Random maps and continuum random 2-dimensional geometries

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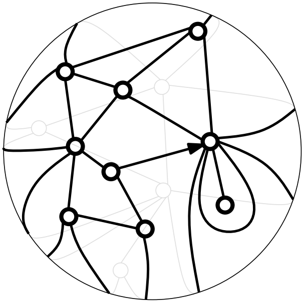
Plane maps

Definition

A plane map is an embedding of a connected, finite (multi)graph into the 2-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere.



A rooted map: distinguish one oriented edge.

- $V(\mathbf{m})$ Vertices
- $E(\mathbf{m})$ Edges
- $F(\mathbf{m})$ Faces
- $d_{\mathbf{m}}(u, v)$ combinatorial graph distance
- The degree of a face is the number of incident corners
- A p-angulation has only faces of degree p.

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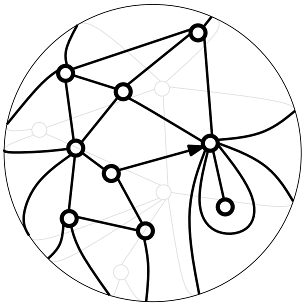

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Motivation

- Maps are seen as discretized 2D Riemannian manifolds.
- This comes from 2D quantum gravity, in which a basic object is the partition function

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \exp(-\alpha \,\mathrm{Area}_g(M))$$

  - M is a 2-dimensional orientable manifold,
  - $\mathcal{R}(M)$ is the space of Riemannian metrics on M,
  - $\mathrm{Diff}^+(M)$ the set of orientation-preserving diffeomorphisms,
  - $\mathcal{D}g$ is a "Lebesgue" measure on $\mathcal{R}(M)$ invariant under the action of $\mathrm{Diff}^+(M)$. This, and the induced measure $[\mathcal{D}g]$, are the problematic objects.

How to deal with $[\mathcal{D}g]$?

One can replace

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \longrightarrow \sum_{T \in \mathbf{Tr}(M)} \delta_T$$

where $\mathbf{Tr}(M)$ is the set of triangulations of M.

- Then one tries to take a scaling limit of the right-hand side, in which triangulations approximate a “smooth”, continuum surface.
- Analog to path integrals, in which random walks can be used to approximate Brownian motion.
- The success of this approach comes from the rich literature on enumerative theory of maps, after Tutte’s work or the literature on matrix integrals.
- However, metric aspects of maps could only be dealt with recently, using bijective approaches.

The quantum Liouville theory approach

- Another approach is quantum Liouville theory (Polyakov, David, ...).
- Representing $g = e^{2u} h^* g_0(\mu)$ where $u : M \to \mathbb{R}$ (conformal factor) and $g_0(\mu)$ is a metric representing its complex structure: such are parameterized by the moduli μ, finally, $h \in \mathrm{Diff}^+(M)$.
- By performing the formal change of variables $g \mapsto (u, h, \mu)$ one can argue that u should be a Gaussian free field (a random distribution).
- This idea has been developed recently by Duplantier-Sheffield and has spawned interest in the study of random measures of the form $e^{\gamma u - \gamma^2 \langle u^2 \rangle / 2} \mathrm{d}z$, which can be constructed using Kahane's Gaussian multiplicative chaos theory.
- At present, the links between the two approaches are tenuous and far from being well-understood.

