LECTURE II
SCATTERING

RANDOM SCATTERING

$f(p,t)$: density of particles with a state p at time t.

Particles undergo a random change in state (a 'collision') with a certain frequency.

$$S(p,p')\ dp = d\mathcal{P}\{p' \to p\}$$

p': pre-collision state, p: post-collision state

BOLTZMANN VS. BROWN

At each infinitesimal time interval Δt flip a coin.

$$\mathcal{P}[scatter] = \omega(p'), \quad \mathcal{P}[not\ scatter] = 1 - \omega(p'),$$

if $scatter$, choose a new state p from the distribution $S(p,p')\,dp$

This gives

$$f(p,t+\Delta t) = \int S(p,p')\omega(p')f(p',t)\,dp' + \int \delta(p-p')(1-\omega(p'))f(p',t)\,dp'$$

The Boltzmann model :

$\omega(p') = \gamma(p')\Delta t$ with $\gamma(p')$ collision frequency

$\Rightarrow$ Boltzmann equation for $\Delta t \to 0$

$$\partial_t f(p,t) = \int S(p,p')\gamma(p')f(p',t)\ dp' - \gamma(p)f(p,t)$$

$f(x,p,t)$ density of particles with momentum p

Brownian motion:

$\omega = 1$ scatter each time (producing a random walk according to a Wiener process)

But scatter 'only a little bit' $S(p,p') \approx \delta(p-p')$

$$S(p,p') = (A\Delta t)^{-dim(p)/2} R(\frac{p - p'(1+\Delta t B)}{\sqrt{A\Delta t}}, p'), \qquad \int \begin{pmatrix} 1 \\ q \\ |q|^2 \end{pmatrix} R(q,p')\ dq =$$

weak formulation:

$$\int \psi(p) f(p, t{+}\Delta t)\ dp = \int \psi(p'(1{+}\Delta t B){+}\sqrt{\Delta t A q}) R(q, p') f(p', t)\ dqp'$$

gives for $\Delta t \to 0$

$$\int \psi(p) \partial_t f(p, t)\ dp = \int f(p')[Bp' \cdot \nabla \psi(p') + A \Delta t(p')]\ dp'$$

or the Fokker - Planck equation

$$\partial_t f = \nabla_p \cdot [A \nabla_p f - Bp \cdot \nabla_p f]$$

Boltzmann:

$$\partial_t f(p, t) = \int S(p, p') \gamma(p') f(p', t)\ dp' - \gamma(p) f(p, t)$$

Brownian motion:

$$\partial_t f = \nabla_p \cdot [A \nabla_p f - Bp \cdot \nabla_p f]$$

A remark on Markov processes

- The frequency ω is independent of the last time the scattering event has occurred.

- For 'intelligent' particles (traffic, production systems) this is often unrealistic.

$$\mathcal{P}[\Delta T_{scat}] = \gamma e^{-\gamma \Delta T_{scat}}$$

ΔT_{scat}: time between scattering events.

Non - Markov processes:

Enlarge the state space: $p \to (p, \tau)$
τ: time elapsed since the last scattering event

$$p' \to p, \;\; \tau = 0 \text{ with } \mathcal{P} = \gamma(\tau)\Delta t,$$

$$p' \to p, \;\; \tau \to \tau + \Delta t \text{ with } \mathcal{P} = 1 - \gamma(\tau)\Delta t,$$

$$\mathcal{P}[\Delta T] = c \exp[-\int_0^{\Delta T} \gamma(\tau) \; d\tau]$$

Gives the Boltzmann equation

$$\partial_t f + \partial_\tau f = Q[f](p, \tau, t)$$

The Boltzmann equation

$$\partial_t f(p,t)+C[\mathcal{E},f]=Q[f]=\int S(p,p')\gamma(p')f(p')\,dp'-\gamma(p)f(p)\int S(p',p)\,dp'$$

$$C[\mathcal{E},f]=\nabla_p\mathcal{E}\nabla_x\cdot f-\nabla_x\mathcal{E}\cdot\nabla_p f,\quad \mathcal{E}=\frac{|p|^2}{2}+\phi(x)$$

The mean free path $\gamma(p')$: scattering frequency of particle with momentum p'.
$|\nabla\varepsilon(p')|\gamma(p')$: mean free path.

