On the solution of the electromagnetic inverse medium scattering problem

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IPAM workshop on Inverse Problems
Los Angeles, November 18 , 2003

direct electromagnetic medium scattering problem

time harmonic Maxwell equations:

$$\operatorname{curl} E - i\,k\,H = 0 \qquad \operatorname{curl} H + ik\,n(x)\,E = 0$$

notations ∧ assumptions:

$E, H : \mathbb{R}^3 \to \mathbb{C}^3$ electric field ∧ magnetic field

$\epsilon(x) > 0$ electric permittivity, $\epsilon(x) = \epsilon_0$ for $|x| > \rho$

$\sigma(x) \geq 0$ electric conductivity, $\sigma(x) = 0$ for $|x| > \rho$

$my(x) > 0$ magnetic permeability, $my(x) = my_0$ for all x

$n(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\,\sigma\frac{(x)}{\omega}\right)$ refractive index, $n \in C^{1,\alpha}(\mathbb{R}^3), \alpha > 0$

$k = \omega\sqrt{\epsilon_0\, my_0}$ wave number, ω frequency: time dependence $\tilde{E}(t,x) = \Re\left(E(x)\,e^{-i\omega t}\right)$

incident field: $E_i(x) = E_i(x,d,p) = p\,e^{ikx\cdot d}, \quad H_i(x) = H_i(x,d,p) = (ik)^{-1}\operatorname{curl} E(x,d,p)$

$d \in S^2$ propagation direction, $p \in \mathbb{C}^3$ polarization, $p \cdot d = 0$

scattered field: $E_s(x,d,p) = E(x,d,p) - E_i(x,d,p), \quad H_s(x,d,p) = H(x,d,p) - H_i(x,d,p)$

Silver – Müller radiation condition: $\lim_{|x|\to\infty}\left(H_s(x) \times x - |x|\,E_s(x)\right) = 0$

far field pattern

The Silver – Müller radiation condition implies the asymptotic behavior

$$E_s(x) = \frac{\exp(ik\,|x|)}{|x|}\left(E_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right), \qquad |x| \to \infty\ .$$

The amplitude factor $E_\infty = E_\infty(\,\cdot\,, d\,, p): S^2 \to \mathbb{C}^3$ is called the *far field pattern* of E_s . It is a tangential field on the unit sphere , i.e. $\hat{x} \cdot E_\infty(\hat{x}) = 0$ for all $\hat{x} \in S^2$.

By linear superposition , there exists a matrix $e_\infty(\hat{x}\,, d\,) \in \mathbb{C}^{3\times 3}$ such that

$$E_\infty(\hat{x}, d\,, p) = e_\infty(\hat{x}, d\,)\,p \qquad \text{for all } p \in \mathbb{C}^3 \text{ satisfying } p \cdot d = 0\ .$$

$e_\infty(\hat{x}\,, d\,)$ represents a linear mapping from the complexified tangent space to S^2 at d to the complexified tangent space at $\hat{x}$.

We choose $e_\infty(\hat{x}\,, d\,)$ such that $e_\infty(\hat{x}\,, d\,)\,d = 0.$

electromagnetic Lippmann-Schwinger equation

Let $\Phi(x) := \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|}$, $\wedge$ let $a := 1 - n$.

If $E, H \in C^1(R^3)$ is a solution to the direct problem , then

$$E(x) + k^2 \int_{R^3} \Phi(x-y)\, a(y)\, E(y)\, dy$$

$$- \operatorname{grad} \int_{R^3} \Phi(x-y) \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, dy = E_i(x), \quad x \in \mathbb{R}^3 .$$

Vice versa , for any solution $E \in C(\mathbb{R}^3)$ to this integral equation ,
$E \wedge H := \operatorname{curl} E / ik$, are a solution to the direct problem .

the inverse problem

inverse problem : Given the far field pattern $e_{\infty}(\hat{x}, d)$ for all $(\vee$ some$)$ $\hat{x}, d \in S^{2},$ determine the refractive index a !

operator equation formulation : Introducing the operator

$$F : D(F) \subset H_0^s(B_\rho) \to L^2(S^2 \times S^2, \mathbb{C}^{3\times 3}),$$

$s > 5/2$, which maps $a \in D(F)$ to the far field pattern e_∞ corresponding to the refractive index $n = 1 - a$, this problem can be formulated as an operator equation

$$F(a) = e_\infty \ .$$

numerical solution of the direct electromagnetic medium scattering problem

periodization I

We adapt Vainikko's method for the acoustic medium scattering problem (1999) ...

Let $\varphi : R^3 \to C$ denote the 4ρ multi − periodic function satisfying

$$\varphi(x) = \begin{cases} \Phi(x), & x \in B_{2\rho} \\ 0 & x \in G_{2\rho} \setminus B_{2\rho} \end{cases}$$

where $G_{2\rho} = \{x : |x|_\infty \le 2\rho\} \wedge B_{2\rho} = \{x : |x|_2 \le 2\rho\}$.

