Asymptotic methods for the random Schrödinger equation and applications to time reversal

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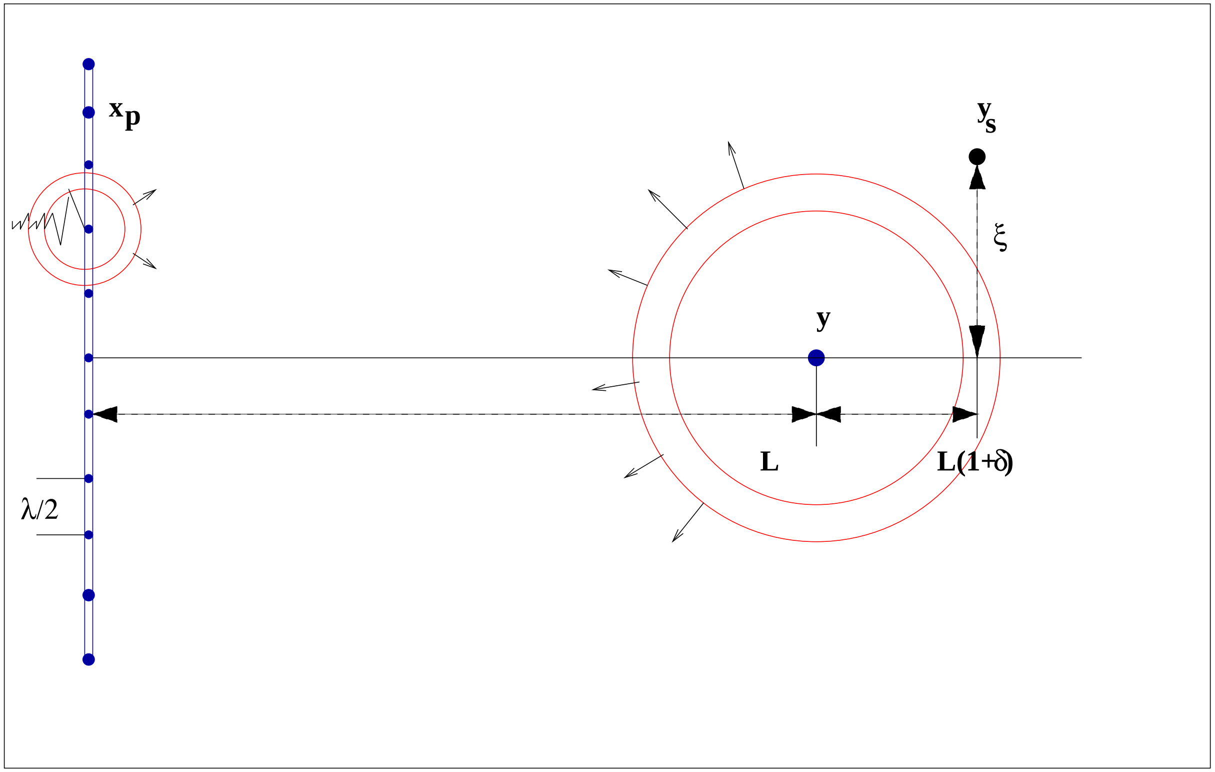
Time Reversal Schematic



Range: L, Carrier wavelength λ, Array size $a = (N-1)\lambda/2$.
Source at y, Search point at y_s, Transducers at x_p.
Remote sensing regime: $\lambda << a << L$.
Random medium: Correlation length $l << L$, fluctuation strength $\sigma << 1$.

