eta + \partial_x Q = 0$ with

$$\eta(\rho, J) = \frac{J^2}{2\rho} + \rho \ln(\rho) + C_\star \rho + 1/e \geq 0,$$
$$Q(\rho, J) = \frac{1}{2}\frac{J^3}{\rho^2} + J \ln(\rho) + (C_\star + 1)J.$$

since $\left(\nabla_U \eta\right)^T \overline{\overline{\nabla_U A}} = \nabla_U Q$.

Relative Entropy, Relative Flux

We set

$$\eta(U|V) = \eta(U) - \eta(V) - \nabla_U \eta(V) \cdot (U - V),$$
$$A(U|V) = A(U) - A(V) - \overline{\overline{\nabla_U A}}\,(U - V).$$

Lemma.

- $U \mapsto \eta(U)$ is convex
- The Relative entropy $\eta(U|V)$ is non negative and vanishes iff $U = V$.
- $|A(U|V)| \leq \eta(U|V)$

Connection to the microscopic quantities

Set $H(f_\varepsilon, v) = f_\epsilon \ln(\epsilon) + \dfrac{v^2}{2} f_\epsilon$ and

$$U_\epsilon = \int \begin{pmatrix} 1 \\ v \end{pmatrix} f_\epsilon \,\mathrm{d}v, \quad A_\epsilon = \int v \begin{pmatrix} 1 \\ v \end{pmatrix} f_\epsilon \,\mathrm{d}v,$$
$$\eta_\epsilon = \int H(f_\varepsilon, v) \,\mathrm{d}v, \quad Q_\epsilon = \int v \; H(f_\varepsilon, v) f_\epsilon \,\mathrm{d}v,$$

Then **Conservation :** $\partial_t U_\epsilon + \partial_x A_\epsilon = 0$ and **Dissipation :** $\partial_t \eta_\epsilon + \partial_x Q_\epsilon \leq 0$

Convergence proof

Let $M_\epsilon = M_{\rho_\epsilon, J_\epsilon/\rho_\epsilon, 1}$, and define the Kinetic Relative Entropy

$$\eta_\epsilon - \eta(U_\epsilon) = \int \left[f_\epsilon \ln\left(\frac{f_\epsilon}{M_\epsilon}\right) - f_\epsilon + M_\epsilon \right] \mathrm{d}v \geq 0.$$

Compute, with U smooth solution of the isothermal Euler system,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \Big(\eta_\epsilon - \eta(U_\epsilon) + \eta(U_\epsilon | U) \Big) \mathrm{d}x = \dots \\
&\leq - \int \partial_x Q(U_\varepsilon | U) \,\mathrm{d}x + \int \eta''(U) \partial_x U \cdot A(U_\varepsilon | U) \,\mathrm{d}x + \int \Big(A_\epsilon - A(U_\epsilon) \Big) \mathrm{d}x \\
&\leq 0 + C(\|U\|_{W^{1,\infty}}) \int \eta(U_\epsilon | U) \,\mathrm{d}x + \int v\sqrt{f_\epsilon} \times \Big((v - u_\epsilon)\sqrt{f_\epsilon} + \partial_v \sqrt{f_\epsilon} \Big) \mathrm{d}v \\
&\leq C(\|U\|_{W^{1,\infty}}) \int \eta(U_\epsilon | U) \,\mathrm{d}x + C\sqrt{\epsilon}
\end{aligned}$$

Conclusion

Theorem.

Let U be a smooth solution of the isothermal Euler system on $[0, T]$ with initial data U_0 such that $\Big(\eta(U_\epsilon|U) + \eta_\epsilon - \eta(U_\epsilon)\Big)(t=0) \to 0$ as $\epsilon \to 0$. Then $\Big(\eta(U_\epsilon|U) + \eta_\epsilon - \eta(U_\epsilon)\Big) \to 0$ uniformly on $[0, T]$.

Comment

We can obtain convergence in more usual spaces by making use of

$$\eta(U_\epsilon|U) = \rho_\epsilon \ln\left(\frac{\rho_\epsilon}{\rho}\right) - \rho_\epsilon + \rho + \rho_\epsilon(u_\epsilon - u)$$

and the Csizar-Kullback inequality

$$\left(\int |\rho - \rho_\epsilon|\,\mathrm{d}y\right)^2 \leq C \int \Big(\rho_\epsilon \ln(\rho/\rho_\epsilon) - \rho_\epsilon + \rho\Big)\,\mathrm{d}y$$

Hence, $\rho_\epsilon \to \rho$, $J_\epsilon = \rho_\epsilon u_\epsilon \to J = \rho u$ in $C^0([0,T]; L^1(\mathbb{T}\times\mathbb{R}))$.

Further comments

- Dealing with higher dimension is not a big deal
- We can also consider "dissipative solutions" of the limit system
- The method applies the same way with $Q(f) = \mathbb{1}_{|u-v|^{2/(\gamma-1)} \leq C_N \rho} - f$ if $\gamma = (N+2)/N$ or $Q(f) = C\left[\dfrac{2\gamma}{\gamma-1}\rho^{\gamma-1} - |v-u|^2\right]_+^{\frac{1}{\gamma-1}-\frac{N}{2}} - f$ otherwise and it leads to the isentropic Euler system.
- The method also applies to mixed fluid/particles models
  [G.-Jabin-Vasseur '04, Berthelin-Vasseur '05, Vasseur '09]
  or to plasma physics
  [Brenier '00, Golse-St Raymond '03, Brenier-Mauser-Puel '03, Puel-St Raymond '04, Bostan-G. '08],
  or to the gas dynamics equations
  [St Raymond '03]