WHAT DID WE LEARN AND STILL MAY LEARN FROM INVERSE SCATTERING ?

by

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Abstract

During the twenty "golden years" of Inverse Problems after World War 2, the most important ideas appeared in the field of linear inverse problems and that of inverse scattering problems, which give a taste of how nonlinearity can modify an inverse problem. Although both were closely connected to problems of real physics, both could be given not only numerical analyses but also exact analyses of how the information which is contained in data can be disentangled : the art of doing it corresponds to the most general meaning of the word "deconvolution". We give here a sketchy review of main ideas which appeared, main remarks, and main directions for future developments in the field of Inverse Scattering.

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1 Introduction

"Solving" an Inverse Problem is discovering the information contained by data. This process should be extended to all data that may appear in the Problem. As so, it must rely on good methods of analysis. The first cases where such methods were derived were linear inverse problems

$$Mx = y \tag{1.1}$$

In (1.1), for instance, $x \in \mathbb{R}^n$, M is a real matrix $m \times n$, $y \in \mathbb{R}^m$ and measures of y are data. As it is well-known, a good method (1.2) of analysis is the so-called "singular value decomposition" (SVD), whose key is the spectrum of M^*M, made of non negative eigenvalues σ_ℓ^2.

$$M^*Mv_\ell = M^*\sigma_\ell u_\ell = \sigma_\ell^2 v_\ell \tag{1.2}$$

The problem (1.1), or preferably the problem $M^*Mx = M^*y$, can be transformed into

$$S^T S x' = S^T y' \tag{1.3}$$

where S is a $(m \times n)$ diagonal matrix, x', resp. y', are obtained from x, resp. y, by rotation in $\mathbb{R}^n$, resp. $\mathbb{R}^m$, of the coordinate systems into the principal axes systems, made of the normalised eigenvectors v_ℓ, resp. u_ℓ. The elements on the diagonal of $S^T S$ are the σ_ℓ^2. Hence the problem (1.1) is clearly analysed on the equation (1.3) : no information on the corresponding components of x' does correspond to a vanishing σ_ℓ, and we can get only little information on those which correspond to small σ_ℓ/σ_1, in addition to a large sensivity to errors. With some more work, most linear inverse problems can be fully analysed, and stable generalised solutions are obtained by shunting small σ_ℓ, for instance the Tikhonov one, [3] which amounts to solving

$$\left(S^T S + \varepsilon^2 I\right) x' = S^T y' \tag{1.4}$$

Thus, our study shows that an essential point in linear problems analysis is the existence of a spectrum made of real eigenvalues as much as the problem linearity itself. This idea is also supported by the "ideal case" of a Fourier transform in $L^2(\mathbb{R})$, defining the direct problem $f \rightarrow F$:

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{-2i\pi\lambda x} f(x) dx \tag{1.5}$$

Its inverse is

$$f(x) = \int_{-\infty}^{+\infty} e^{2i\pi\lambda x} F(\lambda) d\lambda \tag{1.6}$$

and since Parseval theorem guarantees that norm are conserved, errors on data F give equal errors on parameters f. The inverse problem is perfectly well posed. This fact is related to the completeness in $L^2(\mathbb{R})$ of the functions $\exp\left[2i\pi\lambda x\right]$, which are "eigenfunctions" of the operator $-\frac{d^2}{dx^2}$. However, in optical imaging, another sort of uncertainty comes in : F is structurally known only inside a spectral window, say, $|\lambda| \leq \lambda_0$. The inverse problem has an infinity of exact solutions unless imposing a priori information restore uniqueness - for instance by assuming F equal to zero beyond λ_0 (the inverse problem remains well posed), or by assuming f equal to zero outside of a compact support : in this case, F is an entire function completely determined by its exact values inside the window but constructible, outside of it, only by analytic continuation, to the price of stability. Physically this case is called the superresolving case and the resolving power is, in fact, limited by the signal / error ratio.

Many problems can be linearised for some range of their parameters, and direct linear problems were already well understood in the nineteenth century.

Small size inverse linear problems were well understood and easy to work out in 1950, but of course the development of computers made their study easier and that of (very) large size problems possible - arising new problems of numerical analysis.

These few remarks were a preamble citing some ideas afforded by studies of linear inverse problems. Now we go to our subject, showing how inverse scattering introduced nonlinearity, arising new ideas.

2 Scattering

Suppose we know perfectly well an "incident" field of waves or particles, as they are emitted from a source into a set of obstacles, interacting with them, and collectively called a "target". After interaction the incident field is turned into outgoing waves or particles called the "emergent" field, this process being called "Scattering".