Bosons and Fermions:

$$\gamma(p')=\frac{L_{MFP}}{|\nabla\varepsilon(p')|}$$

Maxwellian Molecules:

$$\gamma(p')=\gamma_0,\quad L_{MFP}=\gamma_0|\nabla\varepsilon(p')|$$

CONSERVATION

There are certain quantities which remain unchanged in this process $\kappa(p) = \kappa(p')$

$$\partial_t \int \kappa(p) f(p,t)\ dp = 0$$

Weak formulation:

$$\int \psi(p) Q[f](x,p)\ dp = \int [\psi(p) - \psi(p')] S(p,p') \gamma(p') f(p')\ dpp'$$

$\kappa(p)$ conserved in scattering $\kappa(p) = \kappa(p')$

$$S(p,p') \ \rightarrow \ \delta(\kappa(p) - \kappa(p')) S(p,p')$$

$$\int \kappa(p) Q[f](p)\ dp = \int [\kappa(p) - \kappa(p')] \delta(\kappa - \kappa') S(p,p') \omega(p') f(p')\ dpp' = 0$$

$$\Rightarrow \ \partial_t \int \kappa(p) f(p)\ dp = 0$$

BINARY PARTICLE INTERACTIONS

$x = (x_1, x_2),\ p = (p_1, p_2),$
f: particle of type 1 is at position x_1 with momentum p_1 and particle of type 2 is at position x_2 with momentum p_2.
$\mathcal{E} = \mathcal{E}_1(x_1, p_1) + \mathcal{E}_2(x_2, p_2).$

- ▶ statistical independence: $f(x_1, x_2, p_1, p_2) = f_1(x_1, p_1) f_2(x_2, p_2)$

- ▶ Integrate out the other density functions gives two Boltzmann equations for the two density functions $f_1,\ f_2$

$$\partial_t f_1(x_1,p_1,t) + [\mathcal{E}, f_1] =$$

$$\int S(p,p')\gamma(p')f_1(p_1')f_2(p_2')\,dp_1'p_2'p_2 - f_1(p_1)\int \gamma(p)f_2(p_2)S(p',p)\,dp_1'p_2'p_2$$

$$\partial_t f_2(x_2,p_2,t) + [\mathcal{E}, f_2] =$$

$$\int S(p,p')\gamma(p')f_1(p_1')f_2(p_2')\,dp_1'p_2'p_1 - f_2(p_2)\int \gamma(p)f_1(p_1)\,dp_1'p_2'p_1$$

- Scattering of particles of the same type: (gas dynamics) $f_1 = f_2$, $Q[f,f]$ quadratic integral operator. Electron - electron scattering.

- Scattering of particles of different type. (Two different gases, electron - phonon interaction). Either two BTEs or assume an ansatz for f_2

MICRO - REVERSIBILITY AND MAXWELLIAN KERNELS

The transition $p' \to p$ is as likely as the transition $p \to p'$, meaning the scattering cross section $S\gamma$ is symmetric.

$$S(p,p')\gamma(p') = S(p',p)\gamma(p)$$

This, together with the conservation properties, determines the kernel elements of Q

$$\int \psi(p) Q[f](x,p)\, dp = \frac{1}{2} \int [\psi(p) - \psi(p')] S(p,p')\gamma(p')[f(x,p') - f(x,p)]\, dpp'$$

$$S \approx \delta(\kappa(p) - \kappa(p')) \;\Rightarrow\; Q[f(x,\kappa(p)] = 0$$

For the binary interaction operator we have

$$\kappa(p_1, p_2) = \kappa_1(p_1) + \kappa_2(p_2) \;\Rightarrow\; f_1 f_2 = G(x_1, x_2, \kappa_1 + \kappa_2)$$