Some simulations

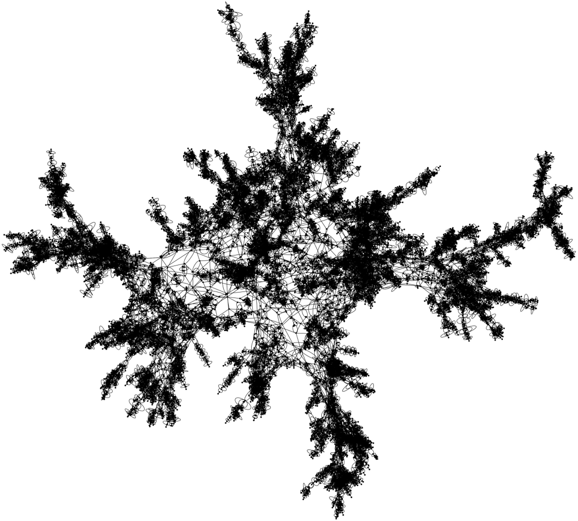

Figure: A random quadrangulation with 30000 vertices, by J.-F. Marckert

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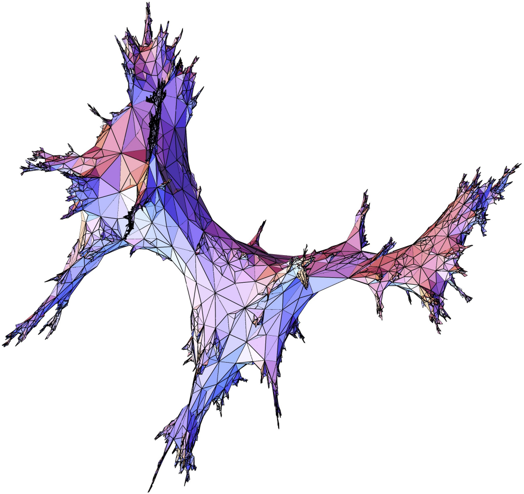
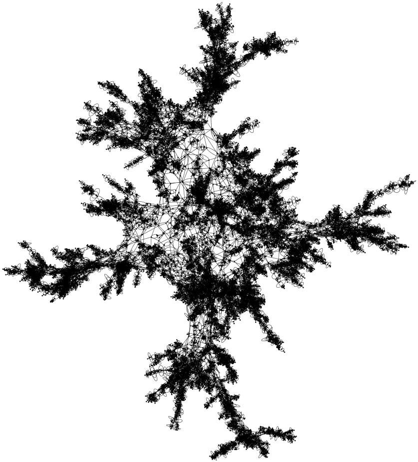


Figure: A random quadrangulation by N. Curien

Simulation of a uniform random plane quadrangulation with 30000 vertices, by J.-F. Marckert



- Q_n uniform random variable in the set $\mathbf{Q}_n$, of rooted plane quadrangulations with n faces.
- The set $V(Q_n)$ of its vertices is endowed with the graph distance d_{Q_n}.
- Typically $d_{Q_n}(u, v)$ scales like $n^{1/4}$ (Chassaing-Schaeffer (2004)).

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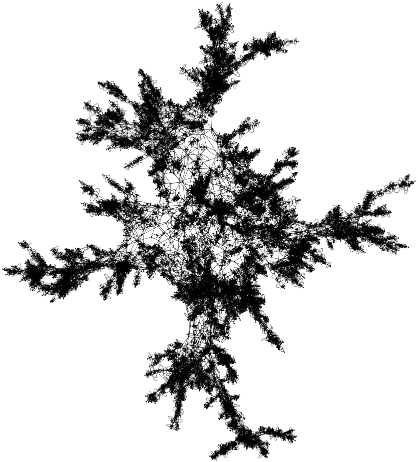

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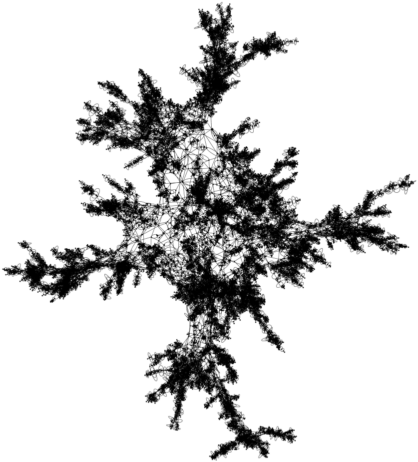

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Convergence to the Brownian map

Theorem

There exists a random metric space (S, D), called the Brownian map, such that the following convergence in distribution holds

$$(V(Q_n), (8n/9)^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} (S, D)$$

as $n \to \infty$, for the Gromov-Hausdorff topology.