Outline

- Resolution in time reversal: $\frac{\lambda L}{a}$, cross-range. It is the same as the Rayleigh resolution of optical instruments

- Super-resolution in random media because of multiple scattering: $\lambda L / a_e$, cross-range. The effective aperture a_e can be much larger that the physical aperture a. In random media, resolution is better than the diffraction limit

- Statistical stability (self-averaging) of time-reversed and back-propagated field. Broad-band and narrow-band signals. Super-resolution is observed only in regimes where there is statistical stability

- Some applications to imaging and communications

The reduced wave equation

The field satisfying the wave equation is written as

$$\int e^{ik(z-c_0 t)}\psi(z,x,k)dk$$

where the complex amplitude ψ solves the "relative" Helmholtz equation

$$\psi_{zz} + 2ik\psi_z + \Delta_x\psi + k^2(n^2-1)\psi = 0$$

Here, z is the coordinate in the direction of propagation (range), $x \in R^2$ are the transverse coordinates (cross-range), $k = \omega/c_0$ is the wave number, $n(z,x) = c_0/c(z,x)$ is the index of refraction with c_0 a reference speed, and

$$n^2(z,x) - 1 = \sigma\mu(\frac{x}{l},\frac{z}{l})$$

The dimensionless random fluctuations $\mu(x,z)$ are stationary with mean $E\{\mu\} = 0$ and covariance $R(z,x) = E\{\mu(z{+}z',x{+}x')\mu(z',x')\}$.

Range: z by L_z, with $L_z = L$ usually
Cross-range: x by L_x, with $L_x = a$ often
Wave number: k by $k_0 = \omega_0 / c_0$, with ω_0 the central frequency

Dimensionless parameters

$$\epsilon = \frac{l}{L_z}\ , \quad \delta = \frac{l}{L_x}\ , \quad \gamma = \frac{1}{k_0 l}$$

Scaled and dimensionless Helmholtz equation

$$\frac{\epsilon^2 \theta^2}{\delta^2} \psi_{zz} + 2ik\theta\psi_z + \theta^2 \Delta_x \psi + \frac{k^2 \delta}{\sqrt{\epsilon}} \mu(\frac{x}{\delta}, \frac{z}{\epsilon})\psi = 0$$

Here,

$$\theta = \frac{L_z}{k_0 L_x^2} = \frac{\gamma \delta^2}{\epsilon} = \text{Fresnel number}$$

and the STD of the fluctuations is chosen so that $\epsilon = \sigma^{2/3} \delta^{2/3}$.

Scaling II

- White noise+Paraxial limit: $\epsilon \to 0$. Effectively defines the remote-sensing regime.

- High frequency limit: $\theta \to 0$. Appropriate in the ladar regime and elsewhere. But often $\theta = O(1)$.

- Broad beam limit: $\delta \to 0$. Transverse dimensions of the medium are wide enough for multipathing.

- Many other limits are possible and important in various applications.

The scattering problem for the scaled relative Helmholtz equation defines a stochastic process that, under suitable hypotheses on the fluctuations μ, converges weakly to the process defined by the Ito-Schrödinger equation

$$2ik\theta d_z\psi + \theta^2\Delta_x\psi dz + \frac{ik^3\delta^2}{4\theta}R_0(0)\psi dz + k^2\delta\psi d_z B(\frac{x}{\delta}, z) = 0$$

where $B(x,z)$ is a Brownian field, that is, a Guassian process with mean zero and covariance

$$E\{B(x,z_1)B(y,z_2)\} = R_0(x-y)\min\{z_1,z_2\}\ , \quad R_0(x) = \int_{-\infty}^{\infty} R(s,x)ds$$

The scattering problem becomes an initial-value problem with $\psi(0,x,k) =$ given. If there is no ψ_{zz} term this is the well-known white-noise limit, motivated by the central limit theorem

$$B^\epsilon(x,z) = \frac{1}{\sqrt{\epsilon}}\int_0^z \mu(x,\frac{s}{\epsilon})ds \to B(x,z)\ , \quad \text{weakly as a random field}$$