Obviously much information on the physical world comes from scattering processes - in fact almost all information available in particle physics. This explains why scattering problems became of outstanding importance[4] after World War II, where they had also proved their interest in many useful devices (radar, etc)[5] first notice that in imaging, and particularly medical imaging, scattering is essential but most inverse problems can be linearised. Opposite cases are geophysical soundings which meet so complicated structures that the example we shall see now much too simples. Now, for physical reasons, several wave equations are linear and involve a Laplacean on space variables, so that the Schrödinger equation in the frequency domain is inintermediate cases a reasonably general example to show how problems are posed. It reads :

$$\Delta\Psi(\mathbf{k},\mathbf{x}) + \left(k^2 - V(\mathbf{x})\right)\Psi(\mathbf{k},\mathbf{x}) = 0 \tag{2.1}$$

where Ψ is the wave function, V is a real function of $\mathbf{x} \in \mathbb{R}^3$, called the potential, $\mathbf{k}$ is the wave number $\left(k^2 = \mathbf{k}\cdot\mathbf{k}\right)$, and $\widehat{k} = \mathbf{k}/k$ (bold letters are used for vectors, italic for their length, hats for elements of the unit sphere). Provided that V decreases fast enough at ∞, we can write down :

$$\Psi(\mathbf{k},\mathbf{x}) = \exp\left[i\mathbf{k}\,\mathbf{x}\right] + x^{-1}\exp\left[ikx\right]F\left(\widehat{x},\widehat{k},k\right) + o\left(x^{-1}\right) \tag{2.2}$$

which shows an incident plane wave plus a scattered wave. F is called the scattering amplitude, and is often used to represent the data in inverse problems, although its determination from real data is usually indirect and always with uncertainties. The simplest case is V with spherical symmetry, (i.e. depending only on r), which gives an axisymmetric scattering around the incident direction, that we choose as the polar axis. Thus

$$\Psi(k,r,\cos\theta) = \sum_{\ell=0}^{\infty}(2\ell+)i^\ell r^{-1}\psi_\ell(r)P_\ell(\cos\theta) \tag{2.3}$$

where the "partial wave" $\psi_\ell(k,r)$ is the regular solution of the "partial wave equation"

$$\frac{d^2\psi_\ell}{dr^2} + \left[k^2 - V(r) - \ell(\ell+1)r^{-2}\right]\psi_\ell = 0 \tag{2.4}$$

which reduces for $V = 0$ to the spherical Bessel function

$$u_\ell(k,r) = \left(\frac{\pi}{2}kr\right)^{1/2} J_{\ell+\frac{1}{2}}(kr) \tag{2.5}$$

As $r \to \infty$, ψ_ℓ is asymptotic to a trigonometric function :

$$\psi_\ell(k,r) = A_\ell(k)\sin\left(kr - \ell\frac{\pi}{2} + \delta_\ell(k)\right) + o(1) \tag{2.6}$$

and the scattering amplitude $F(k,\cos\theta)$ is given in terms of the phase-shifts $\delta_\ell(k)$

$$F(k,\cos\theta) = k^{-1}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell(k)}\sin\delta_\ell(k)P_\ell(\cos\theta) \tag{2.7}$$

Hence, because of the spherical symmetry, the scattering problem becomes an explicit combination of one-dimensional problems.

The scattering problem may also be one-dimensional from its very definition (this case has applications as so in several problems of real physics, e.g. in acoustics and electromagnetics).

Again, the simplest form is (2.1), which reads then

$$\left[-\frac{d^2}{dx^2} + V(x)\right] f(k,x) = k^2 f(k,x) \tag{2.8}$$

We assume $V \in L^1_1(\mathbb{R})$:

$$\int_{-\infty}^{+\infty}(1+|x|)\,|V(x)|\,dx < \infty \tag{2.9}$$

and define the so called "Jost solutions" $f^{\pm}(k,x)$ of (2.8) by their asymptotic behavior ; respectively

$$f^{\pm}(k,x) \sim e^{\pm ikx} \quad \text{as} \quad x \to \pm\infty \tag{2.10}$$

They give the definition of scattering coefficients $T(k)$ (transmission) and $R^{\pm}(k)$ (reflection on the right, resp. left)

$$T(k)f^{-}(k,x) = f^{+}(-k,x) + R^{+}(k)f^{+}(k,x) \tag{2.11}$$

We can study(6) as a direct problem $V \to (T, R^{+})$ and inverse problem $(T, R^{+}) \to V$ (The problem on the left would be the symmetric one).