- $X_n \to X$ for the Gromov-Hausdorff topology if $X'_n \to X'$ in the Hausdorff sense for suitable isometric embeddings of X_n, X in a common space.
- This result has been proved independently by Le Gall (2011) and Miermont (2011), *via* different approaches. Also universality results in Le Gall (2011).
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Some of the previous results on scaling limits for random quadrangulations

- Chassaing-Schaeffer (2004)
  - identify $n^{1/4}$ as the proper scaling and
  - compute limiting functionals for random quadrangulations, including the two-point function.
- Marckert-Mokkadem (2006) introduce the Brownian map.
- Le Gall (2007)
  - Gromov-Hausdorff tightness for rescaled $2p$-angulations
  - the limiting topology is the same as that of the Brownian map.
  - all subsequential limits have Hausdorff dimension 4
- Le Gall-Paulin (2008), and later M. (2008) show that the limiting topology is that of the 2-sphere.
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The Cori-Vauquelin-Schaeffer bijection: coding maps with trees

- Let $\mathbf{T}_n$ be the set of rooted plane trees with n edges,
- $\mathbb{T}_n$ be the set of labeled trees $(\mathbf{t}, \mathbf{l})$ where $\mathbf{l} : V(\mathbf{t}) \to \mathbb{Z}$ satisfies $\mathbf{l}(\mathrm{root}) = 0$ and

$$|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1\,, \qquad u, v \text{ neighbors}\,.$$

Theorem (Cori-Vauquelin 1981, Schaeffer)

The construction to follow yields a bijection between $\mathbb{T}_n \times \{0, 1\}$ and $\mathbf{Q}_n^*$, the set of rooted, pointed plane quadrangulations with n faces.

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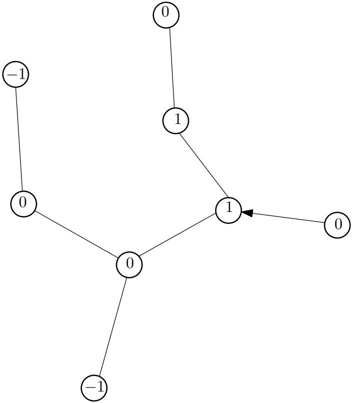

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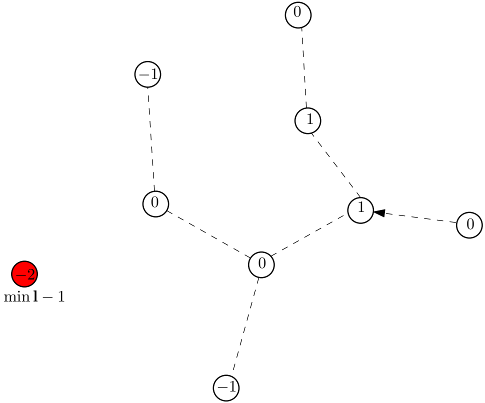

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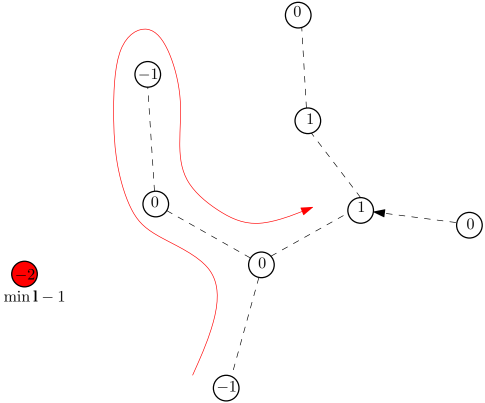