On the plane of the source, at a point with transverse coordinates ξ, the time time harmonic field is

$$\psi^B(L,\xi,k) = \int G_\theta(L,x,\xi;k)\overline{G_\theta(L,\eta,x;k)\psi_0(\eta,k)}\chi_A(x)dxd\eta$$

where G_θ is the (random) Green's function. In the time domain it is

$$\Psi^B(L,\xi,t) = \int e^{-i\omega t}\psi^B(L,\xi,\frac{\omega}{c_0})d\omega$$

Because of the form of this field, and for many other reasons, we introduce and use the Wigner distribution of ψ

$$W_\theta(z,x,p) = \int \frac{dy}{(2\pi)^2}e^{ip\cdot y}\psi(z,x-\frac{\theta y}{2},k)\overline{\psi(z,x+\frac{\theta y}{2},k)}$$

and note that ψ^B can be written entirely in terms of W_θ.

High frequency limit $\theta \to 0$

The Wigner distribution satisfies a linear stochastic equation, the Ito-Wigner equation, that comes from the Ito-Schrödinger equation using the Ito calculus. In the high frequency limit the Wigner process converges weakly to the solution of the Ito-Liouville equation

$$d_z W + (\frac{p}{k} \cdot \nabla_x W - \frac{k^2 D}{2} \Delta_p W) dz + \frac{k}{2} \nabla_p W \cdot \nabla_x d_z B(\frac{x}{\delta}, z) = 0$$

where $D = -R_0''(0)/4$ and the wave number scales out: $W = W(z, x, p/k; k = 1)$. The expected value $E\{W\}$ solves the PDE

$$W_z + \frac{p}{k} \cdot \nabla_x W - \frac{k^2 D}{2} \Delta_p W = 0$$

with given initial conditions $W(0, x, p; k)$.

The process W depends on δ but $E\{W\}$ does not.

The mean of the time-reversed, back-propagated field

If we take a source field that is a directed beam

$$e^{ip_0 \cdot x/\theta}\psi_0(\frac{x}{\sigma_s}, k),$$

with σ_s the lateral extent of the source, then in the white-noise ($\epsilon \to 0$) and high-frequency ($\theta \to 0$) limits we have

$$E\{\psi^B(L,\xi,k)\} = \psi_0(\cdot,-k) * \mathcal{W}(\cdot)(\xi)$$

where $\mathcal{W}$ is the point spread function

$$\mathcal{W}(\eta) = \left(\frac{k}{2\pi L}\right)^2 \widehat{\chi}_A(\frac{\eta k}{L})e^{-\eta^2/(2\sigma_M^2)}$$

and

$$\sigma_M = \frac{L}{ka_e}\ , \quad a_e = \sqrt{\frac{DL^3}{3}}$$

Here $a_e = a_e(L)$ is the effective aperture of the array.

Interpretation of the point spread function

If there is no scattering medium then $D = 0$ and

$$\mathcal{W}(\eta) = \left(\frac{k}{2\pi L}\right)^2 \widehat{\chi}_A(\frac{\eta k}{L})$$

For a square aperture $A = [-\frac{a}{2}, \frac{a}{2}]^2$

$$\mathcal{W}(\eta) = \mathcal{W}(\eta_1, \eta_2) = \frac{1}{\pi^2 \eta_1 \eta_2} \sin(\frac{\eta_1 ka}{2L}) \sin(\frac{\eta_2 ka}{2L})$$

The first zero of the sine function is at

$$\eta_F = \frac{2\pi L}{ka} = \frac{\lambda L}{a} = \text{Rayleigh resolution}$$

If we define $\sigma_F = L/ka$, the Fresnel spot size, then when $\sigma_F << \sigma_M$, or $a >> a_e$, multipathing does not alter the refocused spot size of diffraction theory.

But if $a_e >> a$ then the point spread function is

$$\mathcal{W} \approx \left(\frac{a}{\sqrt{2\pi} a_e}\right)^2 \frac{e^{-\eta^2/(2\sigma_M^2)}}{2\pi\sigma_M^2}$$

Self-averaging

When is the time-reversed, back-propagated field self-averaging? This is a fundamental issue because it determines when super-resolution is observable.