3 Extension of the problem and History

The scheme we presented above on the simplest examples of Schrödinger equation can be extended with nonessential complications to cover not only other scattering problems in quantum mechanics but also in acoustics and electromagnetics (it is particularly useful for radar, lidar, sonor, etc soundings), and even in more exotic problems. This explains the history of scattering studies. Shortly after World War 2, the first ones were either approximated or confined to one-dimensional cases, which were given complete mathematical analysis, the main concern being whether it is possible or not to get the whole information on microphysics by means of scattering experiments. The focus was progressively modified in the seventies and both in quantum mechanics and in the other fields, numerical analyses became dominant. Meanwhile, a completely unexpected field of research had appeared, also called Inverse Scattering (Method or Transform), where solutions of one dimensional inverse scattering problems are applied to solving special nonlinear partial differential equations. After 1980, this field became more and more autonomous, quantum scattering became less and less studied, three dimensional scattering problems became the main concern in acoustics and electromagnetics, creating new concepts. Those corresponding to mathematical models will be studied in the present school by D. Colton. I will go through approximate and exact one-dimensional, reductions and try to show what new ideas they afforded, what mathematical beauty they created.

We shall see in chapter 4 and chapter 5 one dimensional analyses of Gelfand-Levitan-Marchenko kind as "extreme studies" justified by the geometry of the scattering problem. A set of completely opposite "extreme studies" is given by two kinds of reductions of the direct and inverse electromagnetic (or acoustic) scattering problems, obtained either if one parameter of the sounding is itself extreme or if, in the conditions of sounding, the target can be characterised by a finite set of parameters.

Examples of the first kind are the approximation known as Rayleigh scattering and Physical Optics approximation. In the Rayleigh scattering, it is assumed that the wave length k^{-1} is much larger that the scatterer characteristic length, say a, so that one obtain the results at $ka \to 0$ by means of electrostatic potential theory, and next orders involve powers of ka. Hence a scatterer can be represented by monopoles, dipoles, multipoles, and this may be convenient to study for instance scattering by a random distribution of small scatterers. In the other extreme case, short wave lengths (compared to a) can be consistent with a "physical optics" approximation, where studies of optical imaging are useful. Some problems are thus reduced to Fourier transforms or to tomographic scanning[2] and solutions of the corresponding inverse problems are well-known. In some way, one may say that the direct and inverse problem were linearised, but this is true only for some formulas : actually the original nonlinear problem was replaced by another one, with some linear correspondences. Although these studies are approximate, they were fruitful in Inverse Theory because of their way of putting emphasis on the dominant aspects of the object image correspondences. Many examples of reductions of the second kind come from radar studies. In these studies, radars are industrial devices, with only a small number of available frequencies. Targets[7] are of complicated shape, and their coverings are very important for energy radiating. Hence the scattering is very difficult to study, but it is often possible to describe what particular aspects of the target radiate more or less energy. And one uses the results in designing the target, for minimizing the observed target cross section (SER), being understood that the illumination angle (the "aspect" of the target) can modify very much the result. In the same way, people who want to detect the target will represent it by a number of parameters to be "observed" in the radar images (their "signature"). In order to take advantage of all information, polarimetry will also try to give a few parameters, etc, and the eternal fight between the hunter and the hunted will emphasize what can be call the "physical parameters" observable by electromagnetic soundings. Emphasis on this representing the object to be analysed by a set of "essential parameters" is probably the best gift of this kind of study to Inverse Theory. In linear Inverse Theory, a similar idea underlies the definition of the "degrees of freedom" in a system. But in Inverse Scattering the selection of "essential parameters" is more guided by physics than by mathematics and nowhere do we learn better that for complicated, realistic, inverse problems, approaches **combining** physics and mathematics are necessary. Both for the hunter and for the prey, modelling becomes a dialog between selecting parameters and looking what information they afford and how it can be identified. There are other inverse problems where "essential" parameters are selected on the basis of extreme properties only, e.g. gravimetry, and all those where one privilegiates a kind of decisive modelling (i.e enabling decisions without identifying all the parameters), but they rely more on mathematics than physics.

4 Reductions of Inverse scattering Problems

Shortly after World War 2, most experiments of particle physics were able to give only few phase-shifts, with small, values, say, $\delta_0(k)$, $\delta_1(k)$, and in a limited range of k values. Expecting for a small phase-shift a small potential, one may try the so-called Born approximate formula

$$\delta_\ell(k) \simeq -\frac{2m}{\hbar^2}\int_0^\infty V(r)\,[u_\ell(kr)]^2\,r\,dr \tag{4.1}$$

where m is the particle mass. For $\ell = 0$, u_ℓ is $\sin kr$ and the formula is a Fourier transform. Hence one expects that the only limitation for our knowledge of rV in (say) $L_1(\mathbb{R}^+)$ (Bargmann class) or in $L^2(\mathbb{R}^+)$ (another class of interest) is the limitation on k values. That the conclusion is wrong was shown by Bargmann in 1949[8], who showed that several potentials V can fit the same phase-shift $\delta_0(k)$, supposedly known exactly on $\mathbb{R}^+$. One needs more information. In fact, there are potentials V for which the Schrödinger equation has solutions in $L^2(\mathbb{R}^3)$ called bound states, and one needs the corresponding informations. The surprise generated by Bargmann's result was a trigger for studies of inverse problems in quantum mechanics, and many other ones in mathematics, which made the years between 1950 and 1965 the most fruitful ones[9]. However, although the one-dimensional problem (2.8) came a few years later in History, we like it better to explain on this example the same questions, both because they seem easier to understand and because their consequences on Science were eventually more important.