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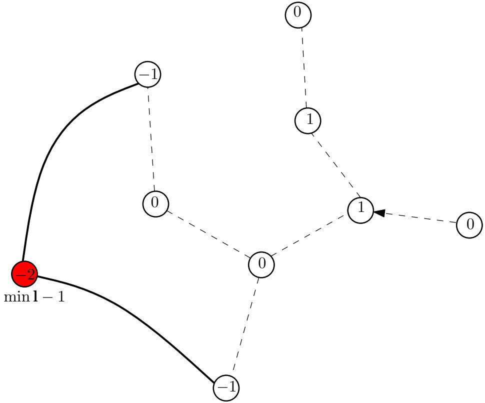

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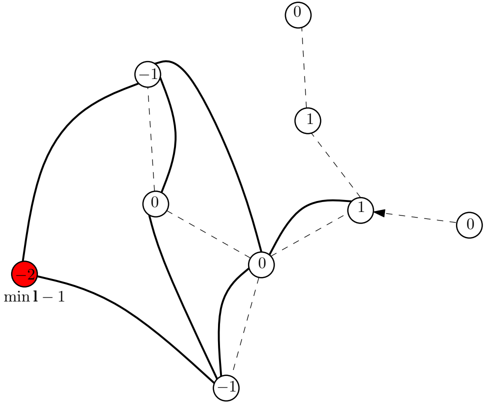

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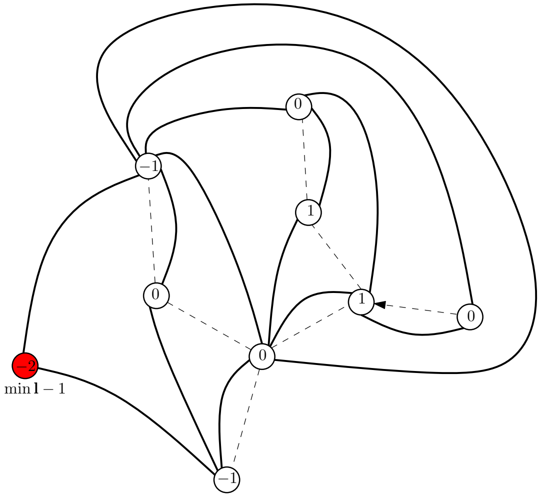

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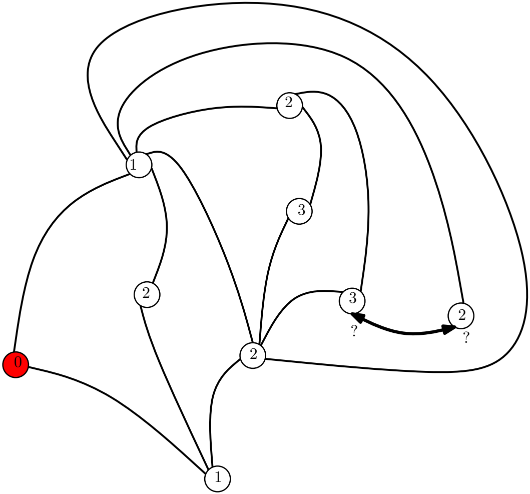


Note that the labels are geodesic distances in the map. Key formula:

$$d_{\mathbf{q}}(v_*, v) = \mathbf{l}(v) - \inf \mathbf{l} + 1$$

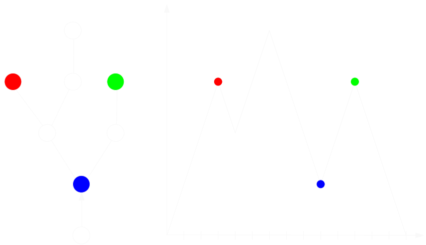
Scaling limits for plane trees: Aldous' CRT

- The Brownian tree arises as the scaling limit of many discrete random tree models, e.g. uniform random element T_n of $\mathbf{T}_n$:

$$(V(T_n), (2n)^{-1/2} d_{T_n}) \to \mathcal{T} \,,$$

for the Gromov-Hausdorff distance.