Then a function $E : B_\rho \to \mathbb{C}^3$ satisfies

$$E(x) + k^2 \int_{B_\rho} \Phi(x-y)\, a(y)\, E(y)\, dy$$

$$(1) \qquad - \operatorname{grad} \int_{B_\rho} \Phi(x-y) \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, dy = E_i(x), \qquad x \in B_\rho$$

if $\wedge$ only if

$$E(x) + k^2 \int_{B_\rho} \varphi(x-y)\, a(y)\, E(y)\, dy$$

$$(2) \qquad - \operatorname{grad} \int_{B_\rho} \varphi(x-y) \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, dy = E_i(x), \qquad x \in B_\rho \, .$$

periodization II

Let f denote a smooth , $4\rho-$periodic function satisfying $f(x)=E_i(x)$ for $x\in B_\rho$. Then a solution $E=u:B_\rho\to\mathbb{C}$ of eq. (2) can be extended to a smooth $4\rho-$periodic function by

$$u(x):=f(x)-k^2\int_{B_\rho}\varphi(x-y)\,a(y)\,E(y)\,dy$$

$$+\operatorname{grad}\int_{B_\rho}\varphi(x-y)\frac{1}{n(y)}\operatorname{grad} n(y)\cdot E(y)\,dy, \qquad x\in R^3 .$$

We will numerically solve the multi $-$ periodic integral equation

$$u(x)+k^2\int_{B_\rho}\varphi(x-y)\,a(y)\,u(y)\,dy$$

$$-\operatorname{grad}\int_{B_\rho}\varphi(x-y)\underbrace{\frac{1}{n(y)}\operatorname{grad} n(y)}_{=:b(y)}\cdot u(y)\,dy=f(x), \qquad x\in G_{2\rho} .$$

discretization

Notation :

$$(Ku)(x) := \int_{G_{2\rho}} \varphi(x-y)\,u(y)\,dy$$

P_N : orthogonal L^2 – projection operator onto

span$\{\exp(i\pi(j\cdot x)/2\rho) : j\in\mathbb{Z}^{3,} -N/2 \le j_1, j_2, j_3 \le N/2\}$

Q_N : trigonometric interpolation operator on equidistant

$N\times N\times N$ – grid in $G_{2\rho}$

We approximate the periodic electromagnetic Lippmann – Schwinger eq.

$$u + k^2 K(a\,u) - \operatorname{grad} K(b\cdot u) = f$$

by

$$u + k^2 K\,Q_N(a\,P_N u) - \operatorname{grad} K\,Q_N(b\cdot P_N u) = Q_N(\chi E_i)\,.$$

The operator on the left hand side can be implemented by FFT .

error analysis

$\textbf{Theorem}\,(\textbf{H. 2003})\textbf{:}$ Let $my > 3/2\,,\, 0 \le \lambda - 1 \le my\,, \wedge$ assume that $n \in H^{my+1}$.
Let $u \in H^{my}$ denote the unique solution to the periodic electromagnetic Lippmann $-$ Schwinger equation. Then there exist constants $N_0\,,\, c > 0$ such taht the discretized equation has a unique solution u_N for all $N \ge N_0 \wedge$

$$\| u_N - u \|_\lambda \le c\, N^{\lambda - my - 1} \| u \|_{my} \,.$$

features

- ➔ arbitrarily high order of convergence
  (depending on the smoothness of the refractive index n)

- ➔ complexity: $O(N^3 \log(N)^2)$

- ➔ with two-grid acceleration: complexity $O(N^3 \log(N))$

- ➔ natural and exact incorporation of the radiation condition

conditional stability estimates

exponential ill-posedness

Recall that $F(a) = e_\infty$ where $n = 1 - a$ is the refractive index.

Lemma : Let $a \in D(F) \wedge$ let $F'[a]$ denote the Fréchet derivative of F at a . Then the singular values of $F'[a]$ satisfy the estimate

$$\sigma_j(F'[a]) = O(\exp(-cj^{1/4})) \qquad \text{for some } c > 0 .$$

a conditional stability estimate

Theorem (Hähner 2000, H.: 2003):

Assume that $n_1 = 1 - a_1 \wedge n_2 = 1 - a_2$ are two refractive indices such that

$$\| 1/n_i \|_\infty + \| a_i \|_{H^s} \le c$$

for some Sobolev index $s > 3/2 \wedge a_i \in C_0^{1,\alpha}(B_\rho)$.

Let $e_{\infty,1} \wedge e_{\infty,2}$ denote the corresponding matrix far field patterns . Then

$$\| a_1 - a_2 \|_\infty \le C \left(\ln^- \frac{1}{\| e_{\infty,1} - e_{\infty,2} \|_{L^2}} \right)^{-(2s-3)/(2s+11)+\epsilon}$$

for any $\epsilon > 0$ with a constant C depending only on $\rho , c , k , \epsilon , \wedge s$. Here

$$\ln^-(t) = \max(1, \ln t) \wedge \| e_\infty \|_{L^2} := \left(\int_{S^2} \int_{S^2} \| e_\infty(\hat{x}, d) \|_{\text{fro}}^2 \, ds(\hat{x}) \, ds(d) \right)^{1/2} .$$

on the history of the proof

1980 : **Calderon** states the problem if the coefficient γ in the pde $\nabla \gamma \nabla u = 0$ in a bounded domain $\Omega \subset R^d$ is uniquely determined by the Dirichlet − to − Neumann map Λ_γ defined by $\Lambda_\gamma(u_{\partial\Omega}) := \partial u / \partial n$ ∧ solves the linearized problem.