Writing (2.8) and (2.11) for two potentials V and $\widetilde{V}$, one can derive[6] the exact formulas :

$$2ik\frac{R^+(k)-\widetilde{R}^+(k)}{T(k)\widetilde{T}(k)} = \int_{-\infty}^{+\infty}\left[V(x)-\widetilde{V}(x)\right]\widetilde{f}^-(k,x)f^-(k,x)dx \tag{4.2}$$

$$2ik\frac{T(k)-\widetilde{T}(k)}{T(k)\widetilde{T}(k)} = \int_{-\infty}^{+\infty}\left[V(x)-\widetilde{V}(x)\right]\widetilde{f}^+(k,x)f^-(k,x)dx \tag{4.3}$$

Assume now $\widetilde{V}=0$, so that $\widetilde{T}=1$, $\widetilde{R}^+=0$, and assume also V "small". It follows from (4.3) and (4.2) that a linear (Born) approximation of the mapping $V \to \mathbb{R}^+$ is described by the formula

$$2ik\,R^+(k) \simeq \int_{-\infty}^{+\infty} V(x)e^{-2ikx}\,dx \tag{4.4}$$

which suggests in turn the "linearised inverse formula"

$$V(x) \simeq \frac{2i}{\pi}\int_{-\infty}^{+\infty} k\,R^+(k)e^{2ikx}dk \tag{4.5}$$

But let us look (2.8) more carefully. For fixed V, it is a spectral equation for the linear operator $\mathbf{D} = -\frac{\partial^2}{\partial x^2} + V$, corresponding to the spectral value k^2. More precisely, if V is attractive and belongs to $L_1^1(\mathbb{R})$ (see 2.9)

it has been shown that the spectrum of the self adjoint operator $\mathbf{D}$ (i.e. the set of numbers λ for which $(\mathbf{D} - \lambda\mathbf{I})$ is not inversible) is made of the positive half axis (values $\lambda = k^2$) and a finite number of negative eigenvalues $\lambda_n = -\kappa_n^2$, for which the operator D has an eigenfunction ψ_n in $L^2(\mathbb{R})$:

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\psi_n(x) = -\kappa_n^2\psi_n(x) \tag{4.6}$$

The ψ_n definition is achieved by normalising it in $L^2(\mathbb{R})$. Comparing to (2.10), which holds for Im $k \geq 0$, we see that $f^{\pm}(i\kappa_n,\, x)$ is proportional to ψ_n

$$f^{\pm}(i\kappa_n,\, x) = C_n^{\pm}\psi_n(x) \tag{4.7}$$

Now, if instead of $\mathbf{D}$ we were dealing with a symmetric real matrix acting on $\mathbb{R}^n$, we could associate to its eigenvalues λ_m orthonormal eigenvectors w_m and thus make a basis in the image of M, such that for any $x \in \mathbb{R}^n$, the effect of M is represented as a sum of projections

$$Mx = \sum_{m=1}^{n_0} \lambda_m < x,\, w_m > w_m \tag{4.8}$$

(n_0 is the rank). Similarly, let us define a "family of projections" as a function E_λ ($\lambda \in \mathbb{R}$) with values in the orthogonal projectors on the Hilbert space H and such that

(a) $E_{-\infty} = 0$, $E_\infty = I$

(b) $\lambda \leq \mu \Rightarrow (\forall x)(x \in H)\,(\langle (E_\mu - E_\lambda)x, x\rangle \geq 0)$

(c) $(\forall x)(x \in H)\lim_{\varepsilon \to 0^+} \|E_{\lambda+\varepsilon}x - E_\lambda x\| = 0$

With these definitions, if A is a self-adjoint operator on H, domain $\mathcal{D}_A$, the so-called "spectral theorem" says that there exists a family E_λ with the following properties

(a) x belongs to $\mathcal{D}_A$ if and only if $\int_{-\infty}^{+\infty} \lambda^2 d\,\langle E_\lambda x, x\rangle < \infty$

$$\text{(b)}\,(\forall x\,,\, x \in \mathcal{D}_A)\left(Ax = \int_{-\infty}^{+\infty} \lambda\, dE_\lambda x\right)\left(\|Ax\|^2 = \int_{-\infty}^{+\infty} \lambda^2 d\,\langle E_\lambda x, x\rangle\right) \tag{4.9}$$