- Note that a tree with n edges can be encoded by a walk (Harris encoding): let $u_i, 0 \leq i \leq 2n$ be the $i+1$-th explored vertex in contour order (started at the root). Let C_i the height of u_i.



The Harris walk is a random walk conditioned to be non-negative and to be at 0 at time $2n$.

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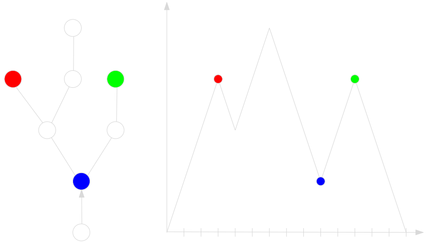

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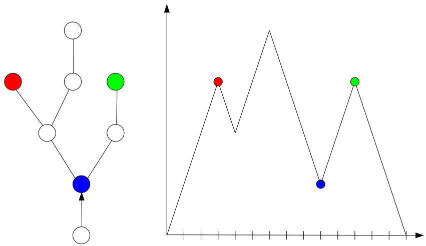

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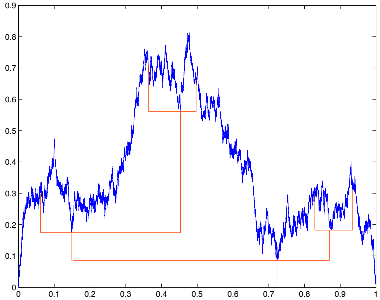
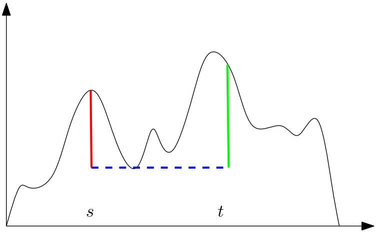
The Brownian CRT

- Let T_n be uniform in $\mathbf{T}_n$, and C^n be its contour process. As $n \to \infty$, the process $((2n)^{-1/2} C^n_{[2nt]}, 0 \leq t \leq 1)$ converges in distribution to a normalized Brownian excursion $(\mathbb{e}_t, 0 \leq t \leq 1)$.
- Define

$$d_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2 \inf_{s \wedge t \leq u \leq s \vee t} \mathbb{e}_u .$$

This is a pseudo-distance on $[0,1]$. The continuum random tree is the quotient space $\mathcal{T}_{\mathbb{e}} = [0,1]/\sim_{\mathbb{e}}$, where $s \sim t \iff d_{\mathbb{e}}(s,t) = 0$. It defines an $\mathbb{R}$-tree.





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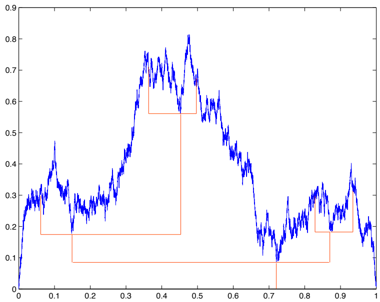

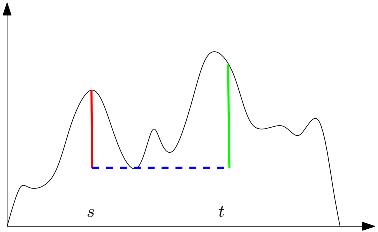

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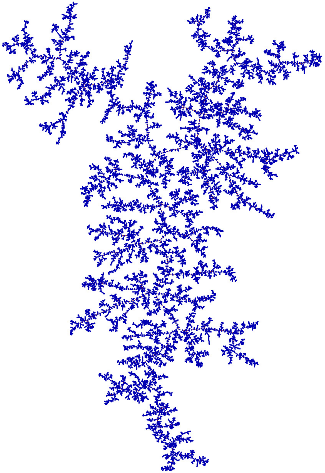