1987 : **Sylvester** ∧ **Uhlmann** solve the Calderon problem for $d \geq 3 \wedge \gamma \in C^\infty(\overline{\Omega})$.

1988 : **Nachman , Novikov ,** ∧ **Ramm** independently prove uniqueness for the inverse acoustic medium scattering problem for $d \geq 3$.

1988 : **Alessandrini** proves a stability estimate for Calderon ' s problem for $d \geq 3$.

1990 : **Stefanov** proves a logarithmic stability estimate for the inverse acoustic medium scattering problem with a norm for the far field patterns involving exponentially growing weights on the Fourier coefficients.

1992 : **Colton** ∧ **Päivärinta** show uniqueness of the inverse electromagnetic medium scattering problem.

2000 : **Hähner** simplifies Stefanov ' s proof , applies it to the electromagnetic problem , ∧ obtains explicit exponents.

2003 : **H.** obtains logarithmic stability estimate with the L^2 − norm for the far field patterns ∧ globalizes Hähner ' s result.

stability for near-field data

Given $R > \rho$, let $\tilde{E}_s^R(x, d, p)$ denote the scattered field corresponding to the incident field given by the electric dipole $E_i(x) = \text{curl curl}\ p\,\Phi(x, Rd)$ where $d \in S^2, p \in C^3, \wedge\ p \cdot d = 0$.

By linear superposition, the tangential component of $\tilde{E}_s^R$ on the sphere $R\,S^2$ is given by

$$\hat{x} \times (\tilde{E}_s^R(R\hat{x}, d, p) \times \hat{x}) = e_R(\hat{x}, d)\, p$$

for some $3 \times 3 -$ matrix $e_R(\hat{x}, d)$.

Theorem: If $a_1 \wedge a_2$ satisfy the assumptions of the previous theorem, then there exists a constant C depending only on $\rho, c, k, R, \wedge\ s$ such that

$$\| a_1 - a_2 \|_\infty \le C \left(\ln^- \frac{1}{\| e_{R,1} - e_{R,2} \|_{L^2}} \right)^{-(2s-3)/(2s+11)} .$$

estimating near field by far field data

Lemma : (Abstract linear stability result)

Let $T : X \rightarrow Y$ be a bounded linear operator between Hilbert spaces $X \wedge Y$.

Assume that $g \in C([0, \| T^* T \|])$ is monotonicly increasing with $g(0) = 0$,

$\wedge$ that the function $\phi_g(\xi) := \xi \cdot (g \cdot g)^{-1}(\xi)$ is convex.

Then the source condition $w = g(T^* T)v, \quad \| v \| \leq c$ implies the stability estimate

$$\| w \|^2 \leq c^2 \phi_g^{-1}\left(\frac{\| Tw \|^2}{c^2} \right) .$$

Specifications :

$$T : L^2(RS^2 \times RS^2) \rightarrow L^2(S^2 \times S^2), \qquad Te_R := e_\infty$$

near field to far field map

$$g_\theta(\lambda) := \exp(-1/2(-\ln \lambda)^\theta), \qquad 0 < \theta < 1$$

weaker than Hölder , but stronger than logarithmic source conditions

A computation shows that $e_{R,1} - e_{R,2}$ satisfies the source condition if $R > \rho$.

a preconditioned Newton method

iterative regularization methods

noisy measured data: e_∞^δ where $\| e_\infty^\delta - e_\infty \| \leq \delta$

Landweber iteration:

$$a_{n+1} = a_n - my\, F'[a_n]^* \left(F(a_n) - e_\infty^\delta\right)$$

$my > 0$ is a scaling parameter such that $my \, \| F'[a_n]^* F'[a_n] \| \leq 1.$

iteratively regularized Gauss – Newton method (IRGNM):

$$\Delta a_n = \operatorname{argmin}_{\Delta \mathrm{a} \in \mathrm{X}} \left(\| F'[a_n] \Delta a + F(a_n) - e_\infty^\delta \|^2 + \gamma_n \| \Delta a + a_n - a_0 \|^2 \right)$$

$$\Leftrightarrow \left(\gamma_n I + F'[a_n]^* F'[a_n] \right) \Delta a_n = F'[a_n]^* (e_\infty^\delta - F(a_n)) + \gamma_n (a_0 - a_n)$$

$$a_{n+1} = a_n + \Delta a_n$$

regularization parameters $\gamma_n = \gamma_0 q^{-n}, \qquad q > 1$

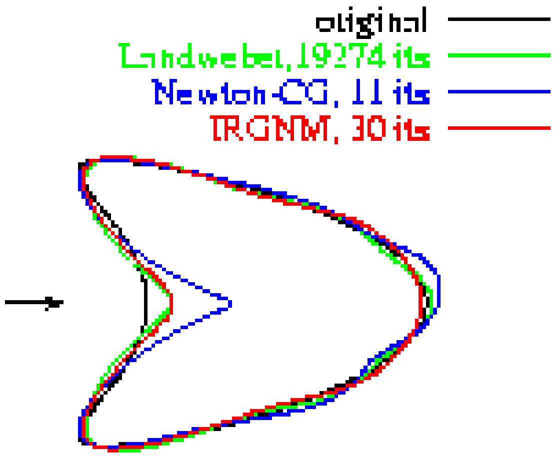
stopping rule (discrepancy principle): Stop the iteration at the first index N for which $\quad \| F(a_N) - e_\infty^\delta \| \leq \tau \delta \quad$ where $\tau > 1$ is a fixed constant .