It is readily seen that the first equality in (4.9) is nothing but the extension to this general case of the formula (4.8), where a finite number of projectors exist, so that $dE_\lambda x$ reduces to a sum of Dirac measures. Now when these results are written for our operator $\mathbf{D}$, the family E_λ can be constructed by means of the κ_m and the C_m for the discrete spectrum and by means of $R^+(k)$

for the continuous spectrum. It follows that knowing $R^+(k)$ is not sufficient to identify V. One also needs the information related to the discrete spectrum ! As we see, a curious feature of this inverse problem is that it is nonlinear but the spectrum of a linear operator containing the parameter (and not independent of it as in linear inverse problems) is its key. But it remains the problem of constructing the linear operator, and therefore V, from the spectrum. The theory which does it is called a Gelfand, Levitan, Marchenko theory because these authors were the first to give it, the two first authors for Sturm Liouville problems and Marchenko for Scattering problems, all these problems centered on the operator $\mathbf{D}$. In the problem (2.8), the method was adapted[6] by Kay and Faddeyev later. The key of GLM theory is the existence for $V \in L_1^1(\mathbb{R})$ of a "transformation kernel" $K(x,y)$ such that $f^+(k,x)$, solution of (2.8) for V, can be obtained from $\exp[ikx]$, solution of (2.8) for 0, by a simple integration

$$f^+(k,x) = \exp[ikx] + \int_x^\infty K(x,y)\exp[iky]\,dy \tag{4.10}$$

As a matter of fact, the existence of K is not surprising (take the Fourier transform of other terms for defining K). The only point is that K vanishes for $y < x$. This result follows from analytic and asymptotic properties of $f^+(k,x)$, in the half plane $\text{Im}k \geq 0$, which can be derived by transforming (2.8) into an integral equation. By the same token[6] one shows that K can be constructed from V by solving a Volterra nonlinear integral equation and that

$$K\left(x,x^+\right) = \frac{1}{2}\int_x^\infty V(s)ds \tag{4.11}$$

The derivation of the GLM equation follows similar steps[6]. It reads :

$$K(x,y) + M(x+y) + \int_x^\infty dz\,K(x,z)M(z+y) = 0 \tag{4.12}$$

where

$$M(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\,R^+(k)\exp[ikx] + \sum_{p=1}^{N}\left(C_p^+\right)^{-2}\exp[-\kappa_p x] \tag{4.13}$$

$M(x)$ contains all the informations we found necessary to identify V. The GLM equation (4.13) is at fixed x a Fredholm integral equation and it is possible to prove that if $R^+(k)$ is indeed the reflection coefficient corresponding to a potential in L_1^1, (4.12) has a unique solution, whose limit $K\left(x,x^+\right)$ yields (4.11), the integral of the potential. Hence $R^+, \{C_p\}, \{\kappa_p\}$ are data (called "spectral data") sufficient to determine the potential and to construct it ; but notice the (weakly) instable step $K \to V$.

Thus, the information we learned from this inverse scattering problem can be described in several different ways. We can say equivalently

(a) that a necessary part of the information escapes all scattering experiments !

(b) that if we consider only scattering data, the solution V is non unique but the non uniqueness can be described on the basis of other physical informations.

(c) that if $R^+ = 0$, there are "transparent" potentials, that cannot be identified by scattering experiments and nevertheless do not vanish. As an example, one eigenvalue only, with data c and κ gives

$$V_0(x) = -\frac{2\kappa^2}{ch^2\left[\kappa\left(x - x_0\right)\right]} \tag{4.14}$$

where

$$x_0 = -(2\kappa)^{-1} \log\left[2\kappa c^2\right] \tag{4.15}$$

Notice that $V(x)$ decreases rapidly at ∞ and that increasing K makes V sharper and stronger.

As in many inverse problems, restoring the uniqueness by means of an a priori assumption on V is tempting. We can see on the linearised problem (4.4) that assuming V has a compact support does the job, to the price of stability, i.e., no matter how small is $\varepsilon > 0$ an error "smaller" than ε can give uncontrolled errors on V. It is interesting to see in the exact nonlinear problem that allowing $V \neq 0$ but $\int_{|t|>x_1} |V(t)|\, dt$ smaller than ε outside of the compact support $[-x_1, x_1]$ is consistent with a nonuniqueness due to potentials like that of (4. 14) provided that x_0 is inside the support and $\kappa^2 \left(= -\frac{1}{2}V\left(x_0\right)\right)$ is large enough.* As a tutorial conclusion, again, **beware** the restoring of uniqueness by mathematicians.

The Gelfand Levitan Marchenko method was generalised and adapted to many inverse problems governed by linear differential systems of order 2. In a time domain approach, one can see that it is related to cansal arguments[10]. One may also notice that the form of its integral equation recalls a layer stripping approach, where scattering information coming from successive layers of the target is sequentially identified. Although the GLM approaches, and particularly that of the one dimensional Schrödinger problem, are now most often associated to inverse transform, they are still interacting to manage in practical problems. For instance they are a possible key for the inverse problems of speech and vocal tract currently studied by Forbes and Pike[11].