In the present setting there are two results:

- If the source is localized, $\sigma_s \sim \theta$, then, in the limit $\delta \to 0$, the time harmonic field ψ^B is self-averaging

$$\lim_{\delta \to 0} E\{(\psi^B - E\{\psi^B\})^2\} = 0$$

- If the source is distributed, $\sigma_s >> \theta$, then only in the time domain, that is for $\Psi^B(L, \xi, t)$, we have self-averaging in mean square sense as $\delta \to 0$.

- What does $\delta \to 0$ mean? Provides cross-range diversity in multipathing.

The self-averaging is based on the following theorem for the Ito-Liouville process (with $k = 1$) defined by

$$d_z W + (p \cdot \nabla_x W - \frac{D}{2} \Delta_p W) dz + \frac{1}{2} \nabla_p W \cdot \nabla_x d_z B(\frac{x}{\delta}, z) = 0$$

with $W(0, x, p) = \chi_A(x)$:

For any $z > 0$ the integral

$$J_\delta(z, x) = \int W_\delta(z, x, p) dp$$

exists and

$$\lim_{\delta \to 0} E\{(J_\delta - E\{J_\delta\})^2\} = 0$$

where $E\{J_\delta\}$ is independent of δ.

This is proved by using properties of the SDE's (random characteristics) through which the Ito-Liouville equation can be solved.

In the time domain and for a distributed source, the self-averaging field, in the white-noise and high-frequency limit, is given by

$$\begin{aligned}\Psi^B(L,\xi,t) &= e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi)\\ &\cdot\int_{\{|\omega|<\Omega\}}\frac{d\omega}{2\pi}e^{-i\omega t}\widehat{g}(-\omega)\ \chi_A * \left(\frac{e^{-x^2/2a_e^2}}{2\pi a_e^2}\right)\left(\frac{Lc_0p_0}{\omega_0+\omega}\right)\end{aligned}$$