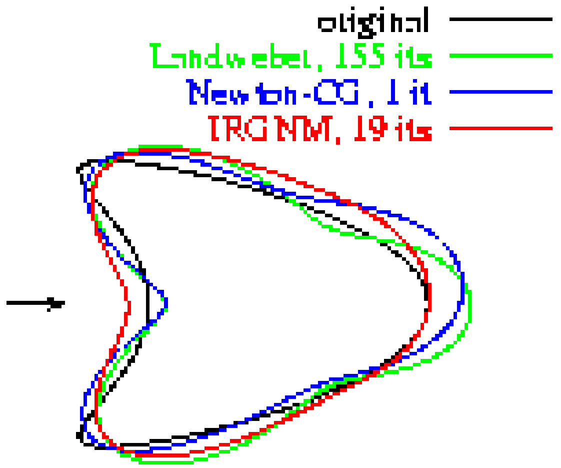
comparison of methods

1% noise

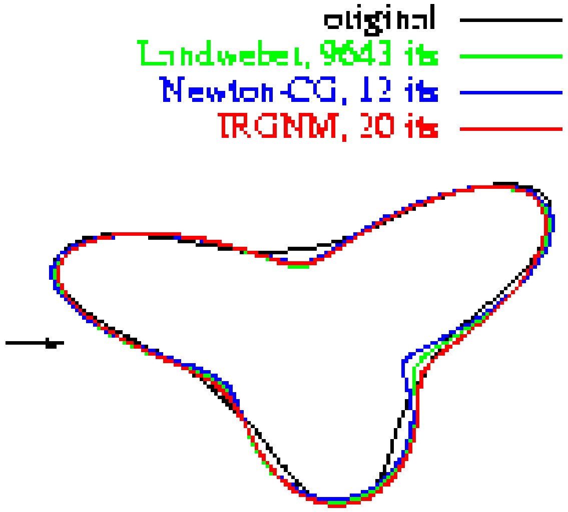
10% noise

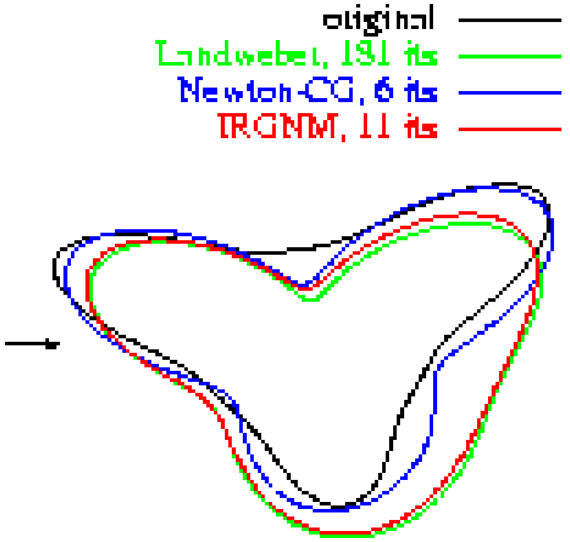
Dirichlet b.c.





Neumann b.c.





a convergence rate result

Corollary :

Assume the iterative regularization method converges in the image space , i.e. $F(a_n) \to e_\infty$ for exact data ∧ that the iterates remain bounded , i.e. $\| a_n^\delta - a \|_{H^s} \le c$ for $n \le N(\delta, e_\infty^\delta)$. Both assumptions hold true under certain nonlinearity assumptions on the operator F .

Moreover , assume that the discrepancy principle is used as a stopping rule. Then the final iterates $a_{N(\delta, e_\infty^\delta)}$ satisfy the error estimate