5 On ill-posed inverse scattering problems

The occurence of a discrete spectrum that cannot be "seen" by scattering experiments is a source of ill-posedness due to nonlinearity, but there are others,

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which remain even if the problems are linearised. Suppose for instance we study the scattering problem for a spherical potential, at fixed energy. For "small" V the formulas (2.7) and (4.1) yield the linear "Born approximation" of the scattering amplitude :

$$F_B(k,\theta) = \widetilde{F}\,(2k\sin\theta/2) = -\int_0^\infty \frac{\sin\left[2kr\sin\theta/2\right]}{2k\sin\theta/2} V(r) r\,dr \qquad (5.1)$$

k, proportional to the root square of the Energy, is fixed. Let $t = 2k\sin\theta/2$. The Fourier transform of V is known only through a "structural" filter of band width $2k$. The general solution of the inverse problem in

$$L_2^2 = \left\{ V / \int_0^\infty r^2 V^2(r) dr < \infty \right\} \qquad (5.2)$$

is

$$rV(r) = -\frac{2}{\pi}\int_0^{2k} \sin tr\,\widetilde{F}(t) t\,dt + \int_{2k}^\infty \sin tr\,\alpha(t) dt \qquad (5.3)$$

where α is any function chosen in $L^2(2k,\infty)$. Hence, in this approximation, there are "transparent potentials" that cannot be seen by a sounding at this energy. One also sees that there are two ways to restore uniqueness - the first one, being to fix α, keeps also stability. The second one, being to assume a priori that V decreases so rapidly at ∞ that $\widetilde{F}$ is analytic and is therefore uniquely determined by its values on $(0,2k)$. In this "superresolving case", instabilities appear because of the continuation of $\widetilde{F}$, and hence they are definitely associated to the uniqueness theorems for potentials of compact support.

As a tutorial, one must keep in mind that if any method giving a potential from the scattering amplitude applies in a domain that contains the domain of "small V" (and this is indeed the general case), the illposedness seen on Born formulas will remain, and limitations of the method can be predicted from it. As an example, instabilities will be present if uniqueness is restored by a compact support assumption. The fourty years old "Newton-Sabatier" method is related to the first way of restoring uniqueness, by choosing α (or, equivalently, choosing a given interpolation of δ_ℓ as a function of ℓ). Some freedom has been allowed on the choice of α, so that the method can also be used to construct examples of exact transparent potentials. The method proceeds by inversing matrices, and has a limited range of validity, the range where Fredholm series converge and a Fredholm determinant does not vanish on $\mathbb{R}^+$. It had the historical value of showing that the inverse problem at fixed energy can be solved exactly and that the solution is not unique.

As a matter of fact, scattering of particles in physics is done on a wide range of energy. With the spherical symmetry and the invarient V assumptions, it is easy to see that knowing the scattering amplitude F at all k and all angles is too much (the left hand side of (5.1) would be enough as $k \to \infty$, or giving $\delta_\ell(k)$ for ℓ large enough that no bound state holds. The truly physical problem would be using all $\delta_\ell(k)$ in a range where it is reasonably assumed that

(1) the model (2.1) is valid

(2) V depends on r only (and not on k)

Curiously mathematicaly physicists and applied mathematicians did not work so much on this problem (although it can obviously be managed by optimisation methods). Similar remarks hold for cases without symmetry.

6 The Inverse Scattering Transform

There are many ways to introduce the IST, including purely algebraic ones and others, where scattering does not appear ! Instead of giving the historical approaches, which are close to inverse scattering but found very "unexpectedly", I will give a more recent one, which I like, and where scattering appears in a natural way **and** from several points of view. We try to find a way of solving special nonlinear partial differential equations. The nonlinear equations are consistency conditions between two *linear* equations, both associated to a scattering and an inverse scattering problem and this fact is the key of the results. We see it now on the most famous historical example.

The nonlinear KdV equation

$$\frac{\partial V}{\partial t} + \frac{1}{4}V''' - \frac{3}{2}VV' = 0 \tag{6.1}$$

where "prime" denotes $x-$derivative, is the existence condition for continuous double-derivatives of 2-vector solutions F of two linear (Lax) equations :

$$\frac{\partial F}{\partial x} = \mathbf{M}\,F \quad ; \quad \frac{\partial F}{\partial t} = \mathbf{N}\,F \tag{6.2}$$

where the matrices $\mathbf{M}$ and $\mathbf{N}$ depend on k, x, t, as so :