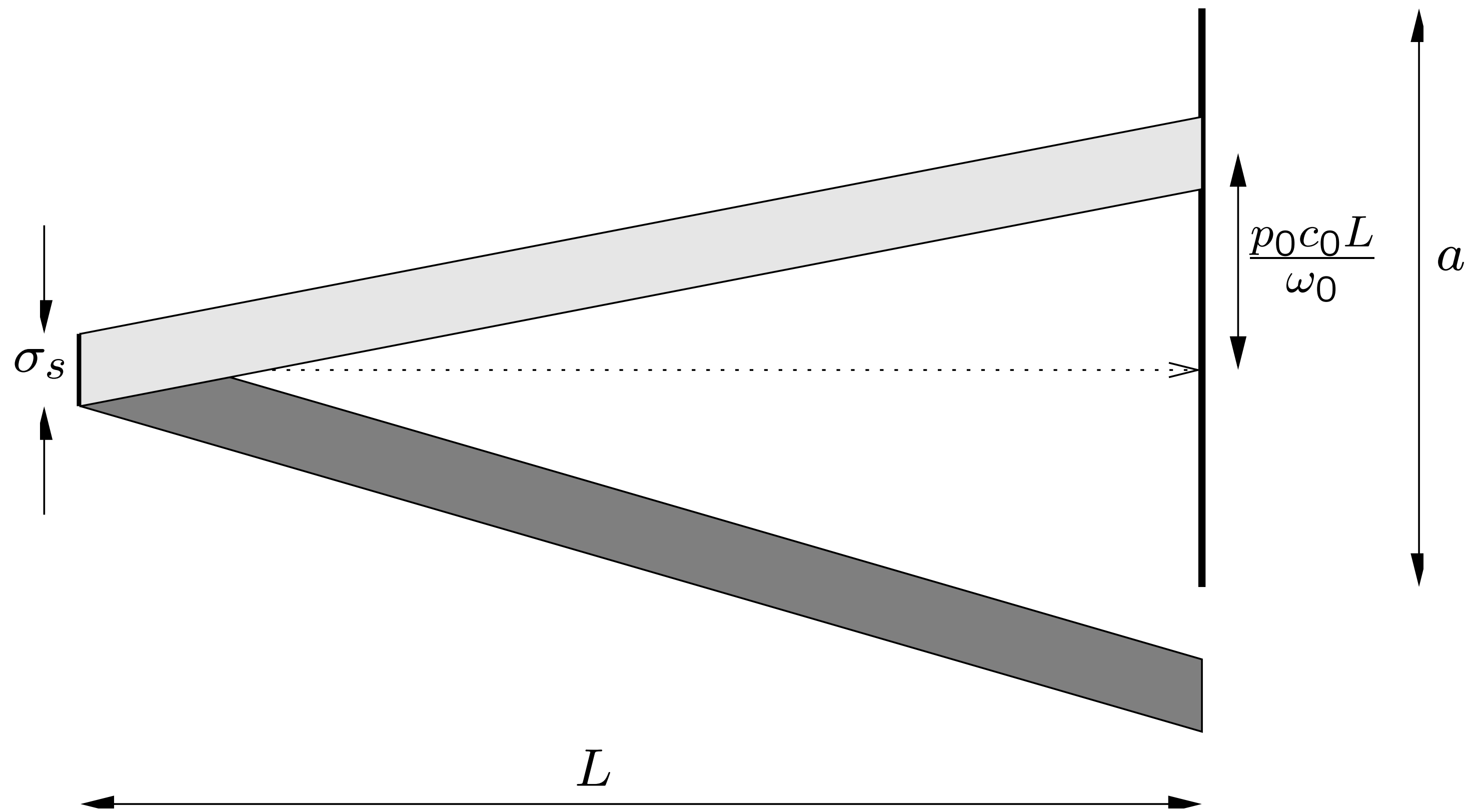
When $a_e << a$, that is, no multipathing, then

$$\begin{aligned}\Psi^B(L,\xi,t) &\sim e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi)\\ &\cdot\int_{\{|\omega|<\Omega\}}\frac{d\omega}{2\pi}e^{-i\omega t}\widehat{g}(-\omega)\ \chi_A\left(\frac{Lc_0p_0}{\omega_0+\omega}\right)\end{aligned}$$

In this case, if the beam lands entirely withing the TRM then the time-reversed and back-propagated pulse is

$$e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi)g(-t)$$

Time-reversed, back-propagated pulse schematic



A directed field propagates from a distributed source of size σ_s toward the time reversal mirror of size a. The time-reversed, back-propagated field depends on the location of the mirror relative to the direction of the propagating beam.

When multipathing is strong, $a_e >> a$, then the self-averaging time-reversed and back-propagated pulse is given by

$$\Psi^B(L,\xi,t) \sim e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \left(\frac{a}{\sqrt{2\pi}a_e}\right)^2 \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\ e^{-\frac{1}{2}\left(\frac{Lc_0p_0}{a_e(\omega_0+\omega)}\right)^2}$$

Note that, remarkably, this expression **is almost independent of the time reversal mirror**!

Use this formula to estimate the most important quantity in time reversal with strong multipathing: the effective aperture a_e.

Point the beam in different directions toward the TRM, measure the time reversed pulse and estimate a_e by fitting to the formula.

Summary and conclusions

- Time reversal in a random medium is important because of super-resolution and self-averaging, which are phenomena that are difficult to analyze and understand quantitatively, and require new and interesting mathematics.

- Applications abound, are enormously exciting and limited only by the **hardware**, our imagination, and also our analytical understanding: Direct TR applications, Imaging, Communications.

- Current research: the ergodic (asymptotic) theory of SPDE (like the $\delta \to 0$ limit), array imaging that uses self-averaging to minimize the effect of clutter, and communications in rich multipathing environments using TR protocols.