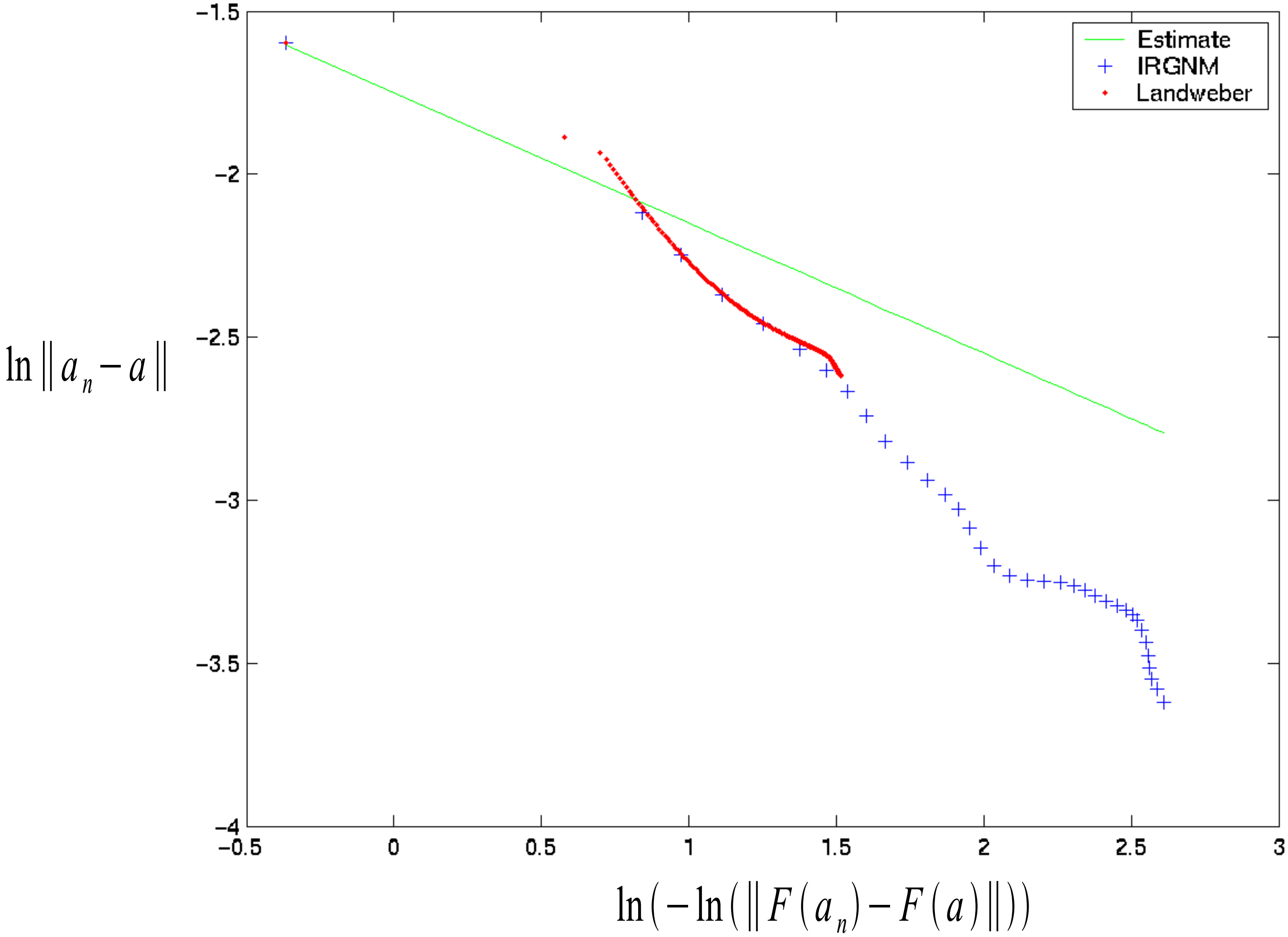
$$\| a_{N(\delta, e_\infty^\delta)} - a \|_\infty \le C \left(\ln (\tau \delta)^{-1} \right)^{-(2s-3)/(2s+11)+\epsilon} \qquad \text{for any } \epsilon < 0 \,,$$

∧ for exact data we have the estimate

$$\| a_n - a \|_\infty \le C \left(\ln \| F(a_n) - F(a) \|^{-1} \right)^{-(2s-3)/(2s+11)+\epsilon}$$

Using the stability estimate we can replace a logarithmic source condition by the natural smoothness assumption $a \in H_0^s(B_\rho)$.

experimental vs theoretical convergence rates



need of a preconditioner

In each step of the IRGNM $\wedge$ the Levenberg – Marquardt algorithm we have to solve a linear operator equation of the form

$$(1) \qquad (F'[a_n]^* F'[a_n] + \gamma_n I)\,\Delta a_n = g_n$$

where the regularization parameter satisfy $\gamma_n \to 0$ as $n \to \infty$. We have

$$\mathrm{cond}\,(F'[a_n]^* F'[a_n] + \gamma_n I) = \frac{\| F'[a_n] \|^2}{\gamma_n} \to \infty \qquad \text{as } n \to \infty \,.$$

Hence, the number of CG steps to solve (1) to a given accuracy typically explodes as $\gamma_n \to 0$.

the conjugate gradient method

$$\underbrace{\left(F'[a_m]^* F'[a_m] + \gamma_n I\right)}_{=:\, A_n} \Delta a_n = g_n\,, \qquad m \leq n$$

```
$k = 0;\ \Delta a_n^0 = 0;\ r_0 = g_n;$
while $(\|r_k\| > \epsilon \gamma_n \|\Delta a_n^k\|)$
   $k = k + 1;$
   if $(k = 1)$
      $p^1 = r^0;$
   else
      $\beta_k = \|r_{k-1}\|^2 / \|r_{k-2}\|^2;$
      $p_k = r_{k-1} + \beta_k p_{k-1};$
   $\alpha_k = \|r_{k-1}\|^2 / \langle p_k, A_n p_k \rangle;$
   $\Delta a_n^k = \Delta a_n^{k-1} + \alpha_k p_k;$
   $r_k = r_{k-1} - \alpha_k A p_k;$
```

properties:

$\langle r_k, r_l \rangle = 0 \quad$ for $k \neq l$

$r_k = g_n - A_n \Delta a_n^k$

Proposition:
The iteration terminates after a finite number K of iterations, ∧

$$\|\Delta a_n^K - \Delta a_n\| \;\leq\; \frac{\epsilon}{1-\epsilon} \|\Delta a_n\|\,.$$

the Lanczos method

$$A_n R = RB^T B + rem$$

$$B := \begin{pmatrix} \sqrt{1/\alpha_1} & -\sqrt{\beta_1/\alpha_1} & 0 & \cdots & 0 \\ 0 & \sqrt{1/\alpha_2} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -\sqrt{\beta_{k-2}/\alpha_{k-2}} \\ 0 & \cdots & \cdots & 0 & \sqrt{1/\alpha_{k-1}} \end{pmatrix}$$

$$R := \left(\frac{r_0}{\| r_0 \|}, \dots, \frac{r^{k-1}}{\| r^{k-1} \|} \right)$$

If $rem = 0 \wedge B^T B = VDV^T$ with a diagonal matrix $D \wedge$ an orthogonal matrix V , then

$$A_n(RV) = (RV)D,$$

i.e. the columns of RV are eigenvectors of A_n, $\wedge$ the diagonal elements of D are eigenvalues of A_n .

construction of a preconditioner

Preconditioned equation : $C_n^{-1} A_n C_n^{-1} \Delta \tilde{a}_n = C_n^{-1} g_n$

$$\Delta \tilde{a}_n = C_n \Delta a_n, \quad A_n := F'[a_m]^* F'[a_m] + \gamma_n I$$

Let $F'[a_m]^* F'[a_m] v_j = \lambda_j v_j$, $\wedge$ assume that the largest K eigenvalues $\wedge$ $-$ vectors are known. Define

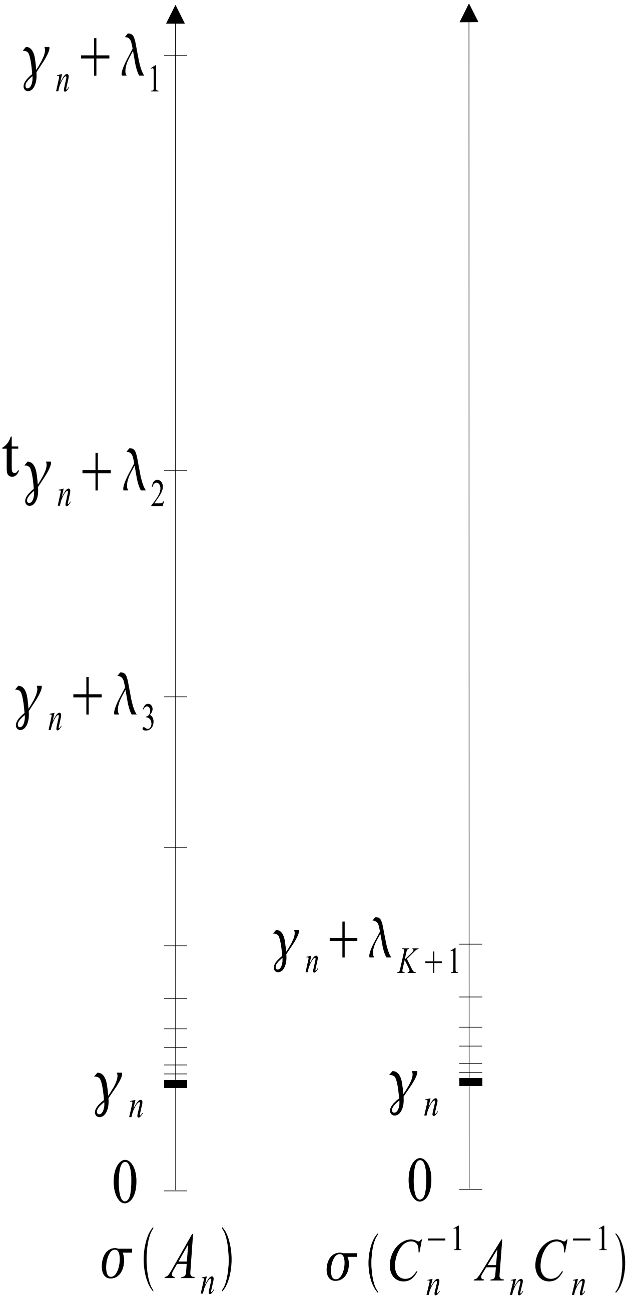
$$C_n^{-2} a := a + \sum_{j=1}^{K} \left(\frac{\gamma_n}{\gamma_n + \lambda_j} - 1 \right) \langle x, v_j \rangle v_j .$$