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{V}(x,t) =: \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V(x,t) & 0 \end{pmatrix} \tag{6.3}$$

$$\mathbf{N} = k^2\mathbf{M}_0 + \begin{pmatrix} V_1 & V_0 \\ k^2V_0 + V_2 & -V_1 \end{pmatrix} =: k^2\mathbf{M}_0 + \mathbf{W} \tag{6.4}$$

$$V_0 = \frac{1}{2}V(x,t) \quad V_1 = -\frac{1}{4}V' \quad V_2 = \frac{1}{2}V^2 - \frac{1}{4}V'' \tag{6.5}$$

It is clear from (6.2) that if $\mathbf{F}$ exists, it has the form

$$F = \begin{pmatrix} f \\ f' \end{pmatrix} \tag{6.6}$$

and that the first equation in (6.2) is nothing but (2.8). This suggests looking for scattering problems. Notice first that since M and N are zero-trace, the matrix determinant $(\mathbf{F}, \mathbf{G})$ of two solutions of (6.2) does not depend on x or t. Thus, for a given function $V(x,t)$, solution of (1.1), the space $\mathcal{F}$ of solutions F of (1.2) is two-dimensional. Assuming now on demand that $V(x,t)$ goes to zero "rapidly enough" as $x \to \pm\infty$, fixed t or $t \to \pm\infty$, fixed x, we can define "scattering problems" if we can construct "Jost solutions" F with given asymptotic properties, referred to the function $E(k,x,t) = \begin{pmatrix} 1 \\ ik \end{pmatrix} \exp\left[i\left(kx + k^3 t\right)\right] = \begin{pmatrix} e \\ e' \end{pmatrix}$.

$$\overleftarrow{F}(k,x,t) - E(k,x,t) = o\left(|x|^{-1}\right) \quad \text{fixed } t,\ x \to \infty \tag{6.7}$$

$$\widehat{F}(k,x,t) - E(-k,x,t) = o\left(|x|^{-1}\right) \quad \text{fixed } t,\ x \to -\infty \tag{6.8}$$

$$\overrightarrow{F}(k,x,t) - E(k,x,t) = o\left(|t|^{-1}\right) \quad \text{fixed } x,\ t \to \infty \tag{6.9}$$

For $k \neq 0$, $F(k,)$ and $F(-k,)$ are a basis in the space $\mathcal{F}$, so that :

$$\widehat{F}(k,) = \widehat{\eta}(k)\overleftarrow{F}(-k,) + \widehat{\xi}(k)\overleftarrow{F}(k,) \tag{6.10}$$

$$\overleftarrow{F}(k,) = \eta(k)\overrightarrow{F}(k,) + \xi(k)\overrightarrow{F}(-k,) \tag{6.11}$$

Each of these relations is associated to a reverse one, for example (6.11) to

$$\overrightarrow{F}(k,) = \eta(-k)\overleftarrow{F}(k,) - \xi(k)\overleftarrow{F}(-k,) \tag{6.12}$$

and inserting the reverse relation in the direct relation yields the unitarity relation, here :

$$\eta(k)\eta(-k) - \xi(k)\xi(-k) = 1 \tag{6.13}$$

Similar ones exist for the other coefficients.

Rétracing now from $\widehat{F}$ and $\overleftarrow{F}$ the usual Jost solutions along fixed axes (by comparing the asymptotic behaviors) we see that $\widehat{\eta}(k)$ and $\widehat{\xi}(k)$ are related to the usual scattering coefficients by

$$\widehat{\eta}(k) = [T(k)]^{-1} \tag{6.14}$$

$$\widehat{\xi}(k) = [T(k)]^{-1} R^{+}(k,t)e^{2ik^3t} \tag{6.15}$$

and these formulas readily show two essentially new résultats :

(1) in the motion described by the nonlinear KdV equation, $T(k)$, and therefore its poles which are nothing but the eigenvalues $-\kappa_n^2$ are conserved. The motion is therefore isospectral !

(2) the evolution of $R^+(k,t)$ is trivially given by (6.15). It follows that solving the inverse problem $R^+ \to V$ gives the evolution of $V(x,t)$ between two times t_0 and t : we have the solution of KdV given the initial input $V(x,0)$ in $L_1^1(\mathbb{R})$ by solving successively the direct problem $V(x,0) \to R^+(k,0)$ and the inverse problem $R^+(k,t) \to V(x,t)$.

Of course, a similar approach can be done on $x-$axes, with similar results(12) for the evolution of R, e^{2ikx} replacing e^{2ik^3t}. But the reflection coefficient R is a matrix, and the equation replacing (2.8) involves 6−vectors, so that the direct and inverse problems, which can still be solved are much more complicated.