For particle - particle scattering $f_1 = f_2$ this gives the Maxwellian

$$f_1(x_1, p_1) = \rho(x_1)\,\mathrm{exp}(\mu(x_1) \cdot \kappa(p_1))$$

For particle - background scattering we assume the ansatz $f_2(x_2, p_2) = \mathrm{exp}(-\mu \cdot \kappa_2(p_2))$ this gives

$$f_1(x_1, p_1) = \rho(x)\,\mathrm{exp}[-\mu \cdot \kappa_1(p_1)]$$

ENTROPY AND INFORMATION LOSS

$S(f)$ convex functional that is decreased by $Q[f]$.

$$\partial_t f = Q[f], \quad DS(f)(Q[f]) \leq 0, \; \forall f$$

$$\partial_t S(f) = DS(f)(\partial_t f) = DS(f)(Q[f])$$

$S(f)$ acts like the nonlinear version of a norm and guarantees stability. It also drives the solution towards the kernel f_{eq} in the long time limit.

A PROBABILISTIC DEFINITION OF ENTROPY

Differential amount of information gained by observing an experiment.

An argument for the logarithm: The Von Neumann entropy

- Consider N balls in a bag, either black (b) with probability p or white (w) with probability $1 - p$.

$$\Delta I(\mathcal{E}) = \phi(\mathcal{P}(\mathcal{E}))$$

$$\Delta I(\mathcal{E}_1 \cup \mathcal{E}_2) = \Delta I(\mathcal{E}_1) + \Delta I(\mathcal{E}_2)$$

$$\mathcal{P}(\mathcal{E}_1 \cup \mathcal{E}_2) = \mathcal{P}(\mathcal{E}_1)\mathcal{P}(\mathcal{E}_2)$$

$$\Rightarrow \ \phi(P_1P_2) = \phi(P_1) + \phi(P_2) \ \Rightarrow \ \phi(P) = -K\,\mathrm{ln}(P)$$

Entropies of the Boltzmann equation

I. Particle - particle scattering:

For electron - electron scattering $f_1 = f_2$ we have to have

$$S(p_1, p_2, p_1', p_2')\gamma(p_1', p_2') = S(p_2, p_1, p_2', p_1')\gamma(p_2', p_1')$$

and therefore in the weak formulation

$$\int \psi(p_1) Q[f](p_1)\ dp_1 =$$

$$\frac{1}{2}[\psi(p_1) - \psi(p_1')] S(p, p')\gamma(p')[f(p_1')f(p_2') - f(p_1)f(p_2)]\ dp_1 p_2 p_1' p_2'$$

$$= \frac{1}{4}[\psi_1 + \psi_2 - \psi_1' - \psi_2'] S(p, p')\gamma(p')[f_1' f_2' - f_1 f_2]\ dp_1 p_2 p_1' p_2'$$

Setting $\psi = \ln(f)$ implies

$$DS(f)(Q[f]) = \int \ln(f)Q[f]\ dp_1$$

$$= \frac{1}{4}[\ln(f_1f_2) - \ln(f_1'f_2')]S(p,p')\gamma(p')[f_1'f_2' - f_1f_2]\ dp_1p_2p_1'p_2' \ \leq 0$$

Therefore the entropy functional S is given by

$$S(f) = \int f(\ln f - 1)\ dp$$

II. Particle - background scattering:

The Boltzmann operator can be written in weak form as

$$\int \psi_1 Q[f_1]\ dp_1 =$$

$$\int (\psi_1 - \psi_1') S(p,p') \gamma(p') e^{-\mu(\kappa_1+\kappa_2)} (f_1' e^{\mu\kappa_1'} - f_1 e^{\mu\kappa_1})\ dpp'$$

Therefore

$$\int g'(f_1 e^{\kappa_1}) Q[f_1]\ dp_1 \leq 0$$

holds for any convex function g and monotone function g' and

$$DS(f)\delta f = \int g'(f e^{\mu\kappa_1}) \delta f\ dp$$

for any convex g.

THE RELATIVE ENTROPY

$$\int g'(f_1 e^{\kappa_1}) Q[f_1]\ dp_1 \leq 0$$

$S(f) =$ has to be conserved by the transport operator (the Poisson bracket).

$$\int g'(f e^{\mu\kappa_1}) [\mathcal{E}, f]\ dxp = 0$$