Figure: Picture by I. Kortchemski

Brownian labels on the Brownian tree

- Once the tree is build, one can consider a white noise supported by the tree, or, equivalently, branching Brownian paths.
- Informally, we let Z be a centered Gaussian process run on $\mathcal{T}_{\mathbb{e}}$, with covariance function

$$\mathrm{Cov}\left(Z_a, Z_b\right) = d_{\mathcal{T}_{\mathbb{e}}}(\mathrm{root}, a \wedge b)\,,$$

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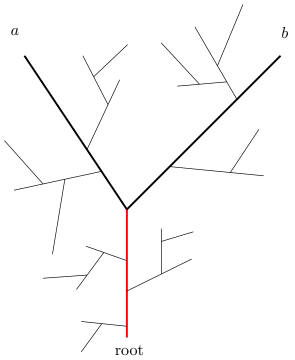

Convergence of labeled trees and Brownian map

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$$\left(\frac{1}{\sqrt{2n}} T_n, \left(\frac{9}{8n}\right)^{1/4} \ell_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_{\mathbb{e}}, Z)\,,$$

  e.g. in the sense of convergence of contour encoding functions.
- We want to apply to $(\mathcal{T}_{\mathbb{e}}, Z)$ a similar construction as the CVS bijection.
- The Brownian map is a quotient of $\mathcal{T}_{\mathbb{e}}$ by the equivalence relation generated by

$$\{(a,b) : Z_a = Z_b = \min_{[a,b]} Z\}\,,$$

  where $[a,b]$ is the interval from a to b around $\mathcal{T}_{\mathbb{e}}$.
- The resulting quotient set is endowed with a distance D such that

$$D(a,a^*) = Z_a - \inf Z$$

  if $a \in \mathcal{T}_{\mathbb{e}}$ and $a^* = \mathrm{argmin}(Z)$. Hard: Other distances $D(a,b), a,b \neq a_*$. Needs to control the geometry of geodesics.

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$$\{(a,b) : Z_a = Z_b = \min_{[a,b]} Z\}\,,$$

  where $[a,b]$ is the interval from a to b around $\mathcal{T}_{\mathbb{e}}$.
- The resulting quotient set is endowed with a distance D such that

$$D(a, a^*) = Z_a - \inf Z$$

  if $a \in \mathcal{T}_{\mathbb{e}}$ and $a^* = \mathrm{argmin}(Z)$. Hard: Other distances $D(a,b), a, b \neq a_*$. Needs to control the geometry of geodesics.

Convergence of labeled trees and Brownian map

- Let (T^n, ℓ^n) be uniform in $\mathbb{T}_n$. Then

$$\left(\frac{1}{\sqrt{2n}} T_n, \left(\frac{9}{8n}\right)^{1/4} \ell_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_{\mathbb{e}}, Z),$$

  e.g. in the sense of convergence of contour encoding functions.
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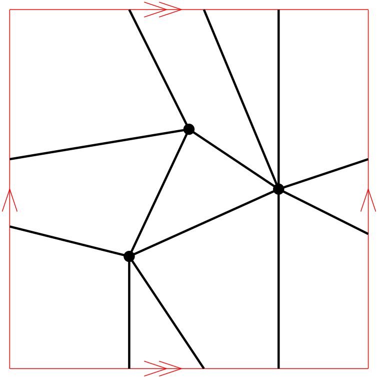
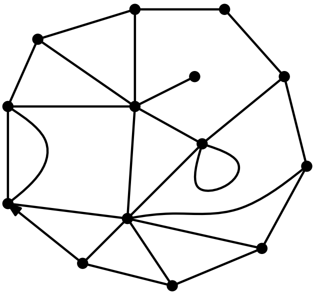
Other topologies

Definition

A map on an (orientable) surface *M* is an embedding of a locally finite graph on *M*, that dissects the latter into topological polygons, and considered up to direct homeomorphisms of *M*.