What about solving KdV in the quarterplane ($x > 0$, $t > 0$) from a knowledge of V on the elbow ($t \geq 0$, $x = 0$; $t = 0$, $x \geq 0$) ? The equation (6.12) still shows the conserved quantities $\eta(k)$ and $\xi(k)$, so that, again, the evolution of a **conveniently defined** reflection coefficients R is trivial. But it remains to solve the direct problem ($V(x,0)$, $V(0,t) \to R$) and the inverse one. $V(x,0)$ gives $\overleftarrow{F}(k,x,0)$. If we tried to construct $\overrightarrow{F}(k,0,t)$, it would require a knowledge of $V(0,t)$, but also $V'(0,t)$ and $V''(0,t)$. Once it is done, it would give $\overrightarrow{F}$, and $\eta(k)$, $\xi(k)$ by using $\overleftarrow{F}(k,0,0)$. Then $V(x,t)$ could be constructed either at fixed t from $\overrightarrow{F}(k,0,t)$, and the value of R at $(0,t)$, or at fixed x from $\overleftarrow{F}(k,x,0)$ and the value of R at $(x,0)$, so that fulfilling strong consistency conditions would be necessary for this "double scanning" of V. This is why eventually the knowledge of $V(0,t)$ is "sufficient", as it had been proved previously(13). But how to use it in the construction ? It has been done in the linearised case and give the solution of the linearised KdV equation in the quarterplane(14). Extending it to the nonlinear case has been possible as yet only in small domains where iterative series converge(12).

We see there a wide class of open problems in Inverse Scattering : find inverse scattering transforms that work for boundary conditions given on arbitrary "contour" in the x, t plane. For NLS as well as KdV, in the last five years all attempts to get it failed. No attempt was done for more complicated integrable equations.

7 Nonlinear coherent structures

Among the solutions of KdV which are calculated by Inverse Scattering Transform, those which correspond to transparent potentials, i.e. to the discrete spectrum only, are distinguished ones. If we look the simplest one, it is given by (4.10) with $x_0 = ct$ where c, related to κ, is a constant, i.e. it looks like a bump propagating at constant speed - the famous soliton. Others are (nonlinear) superposition of solitons. Their discovery was the beginning of a series of results identifying coherent nonlinear structures (solitons, kinks, boomerons, breathers, etc) that can be described by solutions of nonlinear integrable equations, KdV, NLS, KP1, KP2, etc. In these solutions, compared to those of linearised equation, linear features as dispersion are balanced in one way or another one by nonlinear features. Furthermore it has been possible to show[15] that a large class of nonlinear partial differential equations has approximate solutions, corresponding to different scales of their variables, which are eventually also solutions of integrable evolution equations. In other words, integrable evolution equations can be derived as approximations of more complicated ones in special domains : the most simpler approximations beyond linearised equations. Hence an unexpected consequence of the study of Schrödinger inverse problem on the line has been the classification of several features in nonlinear physics !

8 Other approaches to Inverse Scattering.

Inverse scattering has been the object of several approaches which are or may be useful for many other inverse problems. We only cite layer stripping and invariant imbedding[17], which are classical and well known, but will say more words on time reversal techniques. Although they existed in some problems of optics, these techniques were introduced only a few years ago in acoustics and electromagnetics. The idea[7] is that a signal is recorded by transducers, time reversed, then retransmitted back to the medium. Hence, a time reversal cavity (TRC) or mirror (TRM) manages the inverse source problem and gives ways for analysing inverse scattering problems.

A TRC is a 2d transducer array that samples the wavefield ; a receiving amplifier, a storage memory and a programmable transmitter are able to *synthesize a time reversed version of the stored signal.* **Surrounding the source**, a TRC reconcentrates the signal into it, solving the inverse problem.

Using a TRM and time windowing separates waves reflected by a target according to their velocities and make possible concentrating on their apparent generation points ("*secondary sources*") in the target. Hence the **first reflected echo** (which determines the contour of a rigid target), the **elastic echos** (which correspond to various elastic modes) can be analysed.

9 Inverse Scattering and open problems

On the way, we have seen several technical open problems.

As a terse checklist of them and others :

(1) In wave scattering by complicated structures, identify the solution among secondary minima of a cost functional, and go rapidly to it !

(2) In (stealth) targets e.m. soundings, model and identify essential param-

eters.

(3) In nonlinearly propagation signals, identify nonlinear structures. This is currently done by Osborne[4] on waterwaves.

(4) Use analyses of the diffraction spots to improve the resolving power of instruments

(5) Find an inverse scattering transform solving boundary value problems of integrable n.l.p.d.e.

The technical problems should never conceal the **deep open problem** of all studies on nonlinear inverse problems which is to identify class of solutions pertaining to a new physics. This was done with the discovery of solitons and other coherent structures. It goes very deep, since it suggests a nonlinear processing of signals that propagate according to an integrable evolution equation. We are at the very beginning of nonlinear studies and certainly other successes can be expected in this direction.[18].

As a last but not least tutorial, never forget that modelling and solving inverse problems are linked studies and that they require interdisciplinary concepts and relations !

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