Then $C_n^{-1} A_n C_n^{-1} v_k = \gamma_n v_k$ for $k \leq K$

$\wedge$ $C_n^{-1} A_n C_n^{-1} v_k = A_n v_k$ else . Consequently ,

$$\sigma(C_n^{-1} A_n C_n^{-1}) = \{\gamma_n\} \cup \{\gamma_n + \lambda_{K+1}, \gamma_n + \lambda_{k+2}, \ldots\}$$

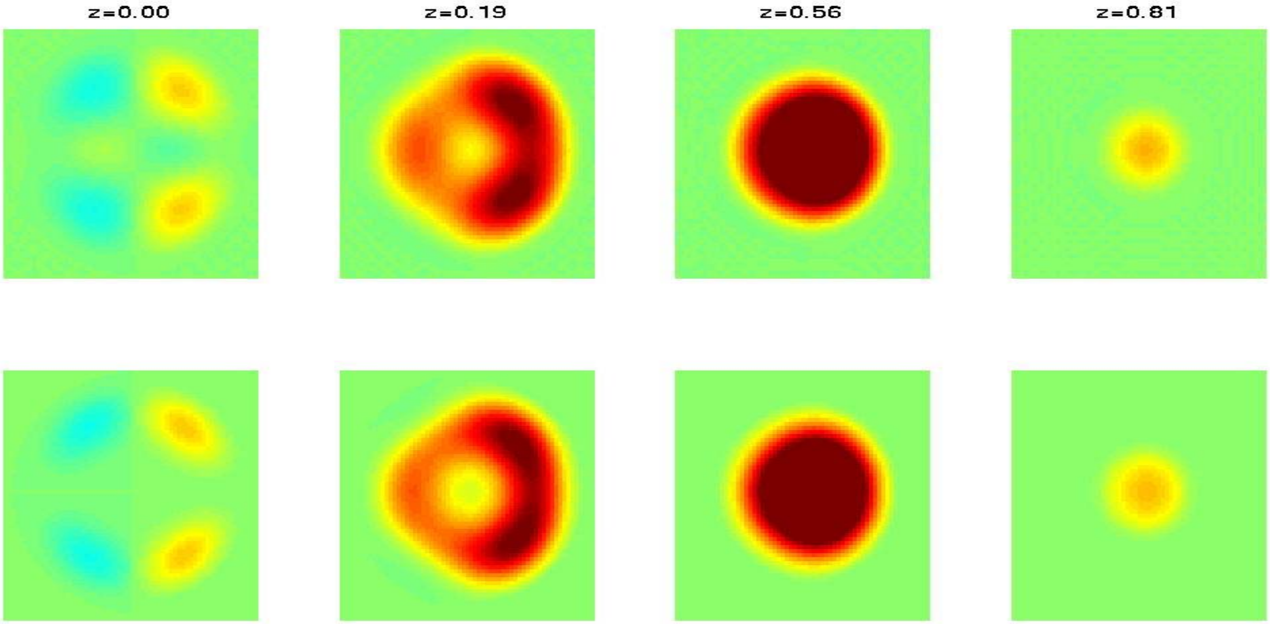
$$\mathrm{cond}(C_n^{-1} A_n C_n^{-1}) = 1 + \lambda_{K+1} / \gamma_n$$



the algorithm

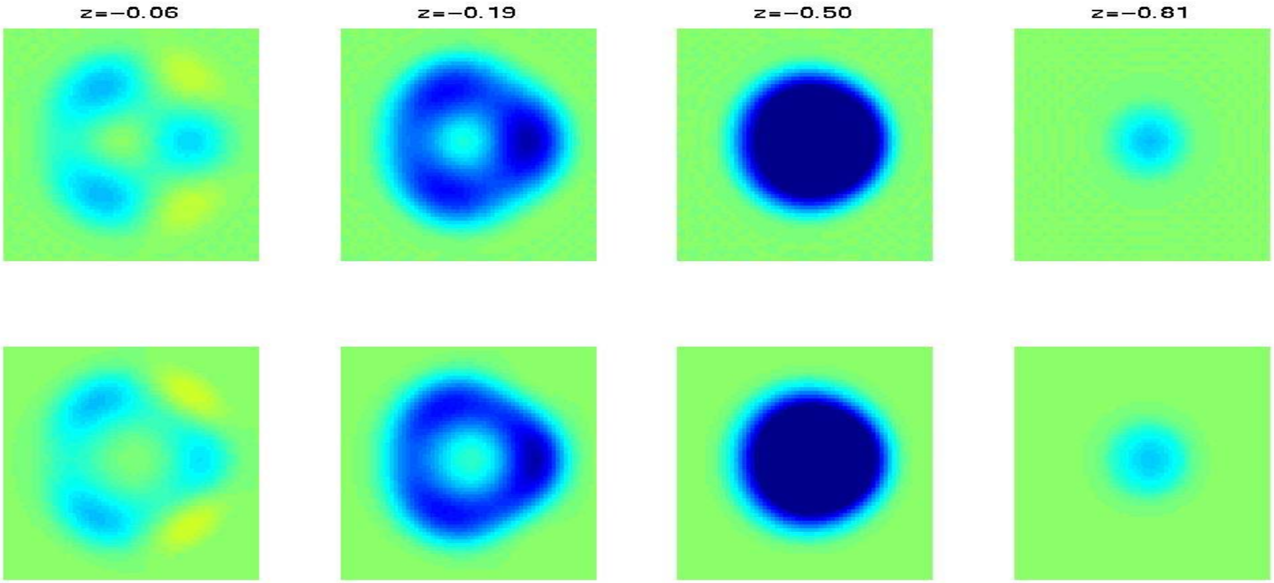
```
Compute $F(a_0)$;
$n = 0$;
while $(\| F(a_n) - y^\delta \| \geq \tau\,\delta)$
    if $(\sqrt{n+1} \in N)$
        $m = n$;
        Solve $A_n \Delta a_n = g_n$ by CG method;
        Compute largest eigenvalues $\wedge -$ vectors of $F'[a_m]^* F'[a_m]$;
    else
        Solve $C_n^{-1} A_n C_n^{-1} (C_n \Delta a_n) = C_n^{-1} g_n$ by CG method;
    $a_{n+1} = a_n + \Delta a_n$;
    $n = n + 1$;
    Compute $F(a_n)$;
```