On the torus



On a disk

Other topologies

- Let $\mathbf{Q}_n^{(g)}$ be the set of rooted, bipartite quadrangulations of the g-torus $\mathbb{T}_g$.
- Let $\mathbf{Q}_{n,m}^{\partial}$ be the set of rooted plane maps, with all faces of degree 4 except the root face, that has degree $2m$.

In ongoing joint work with Jérémie Bettinelli, we show:

Theorem

Let Q_n be uniform in $\mathbf{Q}_n^{(g)}$. Then

$$(V(Q_n), n^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} M^{(g)}\,.$$

If Q_n is uniform in $\mathbf{Q}_{n,m}^{\partial}$ and $m = m(n)$ satisfies $m/\sqrt{n} \to \lambda \in (0, \infty)$, then

$$(V(Q_n), n^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} M_\lambda^{\partial}\,.$$

The convergences hold for the Gromov-Hausdorff topology.

Other topologies

$$\left(V(Q_n^{(g)}), n^{-1/4} d_{Q_n}\right) \xrightarrow[n\to\infty]{(d)} M^{(g)} .$$

$$\left(V(Q_{n,m}^{\partial}), n^{-1/4} d_{Q_n}\right) \xrightarrow[n\to\infty]{(d)} M_\lambda^{\partial} .$$

Previous results by Bettinelli (2011, 2012) show that a.s.,

- $M^{(g)}$ is homeomorphic to $\mathbb{T}_g$ and M_λ^∂ is homeomorphic to the unit disk.
- Moreover, $\dim(M^{(g)}) = \dim(M_\lambda^\partial) = 4$ and $\dim(\partial M_\lambda^\partial) = 2$.

For the case $\lambda \in \{0, \infty\}$:

Theorem (Bettinelli, 2012)

- *If $m \ll \sqrt{n}$, the space $n^{-1/4} Q_{n,m}^\partial$ converges to the Brownian map.*
- *If $\sqrt{n} \ll m$, the space $m^{-1/2} Q_{n,m}^\partial$ converges to the Brownian continuum random tree*

Other natural random families of maps: Boltzmann random maps

- We only consider bipartite plane maps (with faces of even degree), mostly for technical simplicity.
- Boltzmann distribution: let $w = (w_k, k \geq 1)$ be a non-negative non-zero sequence, $w_1 < 1$. Define a measure by

$$\mathbb{B}_w(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} w_{\deg(f)/2}\,, \qquad \mathbf{m} \text{ rooted, bipartite}$$

- Let

$$\mathbb{B}_w^n(\cdot) = \mathbb{B}_w\Big(\cdot \,\Big|\, \{\mathbf{m} \text{ with } n \text{ vertices}\}\Big)\,,$$

defining a probability measure. Uniform on $2p$-angulations with n vertices if $w_k = \delta_{kp}$.

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Brownian map and stable maps universality classes

For Boltzmann maps $\mathbb{B}_w^n$ sampled accoding to "generic" sequences of weights, the Brownian map still prevails in the limit.

Theorem (Universality of the Brownian map, Le Gall 2011)

If $(w_k, k \geq 2)$ is a weight sequence with finite support, then if M_n has law $\mathbb{B}_w^n$, there exists a constant b_w such that $(V(M_n), b_w n^{-1/4} d_{M_n})$ converges in distribution to the Brownian map.

But under certain conditions called non-generic, implying in particular that $w_k \sim C\rho^k k^{-a}$ for some $a \in (3/2, 5/2)$, the limit is different.

Theorem (Le Gall-Miermont 2009)

For non-generic weights, if M_n has law $\mathbb{B}_w^n$, the sequence $(V(M_n), n^{-1/(2a-1)} d_{M_n})$ converges in distribution, at least along some extraction, to a random metric space (S_a, d_a) with Hausdorff dimension a.s. equal to $2a - 1$, the stable map with index a.