a reconstruction with "exact" data



reconstruction

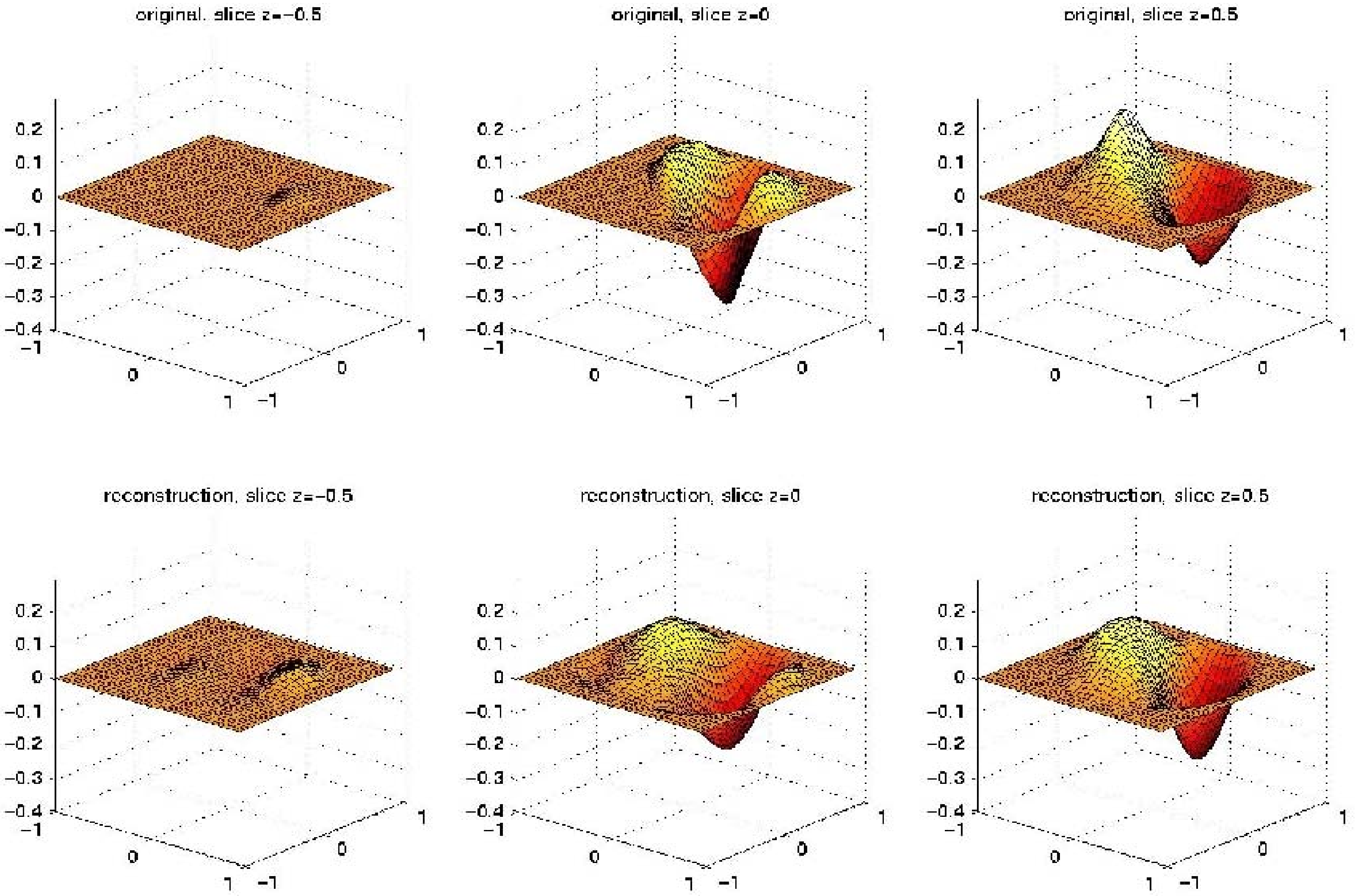
original



reconstruction

original

reconstruction with 1% white noise in data



number of preconditioned inner CG-iterations