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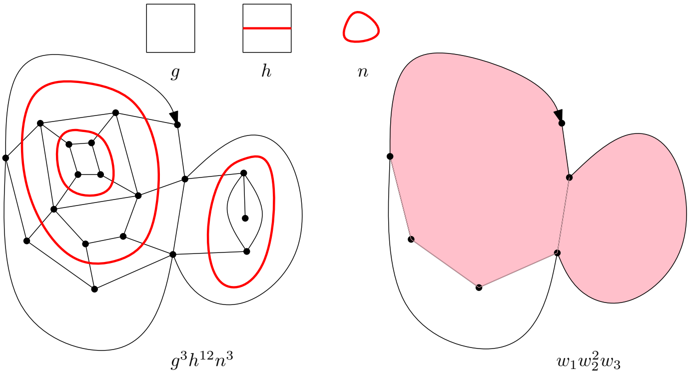
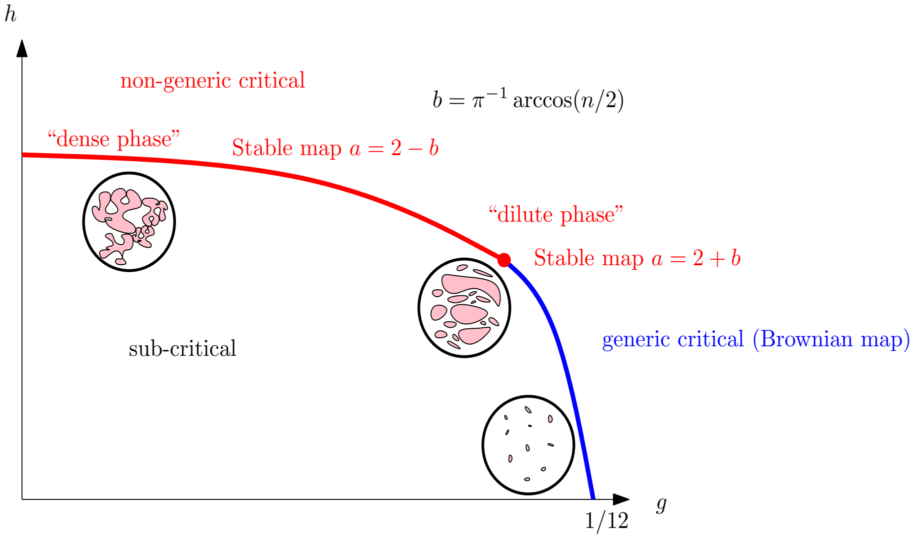
Non-genericity in gaskets of loop models



- Decorate a quadrangulation with simple and mutually avoiding dual loops: weight $W^{(n)}_{g,h}(\mathbf{q}) = g^{\#quad} h^{|loops|} n^{\#loops}$
- Emptying the interior of the loops, gives a Boltzmann random map

$$w_k = g\delta_{k2} + nh^{2k} \sum_{|\partial \mathbf{q}|=2k} W^{(n)}_{g,h}(\mathbf{q}) \,.$$

Phase diagram at fixed $n \in (0, 2]$ [Borot-Bouttier-Guitter 2011]



Some future directions and open questions

- Most important question: can one reconcile random maps and Liouville quantum gravity as different approches to the same object?
- What is the asymptotic law of the moduli of a random uniform triangulation of $\mathbb{T}_g$? Can it be recovered from $M^{(g)}$?
- Is there a “canonical embedding” of the Brownian map in $\mathbb{S}^2$?
- Topology of stable maps (in progress):
  - if $a \in [2, 5/2)$ then (S_a, d_a) is a random Sierpinsky carpet (holes have simple, mutually avoiding boundaries).
  - if $a \in [2, 5/2)$ then holes have self and mutual intersections.
- This phenomenon recalls the phases of Schramm-Loewner Evolutions and Conformal Loop Ensembles (Sheffield-Werner). This is no coincidence, as CLEs are the conjectured limits of $O(n)$ loop models on regular lattices.
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